The curvature of contact structures on 3–manifolds

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We study the sectional curvature of plane distributions on 3–manifolds. We show that if a distribution is a contact structure it is easy to manipulate its curvature. As a corollary we obtain that for every transversally oriented contact structure on a closed 3–dimensional manifold, there is a metric such that the sectional curvature of the contact distribution is equal to -1. We also introduce the notion of Gaussian curvature of the plane distribution. For this notion of curvature we get similar results.

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1 Introduction

The problem of prescribing the curvatures of a manifold is one of the central problems in Riemannian geometry. That is, given a smooth function can it be realized as a scalar (Ricci or sectional) curvature of some Riemannian metric on a manifold. The solution of the Yamabe problem is the best known result in prescribing the scalar curvature on a manifold (cf Lee and Parker [4]). There are several results on prescribing the Ricci curvature of a manifold (cf for example Lohkamp [5]). It is natural to ask to what extent it is possible to prescribe the sectional curvature of the plane distribution on a 3-manifold. It turns out that this problem is closely connected with the contactness of the distribution. In fact we have the following:

Theorem A *Let ξ be a transversally orientable contact structure on a closed orientable 3–manifold M. For any smooth strictly negative function f, there is a metric on M such that f is the sectional curvature of ξ.*

If we impose more topological restrictions on the distribution we can obtain an even stronger result:

Theorem B *Let ξ be a transversally orientable contact structure on M with Euler class zero. Then for any smooth function f, there is a metric on M such that f is a sectional curvature of ξ.*

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In [2], Chern and Hamilton studied a similar problem of prescribing the so-called Webster curvature W on a contact three-manifold. The main difference in their approach is that they restrict the class of metrics to the metrics which are adapted to a contact structure, while we deal with the class of all metrics. They prove that in their class one can either find a metric with the constant negative Webster curvature or a metric with strictly positive Webster curvature.

It is a well-known problem whether a foliation on a 3–dimensional manifold admits a simultaneous uniformization of all its leaves. The Reeb stability theorem asserts that on a compact orientable 3–manifold the only foliation with the leaves having positive Gaussian curvature is the foliation of $M = S^2 \times S^1$ by spheres. It is known (see Candel [1]) that if M is atoroidal and aspherical and the foliation is taut, then there is a metric on M such that all leaves have constant negative Gaussian curvature -1. In the case of contact structures we ask a similar question. For this we have to introduce the notion of Gaussian curvature of the plane distribution.

We define the Gaussian curvature of the plane distribution as the sum $K_G(\xi) = K(\xi) + K_e(\xi)$ of the sectional and the extrinsic curvatures of the distribution. In the case of integrable ξ this equation is nothing but the Gauss equation.

Definition 1.1 Let ξ be a plane distribution on M. We say that ξ admits a uniformization if there is a metric on M such that the Gaussian curvature of ξ is constant.

It turns out that unlike the case of foliations, every transversally orientable contact structure on a closed 3–manifold admits a uniformization. We have the following:

Theorem C *Let ξ be a transversally orientable contact structure on a closed orientable 3–manifold M. For any smooth strictly negative function f, there is a metric on M such that f is the Gaussian curvature of ξ.*

This paper is organized as follows. In Section 2 we recall basic facts about the geometry of plane distributions. In Section 3 we prove the main technical lemma. Section 4 is devoted to the proof of Theorem A and Theorem B. We prove Theorem C in Section 5.

Acknowledgment I would like to thank Patrick Massot for pointing out Corollary 3.6. This led to a much stronger and natural formulation of Theorem B.

2 Basic definitions and notation

Throughout this paper M will be a closed orientable 3–manifold. A distribution on M is a two dimensional subbundle of the tangent bundle of M. That is, at each point p

in M there is a plane ξ_p in the tangent space $T_p M$. A distribution is called integrable, if there is a foliation on M which is tangent to it. The following Frobenius theorem gives necessary and sufficient conditions for ξ to be integrable.

