Sign refinement for combinatorial link Floer homology

ÉTIENNE GALLAIS

Link Floer homology is an invariant for links which has recently been described entirely in a combinatorial way. Originally constructed with mod 2 coefficients, it was generalized to integer coefficients thanks to a sign refinement. In this paper, thanks to the spin extension of the permutation group we give an alternative construction of the combinatorial link Floer chain complex associated to a grid diagram with integer coefficients. In particular we prove that the sign refinement comes from a 2–cohomological class corresponding to the spin extension of the permutation group.

1 Introduction

Heegaard–Floer homology (Ozsváth–Szabó [9]) is an invariant for closed oriented 3-manifolds which was extended to give an invariant for null-homologous oriented links in such manifolds called link Floer homology (Ozsváth–Szabó [8; 10], Rasmussen [11]). It gives the Seifert genus $g(K)$ of a knot $K$ (Ozsváth–Szabó [7]), detects fibered knots (Ghiggini [2] in the case where $g(K) = 1$ and Ni [6] in general) and its graded Euler characteristic gives the Alexander polynomial [8; 11]. Recently, a combinatorial description of link Floer homology was given (Manolescu–Ozsváth–Sarkar [4]) and its topological invariance was proved in a purely combinatorial way (Manolescu–Ozsváth–Sarkar–Thurston [5]). The purpose of this paper is to give an alternative description of combinatorial link Floer homology with $\mathbb{Z}$ coefficients. This point of view was recently used by Audoux [1] to describe combinatorial Heegaard–Floer homology for singular knots.

Let first recall the context of combinatorial link Floer homology: we follow conventions of [5]. A planar grid diagram $G$ lies in a square on the plane with $n \times n$ squares where $n$ is the complexity of $G$. Each square is decorated with an $X$, an $O$ or nothing in such a way that each row and each column contains exactly one $X$ and one $O$. We number the $X$ and the $O$ from 1 to $n$ and denote $X$ the set $\{X_i\}_{i=1}^n$ and $O$ the set $\{O_i\}_{i=1}^n$. 

Published: 15 September 2008 DOI: 10.2140/agt.2008.8.1581
Given a grid diagram $G$, we place it in standard position on the plane as follows: the bottom left corner is at the origin and each cell is a square of length one. We construct a planar link projection by drawing horizontal segments from the $O$ to the $X$ in each row and vertical segments from the $X$ to the $O$ in each column. At each intersection point, the vertical segment is over the horizontal one. This gives an oriented link $\tilde{L}$ in $S^3$ and we say that $\tilde{L}$ has a grid presentation given by $G$.

![Figure 1: Grid presentation of the Hopf link.](image)

We place the grid diagram on the oriented torus $T$ by making the usual identification of the boundary of the square. We endow $T$ with the orientation induced by the planar orientation. Let $\mathcal{H}$ be the collection of the horizontal circles and the collection of the vertical ones. We associate with $G$ a chain complex $\mathcal{C}$; it is the group ring of $\mathfrak{S}_n$ over $\mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$, where $\mathfrak{S}_n$ is the permutation group of $n$ elements. A generator $x \in \mathfrak{S}_n$ is given on $G$ by its graph: we place dots in points $(i, x(i))$ for $i = 0, \ldots, n-1$ (thus the fundamental domain of $G$ is the square minus the right vertical segment and the top horizontal segment).

For $A, B$ two finite sets of points in the plane we define $I(A, B)$ to be the number of pairs $(a, b) \in A \times B$ such that $a < b$. Let $J(A, B) = (I(A, B) + I(B, A))/2$. We provide the set of generators with a Maslov degree $M$ given by

$$M(x) = J(x - \emptyset, x - \emptyset) + 1$$

where we extend $J$ by bilinearly over formal sums (or differences) of subsets. Each variable $U_{O_i}$ has a Maslov degree equal to $-2$ and constants have Maslov degree equal to zero. Let $M_S(x)$ be the same as $M(x)$ with the set $S$ playing the role of $\emptyset$.

We provide the set of generators with an Alexander filtration $A$ given by $A(x) = (A_1(x), \ldots, A_l(x))$ with

$$A_i(x) = J(x - \frac{1}{2}(X + \emptyset), X_i - \emptyset_i) - \frac{n_i - 1}{2}$$
where when we number the components of \( \mathcal{L} \) from 1 to \( \ell \), \( O_i \subset \mathcal{O} \) (resp. \( X_i \subset \mathcal{X} \)) is the subset of \( \mathcal{O} \) (resp. \( \mathcal{X} \)) which belongs to the \( i \)th component of \( \mathcal{L} \) and \( n_i \) is the number of horizontal segments which belongs to the \( i \)th component. We let \( A(U_{O_j}) = (0, \ldots, -1, 0, \ldots, 0) \) where \(-1\) corresponds to the \( i \)th coordinate if \( O_j \) belongs to the \( i \)th component.

