

Computing knot Floer homology in cyclic branched covers

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We use grid diagrams to give a combinatorial algorithm for computing the knot Floer homology of the pullback of a knot $K \subset S^3$ in its m -fold cyclic branched cover $\Sigma_m(K)$, and we give computations when $m = 2$ for over fifty three-bridge knots with up to eleven crossings.

57R58; 57M12, 57M27

1 Introduction

Heegaard Floer knot homology, developed by Ozsváth and Szabó [15] and independently by Rasmussen [18], associates to a nulhomologous knot K in a three-manifold Y a group $\widehat{\text{HF}}K(Y, K)$ that is an invariant of the knot type of K . If K is a knot in S^3 , then the inverse image of K in $\Sigma_m(K)$, the m -fold cyclic branched cover of S^3 branched along K , is a nulhomologous knot \tilde{K} whose knot type depends only on the knot type of K , so the group $\widehat{\text{HF}}K(\Sigma_m(K), \tilde{K})$ is a knot invariant of K . In this paper, we describe an algorithm that can compute $\widehat{\text{HF}}K(\Sigma_m(K), \tilde{K})$ (with coefficients in $\mathbb{Z}/2$) for any knot $K \subset S^3$, and we give computations for a large collection of knots with up to eleven crossings.

Any knot $K \subset S^3$ can be represented by means of a *grid diagram*, consisting of an $n \times n$ grid in which the centers of certain squares are marked X or O , such that each row and each column contains exactly one X and one O . To recover a knot projection, draw an arc from the X to the O in each column and from the O to the X in each row, making the vertical strand pass over the horizontal strand at each crossing. We may view the diagram as lying on a standardly embedded torus $T^2 \subset S^3$ by making the standard edge identifications; the horizontal grid lines become α circles and the vertical ones β circles. Manolescu, Ozsváth, and Sarkar [12] showed that such diagrams can be used to compute $\widehat{\text{HF}}K(S^3, K)$ combinatorially; we shall use them to give a combinatorial description of the chain complex for $\widehat{\text{HF}}K(\Sigma_m(K), \tilde{K})$ for any knot $K \subset S^3$.

Let $m \geq 2$ and let \tilde{T} be the surface obtained by gluing together m copies of T (denoted T_0, \dots, T_{m-1}) along branch cuts connecting the X and the O in each

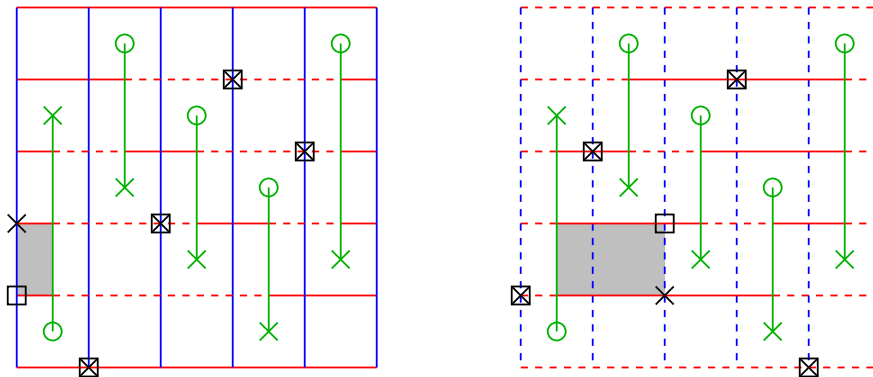


Figure 1: Heegaard diagram $\tilde{D} = (\tilde{T}, \tilde{\alpha}, \tilde{\beta}, \tilde{w}, \tilde{z})$ for $(\Sigma_2(K), \tilde{K})$, where K is the right-handed trefoil. The solid and dashed lines represent different lifts of the α (horizontal/red) and β (vertical/blue) circles. The black squares and crosses represent two generators of $\tilde{C} = \text{CFK}(\tilde{D})$, and the shaded region is a disk that contributes to the differential.

column. Specifically, in each column, if the X is above the O , then glue the left side of the branch cut in T_k to the right side of the same cut in T_{k+1} (indices modulo m); if the O is above the X , then glue the left side of the branch cut in T_k to the right side of the same cut in T_{k-1} . The obvious projection $\pi: \tilde{T} \rightarrow T$ is an m -fold cyclic branched cover, branched around the marked points. Each α and β circle in T intersects the branch cuts a total of zero times algebraically and therefore has m distinct lifts to \tilde{T} , and each lift of each α circle intersects exactly one lift of each β circle. (We will describe these intersections more explicitly in Section 4.)

Denote by \mathcal{R} the set of embedded rectangles in T whose lower and upper edges are arcs of α circles, whose left and right edges are arcs of β circles, and which do not contain any marked points in their interior. Each rectangle in \mathcal{R} has m distinct lifts to \tilde{T} (possibly passing through the branch cuts as in Figure 1); denote the set of such lifts by $\tilde{\mathcal{R}}$.

Let \mathcal{S} be the set of unordered mn -tuples \mathbf{x} of intersection points between the lifts of α and β circles such that each such lift contains exactly one point of \mathbf{x} . (We will give a more explicit characterization of the elements of \mathcal{S} later.) Let C be the $\mathbb{Z}/2$ -vector space generated by \mathcal{S} . Define a differential ∂ on C by making the coefficient of \mathbf{y} in $\partial\mathbf{x}$ nonzero if and only if the following conditions hold.

- All but two of the points in \mathbf{x} are also in \mathbf{y} .

- There is a rectangle $R \in \tilde{\mathcal{R}}$ whose lower-left and upper-right corners are in \mathbf{x} , whose upper-left and lower-right corners are in \mathbf{y} , and which does not contain any X , O , or point of \mathbf{x} in its interior.

In Section 4, we shall define two gradings (Alexander and Maslov) on C , as well as a decomposition of C as a direct sum of complexes corresponding to spin^c structures on $\Sigma_m(K)$. We shall prove the following theorem.

Theorem 1.1 *The homology of the complex (C, ∂) is isomorphic as a bigraded group to $\widehat{\text{HF}}\text{K}(\Sigma_m(K), \tilde{K}; \mathbb{Z}/2) \otimes V^{\otimes n-1}$, where $V \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ with generators in bigradings $(0, 0)$ and $(-1, -1)$.*

In Section 2, we review the construction of knot Floer homology using multi-pointed Heegaard diagrams. In Section 3, we show how to obtain a Heegaard diagram for $(\Sigma_m(K), \tilde{K})$ given one for (S^3, K) , and we apply that discussion to grid diagrams in Section 4, proving Theorem 1.1. In Section 5, we give the values of $\widehat{\text{HF}}\text{K}(\Sigma_m(K), \tilde{K})$ for over fifty knots with up to eleven crossings. (Grigsby [6] has shown how to compute these groups for two-bridge knots, so our tables only include knots that are not two-bridge.) Finally, we make some observations and conjectures about these results in Section 6.

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2 Review of knot Floer homology

Let us briefly recall the basic construction of knot Floer homology using multiple basepoints (Ozsváth–Szabó [15], Manolescu–Ozsváth–Sarkar [12] and Sarkar–Wang [20]). For simplicity, we work with coefficients modulo 2. A *multi-pointed Heegaard diagram* $\mathcal{D} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ consists of an oriented surface Σ ; two sets of closed, embedded, piecewise disjoint curves $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_{g+n-1}\}$ and $\boldsymbol{\beta} = \{\beta_1, \dots, \beta_{g+n-1}\}$ (where $g = g(\Sigma)$ and $n \geq 1$), each of which spans a g -dimensional subspace of $H_1(\Sigma; \mathbb{Z})$; and two sets of basepoints, $\mathbf{w} = \{w_1, \dots, w_n\}$ and $\mathbf{z} = \{z_1, \dots, z_n\}$, such that each component of $\Sigma - \bigcup \alpha_i$ and each component of $\Sigma - \bigcup \beta_i$ contains exactly one point of \mathbf{w} and one point of \mathbf{z} . We call the components of $\Sigma - \bigcup \alpha_i - \bigcup \beta_i$ *regions* and denote them R_1, \dots, R_N . The α and β curves specify a Heegaard decomposition

$H_\alpha \cup_\Sigma H_\beta$ for a 3-manifold Y , oriented so that Σ acquires its orientation as ∂H_α . We obtain a knot or link K by connecting the w (resp. z) basepoints to the z (resp. w) basepoints with arcs in the complement of the α (resp. β) curves and push those arcs into H_α (resp. H_β). The orientations are such that K intersects Σ positively at the z basepoints and negatively at the w basepoints. In terms of Morse theory, the Heegaard diagram corresponds to a self-indexing Morse function f on Y with n critical points of index 0, $g + n - 1$ of index 1, $g + n - 1$ of index 2, and n of index 3. Given a Riemannian metric g , the knot K is given as a union of gradient flowlines connecting the index 0 and 3 critical points through the w and z basepoints. We shall always assume that the knot K is nulhomologous.

Let $\widehat{\text{CFK}}(\mathcal{D})$ be the $\mathbb{Z}/2$ -vector space generated by the intersection points between the $(g + n - 1)$ -dimensional tori $\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_{g+n-1}$ and $\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_{g+n-1}$ in the symmetric product $\text{Sym}^{g+n-1}(\Sigma)$. The differential ∂ is defined by taking counts of holomorphic disks connecting intersection points:

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu(\phi) = 1 \\ n_w(\phi) = n_z(\phi) = 0}} \#(\widehat{\mathcal{M}}(\phi)) \mathbf{y}.$$

Each homotopy class of Whitney disks $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ has an associated domain in Σ : a 2-chain $D(\phi) = \sum a_i R_i$, such that ∂D is made of arcs of α curves that connect each point of \mathbf{x} to a point of \mathbf{y} and arcs of β curves that connect each point of \mathbf{y} to a point of \mathbf{x} . Then $n_w(\phi)$ (resp. $n_z(\phi)$) is the sum of the multiplicities of the regions containing points of \mathbf{w} (resp. \mathbf{z}). The Maslov index $\mu(\phi)$ can be given by formula due to Lipshitz [10]:

$$\mu(\phi) = e(D) + p_x(D) + p_y(D),$$

where $e(D)$ is the Euler measure of D and $p_x(D)$ (resp. $p_y(D)$) is the sum, taken over all points $x \in \mathbf{x}$ (resp. $y \in Y$), of the average of the multiplicities of the four domains that come together at x (resp. y). The coefficient of \mathbf{y} represents the number of holomorphic representatives of ϕ and generally depends on the choice of almost complex structure on Σ .

Each generator \mathbf{x} has an associated spin^c structure $\mathfrak{s}_w(\mathbf{x}) \in \text{Spin}^c(Y)$, obtained by considering the gradient of a compatible Morse function outside of regular neighborhoods of flowlines through the points of \mathbf{x} and \mathbf{w} . Given two generators \mathbf{x} and \mathbf{y} , let $\gamma_{\mathbf{x}, \mathbf{y}}$ be any 1-cycle obtained by connecting \mathbf{x} to \mathbf{y} along the α circles and \mathbf{y} to \mathbf{x} along the β circles, and let $\epsilon(\mathbf{x}, \mathbf{y})$ be its image in

$$H_1(Y) \cong H_1(\Sigma) / \text{Span}([\alpha_i], [\beta_i] \mid i = 1, \dots, g + n - 1).$$

Then $\mathfrak{s}_w(\mathbf{x}) = \mathfrak{s}_w(\mathbf{y})$ if and only if $\epsilon(\mathbf{x}, \mathbf{y}) = 0$. The complex $\widehat{\text{CFK}}(\mathcal{D})$ splits as a direct sum over $\mathfrak{s} \in \text{Spin}^c(Y)$ of subcomplexes $\widehat{\text{CFK}}(\mathcal{D}, \mathfrak{s})$, each generated by those $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $\mathfrak{s}_w(\mathbf{x}) = \mathfrak{s}$.

The restriction of $\mathfrak{s}_w(\mathbf{x})$ to $Y - K$ extends uniquely to a spin^c structure $\underline{\mathfrak{s}}_{w,z}(\mathbf{x})$ on the zero-surgery $Y_0(K)$. Given a Seifert surface F for K , we define the *Alexander grading* of \mathbf{x} (relative to F) as $A(\mathbf{x}) = \frac{1}{2} \left\langle c_1(\underline{\mathfrak{s}}_{w,z}(\mathbf{x})), [\widehat{F}] \right\rangle$, where \widehat{F} is an extension of F to $Y_0(K)$. This quantity is independent of the choice of F up to an additive constant, and it is completely well-defined if Y is a rational homology sphere. The relative Alexander grading between two generators \mathbf{x} and \mathbf{y} , $A(\mathbf{x}, \mathbf{y}) = A(\mathbf{x}) - A(\mathbf{y})$, can also be given as the linking number of $\gamma_{\mathbf{x},\mathbf{y}}$ and K (ie the intersection number of $\gamma_{\mathbf{x},\mathbf{y}}$ with F), or by the formula $A(\mathbf{x}, \mathbf{y}) = n_z(D) - n_w(D)$ when \mathbf{x} and \mathbf{y} are in the same spin^c structure and \mathcal{D} is any domain connecting \mathbf{x} to \mathbf{y} . The latter formula shows that the complex $\widehat{\text{CFK}}(\mathcal{D})$ splits according to Alexander gradings.