Theorem 2.1 *Let ξ be a distribution on M. Then ξ is integrable if and only if for any two sections S and T of ξ its Lie bracket belongs to ξ.*

Definition 2.2 A distribution ξ is called a contact structure if for any linearly independent sections S and T of ξ and for any $p \in M$ the Lie bracket $[S, T]$ at p does not belong to ξ_p.

A distribution ξ is called transversally oriented if there is a globally defined 1–form α such that $\xi = \mathrm{Ker}(\alpha)$. This is equivalent to say that there exists a globally defined vector field n which is transverse to ξ. It is an easy consequence of Frobenius Theorem that ξ is a contact structure if and only if

$$\alpha \wedge d\alpha \neq 0.$$

Fix some orientation on M. A contact structure is said to be positive (resp. negative) if the orientation induced by $\alpha \wedge d\alpha$ coincides (resp. is opposite to) the orientation on M.

A contact structure ξ is called overtwisted, if there is an embedded disk such that $TD|_{\partial D} = \xi|_{\partial D}$. If ξ is not overtwisted, it is called tight.

The Euler class $e(\xi) \in H^2(M, \mathbb{Z})$ of a plane distribution is the Euler class of the bundle $\xi \to M$. It is known that if ξ is a 2–dimensional plane distribution on M with vanishing Euler class then ξ is trivial. Recall, that a framing of M is the presentation of the tangent bundle of M as a product $TM \simeq M \times \mathbb{R}^3$. A framing on M consists of three linearly independent vector fields. It is known that every closed orientable 3–manifold admits a framing.

A bi-contact structure on M is a pair (ξ, η) of transverse contact structures which define opposite orientation on M.

Assume that M is a Riemannian manifold with the metric $\langle \cdot, \cdot \rangle$ and the Levi-Civita connection ∇. Let n be a local unit vector field orthogonal to ξ. We are now going to define the second fundamental form of ξ. The definition is due to Reinhart [7].

Definition 2.3 The second fundamental form of ξ is a symmetric bilinear form, which is defined in the following way:

$$B(S, T) = \frac{1}{2} \langle \nabla_S T + \nabla_T S, n \rangle$$

for all sections S and T of ξ.

Remark 2.4 If ξ is integrable, then B restricted to the leaf of ξ agrees with the second fundamental form of the leaf.

Let S and T be two linearly independent sections of ξ.

Definition 2.5 We call the function

$$K_e(\xi) = \frac{B(S,S)B(T,T) - B(S,T)^2}{\langle S,S\rangle\langle T,T\rangle - \langle S,T\rangle^2}$$

an extrinsic curvature of ξ.

It is easy to verify that $K_e(\xi)$ depends only on ξ, not on the actual choice of S, T and n.

Definition 2.6 Consider the function $K(\xi)$ which assigns to a point $p \in M$ the sectional curvature of the plane ξ_p. We call this function the sectional curvature of ξ.

Definition 2.7 We call the sum $K_G(\xi) = K(\xi) + K_e(\xi)$ the Gaussian curvature of ξ.

Let S, T and U be the local sections of TM. Recall the Koszul formula for the Levi-Civita connection of $\langle\cdot,\cdot\rangle$:

$$\begin{aligned}2\langle\nabla_S T, U\rangle = S\langle T,U\rangle + T\langle U,S\rangle - U\langle S,T\rangle \\ + \langle[S,T],U\rangle - \langle[S,U],T\rangle - \langle[T,U],S\rangle\end{aligned}$$

3 The deformation of metric

In this section we will give the proof of the main technical results we will need throughout the paper.

Let ξ be a transversally orientable plane distribution on a 3–dimensional Riemannian manifold $(M,\langle\cdot,\cdot\rangle)$. Fix a unit normal vector field n. Suppose a is a strictly positive smooth function on M. A stretching of $\langle\cdot,\cdot\rangle$ along n by the function a is the following Riemannian metric on M:

$$\langle\cdot,\cdot\rangle_a = a\langle\cdot,\cdot\rangle|_n \oplus \langle\cdot,\cdot\rangle|_\xi$$

Our aim is to calculate the sectional curvature of ξ in the stretched metric in terms of the initial metric.