Given two generators \( x \) and \( y \) and an immersed rectangle \( r \) in the torus whose edges are arcs in the horizontal and vertical circles, we say that \( r \) connects \( x \) to \( y \) if \( y \cdot x^{-1} \) is a transposition, if all four corners of \( r \) are intersection points in \( x \cup y \), and if we traverse each horizontal boundary component of \( r \) in the direction dictated by the orientation of \( r \) induced by \( T \), then the arc is oriented from a point in \( x \) to the point in \( y \). Let \( \text{Rect}(x, y) \) be the set of rectangles connecting \( x \) to \( y \): either it is the empty set or it consists of exactly two rectangles. Here a rectangle \( r \in \text{Rect}(x, y) \) is said to be empty if there is no point of \( x \) in its interior. Let \( \text{Rect}^e(x, y) \) be the set of empty rectangles connecting \( x \) to \( y \).

The differential \( \partial^- : C^-(G) \to C^-(G) \) is given on the set of generators by

\[
\partial^- x = \sum_{y \in \mathcal{G}} \sum_{r \in \text{Rect}^e(x, y)} U_{O_1(r)}^{O_1} \ldots U_{O_n(r)}^{O_n} \cdot y
\]

where \( O_i(r) \) is the number of times \( O_i \) appears in the interior of \( r \).

**Theorem 1.1** (Manolescu–Ozsváth–Sarkar [4]) \( (C^-(G), \partial^-) \) is a chain complex for \( CF^-(S^3) \) with homological degree induced by \( M \) and filtration level induced by \( A \) which coincides with the link filtration of \( CF^-(S^3) \).

In [5], the authors define a sign assignment for empty rectangles \( S : \text{Rect}^e \to \{ \pm 1 \} \). Then, by considering \( C^-(G) \) the group ring of \( \mathcal{G} \) over \( \mathbb{Z}[U_{O_1}, \ldots, U_{O_n}] \) and the
differential $\partial^- : C^-(G) \to C^-(G)$ given by
\[
\partial^- x = \sum_{y \in \mathfrak{g}_n} \sum_{r \in \text{Rec}^0(x,y)} S(r).U_{O_i}^{O_1(r)} \ldots U_{O_n}^{O_n(r)}.y
\]
they obtain the following result.

**Theorem 1.2** (Manolescu–Ozsváth–Szabó–Thurston [5]) Let $\overrightarrow{L}$ be an oriented link with $\ell$ components. We number the $\mathfrak{g}_0$ so that $O_1, \ldots, O_\ell$ correspond to the different components of $\overrightarrow{L}$. Then the filtered quasi-isomorphism type of $C^-(G) / \partial^-$ over $\mathbb{Z} [U_{O_1}, \ldots, U_{O_\ell}]$ is an invariant of the link.

In this paper, we give a way to refine the complex over $\mathbb{Z}$ thanks to $\mathfrak{S}_n$ the spin extension of $\mathfrak{S}_n$ which is a non-trivial central extension of $\mathfrak{S}_n$ by $\mathbb{Z} / 2\mathbb{Z}$. In Section 2 we define the spin extension $\mathfrak{S}_n$ and make some algebraic calculus. Let $z$ be the unique non-trivial central element of $\mathfrak{S}_n$ and $\Lambda = \mathbb{Z} [U_{O_1}, \ldots, U_{O_n}]$. In Section 3 we define a filtered chain complex $(\widetilde{C}^-(G), \widetilde{\partial}^-)$ where $\widetilde{C}^-(G)$ is the quotient module of the free $\Lambda$–module with generating set $\mathfrak{S}_n$ by the submodule generated by $\{z + 1\}$. Finally, in Section 4, we prove that our chain complex defines a sign assignment in the sense of [5] and that $(\widetilde{C}^-(G), \widetilde{\partial}^-)$ is filtered quasi-isomorphic to $(C^-(G), \partial^-)$ with coefficients in $\mathbb{Z}$.