When Y is a rational homology sphere, the complex $\widehat{\text{CFK}}(\mathcal{D})$ admits an absolute \mathbb{Q} -grading, the *Maslov grading*, which restricts to a relative \mathbb{Z} -grading on each $\widehat{\text{CFK}}(\mathcal{D}, \mathfrak{s})$.¹ The relative Maslov grading between two generators \mathbf{x} and \mathbf{y} with $\mathfrak{s}_w(\mathbf{x}) = \mathfrak{s}_w(\mathbf{y})$ is given by the integer $M(\mathbf{x}, \mathbf{y}) = \mu(D) - 2n_w(D)$, where D is any domain connecting \mathbf{x} to \mathbf{y} . The differential lowers this grading by 1, so the grading descends to $\widehat{\text{HFK}}(Y, K)$. The relative \mathbb{Q} -grading between generators in different spin^c structures can be computed using a formula of Lipshitz and Lee [9].

Theorem 2.1 ([15; 12; 20]) *For a suitable choice of complex structure, the homology of the complex $(\widehat{\text{CFK}}(\mathcal{D}), \partial)$ is isomorphic to $\widehat{\text{HFK}}(Y, K) \otimes V^{\otimes n-1}$, where $V \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ with generators in bigradings $(-1, -1)$ and $(0, 0)$, and $\widehat{\text{HFK}}(Y, K)$ is an invariant of the knot type of $K \subset Y$.*

Call a diagram \mathcal{D} *nice* if every elementary domain that does not contain a basepoint is either a bigon or a square. According to results of Manolescu–Ozsváth–Sarkar [12] and Sarkar–Wang [20], the holomorphic disks are easy to describe when \mathcal{D} is nice.

Theorem 2.2 *Let \mathcal{D} be a nice diagram, and let $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ be a Whitney disk in \mathcal{D} with $\mu(\phi) = 1$. Then ϕ admits a holomorphic representative if and only if $D(\phi)$ is either a bigon or a rectangle without any basepoint or point of \mathbf{x} in its interior.*

It follows that when \mathcal{D} is nice, the coefficients $\#\widehat{\mathcal{M}}(\phi)$ in the boundary map can be determined from the combinatorics of the diagram, without reference to the choice of complex structure on Σ , so $\widehat{\text{HFK}}(Y, K)$ can be computed algorithmically.

¹More generally, such a grading can be defined on $\widehat{\text{CFK}}(\mathcal{D}, \mathfrak{s})$ whenever $c_1(\mathfrak{s})$ is torsion.

If K is a knot in S^3 , then a grid diagram for K , drawn on a torus as in Section 1, yields a Heegaard diagram $\mathcal{D} = (T^2, \alpha, \beta, \mathbf{w}, \mathbf{z})$ for the pair (S^3, K) , where the α circles are the horizontal lines of the grid, the β circles are the vertical lines, and the w and z basepoints are the points marked O and X , respectively. Every region of this diagram is a rectangle, so $\widehat{\text{HFK}}(S^3, K)$ can be computed combinatorially as above. Specifically, the generators correspond to permutations of the set $\{1, \dots, n\}$, and the Alexander and Maslov gradings of each generator can be given by simple formulae (discussed later). Using this diagram, Baldwin and Gillam [1] have computed $\widehat{\text{HFK}}(S^3, K)$ for all knots with up to 12 crossings. Additionally, Manolescu, Ozsváth, Szabó, and Thurston [13] give a self-contained proof that this construction yields a knot invariant. (See also Sarkar and Wang [20], who show how to obtain good diagrams for knots in arbitrary 3-manifolds.)

3 Heegaard diagrams for cyclic branched covers of knots

Given a knot $K \subset S^3$ and an integer $m \geq 2$, the cover of $S^3 - K$ corresponding to the canonical homeomorphism $\pi_1(S^3 - K) \rightarrow \mathbb{Z}/m$ extends to an m -sheeted branched cover $\pi: \Sigma_m(K) \rightarrow S^3$, the m -fold cyclic branched cover, whose downstairs branch locus is K and whose upstairs branch locus is a knot $\tilde{K} \subset \Sigma_m(K)$. The manifold $\Sigma_m(K)$ can be constructed explicitly from m copies of $S^3 - \text{int } F$, where F is a Seifert surface for K , by connecting the negative side of a bicollar of F in the k th copy to the positive side in the $(k+1)$ th (indices modulo m). The inverse image of K in $\Sigma_m(K)$ is a knot \tilde{K} , which is nulhomologous because it bounds a Seifert surface (any of the lifts of the original Seifert surface F). This construction does not depend on the choice of Seifert surface. For details, see Rolfsen [19, chapters 6, 10].

The group of covering transformations of $\Sigma_m(K) \rightarrow S^3$ is cyclic of order m , generated by a map $\tau_m: \Sigma_m(K) \rightarrow \Sigma_m(K)$ that takes the k th copy of $S^3 - \text{int } F$ to the $(k+1)$ th (indices modulo m). If γ is a 1-cycle in S^3 , then by using transfer homomorphisms, we see that for any lift $\tilde{\gamma}$, the equation

$$(1) \quad \sum_{k=0}^{m-1} \tau_{m*}^k(\tilde{\gamma}) = 0$$

holds in $H_1(\Sigma_m(K); \mathbb{Z})$. In particular, when $m = 2$, we have $\tau_{2*}(\tilde{\gamma}) = -\tilde{\gamma}$.

When m is a power of a prime p , the group $H_1(\Sigma_m(K); \mathbb{Z})$ is then finite and contains no p^r -torsion for any r (Gordon [5, page 16]). The order of $H_1(\Sigma_m(K))$ is equal to $\prod_{j=0}^{m-1} \Delta_K(\omega^j)$, where Δ_K is the Alexander polynomial of K , and ω is a primitive m th root of unity (Fox [4, page 149]). In particular, note that the action of the deck

transformation group on $H_1(\Sigma_m(K); \mathbb{Z})$ has no nonzero fixed points: if $\tau_{m*}(\alpha) = \alpha$, then

$$0 = \alpha + \tau_{m*}(\alpha) + \cdots + \tau_{m*}^{m-1}(\alpha) = m\alpha,$$

by (1), so $\alpha = 0$.

Let $\mathcal{D} = (S, \alpha, \beta, \mathbf{w}, \mathbf{z})$ be a multi-pointed Heegaard diagram for $K \subset S^3$ with genus g and n basepoint pairs.² If $f: S^3 \rightarrow \mathbb{R}$ is a self-indexing Morse function compatible with \mathcal{D} , then $\tilde{f} = f \circ \pi: \Sigma_m(K) \rightarrow \mathbb{R}$ is a self-indexing Morse function for the pair $(\Sigma_m(K), \tilde{K})$ whose critical points are simply the inverse images of the critical points of f . This function induces a Heegaard splitting $\Sigma_m(K) = \tilde{H}_\alpha \cup_{\tilde{S}} \tilde{H}_\beta$ that projects onto the Heegaard splitting of S^3 . A simple Euler characteristic argument shows that the genus of the new Heegaard surface $\tilde{S} = \pi^{-1}(S)$ is $h = mg + (m-1)(n-1)$. Each α and β circle in S bounds a disk in $S^3 - K$ and hence has m distinct preimages in $\Sigma_m(K)$. Thus, we obtain a Heegaard diagram $\tilde{\mathcal{D}} = (\tilde{S}, \tilde{\alpha}, \tilde{\beta}, \tilde{\mathbf{w}}, \tilde{\mathbf{z}})$, where \tilde{S} is a surface of genus h and $\tilde{\alpha}, \tilde{\beta}, \tilde{\mathbf{w}}$, and $\tilde{\mathbf{z}}$ are the inverse images of the corresponding objects under the covering map.

The generators of the complex $\widehat{\text{CFK}}(\tilde{\mathcal{D}})$ may be described as follows.

Lemma 3.1 *Any generator \mathbf{x} of $\widehat{\text{CFK}}(\tilde{\mathcal{D}})$ can be decomposed (non-uniquely) as $\mathbf{x} = \tilde{\mathbf{x}}_1 \cup \cdots \cup \tilde{\mathbf{x}}_m$, where $\mathbf{x}_1, \dots, \mathbf{x}_m$ are generators of $\widehat{\text{CFK}}(\mathcal{D})$, and $\tilde{\mathbf{x}}_i$ is a lift of \mathbf{x}_i to $\tilde{\mathcal{D}}$.*

Proof Given a generator \mathbf{x} of $\widehat{\text{CFK}}(\tilde{\mathcal{D}})$, let $G_{\mathbf{x}}$ be a graph with vertices denoted $\{a_1, \dots, a_{g+n-1}, b_1, \dots, b_{g+n-1}\}$ and edges $\{e_x \mid x \in \mathbf{x}\}$, where e_x connects a_i to b_j if x is an intersection point between lifts of α_i and β_j . This is clearly a bipartite graph in which each vertex has incidence number m . By König's Theorem [3, Proposition 5.3.1], the edges of $G_{\mathbf{x}}$ can be partitioned (non-uniquely) into m perfect pairings, each of which corresponds to a lift of a generator of $\widehat{\text{CFK}}(\mathcal{D})$. \square

Example 3.2 As will be explained in Section 4, the diagram $\tilde{\mathcal{D}}$ in Figure 1 is the double branched cover of a grid diagram \mathcal{D} for the right-handed trefoil in S^3 . The generator \mathbf{x} of $\widehat{\text{CFK}}(\tilde{\mathcal{D}})$ indicated by the crosses can be decomposed either as lifts of the generators $\mathbf{x}_1 = (20143)$ and $\mathbf{x}_2 = (13240)$ or as lifts of $\mathbf{x}'_1 = (23140)$ and $\mathbf{x}'_2 = (10243)$ (where we identify generators of \mathcal{D} with permutations of $\{0, 1, 2, 3, 4\}$ as described by Manolescu, Ozsváth, and Sarkar [12]). This provides an example of the non-uniqueness of decompositions beyond reordering of the \mathbf{x}_i .

²We denote the Heegaard surface by S rather than Σ to avoid confusion with the notation $\Sigma_m(K)$.

Given a generator \mathbf{x}_0 of $\widehat{\text{CFK}}(\mathcal{D})$, let $L(\mathbf{x}_0)$ denote the generator of $\widehat{\text{CFK}}(\tilde{\mathcal{D}})$ consisting of all m lifts of each point of \mathbf{x}_0 . Using the action of the deck transformation τ_m on \mathcal{D} , we may write $L(\mathbf{x}_0) = \tilde{\mathbf{x}}_0 \cup \tau_m(\tilde{\mathbf{x}}_0) \cup \cdots \cup \tau_m^{m-1}(\tilde{\mathbf{x}}_0)$, where $\tilde{\mathbf{x}}_0$ is any lift of \mathbf{x}_0 to $\tilde{\mathcal{D}}$.

Lemma 3.3 *All generators of $\widehat{\text{CFK}}(\tilde{\mathcal{D}})$ of the form $\mathbf{x} = L(\mathbf{x}_0)$ are in the same spin^c structure, denoted \mathfrak{s}_0 and called the canonical spin^c structure on $\Sigma_m(K)$.*

Proof (Adapted from Grigsby [7].) Let \mathbf{x}_0 and \mathbf{y}_0 be generators of $\widehat{\text{CFK}}(\mathcal{D})$; we shall show that $L(\mathbf{x}_0)$ and $L(\mathbf{y}_0)$ are in the same spin^c structure. Let $\gamma_{\mathbf{x}_0, \mathbf{y}_0}$ be a 1-cycle joining \mathbf{x}_0 and \mathbf{y}_0 as in Section 2, and let $\tilde{\gamma}_{\mathbf{x}_0, \mathbf{y}_0}$ be a lift of $\gamma_{\mathbf{x}_0, \mathbf{y}_0}$ to $\tilde{\mathcal{S}}$. Then the 1-cycle

$$\tilde{\gamma}_{\mathbf{x}_0, \mathbf{y}_0} + \tau_{m*}(\tilde{\gamma}_{\mathbf{x}_0, \mathbf{y}_0}) + \cdots + \tau_{m*}^{m-1}(\tilde{\gamma}_{\mathbf{x}_0, \mathbf{y}_0})$$

connects $L(\mathbf{x}_0)$ and $L(\mathbf{y}_0)$. Then $\epsilon(L(\mathbf{x}_0), L(\mathbf{y}_0)) = 0$ by (1), so $L(\mathbf{x}_0)$ and $L(\mathbf{y}_0)$ are in the same spin^c structure. \square

Remark 3.4 Note that the spin^c structure \mathfrak{s}_0 is fixed under the action of τ_m . To see this, if $f: S^3 \rightarrow \mathbb{R}$ is a self-indexing Morse function for (S^3, K) , its pullback $\tilde{f}: \Sigma_m(K) \rightarrow \mathbb{R}$ is τ_m -invariant. Using a Riemannian metric on $\Sigma_m(K)$ that is the pullback of a metric on S^3 , the gradient $\vec{\nabla} \tilde{f}$ is τ_m -invariant and projects onto $\vec{\nabla} f$, and the flowlines for \tilde{f} are precisely the lifts of flowlines for f . If N is the union of neighborhoods of flowlines through the points of \mathbf{x}_0 and \mathbf{w} , where \mathbf{x}_0 is a generator of $\widehat{\text{CFK}}(\mathcal{D})$, then $\tilde{N} = \pi^{-1}(N)$ is the union of neighborhoods of flowlines through the points of $L(\mathbf{x}_0)$ and $\tilde{\mathbf{w}}$. By suitably modifying $\vec{\nabla} \tilde{f}$ on \tilde{N} , we may obtain a τ_m -invariant vector field that determines $\mathfrak{s}_{\tilde{\mathbf{w}}}(L(\mathbf{x}_0)) = \mathfrak{s}_0$.³ Now, if m is a prime power, then this property uniquely characterizes \mathfrak{s}_0 , for if \mathfrak{s}'_0 is another spin^c structure fixed under the action of τ_m , then the difference between \mathfrak{s}_0 and \mathfrak{s}'_0 is a class in $H_1(\Sigma_m(K); \mathbb{Z})$ that is fixed by τ_m and hence equals zero. For more about the significance of \mathfrak{s}_0 , see Grigsby, Ruberman, and Strle [8].