Consider an open subset $U \subset M$ such that $\xi|_U$ is a trivial fibration. Let X and Y be a pair of orthonormal sections of $\xi|_U$. The triple (X, Y, n) is an orthonormal framing on U with respect to $\langle \cdot, \cdot \rangle$.

In the stretched metric this frame is orthogonal, vector fields X and Y are unit and the length of n is equal to $\sqrt{a}$. Denote by ∇ the Levi-Civita connection of $\langle \cdot, \cdot \rangle_a$.

Lemma 3.1 *The sectional curvature of ξ with respect to $\langle \cdot, \cdot \rangle_a$ can be calculated by the following formula:*

$$K(\xi) = -\frac{3}{4}a\langle [X, Y], n\rangle^2 + P + \frac{1}{a}Q$$

where
$$P = X\langle [X, Y], Y\rangle - Y\langle [X, Y], X\rangle - \langle [X, Y], X\rangle^2 - \langle [X, Y], Y\rangle^2 + \frac{1}{2}\langle [X, Y], n\rangle(-\langle [n, Y], X\rangle + \langle [n, X], Y\rangle))$$
and
$$Q = \frac{1}{4}(\langle [X, n], Y\rangle + \langle [Y, n], X\rangle)^2 - \langle [Y, n], Y\rangle\langle [X, n], X\rangle$$

Proof Since X and Y are unit, the sectional curvature of ξ is calculated by the formula:

$$K(\xi) = \langle R(X, Y)Y, X\rangle_a = \langle \nabla_X \nabla_Y Y, X\rangle_a - \langle \nabla_Y \nabla_X Y, X\rangle_a - \langle \nabla_{[X,Y]} Y, X\rangle_a$$

The first summand can be rewritten:

$$\langle \nabla_X \nabla_Y Y, X\rangle_a = X\langle \nabla_Y Y, X\rangle_a - \langle \nabla_Y Y, \nabla_X X\rangle_a$$

Apply the Koszul formula to $X\langle \nabla_Y Y, X\rangle_a$. We get:

$$\begin{aligned} X\langle \nabla_Y Y, X\rangle_a &= \frac{1}{2}X(2Y\langle Y, X\rangle_a - X\langle Y, Y\rangle_a + \langle [Y, Y], X\rangle_a - 2\langle [Y, X], Y\rangle_a) \\ &= -X\langle [Y, X], Y\rangle_a = -X\langle [Y, X], Y\rangle \end{aligned}$$

Decompose the vector field $\nabla_Y Y$ with respect to the frame $(X, Y, n/\sqrt{a})$ orthonormal in the stretched metric $\langle \cdot, \cdot \rangle_a$:

$$\nabla_Y Y = \langle \nabla_Y Y, \frac{n}{\sqrt{a}}\rangle_a \frac{n}{\sqrt{a}} + \langle \nabla_Y Y, Y\rangle_a Y + \langle \nabla_Y Y, X\rangle_a X$$

Substituting these expressions into $\langle \nabla_X \nabla_Y Y, X\rangle_a$, we obtain:

$$\begin{aligned} \langle \nabla_X \nabla_Y Y, X\rangle_a = -X\langle [Y, X], Y\rangle - \Big\langle \langle \nabla_Y Y, n\rangle_a \frac{n}{a} + \langle \nabla_Y Y, Y\rangle_a Y \\ + \langle \nabla_Y Y, X\rangle_a X, \nabla_X X\Big\rangle_a \end{aligned}$$

Since X and Y are of unit length this reduces to:

$$\langle \nabla_X \nabla_Y Y, X\rangle_a = -X\langle [Y,X], Y\rangle - \frac{1}{a}\langle \nabla_Y Y, n\rangle_a \langle \nabla_X X, n\rangle_a$$

