The curvature of contact structures on 3–manifolds

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We study the sectional curvature of plane distributions on 3–manifolds. We show that if a distribution is a contact structure it is easy to manipulate its curvature. As a corollary we obtain that for every transversally oriented contact structure on a closed 3–dimensional manifold, there is a metric such that the sectional curvature of the contact distribution is equal to $-1$. We also introduce the notion of Gaussian curvature of the plane distribution. For this notion of curvature we get similar results.

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1 Introduction

The problem of prescribing the curvatures of a manifold is one of the central problems in Riemannian geometry. That is, given a smooth function can it be realized as a scalar (Ricci or sectional) curvature of some Riemannian metric on a manifold. The solution of the Yamabe problem is the best known result in prescribing the scalar curvature on a manifold (cf Lee and Parker [4]). There are several results on prescribing the Ricci curvature of a manifold (cf for example Lohkamp [5]). It is natural to ask to what extent it is possible to prescribe the sectional curvature of the plane distribution on a 3-manifold. It turns out that this problem is closely connected with the contactness of the distribution. In fact we have the following:

**Theorem A**  Let $\xi$ be a transversally orientable contact structure on a closed orientable 3–manifold $M$. For any smooth strictly negative function $f$, there is a metric on $M$ such that $f$ is the sectional curvature of $\xi$.

If we impose more topological restrictions on the distribution we can obtain an even stronger result:

**Theorem B**  Let $\xi$ be a transversally orientable contact structure on $M$ with Euler class zero. Then for any smooth function $f$, there is a metric on $M$ such that $f$ is a sectional curvature of $\xi$.
In [2], Chern and Hamilton studied a similar problem of prescribing the so-called Webster curvature $W$ on a contact three-manifold. The main difference in their approach is that they restrict the class of metrics to the metrics which are adapted to a contact structure, while we deal with the class of all metrics. They prove that in their class one can either find a metric with the constant negative Webster curvature or a metric with strictly positive Webster curvature.

It is a well-known problem whether a foliation on a 3–dimensional manifold admits a simultaneous uniformization of all its leaves. The Reeb stability theorem asserts that on a compact orientable 3–manifold the only foliation with the leaves having positive Gaussian curvature is the foliation of $M = S^2 \times S^1$ by spheres. It is known (see Candel [1]) that if $M$ is atoroidal and aspherical and the foliation is taut, then there is a metric on $M$ such that all leaves have constant negative Gaussian curvature $−1$. In the case of contact structures we ask a similar question. For this we have to introduce the notion of Gaussian curvature of the plane distribution.

We define the Gaussian curvature of the plane distribution as the sum $K_G(\xi) = K(\xi) + K_e(\xi)$ of the sectional and the extrinsic curvatures of the distribution. In the case of integrable $\xi$ this equation is nothing but the Gauss equation.

**Definition 1.1** Let $\xi$ be a plane distribution on $M$. We say that $\xi$ admits a uniformization if there is a metric on $M$ such that the Gaussian curvature of $\xi$ is constant.

It turns out that unlike the case of foliations, every transversally orientable contact structure on a closed 3–manifold admits a uniformization. We have the following:

**Theorem C** Let $\xi$ be a transversally orientable contact structure on a closed orientable 3–manifold $M$. For any smooth strictly negative function $f$, there is a metric on $M$ such that $f$ is the Gaussian curvature of $\xi$.

This paper is organized as follows. In Section 2 we recall basic facts about the geometry of plane distributions. In Section 3 we prove the main technical lemma. Section 4 is devoted to the proof of Theorem A and Theorem B. We prove Theorem C in Section 5.

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## 2 Basic definitions and notation

Throughout this paper $M$ will be a closed orientable 3–manifold. A distribution on $M$ is a two dimensional subbundle of the tangent bundle of $M$. That is, at each point $p$...
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There is a plane \( \xi_p \) in the tangent space \( T_p M \). A distribution is called integrable, if there is a foliation on \( M \) which is tangent to it. The following Frobenius theorem gives necessary and sufficient conditions for \( \xi \) to be integrable.