## 2 Algebraic preliminaries

Let $\mathfrak{S}_n$ be the group of bijections of a set with $n$ elements numbered from 0 to $n - 1$. It is given in terms of generators and relations where the set of generators is $\{\tau_i\}_{i=0}^{n-2}$ with $\tau_i$ the transposition which exchanges $i$ and $i+1$ and relations are

\[
\begin{align*}
\tau_i^2 &= 1 \quad 0 \leq i \leq n - 2 \\
\tau_i \tau_j &= \tau_j \tau_i \quad |i - j| > 1, \quad 0 \leq i, j \leq n - 2 \\
\tau_i \tau_{i+1} \tau_i &= \tau_{i+1} \tau_i \tau_{i+1} \quad 0 \leq i \leq n - 3.
\end{align*}
\]

**Theorem 2.1** The group given by generators and relations

\[
\mathfrak{S}_n = \langle \tau_0, \ldots, \tau_{n-2}, z \mid z^2 = 1, z\tau_i = \tau_i z, \tau_i^2 = z, \quad 0 \leq i \leq n - 2; \tau_i \tau_j = z \tau_j \tau_i \quad |i - j| > 1, \quad 0 \leq i, j \leq n - 2; \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \quad 0 \leq i \leq n - 3 \rangle
\]
is a non-trivial central extension $(n \geq 4)$ of $\mathfrak{S}_n$ by $\mathbb{Z} / 2\mathbb{Z}$ called the spin extension of $\mathfrak{S}_n$.
Remark 2.2  A proof of this theorem can be found in Karpilovsky [3, Theorem 2.12.3]. To see that it is a non-trivial extension, one can notice the following: let $\mathbb{Q}_8$ be the subgroup of $\mathbb{H}/\mathbb{Z}$ generated by $\overline{e}_0, \overline{e}_2, z$. Then $\mathbb{Q}_8$ is isomorphic to the unit sphere in the space of quaternions intersected with the lattice $\mathbb{Z}^4$ by a morphism $\Phi$ such that $\Phi(\overline{e}_0) = i$, $\Phi(\overline{e}_2) = j$, $\Phi(\overline{e}_0, \overline{e}_2) = k$ and $\Phi(z) = -1$. Therefore $\mathbb{H}/\mathbb{Z}$ is non-trivial.

Remark 2.3  Cases $n = 2$ and $n = 3$ are not interesting in our situation: the only knot which can be represented by a grid diagram of complexity 2 or 3 is the trivial knot. Nevertheless, the group given by generators and relations above still exists: in the case $n = 2$, it is isomorphic to $\mathbb{Z}/4\mathbb{Z}$, in the case $n = 3$, it is isomorphic to a subgroup of $GL(2, \mathbb{C})$ (see [3, Lemma 2.12.2]).

For $i < j$, define

$$\overline{e}_{i,j} = \overline{e}_i \cdot \overline{e}_{i+1} \cdot \ldots \cdot \overline{e}_{j-2} \cdot \overline{e}_{j-1} \cdot \overline{e}_j \cdot \overline{e}_{j+1} \cdot \overline{e}_l$$

and $\overline{e}_{j,i} = z \cdot \overline{e}_{i,j}$.

Let $\varepsilon: \mathcal{G}_n \to \{0, 1\}$ be the signature morphism.

Lemma 2.4  Let $\overline{x} = \overline{e}_{i_1}, \overline{e}_{i_2}, \ldots, \overline{e}_{i_k}$ be an element in $\mathcal{G}_n$ and $x = p(\overline{x}) \in \mathcal{G}_n$. Then for any $0 \leq i \neq j \leq n - 1$

$$\overline{x} \cdot \overline{e}_{i,j} \cdot \overline{x}^{-1} = z^{\varepsilon(x)} \overline{e}_{x(i),x(j)}$$

Proof  Since $\overline{x} = \overline{e}_{i_1}, \overline{e}_{i_2}, \ldots, \overline{e}_{i_k}$, $\overline{x}^{-1} = z^{\varepsilon(x)} \overline{e}_{i_k}, \ldots, \overline{e}_{i_1}$. We prove by induction on $k \geq 1$ that for any $i, j \in \{0, \ldots, n - 1\}$ we have $\overline{x} \cdot \overline{e}_{i,j} \cdot \overline{x}^{-1} = z^{\varepsilon(x)} \overline{e}_{x(i),x(j)}$.

- **Initialization** Let $\overline{x} = \overline{e}_l$ and $0 \leq i < j \leq n - 1$. So $\overline{e}_{l}^{-1} = z \cdot \overline{e}_l$ and $\varepsilon(x) = 1$.