Apply the Koszul formula to the term $\langle \nabla_Y Y, n\rangle_a \langle \nabla_X X, n\rangle_a$. Finally, we have:

$$\begin{aligned}\langle \nabla_X \nabla_Y Y, X\rangle_a &= -X\langle [Y,X], Y\rangle - \frac{1}{a}\langle [Y,n], Y\rangle_a \langle [X,n], X\rangle_a \\ &= -X\langle [Y,X], Y\rangle - \frac{1}{a}\langle [Y,n], Y\rangle \langle [X,n], X\rangle\end{aligned}$$

The second summand is equal to:

$$\begin{aligned}-\langle \nabla_Y \nabla_X Y, X\rangle_a &= -Y\langle \nabla_X Y, X\rangle_a + \langle \nabla_X Y, \nabla_Y X\rangle_a \\ &= Y\langle Y, \nabla_X X\rangle_a + \Big\langle \langle \nabla_X Y, n\rangle_a \frac{n}{a} + \langle \nabla_X Y, Y\rangle_a Y \\ &\qquad\qquad + \langle \nabla_X Y, X\rangle_a X, \nabla_Y X\Big\rangle_a \\ &= -Y\langle [X,Y], X\rangle_a + \frac{1}{a}\langle \nabla_X Y, n\rangle_a \langle \nabla_Y X, n\rangle_a\end{aligned}$$

Write the equations for the terms $\langle \nabla_X Y, n\rangle_a$ and $\langle \nabla_Y X, n\rangle_a$:

$$\begin{aligned}2\langle \nabla_X Y, n\rangle_a &= \langle [X,Y], n\rangle_a - \langle [X,n], Y\rangle_a - \langle [Y,n], X\rangle_a \\ &= a\langle [X,Y], n\rangle - \langle [X,n], Y\rangle - \langle [Y,n], X\rangle\end{aligned}$$

$$\begin{aligned}2\langle \nabla_Y X, n\rangle_a &= \langle [Y,X], n\rangle_a - \langle [Y,n], X\rangle_a - \langle [X,n], Y\rangle_a \\ &= a\langle [Y,X], n\rangle - \langle [Y,n], X\rangle - \langle [X,n], Y\rangle\end{aligned}$$

Inserting the above equations into the second summand we have:

$$\begin{aligned}-\langle \nabla_Y \nabla_X Y, X\rangle_a = -Y\langle [X,Y], X\rangle_a + \frac{1}{4a}\big(-a\langle [X,Y], n\rangle + \langle [X,n], Y\rangle + \langle [Y,n], X\rangle\big) \\ \cdot\big(-a\langle [Y,X], n\rangle + \langle [Y,n], X\rangle + \langle [X,n], Y\rangle\big)\end{aligned}$$

The last summand is:

$$\begin{aligned}-\langle \nabla_{[X,Y]} Y, X\rangle_a &= -\langle \nabla_{\langle [X,Y],n\rangle n + \langle [X,Y],X\rangle X + \langle [X,Y],Y\rangle Y} Y, X\rangle_a \\ &= -\langle [X,Y], n\rangle \langle \nabla_n Y, X\rangle_a - \langle [X,Y], X\rangle \langle \nabla_X Y, X\rangle_a \\ &\quad - \langle [X,Y], Y\rangle \langle \nabla_Y Y, X\rangle_a\end{aligned}$$

The term $\langle \nabla_n Y, X\rangle_a$ is equal to

$$\begin{aligned}\langle \nabla_n Y, X\rangle_a &= -\frac{1}{2}\big(-\langle [n,Y],X\rangle_a + \langle [n,X],Y\rangle_a + \langle [Y,X],n\rangle_a\big)\\ &= -\frac{1}{2}\big(-\langle [n,Y],X\rangle + \langle [n,X],Y\rangle + a\langle [Y,X],n\rangle\big)\end{aligned}$$

which gives us:

$$\begin{aligned}-\langle \nabla_{[X,Y]} Y, X\rangle_a &= -\langle [X,Y],n\rangle\langle \nabla_n Y, X\rangle_a - \langle [X,Y],X\rangle\langle \nabla_X Y, X\rangle_a\\ &\qquad - \langle [X,Y],Y\rangle\langle \nabla_Y Y, X\rangle_a\\ &= \frac{1}{2}\langle [X,Y],n\rangle\big(-\langle [n,Y],X\rangle + \langle [n,X],Y\rangle + a\langle [Y,X],n\rangle\big)\\ &\qquad - \langle [X,Y],X\rangle^2 - \langle [X,Y],Y\rangle^2\end{aligned}$$