**Theorem 2.1** Let \( \xi \) be a distribution on \( M \). Then \( \xi \) is integrable if and only if for any two sections \( S \) and \( T \) of \( \xi \) its Lie bracket belongs to \( \xi \).

**Definition 2.2** A distribution \( \xi \) is called a contact structure if for any linearly independent sections \( S \) and \( T \) of \( \xi \) and for any \( p \in M \) the Lie bracket \([S, T]\) at \( p \) does not belong to \( \xi_p \).

A distribution \( \xi \) is called transversally oriented if there is a globally defined 1–form \( \alpha \) such that \( \xi = \text{Ker}(\alpha) \). This is equivalent to say that there exists a globally defined vector field \( n \) which is transverse to \( \xi \). It is an easy consequence of Frobenius Theorem that \( \xi \) is a contact structure if and only if

\[\alpha \wedge d\alpha \neq 0.\]

Fix some orientation on \( M \). A contact structure is said to be positive (resp. negative) if the orientation induced by \( \alpha \wedge d\alpha \) coincides (resp. is opposite to) the orientation on \( M \).

A contact structure \( \xi \) is called overtwisted, if there is an embedded disk such that \( TD|_{\partial D} = \xi|_{\partial D} \). If \( \xi \) is not overtwisted, it is called tight.

The Euler class \( e(\xi) \in H^2(M, \mathbb{Z}) \) of a plane distribution is the Euler class of the bundle \( \xi \to M \). It is known that if \( \xi \) is a 2–dimensional plane distribution on \( M \) with vanishing Euler class then \( \xi \) is trivial. Recall, that a framing of \( M \) is the presentation of the tangent bundle of \( M \) as a product \( TM \simeq M \times \mathbb{R}^3 \). A framing on \( M \) consists of three linearly independent vector fields. It is known that every closed orientable 3–manifold admits a framing.

A bi-contact structure on \( M \) is a pair \((\xi, \eta)\) of transverse contact structures which define opposite orientation on \( M \).

Assume that \( M \) is a Riemannian manifold with the metric \(<\cdot,\cdot>\) and the Levi-Civita connection \( \nabla \). Let \( n \) be a local unit vector field orthogonal to \( \xi \). We are now going to define the second fundamental form of \( \xi \). The definition is due to Reinhart [7].

**Definition 2.3** The second fundamental form of \( \xi \) is a symmetric bilinear form, which is defined in the following way:

\[B(S, T) = \frac{1}{2} \langle \nabla_S T + \nabla_T S, n \rangle\]

for all sections \( S \) and \( T \) of \( \xi \).
Remark 2.4 If \( \xi \) is integrable, then \( B \) restricted to the leaf of \( \xi \) agrees with the second fundamental form of the leaf.

Let \( S \) and \( T \) be two linearly independent sections of \( \xi \).

Definition 2.5 We call the function
\[
K_\varepsilon(\xi) = \frac{B(S,S)B(T,T) - B(S,T)^2}{\langle S, S \rangle \langle T, T \rangle - \langle S, T \rangle^2}
\]
an extrinsic curvature of \( \xi \).

It is easy to verify that \( K_\varepsilon(\xi) \) depends only on \( \xi \), not on the actual choice of \( S, T \) and \( n \).

Definition 2.6 Consider the function \( K(\xi) \) which assigns to a point \( p \in M \) the sectional curvature of the plane \( \xi_p \). We call this function the sectional curvature of \( \xi \).

Definition 2.7 We call the sum \( K_G(\xi) = K(\xi) + K_\varepsilon(\xi) \) the Gaussian curvature of \( \xi \).

Let \( S, T \) and \( U \) be the local sections of \( TM \). Recall the Koszul formula for the Levi-Civita connection of \( \langle \cdot, \cdot \rangle \):
\[
2\langle \nabla_S T, U \rangle = S\langle T, U \rangle + T\langle U, S \rangle - U\langle S, T \rangle
+ \langle [S, T], U \rangle - \langle [S, U], T \rangle - \langle [T, U], S \rangle
\]

3 The deformation of metric

In this section we will give the proof of the main technical results we will need throughout the paper.