Proposition 3.5 *If $\mathbf{x} = \tilde{\mathbf{x}}_1 \cup \cdots \cup \tilde{\mathbf{x}}_m$ as in Lemma 3.1, then the Alexander grading of \mathbf{x} (computed with respect to a lift of a Seifert surface for K) is equal to the average of the Alexander gradings of $\mathbf{x}_1, \dots, \mathbf{x}_m$.⁴ In particular, for any generator \mathbf{x}_0 of $\widehat{\text{CFK}}(\mathcal{D})$, we have $A(\mathbf{x}_0) = A(L(\mathbf{x}_0))$.*

³In general, spin^c structures can always be pulled back under a local diffeomorphism using the vector field interpretation. Specifically, if $F: M \rightarrow N$ is a local diffeomorphism and ξ is a nonvanishing vector field on N that determines a given spin^c structure $\mathfrak{s} \in \text{Spin}^c(N)$, then $F^*(\mathfrak{s}) \in \text{Spin}^c(M)$ is determined by the vector field $(F_*)^{-1}(\xi)$. The first Chern class is natural under this pullback.

⁴Note that we have specified a Seifert surface in order to define the Alexander grading. When m is a prime power, however, $\Sigma_m(K)$ is a rational homology sphere, so the Alexander grading does not depend at all on the choice of Seifert surface.

Proof We first consider the relative Alexander gradings. Let $F \subset S^3$ be a Seifert surface for K , and let \tilde{F} be a lift of F to $\Sigma_m(K)$. The translates $\tilde{F}, \tau_m(\tilde{F}), \dots, \tau_m^{m-1}(\tilde{F})$ are all Seifert surfaces for \tilde{K} . The relative Alexander grading between two generators does not depend on the choice of Seifert surface, so for generators \mathbf{x}, \mathbf{y} of $\widehat{\text{CFK}}(\tilde{\mathcal{D}})$, we have

$$mA(\mathbf{x}, \mathbf{y}) = \gamma_{\mathbf{x}, \mathbf{y}} \cdot \tilde{F} + \gamma_{\mathbf{x}, \mathbf{y}} \cdot \tau_m(\tilde{F}) + \dots + \gamma_{\mathbf{x}, \mathbf{y}} \cdot \tau_m^{m-1}(\tilde{F}),$$

where $\gamma_{\mathbf{x}, \mathbf{y}}$ is a 1-cycle connecting \mathbf{x} and \mathbf{y} as above. The projection $\pi_*(\gamma_{\mathbf{x}, \mathbf{y}})$ is a 1-cycle in S that goes from points of $\pi(\mathbf{x})$ to points of $\pi(\mathbf{y})$ along α circles and from points of $\pi(\mathbf{y})$ to points of $\pi(\bar{\mathbf{x}})$ along β circles. Every intersection point of $\gamma_{\mathbf{x}, \mathbf{y}}$ with one of the lifts of F corresponds to an intersection point of $\pi_*(\gamma_{\mathbf{x}, \mathbf{y}})$ with F , so

$$\gamma_{\mathbf{x}, \mathbf{y}} \cdot \tilde{F} + \gamma_{\mathbf{x}, \mathbf{y}} \cdot \tau_m(\tilde{F}) + \dots + \gamma_{\mathbf{x}, \mathbf{y}} \cdot \tau_m^{m-1}(\tilde{F}) = \pi_*(\gamma_{\mathbf{x}, \mathbf{y}}) \cdot F.$$

The restriction of $\pi_*(\gamma_{\mathbf{x}, \mathbf{y}})$ to any α or β circle consists of m (possibly constant or overlapping) arcs. By perhaps adding copies of the α or β circle, we can arrange that these arcs connect a point of \mathbf{x}_1 with a point of \mathbf{y}_1 , a point of \mathbf{x}_2 with a point of \mathbf{y}_2 , and so on. In other words,

$$\pi_*(\gamma_{\mathbf{x}, \mathbf{y}}) \equiv \gamma_{\mathbf{x}_1, \mathbf{y}_1} + \dots + \gamma_{\mathbf{x}_m, \mathbf{y}_m}$$

modulo the α and β circles in \mathcal{D} , whose intersection numbers with F are zero. We have:

$$\begin{aligned} A(\mathbf{x}, \mathbf{y}) &= \frac{1}{m}(\gamma_{\mathbf{x}_1, \mathbf{y}_1} + \dots + \gamma_{\mathbf{x}_m, \mathbf{y}_m}) \cdot F \\ &= \frac{1}{m}(A(\mathbf{x}_1, \mathbf{y}_1) + \dots + A(\mathbf{x}_m, \mathbf{y}_m)). \end{aligned}$$

Thus, the Alexander grading of a generator of $\widehat{\text{CFK}}(\tilde{\mathcal{D}})$ is given up to an additive constant by the average Alexander grading of its parts.

To pin down the additive constant, note that the branched covering map $\pi: \Sigma_m(K) \rightarrow S^3$ extends to an *unbranched* covering map from the zero-surgery on \tilde{K} to the zero-surgery on K , $\pi_0: Y_0(\tilde{K}) \rightarrow S_0^3(K)$. Since this is a local diffeomorphism, it is possible to pull back spin^c structures. Let \mathbf{x}_0 be a generator of $\widehat{\text{CFK}}(\mathcal{D})$ in Alexander grading 0, and let $\mathbf{x} = L(\mathbf{x}_0)$. (The symmetry $\widehat{\text{HF}}K(S^3, K, i) \cong \widehat{\text{HF}}K(S^3, K, -i)$ and the fact that $\text{rank } \widehat{\text{HF}}K(S^3, K) \equiv \det(K) \equiv 1 \pmod{2}$ [15] imply that such $\widehat{\text{HF}}K(S^3, K, 0)$ has odd rank, so such a generator \mathbf{x}_0 always exists.) As in the discussion following Lemma 3.3, we may find a nonvanishing vector field that determines $\mathfrak{s}_{\tilde{\mathbf{w}}}(\mathbf{x}) = \mathfrak{s}_0$ and is τ_m -equivariant. The unique extension (up to isotopy) of this vector field to $\Sigma_m(K)_0$ can also be made τ_m -invariant, so it is the pullback of an extension to S_0^3 of a vector field determining $\mathfrak{s}_{\tilde{\mathbf{w}}}(\mathbf{x}_0)$. It follows that $\mathfrak{s}_{\tilde{\mathbf{w}}, \tilde{\mathbf{z}}}(\mathbf{x}) = \pi_0^*(\mathfrak{s}_{\tilde{\mathbf{w}}, \tilde{\mathbf{z}}}(\mathbf{x}_0))$. Now, if $\tilde{F} \subset Y_0(\tilde{K})$

is obtained by capping off \tilde{F} in the zero-surgery, then $\pi_{0*}[\tilde{F}] = [\widehat{F}]$ in $H_2(S_3^0; \mathbb{Z})$. We therefore have:

$$\begin{aligned} A(\mathbf{x}) &= \frac{1}{2} \left\langle c_1(\underline{\mathfrak{s}}_{\tilde{\mathbf{w}}, \tilde{\mathbf{z}}}(\mathbf{x})), [\widehat{F}] \right\rangle \\ &= \frac{1}{2} \left\langle c_1(\pi_0^*(\underline{\mathfrak{s}}_{\mathbf{w}, \mathbf{z}}(\mathbf{x}_0))), [\widehat{F}] \right\rangle \\ &= \frac{1}{2} \left\langle c_1(\underline{\mathfrak{s}}_{\mathbf{w}, \mathbf{z}}(\mathbf{x}_0)), \pi_{0*}[\widehat{F}] \right\rangle \\ &= \frac{1}{2} \left\langle c_1(\underline{\mathfrak{s}}_{\mathbf{w}, \mathbf{z}}(\mathbf{x}_0)), [\widehat{F}] \right\rangle \\ &= 0 = A(\mathbf{x}_0). \end{aligned}$$

Thus, the additive constant C must equal 0. \square

Remark 3.6 When K is a two-bridge knot and $m = 2$, Grigsby [7] shows that for a specific diagram \mathcal{D} , the map L is surjective and preserves the relative Maslov grading. Therefore, for any two-bridge knot K , $\widehat{\text{HF}}\text{K}(\Sigma_2(K), \tilde{K}, \mathfrak{s}_0) \cong \widehat{\text{HF}}\text{K}(S^3, K)$, up to a possible shift in the absolute Maslov grading. It may be possible to extend this result to a wider class of knots, such as alternating knots. However, in general L is neither surjective nor Maslov-grading-preserving.

Finally, we consider the regions in $\tilde{\mathcal{D}}$. First, note that the preimage of any region R in \mathcal{D} consists of either m distinct regions, each of which is projected diffeomorphically onto R , or a single region. (In the former case, we say that R is *evenly covered*.) In particular, when \mathcal{D} is nice, each region of \mathcal{D} that does not contain a basepoint is a simply-connected polygon that misses the branch set, so it is evenly covered. Thus, we obtain the following proposition.

Proposition 3.7 *Let \mathcal{D} be a nice Heegaard diagram for (S^3, K) , and let $\tilde{\mathcal{D}}$ be its m -fold cyclic branched cover. Then $\tilde{\mathcal{D}}$ is nice.*

4 Grid diagrams and cyclic branched covers

Proof of Theorem 1.1 As described in Section 1, any oriented knot $K \subset S^3$ can be represented by means of a grid diagram. By drawing the grid diagram on a standardly embedded torus in S^3 , we may think of the grid diagram as a genus 1, multi-pointed Heegaard diagram $\mathcal{D} = (T^2, \alpha, \beta, \mathbf{w}, \mathbf{z})$ for the pair (S^3, K) , where the α circles are the horizontal lines of the grid, the β circles are the vertical lines, the w basepoints are in the regions marked O , and the z basepoints are in the regions marked X .

Note that the diagram \mathcal{D} is nice, so the differential can be computed combinatorially as described in Section 2. Specifically, the coefficient of \mathbf{y} in $\partial\mathbf{x}$ is 1 if all but two of the points of \mathbf{x} and \mathbf{y} agree and there is a rectangle embedded in the torus with points of \mathbf{x} as its lower-left and upper-right corners, points of \mathbf{y} as its lower-right and upper-left corners, and no basepoints or points of \mathbf{x} in its interior, and 0 otherwise. Note that there cannot be two such rectangles, or else K would be a split link.

A Seifert surface for K may be seen as follows. We may isotope K to lie entirely within H_α by letting the arcs of $K \cap H_\beta$ fall onto the boundary torus. In fact, it lies within a ball contained in H_α since the knot projection in the grid diagram never passes through the left edge of the grid. Take a Seifert surface F contained in this ball, and then isotope F and K so that K returns to its original position. F then intersects the Heegaard surface T^2 in n arcs, one connecting the two basepoints in each column of the grid diagram, and it intersects H_β in strips that lie above these arcs. The orientations of K and S^3 imply that the positive side of a bicollar for F lies on the *right* of one of these strips when the X is above the O and on the *left* when the O is above the X .

If we construct $\Sigma_m(K)$ by gluing together m copies of $S^3 - \text{int } F$ as in Section 3, the Heegaard surfaces in each copy are connected exactly to each other as described in Section 1 to form a surface \tilde{T} . Hence, $\tilde{\mathcal{D}} = (\tilde{T}, \tilde{\alpha}, \tilde{\beta}, \tilde{\mathbf{w}}, \tilde{\mathbf{z}})$ is a Heegaard diagram for $(\Sigma_m(K), \tilde{K})$ for which the results of Section 3 apply. In particular, it is a nice Heegaard diagram.

It remains to show that the domains that count for the differential in $\widehat{\text{CFK}}(\tilde{\mathcal{D}})$ are precisely the lifts of those that count for the differential in $\widehat{\text{CFK}}(\mathcal{D})$, as was asserted in Section 1. Since $\tilde{\mathcal{D}}$ is a nice diagram with no bigons, any domain that counts for the differential is an embedded rectangle R . The projection of R to \mathcal{D} , $\pi(R)$, is an immersed rectangle in $\tilde{\mathcal{D}}$ whose edges are contained in at most two α circles and two β circles. By lifting $\pi(R)$ to the universal cover of T^2 , we see that $\pi(R)$ cannot intersect any α or β circle more than once, or else it would contain an entire column or row of the grid diagram and hence a basepoint. Therefore, $\pi(R)$ is an embedded rectangle that misses the basepoints, so it counts for the differential of $\widehat{\text{CFK}}(\mathcal{D})$. \square

We shall now give a more explicit description of the generators of $\widehat{\text{CFK}}(\tilde{\mathcal{D}})$ and their gradings in order to facilitate computation.