Summing this up, the sectional curvature of ξ is equal to:

$$\begin{aligned}K(\xi) = &- X\langle [Y,X],Y\rangle - \frac{1}{a}\langle [Y,n],Y\rangle\langle [X,n],X\rangle\\ &- \Big(Y\langle [X,Y],X\rangle - \frac{1}{4a}\big(-a\langle [X,Y],n\rangle + \langle [X,n],Y\rangle + \langle [Y,n],X\rangle\big)\\ &\qquad \cdot\big(-a\langle [Y,X],n\rangle + \langle [Y,n],X\rangle + \langle [X,n],Y\rangle\big)\Big)\\ &- \Big(-\frac{1}{2}\langle [X,Y],n\rangle\big(-\langle [n,Y],X\rangle + \langle [n,X],Y\rangle + a\langle [Y,X],n\rangle\big)\\ &\qquad + \langle [X,Y],X\rangle^2 + \langle [X,Y],Y\rangle^2\Big)\end{aligned}$$

It is straightforward to verify that this gives us the desired expression. □

Lemma 3.2 *The extrinsic curvature $K_e(\xi)$ with respect to $\langle\cdot,\cdot\rangle_a$ can be calculated by the following formula:*

$$K_e(\xi) = \frac{1}{a}\Big(\langle [X,n],X\rangle\langle [Y,n],Y\rangle - \frac{1}{4}\big(\langle [X,n],Y\rangle + \langle [Y,n],X\rangle\big)^2\Big)$$

Proof Since X and Y are unit vectors, the extrinsic curvature is given by:

$$K_e(\xi) = B(X,X)B(Y,Y) - B(X,Y)^2$$

By the definition of B, the extrinsic curvature is equal to:

$$K_e(\xi) = \langle \nabla_X X, \frac{n}{\sqrt{a}}\rangle_a \langle \nabla_Y Y, \frac{n}{\sqrt{a}}\rangle_a - \frac{1}{4}\langle \nabla_X Y + \nabla_Y X, \frac{n}{\sqrt{a}}\rangle_a^2$$

Apply the Koszul formula to

$$\langle \nabla_X X, \frac{n}{\sqrt{a}} \rangle_a, \quad \langle \nabla_Y Y, \frac{n}{\sqrt{a}} \rangle_a \quad \text{and} \quad \langle \nabla_X Y + \nabla_Y X, \frac{n}{\sqrt{a}} \rangle_a$$

to obtain:

$$\begin{aligned} K_e(\xi) &= \frac{1}{a}\Big(\langle [X,n], X\rangle_a \langle [Y,n], Y\rangle_a - \frac{1}{4}\big(\frac{1}{2}\langle [X,Y], n\rangle_a - \frac{1}{2}\langle [X,n], Y\rangle_a \\ &\qquad - \frac{1}{2}\langle [Y,n], X\rangle_a - \frac{1}{2}\langle [X,Y], n\rangle_a - \frac{1}{2}\langle [Y,n], X\rangle_a - \frac{1}{2}\langle [X,n], Y\rangle_a\big)^2 \Big) \\ &= \frac{1}{a}\Big(\langle [X,n], X\rangle \langle [Y,n], Y\rangle - \frac{1}{4}\big(\langle [X,n], Y\rangle + \langle [Y,n], X\rangle\big)^2 \Big) \end{aligned}$$

Summing the extrinsic curvature of ξ with the sectional curvature gives us the Gaussian curvature of the plane distribution ξ. □