Let \( \xi \) be a transversally orientable plane distribution on a 3–dimensional Riemannian manifold \( (M, \langle \cdot, \cdot \rangle) \). Fix a unit normal vector field \( n \). Suppose \( a \) is a strictly positive smooth function on \( M \). A stretching of \( \langle \cdot, \cdot \rangle \) along \( n \) by the function \( a \) is the following Riemannian metric on \( M \):
\[
\langle \cdot, \cdot \rangle_a = a \langle \cdot, \cdot \rangle|_n \oplus \langle \cdot, \cdot \rangle|_\xi
\]

Our aim is to calculate the sectional curvature of \( \xi \) in the stretched metric in terms of the initial metric.
Consider an open subset $U \subset M$ such that $\xi|_U$ is a trivial fibration. Let $X$ and $Y$ be a pair of orthonormal sections of $\xi|_U$. The triple $(X, Y, n)$ is an orthonormal framing on $U$ with respect to $\langle \cdot, \cdot \rangle_a$.

In the stretched metric this frame is orthogonal, vector fields $X$ and $Y$ are unit and the length of $n$ is equal to $\sqrt{a}$. Denote by $\nabla$ the Levi-Civita connection of $\langle \cdot, \cdot \rangle_a$.

**Lemma 3.1** The sectional curvature of $\xi$ with respect to $\langle \cdot, \cdot \rangle_a$ can be calculated by the following formula:

$$K(\xi) = \frac{3}{4} a \langle [X, Y], n \rangle^2 + P + \frac{1}{a} Q$$

where

$$P = X \langle [X, Y], Y \rangle - Y \langle [X, Y], X \rangle - \langle [X, Y], X \rangle^2 - \langle [X, Y], Y \rangle^2$$

$$+ \frac{1}{2} \langle [X, Y], n \rangle (\langle n, Y \rangle X + \langle n, X \rangle Y))$$

and

$$Q = \frac{1}{4} (\langle [X, n], Y \rangle + \langle [Y, n], X \rangle)^2 - \langle [Y, n], Y \rangle \langle [X, n], X \rangle$$

**Proof** Since $X$ and $Y$ are unit, the sectional curvature of $\xi$ is calculated by the formula:

$$K(\xi) = \langle R(X, Y)Y, X \rangle_a = \langle \nabla_X \nabla_Y Y, X \rangle_a - \langle \nabla_Y \nabla_X Y, X \rangle_a - \langle \nabla_{[X,Y]} Y, X \rangle_a$$

The first summand can be rewritten:

$$\langle \nabla_X \nabla_Y Y, X \rangle_a = X \langle \nabla_Y Y, X \rangle_a - \langle \nabla_Y Y, \nabla_X X \rangle_a$$

Apply the Koszul formula to $X \langle \nabla_Y Y, X \rangle_a$. We get:

$$X \langle \nabla_Y Y, X \rangle_a = \frac{1}{2} X \langle 2Y(Y, X), X \rangle_a - X \langle Y, Y \rangle_a + \langle [Y, X], X \rangle_a - 2 \langle [Y, X], Y \rangle_a$$

$$= - X \langle [Y, X], Y \rangle_a = - X \langle [Y, X], Y \rangle$$

Decompose the vector field $\nabla_Y Y$ with respect to the frame $(X, Y, n/\sqrt{a})$ orthonormal in the stretched metric $\langle \cdot, \cdot \rangle_a$:

$$\nabla_Y Y = \langle \nabla_Y Y, \frac{n}{\sqrt{a}} \rangle_a \frac{n}{\sqrt{a}} + \langle \nabla_Y Y, Y \rangle_a Y + \langle \nabla_Y Y, X \rangle_a X$$

Substituting these expressions into $\langle \nabla_X \nabla_Y Y, X \rangle_a$, we obtain:

$$\langle \nabla_X \nabla_Y Y, X \rangle_a = - X \langle [Y, X], Y \rangle - \langle \langle \nabla_Y Y, n \rangle_a \frac{n}{a} + \langle \nabla_Y Y, Y \rangle_a Y$$

$$+ \langle \nabla_Y Y, X \rangle_a X, \nabla_X X \rangle_a$$

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Since $X$ and $Y$ are of unit length this reduces to:

$$\langle \nabla_X \nabla_Y X, X \rangle_a = -X \langle [Y, X], Y \rangle - \frac{1}{a} \langle \nabla_Y Y, n \rangle_a \langle \nabla_X X, n \rangle_a$$