In the grid diagram \mathcal{D} , we label the α circles $\alpha_0, \dots, \alpha_{n-1}$ from bottom to top and the β circles $\beta_0, \dots, \beta_{n-1}$ from left to right. Each α circle intersects each β circle exactly once: $\beta_i \cap \alpha_j = \{x_{ij}\}$. Generators of $\widehat{\text{CFK}}(\mathcal{D})$ then correspond to permutations of the index set $\{0, \dots, n-1\}$ via the correspondence $\sigma \mapsto (x_{0,\sigma(0)}, \dots, x_{n-1,\sigma(n-1)})$.

For each grid point x , let $w(x)$ denote the winding number of the knot projection around x . Let p_1, \dots, p_{8n} (repetitions allowed) denote the vertices of the $2n$ squares containing basepoints, and set

$$a = \frac{1-n}{2} + \frac{1}{8} \sum_{i=1}^{8n} w(p_i).$$

According to Manolescu, Ozsváth, and Sarkar [12], the Alexander grading of a generator \mathbf{x} of $\widehat{\text{CFK}}(\mathcal{D})$ is given by the formula

$$(2) \quad A(\mathbf{x}) = a - \sum_{x \in \mathbf{x}} w(x).$$

There is also a formula for the Maslov grading of a generator, but it is not relevant for our purposes.

The generators of $\widehat{\text{CFK}}(\tilde{\mathcal{D}})$ can be described easily as follows. For any $i = 0, \dots, n-1$ and $j = 0, \dots, n-1$, each lift of β_i meets exactly one lift of α_j . Specifically, let $\tilde{\beta}_j^k$ denote the lift of β_j on the k th copy of \mathcal{D} (for $k = 0, \dots, m-1$). Let $\tilde{\alpha}_j^k$ denote the lift of α_j that intersects the leftmost edge of the k th grid diagram ($\tilde{\beta}_0^k$). Let $\tilde{x}_{i,j}^k$ denote the lift of $x_{i,j}$ on the k th diagram. Define a map $g: \mathbb{Z}/n \times \mathbb{Z}/n \times \mathbb{Z}/m \rightarrow \mathbb{Z}/m$ by $g(i, j, k) = k - w(x_{i,j}) \pmod{m}$. The lift of α_j that meets a particular $\tilde{\beta}_i^k$ is given by the following lemma.

Lemma 4.1 *The point $\tilde{x}_{i,j}^k$ is the intersection between $\tilde{\beta}_i^k$ and $\tilde{\alpha}_j^{g(i,j,k)}$.*

Proof We induct on i . For $i = 0$, we have $w(x_{0,j}) = 0$, and by construction $\tilde{\alpha}_j^k$ meets $\tilde{\beta}_0^k$. For the induction step, let S be the segment of α_j from $x_{i,j}$ to $x_{i+1,j}$. Note that $w(x_{i+1,j})$ is equal to $w(x_{i,j}) + 1$ if S passes below the X and above the O in its column, $w(x_{i,j}) - 1$ if it passes above X and below O , and $w(x_{i,j})$ otherwise. Similarly, if $\tilde{x}_{i,j}^k$ lies on $\tilde{\alpha}_j^l$, then by the previous discussion, $\tilde{x}_{i+1,j}^k$ lies on $\tilde{\alpha}_j^{l-1}$ in the first case, on $\tilde{\alpha}_j^{l+1}$ in the second, and on $\tilde{\alpha}_j^l$ in the third (upper indices modulo m). This proves the induction step. \square

We may then identify the generators of $\widehat{\text{CFK}}(\tilde{\mathcal{D}})$ with the set of m -to-one maps

$$\phi: \{0, \dots, n-1\} \times \{0, \dots, m-1\} \rightarrow \{0, \dots, n-1\}$$

such that for each $j = 0, \dots, n-1$, the function $g(\cdot, j, \cdot)$ assumes all m possible values on $\phi^{-1}(j)$. In other words, if we shade the m lifts of each α circle with different colors as in Figure 1 and arrange the copies of \mathcal{D} horizontally, a generator is a

selection of mn grid points so each column contains one point and each row contains m points, one of each color. It is not difficult to enumerate such maps algorithmically.

To split up the generators of $\widehat{\text{CFK}}(\tilde{\mathcal{D}})$ according to spin^c structures, we simply need to express $\epsilon(\mathbf{x}, \mathbf{y})$ in terms of a \mathbb{Z} -module presentation for $H_1(\Sigma_m(K); \mathbb{Z})$. We obtain such a presentation from the Heegaard decomposition of $\Sigma_m(K)$: the generators a_j^k ($0 \leq j \leq n-1, 0 \leq k \leq m-1$) corresponding to the 1-handles dual to the α circles and relations corresponding to the 2-handles spanned by the β circles. By Lemma 4.1, the relations are

$$0 = [\tilde{\beta}_i^k] = \sum_{j=1}^n a_j^{g(i,j,k)} \quad (0 \leq i \leq n-1, 0 \leq k \leq m-1).$$

To express $\epsilon(\mathbf{x}, \mathbf{y})$ in terms of this basis, one simply counts the number of times that a representative $\gamma_{\mathbf{x}, \mathbf{y}}$ crosses the α circles.

To compute the Alexander grading of a generator \mathbf{x} , we decompose it as $\mathbf{x} = \tilde{\mathbf{x}}_1 \cup \dots \cup \tilde{\mathbf{x}}_m$ using Lemma 3.1 and then use Proposition 3.5 and (2) to write:

$$\begin{aligned} A(\mathbf{x}) &= \frac{1}{m}(A(\mathbf{x}_1) + \dots + A(\mathbf{x}_m)) \\ &= \frac{1}{m} \sum_{k=1}^m \left(a - \sum_{x \in \mathbf{x}_k} w(x) \right) \\ &= a - \frac{1}{m} \sum_{k=1}^m \sum_{x \in \tilde{\mathbf{x}}_k} w(\pi(x)) \\ &= a - \frac{1}{m} \sum_{x \in \mathbf{x}} w(\pi(x)). \end{aligned}$$

Computing the relative Maslov grading between two generators in the same spin^c structure requires finding a domain D connecting them, which is simply a matter of linear algebra, and then using the formula $M(\mathbf{x}) - M(\mathbf{y}) = \mu(D) - 2n_w(D)$. The relative Maslov grading between generators in different spin^c structures can be computed similarly using the formula of Lee and Lipshitz [9]. Since all the basepoints in the Heegaard diagrams used in this paper are contained in $4m$ -gonal regions, it is not possible to compute the absolute Maslov gradings or the spectral sequence from $\widehat{\text{HF}}K(\Sigma_2(K), \tilde{K})$ to $\widehat{\text{HF}}(\Sigma_2(K))$ combinatorially. However, when $m = 2$, the groups $\widehat{\text{HF}}(\Sigma_2(K))$, or at least the correction terms $d(\Sigma_2(K), \mathfrak{s})$, can in many instances be computed via other means (Ozsváth and Szabó [16]). In such cases, it is often possible to pin down the absolute Maslov gradings for $\widehat{\text{HF}}K(\Sigma_2(K), \tilde{K})$. Specifically, the relative Maslov \mathbb{Q} -grading and the action of $H_1(\Sigma_2(K))$ on $\text{Spin}^c(\Sigma_2(K))$ usually

provide enough information to match the groups $\widehat{\text{HFK}}(\Sigma_2(K), \tilde{K}, \mathfrak{s})$ up with the rational numbers $d(\Sigma_2(K), \mathfrak{s})$ that are computed via some other means. If there is a spin^c structure \mathfrak{s} in which $\widehat{\text{HFK}}(\Sigma_2(K), \tilde{K}, \mathfrak{s})$ has rank 1, then the absolute Maslov grading of that group equals the corresponding d invariant, and the rest of the absolute gradings are completely determined.

5 Results

The following tables list the ranks for $\widehat{\text{HFK}}(\Sigma_2(K), \tilde{K}; \mathbb{Z}/2)$ by means of the Poincaré polynomials:

$$p_{\mathfrak{s}}(q, t) = \sum_{i,j} \dim_{\mathbb{Z}/2} \widehat{\text{HFK}}_j(\Sigma_2(K), \tilde{K}, \mathfrak{s}, i; \mathbb{Z}/2) t^i q^j.$$

The Maslov \mathbb{Q} -gradings are normalized so that in the canonical spin^c structure \mathfrak{s}_0 , the nonzero elements in Alexander grading $g(K)$ have Maslov grading $g(K)$. For each knot, the first line gives $p_{\mathfrak{s}_0}(q, t)$, and each subsequent line gives $p_{\mathfrak{s}}(q, t)$ for a pair of conjugate spin^c structures. We identify spin^c structures with elements of $H_1(\Sigma_2(K); \mathbb{Z})$, which is either a cyclic group or the sum of two cyclic groups, taking \mathfrak{s}_0 to 0. (Of course, the choice of basis for $H_1(\Sigma_2(K); \mathbb{Z})$ is not canonical.) In each spin^c structure, most of the nonzero groups lie along a single diagonal; the terms corresponding to the groups not on that diagonal are underlined>.

These results were computed using a program written in C++ and *Mathematica*, based on Baldwin and Gillam’s program [1] for computing $\widehat{\text{HFK}}(S^3, K)$. Most of the grid diagrams were obtained using Marc Culler’s program *Gridlink* [2]. Using available computer resources, it was possible to compute $\widehat{\text{HFK}}(\Sigma_2(K), \tilde{K})$ for all the three-bridge knots with up to eleven crossings and arc index ≤ 9 , and for many knots with arc index 10. (Grigsby [6] has a much more efficient algorithm for computing $\widehat{\text{HFK}}(\Sigma_2(K), \tilde{K})$ when K is two-bridge, so we do not list those knots here.)

K	$H_1(\Sigma_2(K); \mathbb{Z})$	\mathfrak{s}	$\sum_{i,j} \dim_{\mathbb{Z}/2} \widehat{\text{HFK}}_j(\Sigma_2(K), \tilde{K}, \mathfrak{s}, i; \mathbb{Z}/2) t^i q^j$
85	$\mathbb{Z}/21$	0	$q^{-3}t^{-3} + 3q^{-2}t^{-2} + 4q^{-1}t^{-1} + 5 + 4qt + 3q^2t^2 + q^3t^3$
		± 1	$q^{5/21}(q^{-2}t^{-2} + 3q^{-1}t^{-1} + 3 + 3qt + q^2t^2)$
		± 2	$q^{20/21}$
		± 3	$q^{8/7}$
		± 4	$q^{17/21}(q^{-1}t^{-1} + 1 + qt)$
		± 5	$q^{20/21}$
		± 6	$q^{4/7}$
		± 7	$q^{2/3}(q^{-1}t^{-1} + 3 + qt)$
		± 8	$q^{5/21}(q^{-2}t^{-2} + 3q^{-1}t^{-1} + 3 + 3qt + q^2t^2)$
		± 9	$q^{2/7}(q^{-2}t^{-2} + 2q^{-1}t^{-1} + 3 + 2qt + q^2t^2)$
		± 10	$q^{17/21}(q^{-1}t^{-1} + 1 + qt)$

K	$H_1(\Sigma_2(K); \mathbb{Z})$	s	$\sum_{i,j} \dim_{\mathbb{Z}/2} \widehat{\text{HF}}K_j(\Sigma_2(K), \tilde{K}, s, i; \mathbb{Z}/2)t^i q^j$
8 ₁₀	$\mathbb{Z}/27$	0	$q^{-3}t^{-3} + 3q^{-2}t^{-2} + 6q^{-1}t^{-1} + 7 + 6qt + 3q^2t^2 + q^3t^3$
		± 1	$q^{7/27}(q^{-2}t^{-2} + 3q^{-1}t^{-1} + 5 + 3qt + q^2t^2)$
		± 2	$q^{1/27}$
		± 3	$q^{1/3}$
		± 4	$q^{4/27}(q^{-1}t^{-1} + 1 + qt)$
		± 5	$q^{13/27}$
		± 6	$q^{1/3}$
		± 7	$q^{-8/27}(q^{-1}t^{-1} + 1 + qt)$
		± 8	$q^{-11/27}(q^{-2}t^{-2} + q^{-1}t^{-1} + 1 + qt + q^2t^2)$
		± 9	$q^{-1}t^{-1} + 1 + qt$
		± 10	$q^{25/27}$
		± 11	$q^{10/27}(2q^{-1}t^{-1} + 3 + 2qt)$
		± 12	$q^{1/3}(q^{-2}t^{-2} + 2q^{-1}t^{-1} + 3 + 2qt + q^2t^2)$
± 13	$q^{22/27}(q^{-1}t^{-1} + 1 + qt)$		
8 ₁₅	$\mathbb{Z}/33$	0	$3q^{-2}t^{-2} + 8q^{-1}t^{-1} + 11 + 8qt + 3q^2t^2$
		± 1	$q^{13/33}(2q^{-1}t^{-1} + 3 + 2qt)$
		± 2	$q^{-14/33}(q^{-2}t^{-2} + q^{-1}t^{-1} + 1 + qt + q^2t^2)$
		± 3	$q^{6/11}$
		± 4	$q^{10/33}$
		± 5	$q^{-5/33}(q^{-1}t^{-1} + 1 + qt)$
		± 6	$q^{2/11}$
		± 7	$q^{10/33}$
		± 8	$q^{7/33}(q^{-1}t^{-1} + 1 + qt)$
		± 9	$q^{10/11}$
		± 10	$q^{13/33}(2q^{-1}t^{-1} + 3 + 2qt)$
		± 11	$q^{2/3}$
		± 12	$q^{-3/11}(q^{-1}t^{-1} + 1 + qt)$
		± 13	$q^{-14/33}(q^{-2}t^{-2} + q^{-1}t^{-1} + 1 + qt + q^2t^2)$
		± 14	$q^{7/33}(q^{-1}t^{-1} + 1 + qt)$
		± 15	$q^{-4/11}(q^{-2}t^{-2} + 2q^{-1}t^{-1} + 3 + 2qt + q^2t^2)$
± 16	$q^{-5/33}(q^{-1}t^{-1} + 1 + qt)$		
8 ₁₆	$\mathbb{Z}/35$	0	$q^{-3}t^{-3} + 4q^{-2}t^{-2} + 8q^{-1}t^{-1} + 9 + 8qt + 4q^2t^2 + q^3t^3$
		± 1	$q^{16/35}(q^{-1}t^{-1} + 1 + qt)$
		± 2	$q^{29/35}$
		± 3	$q^{4/35}(q^{-1}t^{-1} + 1 + qt)$
		± 4	$q^{11/35}(q^{-2}t^{-2} + 3q^{-1}t^{-1} + 5 + 3qt + q^2t^2)$
		± 5	$q^{3/7}(q^{-1}t^{-1} + 3 + qt)$
		± 6	$q^{16/35}(q^{-1}t^{-1} + 1 + qt)$
		± 7	$q^{2/5}(2q^{-1}t^{-1} + 3 + 2qt)$
		± 8	$q^{9/35}$
		± 9	$q^{1/35}$
		± 10	$q^{5/7}$
		± 11	$q^{11/35}(q^{-2}t^{-2} + 3q^{-1}t^{-1} + 5 + 3qt + q^2t^2)$
		± 12	$q^{29/35}$
		± 13	$q^{9/35}$
		± 14	$q^{3/5}$
		± 15	$q^{6/7}(q^{-1}t^{-1} + 1 + qt)$
		± 16	$q^{1/35}$
± 17	$q^{4/35}(q^{-1}t^{-1} + 1 + qt)$		