Lemma 3.3 *The Gaussian curvature $K_G(\xi)$ can be calculated by the formula:*

$$\begin{aligned} K_G(\xi) &= K(\xi) + K_e(\xi) \\ &= -\frac{3}{4}a\langle [X,Y], n\rangle^2 + \big(X\langle [X,Y], Y\rangle - Y\langle [X,Y], X\rangle \\ &\qquad - \langle [X,Y], X\rangle^2 - \langle [X,Y], Y\rangle^2\big) \\ &\qquad + \frac{1}{2}\langle [X,Y], n\rangle\big(-\langle [n,Y], X\rangle + \langle [n,X], Y\rangle\big) \end{aligned}$$

Remark 3.4 If ξ is integrable then $\langle [X,Y], n\rangle = 0$ and

$$K_G(\xi) = X\langle [X,Y], Y\rangle - Y\langle [X,Y], X\rangle - \langle [X,Y], X\rangle^2 - \langle [X,Y], Y\rangle^2$$

is nothing else as the expression of the Gaussian curvature of the leaves of ξ written in the local frame tangent to the leaves.

Lemma 3.5 *Let (X, Y, n) be a framing on M. Assume that distribution spanned by n and Y is a contact structure. Then there is a metric on M such that extrinsic curvature of the distribution spanned by X and Y is strictly less than zero.*

Proof Fix a metric $\langle \cdot,\cdot \rangle$ such that the framing is orthonormal. Let ξ be a distribution spanned by vector fields X and Y. Stretch the metric along X by a constant factor λ^2 and along Y by a constant factor $1/\lambda^2$. Let's denote this metric by $\langle \cdot,\cdot \rangle_\lambda$. Calculate the extrinsic curvature of ξ with respect to this metric:

$$\begin{aligned}
K_e(\eta) &= \langle [n,X],X\rangle_\lambda \langle [n,Y],Y\rangle_\lambda - \frac{1}{4}\big(\langle [n,X],Y\rangle_\lambda + \langle [n,Y],X\rangle_\lambda\big)^2 \\
&= \lambda^2 \langle [n,X],X\rangle \frac{1}{\lambda^2}\langle [n,Y],Y\rangle - \frac{1}{4}\big(\frac{1}{\lambda^2}\langle [n,X],Y\rangle + \lambda^2\langle [n,Y],X\rangle\big)^2 \\
&= \langle [n,X],X\rangle \langle [n,Y],Y\rangle - \frac{1}{4}\big(\frac{1}{\lambda^2}\langle [n,X],Y\rangle + \lambda^2\langle [n,Y],X\rangle\big)^2 \\
&= \langle [n,X],X\rangle \langle [n,Y],Y\rangle - \frac{1}{4\lambda^4}\langle [n,X],Y\rangle^2 - \frac{1}{2}\langle [n,X],Y\rangle\langle [n,Y],X\rangle \\
&\quad - \frac{\lambda^4}{4}\langle [n,Y],X\rangle^2
\end{aligned}$$

Since M is compact there is a positive constant C such that:

$$\Big|\langle [n,X],X\rangle \langle [n,Y],Y\rangle - \frac{1}{2}\langle [n,X],Y\rangle\langle [n,Y],X\rangle\Big| < C$$

We assumed that distribution spanned by vector fields n and Y is a contact structure. The form $\alpha(*) = \langle *, X\rangle$ is a contact form of this distribution, so $\langle [n,Y],X\rangle = \alpha([n,Y]) \neq 0$. Since M is compact there is an ε such that:

$$\Big|\langle [n,Y],X\Big| > \varepsilon$$

This means that
$$K_e(\eta) < C - \frac{\lambda^4\varepsilon^2}{4}.$$

This expression is strictly negative for some sufficiently large λ. □

Corollary 3.6 *Assume that ξ is a transversally orientable contact structure with the Euler class zero on M. Then there is a metric on M such that the extrinsic curvature of ξ is a strictly negative function.*

Proof Let n be a vector field on M transverse to ξ. Since $e(\xi) = 0$, the distribution ξ is trivial and has two nowhere zero sections, say X and Y.