Apply the Koszul formula to the term $\langle \nabla_Y Y, n \rangle_a \langle \nabla_X X, n \rangle_a$. Finally, we have:

$$\langle \nabla_X \nabla_Y Y, X \rangle_a = -X \langle [Y, X], Y \rangle - \frac{1}{a} \langle [Y, n], Y \rangle_a \langle [X, n], X \rangle_a$$

$$= -X \langle [Y, X], Y \rangle - \frac{1}{a} \langle [Y, n], Y \rangle \langle [X, n], X \rangle$$

The second summand is equal to:

$$-(\nabla_Y \nabla_X Y, X)_a = -Y (\nabla_X Y, X)_a + \langle \nabla_X Y, \nabla_Y X \rangle_a$$

$$= Y (\nabla_X X)_a + \langle \langle \nabla_X Y, n \rangle_a n_\frac{1}{a} \rangle + \langle \nabla_X Y \rangle_a Y$$

$$+ \langle \nabla_X X \rangle_a X, \nabla_Y X \rangle_a$$

$$= -Y \langle [Y, X], X \rangle_a + \frac{1}{a} \langle \nabla_X Y, n \rangle_a \langle \nabla_Y X, n \rangle_a$$

Write the equations for the terms $\langle \nabla_X Y, n \rangle_a$ and $\langle \nabla_Y X, n \rangle_a$:

$$2 \langle \nabla_X Y, n \rangle_a = \langle [X, Y], n \rangle_a - \langle [X, n], Y \rangle_a - \langle [Y, n], X \rangle_a$$

$$= a \langle [X, Y], n \rangle - \langle [X, n], Y \rangle - \langle [Y, n], X \rangle$$

$$2 \langle \nabla_Y X, n \rangle_a = \langle [Y, X], n \rangle_a - \langle [Y, n], X \rangle_a - \langle [X, n], Y \rangle_a$$

$$= a \langle [Y, X], n \rangle - \langle [Y, n], X \rangle - \langle [X, n], Y \rangle$$

Inserting the above equations into the second summand we have:

$$-(\nabla_Y \nabla_X Y, X)_a = -Y \langle [X, Y], X \rangle_a + \frac{1}{4a} \left( -a \langle [X, Y], n \rangle + \langle [X, n], Y \rangle + \langle [Y, n], X \rangle \right)$$

$$\cdot \left( -a \langle [X, Y], n \rangle + \langle [Y, n], X \rangle + \langle [X, n], Y \rangle \right)$$

The last summand is:

$$-(\nabla_{[X,Y]} X, X)_a = -(\nabla_{([X,Y],n)} X + [X,Y] X + [X,Y] Y, X)_a$$

$$= -\langle [X, Y], n \rangle \langle \nabla_n Y, X \rangle_a - \langle [X, Y], X \rangle \langle \nabla_X Y, X \rangle_a$$

$$- \langle [X, Y], Y \rangle \langle \nabla_Y Y, X \rangle_a$$

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The term $\langle \nabla_n Y, X \rangle_a$ is equal to

$$\langle \nabla_n Y, X \rangle_a = -\frac{1}{2} \left( -\langle [n, Y], X \rangle_a + \langle [n, X], Y \rangle_a + \langle [Y, X], n \rangle_a \right)$$

which gives us:

$$-\langle \nabla_{[X,Y]} Y, X \rangle_a = -\langle [X, Y], n \rangle \langle \nabla_n Y, X \rangle_a - \langle [X, Y], X \rangle \langle \nabla_X Y, X \rangle_a$$

$$= \frac{1}{2} \langle [X, Y], n \rangle \left( -\langle [n, Y], X \rangle + \langle [n, X], Y \rangle + a \langle [Y, X], n \rangle \right)$$