K	$H_1(\Sigma_2(K); \mathbb{Z})$	s	$\sum_{i,j} \dim_{\mathbb{Z}/2} \widehat{\text{HFK}}_j(\Sigma_2(K), \tilde{K}, s, i; \mathbb{Z}/2) t^i q^j$
8 ₁₇	$\mathbb{Z}/37$	0	$q^{-3}t^{-3} + 4q^{-2}t^{-2} + 8q^{-1}t^{-1} + 11 + 8qt + 4q^2t^2 + q^3t^3$
		± 1	$q^{2/37}$
		± 2	$q^{8/37}(q^{-1}t^{-1} + 3 + qt)$
		± 3	$q^{18/37}$
		± 4	$q^{-5/37}(q^{-1}t^{-1} + 1 + qt)$
		± 5	$q^{13/37}(q^{-2}t^{-2} + 3q^{-1}t^{-1} + 3 + 3qt + q^2t^2)$
		± 6	$q^{-2/37}$
		± 7	$q^{-13/37}(q^{-2}t^{-2} + 3q^{-1}t^{-1} + 3 + 3qt + q^2t^2)$
		± 8	$q^{17/37}(q^{-1}t^{-1} + 1 + qt)$
		± 9	$q^{14/37}$
		± 10	$q^{-22/37}$
		± 11	$q^{-17/37}(q^{-1}t^{-1} + 1 + qt)$
		± 12	$q^{-8/37}(q^{-1}t^{-1} + 3 + qt)$
		± 13	$q^{5/37}(q^{-1}t^{-1} + 1 + qt)$
		± 14	$q^{22/37}$
		± 15	$q^{6/37}$
		± 16	$q^{-6/37}$
		± 17	$q^{-14/37}$
± 18	$q^{-18/37}$		
8 ₁₈	$\mathbb{Z}/3 \oplus \mathbb{Z}/15$	(0, 0)	$q^{-3}t^{-3} + 5q^{-2}t^{-2} + 10q^{-1}t^{-1} + 13 + 10qt + 5q^2t^2 + q^3t^3$
		$\pm(0, 1)$	$q^{7/15}(q^{-1}t^{-1} + 1 + qt)$
		$\pm(0, 2)$	$q^{-2/15}$
		$\pm(0, 3)$	$q^{1/5}(q^{-1}t^{-1} + 1 + qt)$
		$\pm(0, 4)$	$q^{7/15}(q^{-1}t^{-1} + 1 + qt)$
		$\pm(0, 5)$	$q^{-2/3}$
		$\pm(0, 6)$	$q^{-1/5}(q^{-1}t^{-1} + 1 + qt)$
		$\pm(0, 7)$	$q^{-2/15}$
		$\pm(1, 0)$	$q^{-2/3}$
		$\pm(1, 1)$	$q^{7/15}(q^{-1}t^{-1} + 1 + qt)$
		$\pm(1, 2)$	$q^{-7/15}(q^{-1}t^{-1} + 1 + qt)$
		$\pm(1, 3)$	$q^{-7/15}(q^{-1}t^{-1} + 1 + qt)$
		$\pm(1, 4)$	$q^{7/15}(q^{-1}t^{-1} + 1 + qt)$
		$\pm(1, 5)$	$q^{-2/3}$
		$\pm(1, 6)$	$q^{2/15}$
$\pm(1, 7)$	$q^{-2/15}$		
$\pm(1, 8)$	$q^{-7/15}(q^{-1}t^{-1} + 1 + qt)$		
$\pm(1, 9)$	$q^{2/15}$		
$\pm(1, 10)$	$q^{2/3}$		
$\pm(1, 11)$	$q^{2/15}$		
$\pm(1, 12)$	$q^{-7/15}(q^{-1}t^{-1} + 1 + qt)$		
$\pm(1, 13)$	$q^{-2/15}$		
$\pm(1, 14)$	$q^{2/15}$		
8 ₁₉	$\mathbb{Z}/3$	0	$q^{-3}t^{-3} + q^{-2}t^{-2} + q + q^2t^2 + q^3t^3$
		± 1	$q^{2/3}(q^{-1}t^{-1} + 1 + qt)$
8 ₂₀	$\mathbb{Z}/9$	0	$q^{-2}t^{-2} + 2q^{-1}t^{-1} + 3 + 2qt + q^2t^2$
		± 1	$q^{7/9}(q^{-1}t^{-1} + 1 + qt)$
		± 2	$q^{1/9}(q^{-1}t^{-1} + 1 + qt)$
		± 3	1
		± 4	$q^{4/9}$

K	$H_1(\Sigma_2(K); \mathbb{Z})$	s	$\sum_{i,j} \dim_{\mathbb{Z}/2} \widehat{\text{HFK}}_j(\Sigma_2(K), \tilde{K}, s, i; \mathbb{Z}/2)t^i q^j$
821	$\mathbb{Z}/15$	1	$q^{-2}t^{-2} + 4q^{-1}t^{-1} + 5 + 4qt + q^2t^2$
		± 1	$q^{-2/15}(q^{-1}t^{-1} + 1 + qt)$
		± 2	$q^{7/15}$
		± 3	$q^{-1/5}(q^{-1}t^{-1} + 3 + qt)$
		± 4	$q^{-2/15}(q^{-1}t^{-1} + 1 + qt)$
		± 5	$q^{-1/3}$
		± 6	$q^{1/5}$
		± 7	$q^{7/15}$
942	$\mathbb{Z}/7$	0	$q^{-2}t^{-2} + 2q^{-1}t^{-1} + 2 + q + 2qt + q^2t^2$
		± 1	$q^{3/7}$
		± 2	$q^{5/7}(q^{-1}t^{-1} + 3 + qt)$
		± 3	$q^{6/7}(q^{-1}t^{-1} + 1 + qt)$
943	$\mathbb{Z}/13$	0	$q^{-3}t^{-3} + 3q^{-2}t^{-2} + 2q^{-1}t^{-1} + 1 + 2qt + 3q^2t^2 + q^3t^3$
		± 1	$q^{10/13}(q^{-1}t^{-1} + 3 + qt)$
		± 2	$q^{1/13}(q^{-1}t^{-1} + 1 + qt)$
		± 3	$q^{12/13}$
		± 4	$q^{4/13}(q^{-2}t^{-2} + q^{-1}t^{-1} + 1 + qt + q^2t^2)$
		± 5	$q^{16/13}$
± 6	$q^{9/13}(q^{-1}t^{-1} + 1 + qt)$		
944	$\mathbb{Z}/17$	0	$q^{-2}t^{-2} + 4q^{-1}t^{-1} + 7 + 4qt + q^2t^2$
		± 1	$q^{-8/17}$
		± 2	$q^{-15/17}(q^{-1}t^{-1} + 1 + qt)$
		± 3	$q^{-4/17}$
		± 4	$q^{8/17}$
		± 5	$q^{4/17}$
		± 6	$q^{-16/17}$
		± 7	$q^{-1/17}(q^{-1}t^{-1} + 1 + qt)$
± 8	$q^{-2/17}(q^{-1}t^{-1} + 3 + qt)$		
945	$\mathbb{Z}/23$	0	$q^{-2}t^{-2} + 6q^{-1}t^{-1} + 9 + 6qt + q^2t^2$
		± 1	$q^{-8/23}(2q^{-1}t^{-1} + 3 + 2qt)$
		± 2	$q^{-9/23}$
		± 3	$q^{-3/23}(q^{-1}t^{-1} + 3 + qt)$
		± 4	$q^{-13/23}$
		± 5	$q^{7/23}$
		± 6	$q^{11/23}$
		± 7	$q^{-1/23}$
		± 8	$q^{-6/23}(q^{-1}t^{-1} + 1 + qt)$
		± 9	$q^{-4/23}(2q^{-1}t^{-1} + 3 + 2qt)$
		± 10	$q^{-18/23}(q^{-1}t^{-1} + 1 + qt)$
± 11	$q^{-2/23}(q^{-1}t^{-1} + 1 + qt)$		
946	$\mathbb{Z}/3 \oplus \mathbb{Z}/3$	(0, 0)	$2q^{-1}t^{-1} + 5 + 2qt$
		$\pm(0, 1)$	$q^{-2/3}(q^{-1}t^{-1} + 3 + qt)$
		$\pm(1, 0)$	1
		$\pm(1, 1)$	1
		$\pm(1, 2)$	$q^{-4/3}$

K	$H_1(\Sigma_2(K); \mathbb{Z})$	\mathfrak{s}	$\sum_{i,j} \dim_{\mathbb{Z}/2} \widehat{\text{HFK}}_j(\Sigma_2(K), \tilde{K}, \mathfrak{s}, i; \mathbb{Z}/2)t^i q^j$
9 ₄₇	$\mathbb{Z}/3 \oplus \mathbb{Z}/9$	(0, 0) $\pm(0, 1)$ $\pm(0, 2)$ $\pm(0, 3)$ $\pm(0, 4)$ $\pm(1, 0)$ $\pm(1, 1)$ $\pm(1, 2)$ $\pm(1, 3)$ $\pm(1, 4)$ $\pm(1, 5)$ $\pm(1, 6)$ $\pm(1, 7)$ $\pm(1, 8)$	$q^{-3}t^{-3} + 4q^{-2}t^{-2} + 6q^{-1}t^{-1} + 5 + 6qt + 4q^2t^2 + q^3t^3$ $q^{-1/9}(q^{-1}t^{-1} + 3 + qt)$ $q^{-4/9}(q^{-1}t^{-1} + 1 + qt)$ $q^{-1}t^{-1} + 1 + qt$ $q^{-7/9}$ $q^{-1/3}$ $q^{-1/9}(q^{-1}t^{-1} + 3 + qt)$ $q^{-1/9}(q^{-1}t^{-1} + 3 + qt)$ $q^{-1/3}$ $q^{-7/9}$ $q^{-4/9}(q^{-1}t^{-1} + 1 + qt)$ $q^{-1/3}$ $q^{-4/9}(q^{-1}t^{-1} + 1 + qt)$ $q^{-7/9}$
9 ₄₈	$\mathbb{Z}/3 \oplus \mathbb{Z}/9$	(0, 0) $\pm(0, 1)$ $\pm(0, 2)$ $\pm(0, 3)$ $\pm(0, 4)$ $\pm(1, 0)$ $\pm(1, 1)$ $\pm(1, 2)$ $\pm(1, 3)$ $\pm(1, 4)$ $\pm(1, 5)$ $\pm(1, 6)$ $\pm(1, 7)$ $\pm(1, 8)$	$q^{-2}t^{-2} + 7q^{-1}t^{-1} + 11 + 7qt + q^2t^2$ $q^{-4/9}(q^{-1}t^{-1} + 1 + qt)$ $q^{2/9}(2q^{-1}t^{-1} + 3 + 2qt)$ $q^{-1}t^{-1} + 1 + qt$ $q^{-1/9}$ $q^{1/3}$ $q^{2/9}(2q^{-1}t^{-1} + 3 + 2qt)$ $q^{2/9}(2q^{-1}t^{-1} + 3 + 2qt)$ $q^{1/3}$ $q^{-4/9}(q^{-1}t^{-1} + 1 + qt)$ $q^{-1/9}$ $q^{1/3}$ $q^{-1/9}$ $q^{-4/9}(q^{-1}t^{-1} + 1 + qt)$
9 ₄₉	$\mathbb{Z}/5 \oplus \mathbb{Z}/5$	(0, 0) $\pm(0, 1)$ $\pm(0, 2)$ $\pm(1, 0)$ $\pm(1, 1)$ $\pm(1, 2)$ $\pm(1, 3)$ $\pm(1, 4)$ $\pm(2, 0)$ $\pm(2, 1)$ $\pm(2, 2)$ $\pm(2, 3)$ $\pm(2, 4)$	$3q^{-2}t^{-2} + 6q^{-1}t^{-1} + 7 + 6qt + 3q^2t^2$ $q^{-2/5}(q^{-2}t^{-2} + q^{-1}t^{-1} + 1 + qt + q^2t^2)$ $q^{2/5}$ $q^{-2/5}(q^{-2}t^{-2} + q^{-1}t^{-1} + 1 + qt + q^2t^2)$ $q^{-1/5}(q^{-1}t^{-1} + 1 + qt)$ $q^{1/5}(2q^{-1}t^{-1} + 3 + 2qt)$ $q^{1/5}(2q^{-1}t^{-1} + 3 + 2qt)$ $q^{-2/5}(q^{-2}t^{-2} + q^{-1}t^{-1} + 1 + qt + q^2t^2)$ $q^{2/5}$ $q^{1/5}(2q^{-1}t^{-1} + 3 + 2qt)$ $q^{-1/5}(q^{-1}t^{-1} + 1 + qt)$ $q^{2/5}$ $q^{-1/5}(q^{-1}t^{-1} + 1 + qt)$
10 ₁₂₄	{0}	0	$q^{-4}t^{-4} + q^{-3}t^{-3} + t^{-1} + q + q^2t + q^3t^3 + q^4t^4$
10 ₁₂₈	$\mathbb{Z}/11$	0 ± 1 ± 2 ± 3 ± 4 ± 5	$2q^{-3}t^{-3} + 3q^{-2}t^{-2} + q^{-1}t^{-1} + q + qt + 3q^2t^2 + 2q^3t^3$ $q^{8/11}(2q^{-1}t^{-1} + 3 + 2qt)$ $q^{10/11}(q^{-1}t^{-1} + 1 + qt)$ $q^{6/11}(q^{-1}t^{-1} + 1 + qt)$ $q^{-4/11}(q^{-2}t^{-2} + q^{-1}t^{-1} + q + qt + q^2t^2)$ $q^{2/11}(q^{-1}t^{-1} + 1 + qt)$