Choose some positive number ε and consider a distribution η spanned by the vector fields X and $Y + \varepsilon n$. It is obvious that for all ε the distribution η is transverse to ξ and is a contact structure for some sufficiently small ε. Therefore, we can apply Lemma 3.5 to the framing $(X, Y, Y + \varepsilon n)$ to get a desired metric. □

4 Prescribing the sectional curvature of ξ

Theorem A *Let ξ be a transversally orientable contact structure on a closed orientable 3–manifold M. For any smooth strictly negative function f, there is a metric on M such that f is the sectional curvature of ξ.*

Proof Since ξ is transversally orientable, there is a globally defined vector field n which is transverse to ξ. Fix some Riemannian metric $\langle \cdot, \cdot \rangle$ on M such that n is a unit normal vector field. Consider a finite cover of M by the open sets U_α such that for each α there is an open set U'_α for which $\bar{U}_\alpha \subset U'_\alpha$ and $\xi|_{U'_\alpha}$ is a trivial fibration.

In each U'_α choose an orthonormal framing $(X_\alpha, Y_\alpha, n|_{U'_\alpha})$. Consider the stretching $\langle \cdot, \cdot \rangle_a$ of $\langle \cdot, \cdot \rangle$ along n by a positive function a.

According to Lemma 3.1 the sectional curvature $K(\xi)$ on U'_α can be rewritten in the following way:

$$K(\xi) = -\frac{3}{4} a \langle [X_\alpha, Y_\alpha], n \rangle^2 + P_\alpha + \frac{1}{a} Q_\alpha$$

where P_α and Q_α are functions on U'_α independent of a.

Since ξ is a contact structure and U_α has a compact closure, $\langle [X_\alpha, Y_\alpha], n \rangle^2$ is bounded below by some positive constant ε and the functions P_α and Q_α are bounded from above. Therefore there is a sufficiently large D_α such that the equation

$$-\frac{3}{4} a \langle [X_\alpha, Y_\alpha], n \rangle^2 + P_\alpha + \frac{1}{a} Q_\alpha = f D_\alpha$$

has a strictly positive solution $a_\alpha(D_\alpha)$. Notice, that for any $D > D_\alpha$ this equation still has a positive solution $a_\alpha(D)$. Let $D_0 = \max_\alpha \{D_\alpha\}$. Solve the equation above for D_0 in each chart U_α. Let $a_\alpha = a_\alpha(D_0)$.

We claim that a_α constructed this way does not depend on the choice of the orthonormal framing $(X_\alpha, Y_\alpha, n|_{U_\alpha})$. Let $(X'_\alpha, Y'_\alpha, n|_{U_\alpha})$ be any other orthonormal framing on $\xi|_{U_\alpha}$. This defines a map

$$\phi_\alpha \colon U_\alpha \to O(2)$$

which maps a point $p \in U_\alpha$ to the transition matrix $\phi_\alpha(p)$ between two framings (X'_α, Y'_α) and (X_α, Y_α) on ξ. We have

$$\begin{aligned} \langle [X'_\alpha, Y'_\alpha], n \rangle^2 &= (d\eta(X'_\alpha, Y'_\alpha))^2 = (d\eta(\phi_\alpha X_\alpha, \phi_\alpha Y_\alpha))^2 = \det \phi_\alpha^2 (d\eta(X_\alpha, Y_\alpha))^2 \\ &= \det \phi_\alpha^2 \langle [X_\alpha, Y_\alpha], n \rangle^2 = \langle [X_\alpha, Y_\alpha], n \rangle^2, \end{aligned}$$

where η is a 1–form defined by $\eta(*) = \langle *, n \rangle$. Therefore, $\langle [X_\alpha, Y_\alpha], n \rangle^2$ is independent of the choice of orthonormal framing. The expression $(1/a) Q_\alpha = -K_e(\xi)$ also does not depend on the choice of the trivialization. Finally the sectional curvature $K(\xi)$ is independent of the framing. It is obvious that the right hand side of

$$P_\alpha = K(\xi) - \frac{1}{a} Q_\alpha + \frac{3}{4} a \langle [X_\alpha, Y_\alpha], n\rangle^2$$

does not depend on the choice of framing, so does P_α.