Summing this up, the sectional curvature of $\xi$ is equal to:

$$K(\xi) = -X \langle [Y, X], Y \rangle - \frac{1}{a} \langle [Y, n], Y \rangle \langle [X, n], X \rangle$$

$$- \left( Y \langle [X, Y], X \rangle - \frac{1}{4a} \left( -a \langle [Y, X], n \rangle + \langle [X, n], Y \rangle + \langle [Y, n], X \rangle \right) \right)$$

$$= -\frac{1}{2} \langle [X, Y], n \rangle \left( -\langle [n, Y], X \rangle + \langle [n, X], Y \rangle + a \langle [Y, X], n \rangle \right)$$

$$+ \langle [X, Y], X \rangle^2 + \langle [X, Y], Y \rangle^2$$

It is straightforward to verify that this gives us the desired expression.

\[\square\]

**Lemma 3.2** The extrinsic curvature $K_e(\xi)$ with respect to $\langle \cdot, \cdot \rangle_a$ can be calculated by the following formula:

$$K_e(\xi) = \frac{1}{a} \left( \langle [X, n], X \rangle \langle [Y, n], Y \rangle - \frac{1}{4} \langle [X, n], Y \rangle + \langle [Y, n], X \rangle \right)^2$$

**Proof** Since $X$ and $Y$ are unit vectors, the extrinsic curvature is given by:

$$K_e(\xi) = B(X, X) B(Y, Y) - B(X, Y)^2$$

By the definition of $B$, the extrinsic curvature is equal to:

$$K_e(\xi) = \langle \nabla_X Y, \frac{n}{\sqrt{a}} \rangle_a \langle \nabla_Y X, \frac{n}{\sqrt{a}} \rangle_a - \frac{1}{4} \langle \nabla_Y X + \nabla_X Y, \frac{n}{\sqrt{a}} \rangle_a^2$$

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Apply the Koszul formula to 
\[ \langle \nabla_X Y, \frac{n}{\sqrt{a}} \rangle_a, \quad \langle \nabla_Y X, \frac{n}{\sqrt{a}} \rangle_a \quad \text{and} \quad \langle \nabla_X Y + \nabla_Y X, \frac{n}{\sqrt{a}} \rangle_a \]
to obtain:

\[ K_e(\xi) = \frac{1}{a} \left( \langle [X, n], X \rangle_a \langle [Y, n], Y \rangle_a - \frac{1}{4} \langle [X, Y], n \rangle_a - \frac{1}{2} \langle [X, n], Y \rangle_a \right. \]
\[ - \frac{1}{2} \langle [Y, n], X \rangle_a - \frac{1}{2} \langle [X, Y], n \rangle_a - \frac{1}{2} \langle [Y, n], X \rangle_a - \frac{1}{2} \langle [X, n], Y \rangle_a \right)^2 \]
\[ = \frac{1}{a} \left( \langle [X, n], X \rangle \langle [Y, n], Y \rangle - \frac{1}{4} \left( \langle [X, n], Y \rangle + \langle [Y, n], X \rangle \right)^2 \right) \]

Summing the extrinsic curvature of \( \xi \) with the sectional curvature gives us the Gaussian curvature of the plane distribution \( \xi \).

**Lemma 3.3** The Gaussian curvature \( K_G(\xi) \) can be calculated by the formula:

\[ K_G(\xi) = K(\xi) + K_e(\xi) \]
\[ = -\frac{3}{4} a \langle [X, Y], n \rangle^2 + \langle X \langle [X, Y], Y \rangle - Y \langle [X, Y], X \rangle \rangle - \langle [X, Y], X \rangle^2 - \langle [X, Y], Y \rangle^2 \]
\[ + \frac{1}{2} \langle [X, Y], n \rangle \left( - \langle [n, Y], X \rangle + \langle [n, X], Y \rangle \right) \]

**Remark 3.4** If \( \xi \) is integrable then \( \langle [X, Y], n \rangle = 0 \) and

\[ K_G(\xi) = X \langle [X, Y], Y \rangle - Y \langle [X, Y], X \rangle - \langle [X, Y], X \rangle^2 - \langle [X, Y], Y \rangle^2 \]
is nothing else as the expression of the Gaussian curvature of the leaves of \( \xi \) written in the local frame tangent to the leaves.