K	$H_1(\Sigma_2(K); \mathbb{Z})$	s	$\sum_{i,j} \dim_{\mathbb{Z}/2} \widehat{\text{HFK}}_j(\Sigma_2(K), \tilde{K}, s, i; \mathbb{Z}/2)t^i q^j$
10 ₁₂₉	$\mathbb{Z}/25$	0	$2q^{-2}t^{-2} + 6q^{-1}t^{-1} + 9 + 6qt + 2q^2t^2$
		± 1	$q^{-8/25}(q^{-2}t^{-2} + 2q^{-1}t^{-1} + 3 + 2qt + q^2t^2)$
		± 2	$q^{-7/25}(q^{-1}t^{-1} + 1 + qt)$
		± 3	$q^{3/25}(2q^{-1}t^{-1} + 3 + 2qt)$
		± 4	$q^{-3/25}(q^{-1}t^{-1} + 1 + qt)$
		± 5	1
		± 6	$q^{12/25}$
		± 7	$q^{8/25}$
		± 8	$q^{-12/25}$
		± 9	$q^{2/25}(q^{-1}t^{-1} + 3 + qt)$
		± 10	1
		± 11	$q^{7/25}(q^{-1}t^{-1} + 1 + qt)$
± 12	$q^{23/25}(q^{-1}t^{-1} + 1 + qt)$		
10 ₁₃₀	$\mathbb{Z}/17$	0	$2q^{-2}t^{-2} + 4q^{-1}t^{-1} + 5 + 4qt + 2q^2t^2$
		± 1	$q^{4/17}(q^{-2}t^{-2} + 2q^{-1}t^{-1} + 3 + 2qt + q^2t^2)$
		± 2	$q^{16/17}$
		± 3	$q^{19/17}(q^{-1}t^{-1} + 1 + qt)$
		± 4	$q^{13/17}(2q^{-1}t^{-1} + 3 + 2qt)$
		± 5	$q^{15/17}(q^{-1}t^{-1} + 1 + qt)$
		± 6	$q^{8/17}$
		± 7	$q^{9/17}(q^{-1}t^{-1} + 1 + qt)$
± 8	$q^{1/17}(q^{-1}t^{-1} + 1 + qt)$		
10 ₁₃₁	$\mathbb{Z}/31$	0	$2q^{-2}t^{-2} + 8q^{-1}t^{-1} + 11 + 8qt + 2q^2t^2$
		± 1	$q^{-18/31}(q^{-1}t^{-1} + 1 + qt)$
		± 2	$q^{-10/31}(q^{-1}t^{-1} + 1 + qt)$
		± 3	$q^{-7/31}(q^{-1}t^{-1} + 3 + qt)$
		± 4	$q^{-9/31}$
		± 5	$q^{15/31}$
		± 6	$q^{3/31}$
		± 7	$q^{-14/31}(q^{-1}t^{-1} + 1 + qt)$
		± 8	$q^{-5/31}(2q^{-1}t^{-1} + 5 + 2qt)$
		± 9	$q^{-1/31}$
		± 10	$q^{-2/31}(q^{-1}t^{-1} + 1 + qt)$
		± 11	$q^{-8/31}(q^{-2}t^{-2} + 4q^{-1}t^{-1} + 5 + 4qt + q^2t^2)$
		± 12	$q^{-19/31}(q^{-1}t^{-1} + 3 + qt)$
		± 13	$q^{-4/31}(2q^{-1}t^{-1} + 3 + 2qt)$
		± 14	$q^{-25/31}$
± 15	$q^{11/31}$		
10 ₁₃₂	$\mathbb{Z}/5$	0	$q^{-2}t^{-2} + (2q^{-1} + \underline{1})t^{-1} + (2 + \underline{q}) + (2q + \underline{q^2})t + q^2t^2$
		± 1	$q^{2/5}$
		± 2	$q^{3/5}(q^{-1}t^{-1} + 1 + qt)$
10 ₁₃₃	$\mathbb{Z}/19$	0	$q^{-2}t^{-2} + 5q^{-1}t^{-1} + 7 + 5qt + q^2t^2$
		± 1	$q^{-3/19}$
		± 2	$q^{-12/19}(q^{-1}t^{-1} + 1 + qt)$
		± 3	$q^{-8/19}(q^{-1}t^{-1} + 1 + qt)$
		± 4	$q^{9/19}$
		± 5	$q^{1/19}$
		± 6	$q^{-13/19}(q^{-1}t^{-1} + 3 + qt)$
		± 7	$q^{5/19}$
		± 8	$q^{-2/19}(2q^{-1}t^{-1} + 3 + 2qt)$
± 9	$q^{-15/19}$		

K	$H_1(\Sigma_2(K); \mathbb{Z})$	s	$\sum_{i,j} \dim_{\mathbb{Z}/2} \widehat{\text{HFK}}_j(\Sigma_2(K), \tilde{K}, s, i; \mathbb{Z}/2)t^i q^j$
10 ₁₃₄	$\mathbb{Z}/23$	0	$2q^{-3}t^{-3} + 4q^{-2}t^{-2} + 4q^{-1}t^{-1} + 3 + 4qt + 4q^2t^2 + 2q^3t^3$
		±1	$q^{8/23}(q^{-1}t^{-1} + 1 + qt)$
		±2	$q^{9/23}(q^{-2}t^{-2} + q^{-1}t^{-1} + 1 + qt + q^2t^2)$
		±3	$q^{3/23}(q^{-2}t^{-2} + 2q^{-1}t^{-1} + 3 + 2qt + q^2t^2)$
		±4	$q^{-10/23}(q^{-3}t^{-3} + q^{-2}t^{-2} + q + q^2t^2 + q^3t^3)$
		±5	$q^{16/23}(q^{-1}t^{-1} + 1 + qt)$
		±6	$q^{12/23}(q^{-1}t^{-1} + 1 + qt)$
		±7	$q^{1/23}(q^{-2}t^{-2} + q^{-1}t^{-1} + 1 + qt + q^2t^2)$
		±8	$q^{29/23}$
		±9	$q^{4/23}(q^{-1}t^{-1} + 1 + qt)$
		±10	$q^{18/23}(2q^{-1}t^{-1} + 3 + 2qt)$
±11	$q^{25/23}$		
10 ₁₃₅	$\mathbb{Z}/37$	0	$3q^{-2}t^{-2} + 9q^{-1}t^{-1} + 13 + 9qt + 3q^2t^2$
		±1	$q^{14/37}$
		±2	$q^{-18/37}$
		±3	$q^{15/37}(2q^{-1}t^{-1} + 3 + 2qt)$
		±4	$q^{2/37}$
		±5	$q^{17/37}(q^{-1}t^{-1} + 1 + qt)$
		±6	$q^{-14/37}(q^{-2}t^{-2} + 2q^{-1}t^{-1} + 3 + 2qt + q^2t^2)$
		±7	$q^{-17/37}(q^{-1}t^{-1} + 1 + qt)$
		±8	$q^{8/37}(q^{-2}t^{-2} + 3q^{-1}t^{-1} + 5 + 3qt + q^2t^2)$
		±9	$q^{-13/37}(q^{-1}t^{-1} + 1 + qt)$
		±10	$q^{-6/37}$
		±11	$q^{29/37}(q^{-1}t^{-1} + 1 + qt)$
		±12	$q^{18/37}$
		±13	$q^{-2/37}$
		±14	$q^{6/37}(q^{-2}t^{-2} + 2q^{-1}t^{-1} + 3 + 2qt + q^2t^2)$
		±15	$q^{5/37}(2q^{-1}t^{-1} + 3 + 2qt)$
		±16	$q^{-5/37}(q^{-1}t^{-1} + 1 + qt)$
		±17	$q^{13/37}(q^{-1}t^{-1} + 1 + qt)$
±18	$q^{22/37}$		
10 ₁₃₆	$\mathbb{Z}/15$	1	$q^{-2}t^{-2} + 4q^{-1}t^{-1} + 6 + q + 4qt + q^2t^2$
		±1	$q^{7/15}$
		±2	$q^{13/15}(q^{-1}t^{-1} + 3 + qt)$
		±3	$q^{1/5}$
		±4	$q^{7/15}$
		±5	$q^{2/3}(q^{-1}t^{-1} + 1 + qt)$
		±6	$q^{4/5}(2q^{-1}t^{-1} + 3 + 2qt)$
±7	$q^{13/15}(q^{-1}t^{-1} + 3 + qt)$		
10 ₁₃₉	$\mathbb{Z}/3$	0	$q^{-4}t^{-4} + q^{-3}t^{-3} + 2qt^{-1} + 3q^2 + 2q^3t + q^3t^3 + q^4t^4$
		±1	$q^{5/3}(q^{-2}t^{-2} + q^{-1}t^{-1} + 1 + qt + q^2t^2)$
10 ₁₄₀	$\mathbb{Z}/9$	0	$q^{-2}t^{-2} + 2q^{-1}t^{-1} + 3 + 2qt + q^2t^2$
		±1	$q^{11/9}(q^{-1}t^{-1} + 1 + qt)$
		±2	$q^{8/9}$
		±3	1
±4	$q^{5/9}(q^{-1}t^{-1} + 1 + qt)$		
10 ₁₄₂	$\mathbb{Z}/15$	0	$q^{-3}t^{-3} + 2q^{-2}t^{-2} + 2q^{-1}t^{-1} + 1 + 2qt + 3q^2t^2 + 2q^3t^3$
		±1	$q^{1/15}(q^{-2}t^{-2} + q^{-1}t^{-1} + 1 + qt + q^2t^2)$
		±2	$q^{4/15}(q^{-1}t^{-1} + 1 + qt)$
		±3	$q^{-2/5}(q^{-3}t^{-3} + q^{-2}t^{-2} + q + q^2t^2 + q^3t^3)$
		±1	$q^{1/15}(q^{-2}t^{-2} + q^{-1}t^{-1} + 1 + qt + q^2t^2)$
		±6	$q^{2/3}(2q^{-1}t^{-1} + 3 + 2qt)$
		±2	$q^{7/5}$
±2	$q^{4/15}(q^{-1}t^{-1} + 1 + qt)$		