Therefore, the functions a_α agree on the overlaps and define a global function a on M. The sectional curvature of ξ in the metric $\langle\cdot,\cdot\rangle_a$ is $f D_0$. Consider the metric $\langle\cdot,\cdot\rangle_0 = (1/D_0)\langle\cdot,\cdot\rangle_a$. It is easy to calculate, that the sectional curvature of ξ in this metric is equal to f. □

Corollary 4.1 *For any transversally orientable contact structure on a closed orientable 3–manifold, there is a metric on M,such that the sectional curvature of ξ in this metric is equal to -1.*

Theorem B *Let ξ be a transversally orientable contact structure on M with Euler class zero. Then for any smooth function f, there is a metric on M such that f is a sectional curvature of ξ.*

Proof Since the Euler class of ξ is zero, there is a contact structure η, which is transverse to ξ. According to the Corollary 3.6, there is a metric $\langle\cdot,\cdot\rangle$ in which the extrinsic curvature of ξ is a strictly negative function. Let n be a unit normal vector field with respect to this metric.

Consider the stretching of $\langle\cdot,\cdot\rangle$ along n by a positive function a. According to Lemma 3.1, we have to find a to satisfy the equation

$$-\frac{3}{4} a \langle [X,Y], n\rangle^2 + P - \frac{1}{4a} K_e(\xi) = f$$

where P is a function on M which is independent of a.

But since $-K_e(\xi) > 0$ this equation always has a strictly positive solution a. This completes the proof of the theorem. □

Remark 4.2 In the proof of Theorem B it is crucial that ξ is a contact structure. At points where $\langle [X,Y], n\rangle = 0$ the equation may not have any positive solutions.

Example 4.3 (Propeller construction [6]) Consider the following pair of contact structures on $\mathbb{T}^3$:

$$\xi = \mathrm{Ker}(\alpha = \cos z dx - \sin z dy + dz)$$

$$\eta = \mathrm{Ker}(\beta = \cos z dx + \sin z dy)$$

It is easy to verify, that ξ is transverse to η and we get a bi-contact structure. From Theorem B, there is a metric on $\mathbb{T}^3$ such that ξ has a positive sectional curvature. This is an example of a tight contact structure of positive sectional curvature.

Example 4.4 (Overtwisted contact structures of positive sectional curvature) Let ξ be any contact structure with the Euler class zero on M. It is known (see Geiges [3]) that if we apply a full Lutz twist to this contact structure, the resulting contact structure is overtwisted and has Euler class zero. From Theorem B, it has a positive sectional curvature for some choice of metric on M.

5 Uniformization of contact structures on 3–manifolds

The same technique as in Theorem A can be applied to the Gaussian curvature of contact structures on three-manifolds.

Theorem C *Let ξ be a transversally orientable contact structure on a closed orientable 3–manifold M. For any smooth strictly negative function f, there is a metric on M such that f is the Gaussian curvature of ξ.*

Proof Same as Theorem A. The only difference is that in the present case the equation which needs to be solved in each trivializing chart is:

$$K_G(\xi) = -\frac{3}{4}a\langle [X_\alpha, Y_\alpha], n\rangle^2 + P_\alpha = fD_0 \qquad \square$$

Corollary 5.1 (Uniformization of contact structures) *For every transversally orientable contact structure ξ on M, there is a metric such that $K_G(\xi) = -1$.*

Example 5.2 (Contact structure with $K_G(\xi) = 1$) Consider the unit sphere $S^3 \subset \mathbb{C}^2$ with a bi-invariant metric. The standard contact structure on S^3 is defined as the kernel of the 1–form

$$\alpha = \sum_{i=1}^{2}(x_i dy_i - y_i dx_i),$$

restricted from $\mathbb{C}^2$ to S^3. This contact structure is orthogonal to a left-invariant vector field and therefore is left-invariant. Let (X, Y) be a pair of orthonormal left-invariant sections of ξ. Since the metric is bi-invariant,

$$\nabla_S T = \frac{1}{2}[S, T]$$

for any left-invariant vector fields on S^3. Therefore the second fundamental form of ξ vanishes and $K_G(\xi) = K(\xi) = 1$.

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