**Lemma 3.5** Let \( (X, Y, n) \) be a framing on \( M \). Assume that distribution spanned by \( n \) and \( Y \) is a contact structure. Then there is a metric on \( M \) such that extrinsic curvature of the distribution spanned by \( X \) and \( Y \) is strictly less than zero.

**Proof** Fix a metric \( \langle \cdot, \cdot \rangle \) such that the framing is orthonormal. Let \( \xi \) be a distribution spanned by vector fields \( X \) and \( Y \). Stretch the metric along \( X \) by a constant factor \( \lambda^2 \) and along \( Y \) by a constant factor \( 1/\lambda^2 \). Let’s denote this metric by \( \langle \cdot, \cdot \rangle_\lambda \). Calculate
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The extrinsic curvature of $\xi$ with respect to this metric:

$$K_e(\eta) = \langle [n, X], Y \rangle_{\lambda} \langle [n, Y], Y \rangle_{\lambda} - \frac{1}{4} \langle [n, X], Y \rangle_{\lambda} \langle [n, Y], X \rangle_{\lambda}^2$$

$$= \lambda^2 \langle [n, X], X \rangle \frac{1}{\lambda^2} \langle [n, Y], Y \rangle - \frac{1}{4} \left( \frac{1}{\lambda^2} \langle [n, X], Y \rangle + \lambda^2 \langle [n, Y], X \rangle \right)^2$$

$$= \langle [n, X], [n, Y] \rangle \langle [n, Y], Y \rangle - \frac{1}{4} \left( \frac{1}{\lambda^2} \langle [n, X], Y \rangle + \lambda^2 \langle [n, Y], X \rangle \right)^2$$

$$= \langle [n, X], X \rangle \langle [n, Y], Y \rangle - \frac{1}{4\lambda^4} \langle [n, X], Y \rangle^2 - \frac{1}{2} \langle [n, X], Y \rangle \langle [n, Y], X \rangle$$

$$- \frac{\lambda^4}{4} \langle [n, Y], X \rangle^2$$

Since $M$ is compact there is a positive constant $C$ such that:

$$\left| \langle [n, X], X \rangle \langle [n, Y], Y \rangle - \frac{1}{2} \langle [n, X], Y \rangle \langle [n, Y], X \rangle \right| < C$$

We assumed that distribution spanned by vector fields $n$ and $Y$ is a contact structure. The form $\alpha(*) = (*, X)$ is a contact form of this distribution, so $\langle [n, X], Y \rangle = \alpha([n, Y]) \neq 0$. Since $M$ is compact there is an $\varepsilon$ such that:

$$\left| \langle [n, Y], X \rangle \right| > \varepsilon$$

This means that

$$K_e(\eta) < C - \frac{\lambda^4 \varepsilon^2}{4}.$$  

This expression is strictly negative for some sufficiently large $\lambda$. \hfill $\square$

**Corollary 3.6** Assume that $\xi$ is a transversally orientable contact structure with the Euler class zero on $M$. Then there is a metric on $M$ such that the extrinsic curvature of $\xi$ is a strictly negative function.

**Proof** Let $n$ be a vector field on $M$ transverse to $\xi$. Since $e(\xi) = 0$, the distribution $\xi$ is trivial and has two nowhere zero sections, say $X$ and $Y$.

Choose some positive number $\varepsilon$ and consider a distribution $\eta$ spanned by the vector fields $X$ and $Y + \varepsilon n$. It is obvious that for all $\varepsilon$ the distribution $\eta$ is transverse to $\xi$ and is a contact structure for some sufficiently small $\varepsilon$. Therefore, we can apply Lemma 3.5 to the framing $(X, Y, Y + \varepsilon n)$ to get a desired metric. \hfill $\square$
4 Prescribing the sectional curvature of $\xi$

**Theorem A** Let $\xi$ be a transversally orientable contact structure on a closed orientable 3–manifold $M$. For any smooth strictly negative function $f$, there is a metric on $M$ such that $f$ is the sectional curvature of $\xi$.