K	$H_1(\Sigma_2(K); \mathbb{Z})$	s	$\sum_{i,j} \dim_{\mathbb{Z}/2} \widehat{\text{HF}}\tilde{K}_j(\Sigma_2(K), \tilde{K}, s, i; \mathbb{Z}/2)t^i q^j$
10 ₁₄₅	$\mathbb{Z}/3$	0 ± 1	$q^{-2}t^{-2} + (q^{-1} + 2q)t^{-1} + q + 4q^2 + (q + 2q^3)t + q^2t^2$ $q^{4/3}(2q^{-1}t^{-1} + 3 + 2qt)$
10 ₁₄₇	$\mathbb{Z}/27$	0 ± 1 ± 2 ± 3 ± 4 ± 5 ± 6 ± 7 ± 8 ± 9 ± 10 ± 11 ± 12 ± 13	$2q^{-2}t^{-2} + 7q^{-1}t^{-1} + 9 + 7qt + 2q^2t^2$ $q^{7/27}(q^{-1}t^{-1} + 3 + qt)$ $q^{1/27}$ $q^{1/3}(2q^{-1}t^{-1} + 5 + 2qt)$ $q^{4/27}(q^{-2}t^{-2} + 3q^{-1}t^{-1} + 3 + 3qt + q^2t^2)$ $q^{13/27}$ $q^{1/3}$ $q^{19/27}(q^{-1}t^{-1} + 3 + qt)$ $q^{16/27}(q^{-1}t^{-1} + 1 + qt)$ $q^{-1}t^{-1} + 1 + qt$ $q^{25/27}$ $q^{37/27}$ $q^{1/3}$ $q^{22/27}(2q^{-1}t^{-1} + 3 + 2qt)$
10 ₁₅₈	$\mathbb{Z}/45$	0 ± 1 ± 2 ± 3 ± 4 ± 5 ± 6 ± 7 ± 8 ± 9 ± 10 ± 11 ± 12 ± 13 ± 14 ± 15 ± 16 ± 17 ± 18 ± 19 ± 20 ± 21 ± 22	$q^{-3}t^{-3} + 4q^{-2}t^{-2} + 10q^{-1}t^{-1} + 15 + 10qt + 4q^2t^2 + q^3t^3$ $q^{8/45}(q^{-1}t^{-1} + 3 + qt)$ $q^{-13/45}(q^{-2}t^{-2} + 3q^{-1}t^{-1} + 3 + 3qt + q^2t^2)$ $q^{-2/5}$ $q^{38/45}$ $q^{4/9}(q^{-1}t^{-1} + 3 + qt)$ $q^{2/5}$ $q^{-13/45}(q^{-2}t^{-2} + 3q^{-1}t^{-1} + 3 + 3qt + q^2t^2)$ $q^{17/45}(q^{-2}t^{-2} + 3q^{-1}t^{-1} + 3 + 3qt + q^2t^2)$ $q^{2/5}(2q^{-1}t^{-1} + 5 + 2qt)$ $q^{-2/9}(2q^{-1}t^{-1} + 5 + 2qt)$ $q^{-22/45}$ $q^{-2/5}$ $q^{2/45}$ $q^{38/45}$ $q^{-1}t^{-1} + 3 + qt$ $q^{-22/45}$ $q^{17/45}(q^{-2}t^{-2} + 3q^{-1}t^{-1} + 3 + 3qt + q^2t^2)$ $q^{-2/5}$ $q^{8/45}(q^{-1}t^{-1} + 3 + qt)$ $q^{1/9}(q^{-1}t^{-1} + 1 + qt)$ $q^{2/5}$ $q^{2/45}$
10 ₁₆₀	$\mathbb{Z}/21$	0 ± 1 ± 2 ± 3 ± 4 ± 5 ± 6 ± 7 ± 8 ± 9 ± 10	$q^{-3}t^{-3} + 4q^{-2}t^{-2} + 4q^{-1}t^{-1} + 3 + 4qt + 4q^2t^2 + q^3t^3$ $q^{1/21}(q^{-1}t^{-1} + 1 + qt)$ $q^{4/21}(q^{-2}t^{-2} + q^{-1}t^{-1} + 1 + qt + q^2t^2)$ $q^{3/7}(q^{-1}t^{-1} + 1 + qt)$ $q^{16/21}$ $q^{4/21}(q^{-2}t^{-2} + q^{-1}t^{-1} + 1 + qt + q^2t^2)$ $q^{5/7}(2q^{-1}t^{-1} + 3 + 2qt)$ $q^{4/3}$ $q^{1/21}(q^{-1}t^{-1} + 1 + qt)$ $q^{6/7}(q^{-1}t^{-1} + 3 + qt)$ $q^{16/21}$
10 ₁₆₁	$\mathbb{Z}/5$	0 ± 1 ± 2	$q^{-3}t^{-3} + (q^{-2} + 1)t^{-2} + 2qt^{-1} + 3q^2 + 2q^3t + (q^2 + q^4)t^2 + q^3t^3$ $q^{9/5}(2q^{-1}t^{-1} + 3 + 2qt)$ $q^{6/5}(q^{-2}t^{-2} + q^{-1}t^{-1} + 1 + qt + q^2t^2)$

K	$H_1(\Sigma_2(K); \mathbb{Z})$	s	$\sum_{i,j} \dim_{\mathbb{Z}/2} \widehat{\text{HFK}}_j(\Sigma_2(K), \tilde{K}, s, i; \mathbb{Z}/2)t^i q^j$
10 ₁₆₄	$\mathbb{Z}/45$	0	$3q^{-2}t^{-2} + 11q^{-1}t^{-1} + 17 + 11qt + 3q^2t^2$
		± 1	$q^{17/45}(q^{-1}t^{-1} + 1 + qt)$
		± 2	$q^{-22/45}$
		± 3	$q^{2/5}$
		± 4	$q^{2/45}$
		± 5	$q^{4/9}(q^{-1}t^{-1} + 3 + qt)$
		± 6	$q^{-2/5}$
		± 7	$q^{-2/45}$
		± 8	$q^{8/45}(q^{-2}t^{-2} + 3q^{-1}t^{-1} + 5 + 3qt + q^2t^2)$
		± 9	$q^{-2/5}(q^{-2}t^{-2} + 2q^{-1}t^{-1} + 3 + 2qt + q^2t^2)$
		± 10	$q^{-2/9}$
		± 11	$q^{-13/45}(q^{-1}t^{-1} + 1 + qt)$
		± 12	$q^{2/5}$
		± 13	$q^{38/45}$
		± 14	$q^{2/45}$
		± 15	$q^{-1}t^{-1} + 3 + qt$
		± 16	$q^{-13/45}(q^{-1}t^{-1} + 1 + qt)$
		± 17	$q^{8/45}(q^{-2}t^{-2} + 3q^{-1}t^{-1} + 5 + 3qt + q^2t^2)$
		± 18	$q^{2/5}$
		± 19	$q^{17/45}(q^{-1}t^{-1} + 1 + qt)$
		± 20	$q^{1/9}(2q^{-1}t^{-1} + 3 + 2qt)$
		± 21	$q^{-2/5}$
± 22	$q^{38/45}$		
11 _{n12}	$\mathbb{Z}/13$	0	$q^{-2}t^{-2} + (q^{-2} + 4q^{-1})t^{-1} + q^{-1} + 6 + (1 + 4q)t + q^2t^2$
		± 1	$q^{-2/13}$
		± 2	$q^{-8/13}(q^{-1}t^{-1} + 3 + qt)$
		± 3	$q^{-18/13}$
		± 4	$q^{-6/13}$
		± 5	$q^{-11/13}(2q^{-1}t^{-1} + 3 + 2qt)$
± 6	$q^{-7/13}(q^{-1}t^{-1} + 1 + qt)$		
11 _{n19}	$\mathbb{Z}/5$	0	$q^{-3}t^{-3} + 2q^{-2}t^{-2} + (q^{-1} + 1)t^{-1} + q + (q + q^2)t + 2q^2t^2 + q^3t^3$
		± 1	$q^{4/5}(q^{-2}t^{-2} + q^{-1}t^{-1} + 1 + qt + q^2t^2)$
		± 2	$q^{6/5}(q^{-1}t^{-1} + 3 + qt)$
11 _{n20}	$\mathbb{Z}/23$	0	$2q^{-2}t^{-2} + 6q^{-1}t^{-1} + 8 + q + 6qt + 2q^2t^2$
		± 1	$q^{17/23}(q^{-1}t^{-1} + 3 + qt)$
		± 2	$q^{-1/23}$
		± 3	$q^{-8/23}(q^{-2}t^{-2} + 2q^{-1}t^{-1} + 2 + q + 2qt + q^2t^2)$
		± 4	$q^{19/23}(2q^{-1}t^{-1} + 5 + 2qt)$
		± 5	$q^{11/23}$
		± 6	$q^{14/23}(q^{-1}t^{-1} + 1 + qt)$
		± 7	$q^{5/23}(q^{-1}t^{-1} + 3 + qt)$
		± 8	$q^{7/23}$
		± 9	$q^{20/23}(2q^{-1}t^{-1} + 3 + 2qt)$
		± 10	$q^{21/23}(q^{-1}t^{-1} + 3 + qt)$
± 11	$q^{10/23}(q^{-1}t^{-1} + 1 + qt)$		
11 _{n38}	$\mathbb{Z}/3$	0	$q^{-2}t^{-2} + (2q^{-1} + 1)t^{-1} + 2 + 3q + (2q + q^2)t + q^2t^2$
		± 1	$q^{4/3}(q^{-1}t^{-1} + 1 + qt)$
11 _{n49}	{0}	0	$q^{-2}t^{-2} + (4q^{-3} + 2q^{-1})t^{-1} + 9q^{-2} + 2 + (4q^{-1} + 2q)t + q^2t^2$

K	$H_1(\Sigma_2(K); \mathbb{Z})$	s	$\sum_{i,j} \dim_{\mathbb{Z}/2} \widehat{\text{HFK}}_j(\Sigma_2(K), \tilde{K}, s, i; \mathbb{Z}/2)t^i q^j$
11n ₉₅	$\mathbb{Z}/33$	0	$q^{-3}t^{-3} + 5q^{-2}t^{-2} + 7q^{-1}t^{-1} + 7 + 7qt + 5q^2t^2 + q^3t^3$
		±1	$q^{-13/33}(q^{-1}t^{-1} + 1 + qt)$
		±2	$q^{14/33}$
		±3	$q^{5/11}(q^{-1}t^{-1} + 1 + qt)$
		±4	$q^{-10/33}(q^{-2}t^{-2} + q^{-1}t^{-1} + 1 + qt + q^2t^2)$
		±5	$q^{5/33}(2q^{-1}t^{-1} + 3 + 2qt)$
		±6	$q^{-2/11}(q^{-2}t^{-2} + q^{-1}t^{-1} + 1 + qt + q^2t^2)$
		±7	$q^{-10/33}(q^{-2}t^{-2} + q^{-1}t^{-1} + 1 + qt + q^2t^2)$
		±8	$q^{26/33}$
		±9	$q^{1/11}(q^{-1}t^{-1} + 1 + qt)$
		±10	$q^{-13/33}(q^{-1}t^{-1} + 1 + qt)$
		±11	$q^{1/3}(q^{-1}t^{-1} + 1 + qt)$
		±12	$q^{3/11}(2q^{-1}t^{-1} + 3 + 2qt)$
		±13	$q^{14/33}$
		±14	$q^{26/33}$
		±15	$q^{4/11}(q^{-1}t^{-1} + 3 + qt)$
±16	$q^{5/33}(2q^{-1}t^{-1} + 3 + 2qt)$		
11n ₁₀₂	$\mathbb{Z}/3$	0	$q^{-2}t^{-2} + (5q^{-1} + 2q)t^{-1} + 7 + 4q^2 + (5q + 2q^3)t + q^2t^2$
		±1	$q^{1/3}(2q^{-1}t^{-1} + 5 + 2qt)$
11n ₁₁₆	{0}	0	$q^{-2}t^{-2} + (4q^{-3} + 2q^{-1})t^{-1} + 9q^{-2} + 2 + (4q^{-1} + 2q)t + q^2t^2$
11n ₁₁₇	$\mathbb{Z}/35$	0	$3q^{-2}t^{-2} + 9q^{-1}t^{-1} + 11 + 9qt + 3q^2t^2$
		±1	$q^{9/35}(2q^{-1}t^{-1} + 5 + 2qt)$
		±2	$q^{1/35}$
		±3	$q^{11/35}(q^{-1}t^{-1} + 3 + qt)$
		±4	$q^{4/35}(q^{-2}t^{-2} + 3q^{-1}t^{-1} + 3 + 3qt + q^2t^2)$
		±5	$q^{3/7}(q^{-1}t^{-1} + 3 + qt)$
		±6	$q^{9/35}(2q^{-1}t^{-1} + 5 + 2qt)$
		±7	$q^{-2/5}(q^{-2}t^{-2} + 2q^{-1}t^{-1} + 2 + q + 2qt + q^2t^2)$
		±8	$q^{16/35}(q^{-1}t^{-1} + 1 + qt)$
		±9	$q^{29/35}$
		±10	$q^{5/7}(2q^{-1}t^{-1} + 5 + 2qt)$
		±11	$q^{4/35}(q^{-2}t^{-2} + 3q^{-1}t^{-1} + 3 + 3qt + q^2t^2)$
		±12	$q^{1/35}$
		±13	$q^{16/35}(q^{-1}t^{-1} + 1 + qt)$
		±14	$q^{7/5}$
		±15	$q^{6/7}(2q^{-1}t^{-1} + 3 + 2qt)$
		±16	$q^{29/35}$
±17	$q^{11/35}(q^{-1}t^{-1} + 3 + qt)$		
11n ₁₁₈	$\mathbb{Z}/21$	0	$q^{-3}t^{-3} + 4q^{-2}t^{-2} + 4q^{-1}t^{-1} + 3 + 4qt + 4q^2t^2 + q^3t^3$
		±1	$q^{5/21}(q^{-1}t^{-1} + 1 + qt)$
		±2	$q^{20/21}$
		±3	$q^{1/7}(2q^{-1}t^{-1} + 3 + 2qt)$
		±4	$q^{-4/21}(q^{-2}t^{-2} + q^{-1}t^{-1} + 1 + qt + q^2t^2)$
		±5	$q^{20/21}$
		±6	$q^{4/7}$
		±7	$q^{-1/3}(q^{-1}t^{-1} + 1 + qt)$
		±8	$q^{5/21}(q^{-1}t^{-1} + 1 + qt)$
		±9	$q^{2/7}(q^{-1}t^{-1} + 3 + qt)$
±10	$q^{-4/21}(q^{-2}t^{-2} + q^{-1}t^{-1} + 1 + qt + q^2t^2)$		