**Proof** Since $\xi$ is transversally orientable, there is a globally defined vector field $n$ which is transverse to $\xi$. Fix some Riemannian metric $\langle \cdot , \cdot \rangle$ on $M$ such that $n$ is a unit normal vector field. Consider a finite cover of $M$ by the open sets $U_\alpha$ such that for each $\alpha$ there is an open set $U'_\alpha$ for which $\overline{U_\alpha} \subset U'_\alpha$ and $\xi|_{U'_\alpha}$ is a trivial fibration.

In each $U'_\alpha$ choose an orthonormal framing $(X_\alpha, Y_\alpha, n|_{U'_\alpha})$. Consider the stretching $\langle \cdot , \cdot \rangle_a$ of $\langle \cdot , \cdot \rangle$ along $n$ by a positive function $a$.

According to Lemma 3.1 the sectional curvature $K(\xi)$ on $U'_\alpha$ can be rewritten in the following way:

$$K(\xi) = -\frac{3}{4} a \langle [X_\alpha, Y_\alpha], n \rangle^2 + P_\alpha + \frac{1}{a} Q_\alpha$$

where $P_\alpha$ and $Q_\alpha$ are functions on $U'_\alpha$ independent of $a$.

Since $\xi$ is a contact structure and $U_\alpha$ has a compact closure, $\langle [X_\alpha, Y_\alpha], n \rangle^2$ is bounded below by some positive constant $\varepsilon$ and the functions $P_\alpha$ and $Q_\alpha$ are bounded from above. Therefore there is a sufficiently large $D_\alpha$ such that the equation

$$-\frac{3}{4} a \langle [X_\alpha, Y_\alpha], n \rangle^2 + P_\alpha + \frac{1}{a} Q_\alpha = f D_\alpha$$

has a strictly positive solution $a_\alpha(D)$. Notice, that for any $D > D_\alpha$ this equation still has a positive solution $a_\alpha(D)$. Let $D_0 = \max_\alpha \{ D_\alpha \}$. Solve the equation above for $D_0$ in each chart $U_\alpha$. Let $a_\alpha = \hat{a}_\alpha(D_0)$.

We claim that $a_\alpha$ constructed this way does not depend on the choice of the orthonormal framing $(X_\alpha, Y_\alpha, n|_{U_\alpha})$. Let $(X'_\alpha, Y'_\alpha, n|_{U_\alpha})$ be any other orthonormal framing on $\xi|_{U_\alpha}$. This defines a map

$$\phi_\alpha: U_\alpha \to O(2)$$

which maps a point $p \in U_\alpha$ to the transition matrix $\phi_\alpha(p)$ between two framings $(X'_\alpha, Y'_\alpha)$ and $(X_\alpha, Y_\alpha)$ on $\xi$. We have

$$\langle [X'_\alpha, Y'_\alpha], n \rangle^2 = (d\eta(X'_\alpha, Y'_\alpha))^2 = (d\eta(X_\alpha, Y_\alpha))^2 = \det \phi_\alpha^2 (d\eta(X_\alpha, Y_\alpha))^2$$

$$= \det \phi_\alpha^2 \langle [X_\alpha, Y_\alpha], n \rangle^2 = \langle [X_\alpha, Y_\alpha], n \rangle^2,$$

where $\eta$ is a 1–form defined by $\eta(*) = \langle *, n \rangle$. Therefore, $\langle [X_\alpha, Y_\alpha], n \rangle^2$ is independent of the choice of orthonormal framing. The expression $(1/a) Q_\alpha = -K_\varepsilon(\xi)$ also does
not depend on the choice of the trivialization. Finally the sectional curvature $K(\xi)$ is independent of the framing. It is obvious that the right hand side of

$$P_\alpha = K(\xi) - \frac{1}{a} Q_\alpha + \frac{3}{4} \alpha([X_\alpha, Y_\alpha], n)^2$$

does not depend on the choice of framing, so does $P_\alpha$.

Therefore, the functions $a_\alpha$ agree on the overlaps and define a global function $a$ on $M$. The sectional curvature of $\xi$ in the metric $\langle \cdot, \cdot \rangle_a$ is $f D_0$. Consider the metric $\langle \cdot, \cdot \rangle_0 = (1/D_0) \langle \cdot, \cdot \rangle_a$. It is easy to calculate, that the sectional curvature of $\xi$ in this metric is equal to $f$.