K	$H_1(\Sigma_2(K); \mathbb{Z})$	s	$\sum_{i,j} \dim_{\mathbb{Z}/2} \widehat{\text{HFK}}_j(\Sigma_2(K), \tilde{K}, s, i; \mathbb{Z}/2)t^i q^j$
$11n_{122}$	$\mathbb{Z}/27$	0	$2q^{-2}t^{-2} + 7q^{-1}t^{-1} + 9 + 2qt + 2q^2t^2$
		± 1	$q^{13/27}$
		± 2	$q^{-2/27}(2q^{-1}t^{-1} + 3 + 2qt)$
		± 3	$q^{1/3}(2q^{-1}t^{-1} + 5 + 2qt)$
		± 4	$q^{-8/27}(q^{-1}t^{-1} + 1 + qt)$
		± 5	$q^{1/27}$
		± 6	$q^{1/3}$
		± 7	$q^{-11/27}$
		± 8	$q^{-5/27}(q^{-1}t^{-1} + 3 + qt)$
		± 9	$q^{-1}t^{-1} + 1 + qt$
		± 10	$q^{-23/27}$
		± 11	$q^{-20/27}(q^{-2}t^{-2} + 3q^{-1}t^{-1} + 3 + 3qt + q^2t^2)$
		± 12	$q^{1/3}$
± 13	$q^{-17/27}(q^{-1}t^{-1} + 3 + qt)$		
$11n_{138}$	$\mathbb{Z}/15$	0	$2q^{-2}t^{-2} + 4q^{-1}t^{-1} + (q^{-1} + 4) + 4qt + 2q^2t^2$
		± 1	$q^{-7/15}$
		± 2	$q^{-13/15}(q^{-1}t^{-1} + 3 + qt)$
		± 3	$q^{-1/5}((q^{-2} + 2q^{-1})t^{-1} + (q^{-1} + 4) + (1 + 2q)t)$
		± 4	$q^{-7/15}$
		± 5	$q^{-2/3}(q^{-2}t^{-2} + 3q^{-1}t^{-1} + 3 + 3qt + q^2t^2)$
		± 7	$q^{-13/15}(q^{-1}t^{-1} + 3 + qt)$
$11n_{139}$	$\mathbb{Z}/9$	0	$2q^{-1}t^{-1} + 5 + 2qt$
		± 1	$q^{-4/9}$
		± 2	$q^{-16/9}$
		± 4	$q^{-10/9}(q^{-1}t^{-1} + 3 + qt)$
$11n_{141}$	$\mathbb{Z}/21$	0	$5q^{-1}t^{-1} + 11 + 5qt$
		± 1	$q^{-10/21}$
		± 2	$q^{2/21}(2q^{-1}t^{-1} + 5 + 2qt)$
		± 3	$q^{-2/7}(2q^{-1}t^{-1} + 5 + 2qt)$
		± 4	$q^{8/21}(q^{-1}t^{-1} + 3 + qt)$
		± 5	$q^{2/21}(2q^{-1}t^{-1} + 5 + 2qt)$
		± 6	$q^{6/7}$
		± 7	$q^{2/3}(2q^{-1}t^{-1} + 5 + 2qt)$
		± 8	$q^{-10/21}$
		± 9	$q^{10/7}$
± 10	$q^{8/21}(q^{-1}t^{-1} + 3 + qt)$		
$11n_{142}$	$\mathbb{Z}/33$	0	$q^{-2}t^{-2} + 8q^{-1}t^{-1} + 15 + 8qt + q^2t^2$
		± 1	$q^{2/33}(q^{-1}t^{-1} + 3 + qt)$
		± 2	$q^{8/33}(2q^{-1}t^{-1} + 5 + 2qt)$
		± 3	$q^{6/11}(q^{-1}t^{-1} + 3 + qt)$
		± 4	$q^{32/33}$
		± 5	$q^{-16/33}$
		± 6	$q^{2/11}(q^{-1}t^{-1} + 3 + qt)$
		± 7	$q^{32/33}$
		± 8	$q^{-4/33}$
		± 9	$q^{10/11}(q^{-1}t^{-1} + 3 + qt)$
		± 10	$q^{2/33}(q^{-1}t^{-1} + 3 + qt)$
		± 11	$q^{4/3}$
		± 12	$q^{8/11}(2q^{-1}t^{-1} + 5 + 2qt)$
		± 13	$q^{8/33}(2q^{-1}t^{-1} + 5 + 2qt)$
		± 14	$q^{-4/33}$
		± 15	$q^{-4/11}(2q^{-1}t^{-1} + 5 + 2qt)$
± 16	$q^{-16/33}$		

K	$H_1(\Sigma_2(K); \mathbb{Z})$	\mathfrak{s}	$\sum_{i,j} \dim_{\mathbb{Z}/2} \widehat{\text{HFK}}_j(\Sigma_2(K), \tilde{K}, \mathfrak{s}, i; \mathbb{Z}/2) t^i q^j$
$11n_{143}$	$\mathbb{Z}/9$	0	$q^{-3}t^{-3} + (q^{-4} + 3q^{-2})t^{-2} + (2q^{-3} + 3q^{-1})t^{-1} + (2q^{-2} + 3)$ $+ (2q^{-1} + 3q)t + (\underline{1} + 3q^2)t^2 + q^3t^3$ $\pm 1 \quad q^{-10/9}((q^{-1} + 1)t^{-1} + (2 + q) + (1 + q^2)t)$ $\pm 2 \quad q^{-4/9}$ $\pm 3 \quad 1$ $\pm 4 \quad q^{-7/9}(q^{-2}t^{-2} + 3q^{-1}t^{-1} + 3 + 3qt + q^2t^2)$
$11n_{145}$	$\mathbb{Z}/9$	0	$q^{-3}t^{-3} + (2q^{-2} + \underline{1})t^{-2} + (q^{-1} + 4q)t^{-1} + 7q^2 + (q + 4q^3)t$ $+ (2q^2 + t^4)t^2 + q^3t^3$ $\pm 1 \quad q^{10/9}(q^{-2}t^{-2} + 3q^{-1}t^{-1} + 5 + 3qt + q^2t^2)$ $\pm 2 \quad q^{22/9}$ $\pm 3 \quad q^2$ $\pm 4 \quad q^{16/9}(q^{-2}t^{-2} + 3q^{-1}t^{-1} + 5 + 3qt + q^2t^2)$

6 Observations

Grigsby [7] showed that when $K \subset S^3$ is a two-bridge knot, the Heegaard Floer knot homology of $\tilde{K} \subset \Sigma_2(K)$ in the canonical spin^c structure is isomorphic as a bigraded $\mathbb{Z}/2$ -vector space to that of $K \subset S^3$: ie, $\widehat{\text{HFK}}(\Sigma_2(K), \tilde{K}, \mathfrak{s}_0) \cong \widehat{\text{HFK}}(S^3, K)$, up to an overall shift in the Maslov grading. Our results suggest that the same is true for a wider class of knots. Specifically, define the δ -grading on $\widehat{\text{HFK}}(Y, K, \mathfrak{s})$ as the difference between the Alexander and Maslov gradings. We say that $\widehat{\text{HFK}}(Y, K, \mathfrak{s})$ is *thin* if it is supported in a single δ -grading. We make the following conjecture.

Conjecture 6.1 *Let $K \subset S^3$ be a knot for which $\widehat{\text{HFK}}(S^3, K)$ is thin. Then*

$$\widehat{\text{HFK}}(\Sigma_2(K), \tilde{K}, \mathfrak{s}_0) \cong \widehat{\text{HFK}}(S^3, K)$$

as bigraded groups, up to a possible shift in the absolute Maslov grading.

It is well-known (Ozsváth–Szabó [14] or Rasmussen [17]) that $\widehat{\text{HFK}}(S^3, K)$ is thin whenever K is alternating (and hence for all two-bridge knots). More generally, let \mathcal{Q} be the smallest set of link types such that:

- The unknot is in \mathcal{Q} .
- Suppose L admits a projection such that the two resolutions at some crossing, L_0 and L_1 , are both in \mathcal{Q} and satisfy $\det(L_0) + \det(L_1) = \det(L)$. Then L is in \mathcal{Q} .

The links in \mathcal{Q} are called *quasi-alternating*; for instance, any alternating link is quasi-alternating. Manolescu and Ozsváth [11] that whenever L is quasi-alternating, both $\widehat{\text{HFK}}(S^3, L)$ and the Khovanov homology of L are thin. Conjecture 6.1 would then imply that $\widehat{\text{HFK}}(\Sigma_2(K), \tilde{K}, \mathfrak{s}_0)$ is thin whenever K is quasi-alternating.

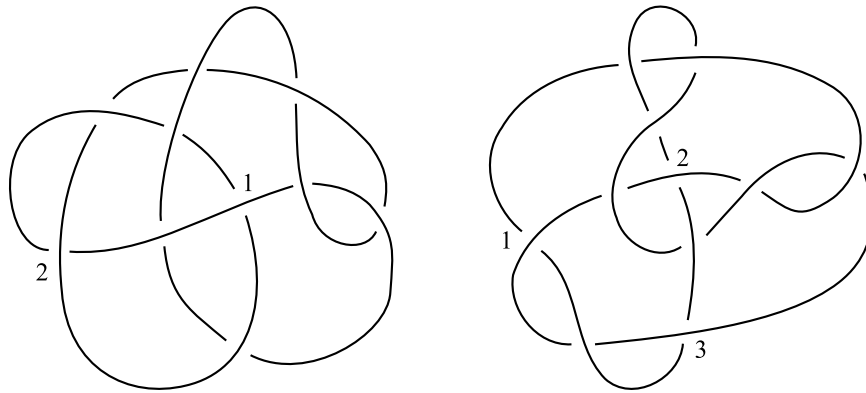


Figure 2: To see that the knots 10_{134} (left) and $11n_{117}$ (right) are quasi-alternating, resolve the marked crossings in the order indicated.

One may also ask under what conditions $\widehat{\text{HFK}}(\Sigma_2(K), \tilde{K}, \mathfrak{s})$ is thin for spin^c structures $\mathfrak{s} \neq \mathfrak{s}_0$. The knots 10_{134} and $11n_{117}$ both satisfy the hypothesis and conclusion of Conjecture 6.1. Indeed, they are both quasi-alternating, as illustrated in Figure 2. However, each one admits a spin^c structure \mathfrak{s} on $\Sigma_2(K)$ in which $\widehat{\text{HFK}}(S^2(K), \tilde{K}, \mathfrak{s})$ is not thin. There are no known examples of alternating knots for which this phenomenon occurs, though.

On the other hand, when $\widehat{\text{HFK}}(S^3, K)$ is not thin, the isomorphism between $\widehat{\text{HFK}}(S^3, K)$ and $\widehat{\text{HFK}}(\Sigma_2(K), \tilde{K}, \mathfrak{s}_0)$ generally fails. A few patterns are worth mentioning. Note that for the knots considered here, in each Alexander grading i , the total rank of $\widehat{\text{HFK}}(\Sigma_2(K), \tilde{K}, \mathfrak{s}_0, i)$ is at least that of $\widehat{\text{HFK}}(S^3, K, i)$, and the two ranks are congruent modulo 2. Some examples in which the ranks fail to be equal are $11n_{49}$, $11n_{102}$, and $11n_{116}$. Even when the total ranks of $\widehat{\text{HFK}}(\Sigma_2(K), \tilde{K}, \mathfrak{s}_0, i)$ and $\widehat{\text{HFK}}(S^3, K, i)$ are the same for all i , the relative Maslov gradings can differ. A common pattern is that the Maslov gradings of all the groups in one δ -grading of $\widehat{\text{HFK}}(S^3, K)$ are shifted by a constant amount in $\widehat{\text{HFK}}(\Sigma_2(K), \tilde{K}, \mathfrak{s}_0)$, such as with the knots 9_{42} and 10_{161} , where the groups are shifted by 2 and 3, respectively. However, there are also examples where the relative Maslov gradings in different Alexander gradings change in different ways. For example, for 10_{145} , the total ranks of $\widehat{\text{HFK}}(S^3, K, i)$ and $\widehat{\text{HFK}}(\Sigma_2(K), \tilde{K}, \mathfrak{s}_0, i)$ are the same for each i , but $\widehat{\text{HFK}}(S^3, K)$ is supported in two δ -gradings while $\widehat{\text{HFK}}(\Sigma_2(K), \tilde{K}, \mathfrak{s}_0)$ is supported in three.

Finally, note that the pretzel knots $8_{20} = P(2, 3, -3)$ and $10_{140} = P(4, 3, -3)$ have identical knot Floer homology but can be distinguished by $\widehat{\text{HFK}}(\Sigma_2(K), \tilde{K})$. The relative Maslov gradings between spin^c structures are necessary in this case. For another such example, see Grigsby [7].

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