**Corollary 4.1** For any transversally orientable contact structure on a closed orientable $3$–manifold, there is a metric on $M$ such that the sectional curvature of $\xi$ in this metric is equal to $-1$.

**Theorem B** Let $\xi$ be a transversally orientable contact structure on $M$ with Euler class zero. Then for any smooth function $f$, there is a metric on $M$ such that $f$ is a sectional curvature of $\xi$.

**Proof** Since the Euler class of $\xi$ is zero, there is a contact structure $\eta$, which is transverse to $\xi$. According to the **Corollary 3.6**, there is a metric $\langle \cdot, \cdot \rangle$ in which the extrinsic curvature of $\xi$ is a strictly negative function. Let $n$ be a unit normal vector field with respect to this metric.

Consider the stretching of $\langle \cdot, \cdot \rangle$ along $n$ by a positive function $a$. According to **Lemma 3.1**, we have to find $a$ to satisfy the equation

$$-\frac{3}{4} a([X, Y], n)^2 + P - \frac{1}{4a} K_\xi(\xi) = f$$

where $P$ is a function on $M$ which is independent of $a$.

But since $-K_\xi(\xi) > 0$ this equation always has a strictly positive solution $a$. This completes the proof of the theorem.

**Remark 4.2** In the proof of **Theorem B** it is crucial that $\xi$ is a contact structure. At points where $\langle [X, Y], n \rangle = 0$ the equation may not have any positive solutions.

**Example 4.3** (Propeller construction [6]) Consider the following pair of contact structures on $\mathbb{T}^3$:

$$\xi = \text{Ker}(\alpha = \cos zdx - \sin zdy + dz)$$

$$\eta = \text{Ker}(\beta = \cos zdx + \sin zdy)$$
It is easy to verify, that $\xi$ is transverse to $\eta$ and we get a bi-contact structure. From Theorem B, there is a metric on $\mathbb{T}^3$ such that $\xi$ has a positive sectional curvature. This is an example of a tight contact structure of positive sectional curvature.

**Example 4.4** (Overtwisted contact structures of positive sectional curvature) Let $\xi$ be any contact structure with the Euler class zero on $M$. It is known (see Geiges [3]) that if we apply a full Lutz twist to this contact structure, the resulting contact structure is overtwisted and has Euler class zero. From Theorem B, it has a positive sectional curvature for some choice of metric on $M$.

## 5 Uniformization of contact structures on 3–manifolds

The same technique as in Theorem A can be applied to the Gaussian curvature of contact structures on three-manifolds.

**Theorem C** Let $\xi$ be a transversally orientable contact structure on a closed orientable 3–manifold $M$. For any smooth strictly negative function $f$, there is a metric on $M$ such that $f$ is the Gaussian curvature of $\xi$.

**Proof** Same as Theorem A. The only difference is that in the present case the equation which needs to be solved in each trivializing chart is:

$$K_G(\xi) = -\frac{3}{4} a([X_a, Y_a], n)^2 + P_a = f D_0$$

**Corollary 5.1** (Uniformization of contact structures) For every transversally orientable contact structure $\xi$ on $M$, there is a metric such that $K_G(\xi) = -1$.

**Example 5.2** (Contact structure with $K_G(\xi) = 1$) Consider the unit sphere $S^3 \subset \mathbb{C}^2$ with a bi-invariant metric. The standard contact structure on $S^3$ is defined as the kernel of the 1–form

$$\alpha = \sum_{i=1}^{2} (x_idy_i - y_idx_i),$$

restricted from $\mathbb{C}^2$ to $S^3$. This contact structure is orthogonal to a left-invariant vector field and therefore is left-invariant. Let $(X, Y)$ be a pair of orthonormal left-invariant sections of $\xi$. Since the metric is bi-invariant,

$$\nabla_S T = \frac{1}{2}[S, T]$$

for any left-invariant vector fields on $S^3$. Therefore the second fundamental form of $\xi$ vanishes and $K_G(\xi) = K(\xi) = 1.$

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References


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