There are several cases.

- **Case 1**: $l < i - 1$ or $l > j$  \[ \overline{x} \cdot \overline{e}_{i,j} \cdot \overline{x} = z \cdot \overline{e}_{i,j}. \]

- **Case 2**: $l = i - 1$  \[ \overline{x} \cdot \overline{e}_{i,j} \cdot \overline{x} = z^{\varepsilon(x)} \overline{e}_{i-1,j} \cdot \overline{e}_{i,j}^{-1} = z^{\varepsilon(x)} \overline{e}_{i-1,j} \text{ by definition.} \]

- **Case 3**: $l = i$  \[ \overline{x} \cdot \overline{e}_{i,j} \cdot \overline{x} = z \cdot \overline{e}_{i+1,j}. \]

- **Case 4**: $i < l < j - 1$  We prove by induction on $l - i \geq 1$ for $i, j$ fixed that $\overline{e}_{i,j} \cdot \overline{e}_{i,j} \cdot \overline{e}_{i,j}^{-1} = z^{\varepsilon(x)} \overline{e}_{x(i),x(j)}$. For $l = i + 1$ then we have

$$\overline{e}_{i+1,j} \cdot \overline{e}_{i,j} \cdot \overline{e}_{i+1,j}^{-1} = z^{\varepsilon(x)} \overline{e}_{i+1,j} \cdot \overline{e}_{i+1,2,j} \cdot \overline{e}_{i,j} \cdot \overline{e}_{i+1,j}^{-1} = z^{\varepsilon(x)} \overline{e}_{i+1,j} \cdot \overline{e}_{i+1,2,j} \cdot \overline{e}_{i+1,j}^{-1} = z^{\varepsilon(x)} \overline{e}_{i,j}.$$
Suppose it is proved until rank \((l - 1) - i\). Then for \(\bar{x} = \bar{t}_l\) with \(l < j - 1\) we have

\[
\bar{x}.\bar{t}_l.\bar{x} = z\bar{t}_j.\bar{t}_{i,j}.\bar{t}_l = z(\bar{t}_j, \ldots, \bar{t}_{j-2}).(\bar{t}_{j-1}j).(\bar{t}_l.\bar{t}_{l-1}.\bar{t}_{l-1}).(\bar{t}_{l-2}j).
\]

\[
= z(\bar{t}_j, \ldots, \bar{t}_{j-2}).(\bar{t}_{j-1}j).(\bar{t}_l.\bar{t}_{l-1}.\bar{t}_{l-1}).(\bar{t}_{l-2}j).\bar{t}_{l-1}.j.(\bar{t}_j.\bar{t}_{j-1}).\bar{t}_{j-1}j.
\]

\[
= z(\bar{t}_j, \ldots, \bar{t}_{j-1}).j.(\bar{t}_j.\bar{t}_{j-1})\text{ by induction}
\]

\[
= z\bar{t}_{l,j}\text{ by case 2.}
\]

- **Case 5:** \(l = j - 1\)

\[
\bar{t}_{l-1}.\bar{t}_{l,j}.\bar{t}_{l-1} = z(\bar{t}_j, \ldots, \bar{t}_{j-3}).\bar{t}_{j-2}.\bar{t}_{j-1}.\bar{t}_{l-1}.\bar{t}_{j-1}.(\bar{t}_{j-3}j)
\]

\[
= z\bar{t}_{l,j-1}.
\]

- **Case 6:** \(l = j\)

\[
\bar{t}_j.\bar{t}_{i,j}.\bar{t}_j = z(\bar{t}_j, \ldots, \bar{t}_{j-2}).\bar{t}_j.\bar{t}_{j-1}.\bar{t}_j.(\bar{t}_{j-2}j)
\]

\[
= z\bar{t}_{l,j+1}.
\]

- **Heredity** Suppose the property is true until rank \(k\). Let \(\bar{x} = \bar{t}_{i_1}.\bar{t}_{i_2}.\ldots.\bar{t}_{i_k}\) and \(\bar{t}_{i,j}\) be two elements in \(\tilde{G}_n\). Denote \(\bar{y} = \bar{t}_{i_2}.\ldots.\bar{t}_{i_k}\). Then \(\bar{x}.\bar{t}_{i,j}.\bar{x}^{-1} = \bar{t}_{i_1}.\bar{y}.\bar{t}_{i,j}.\bar{y}^{-1}.z\bar{t}_{i_1}\). By induction hypothesis,

\[
\bar{y}.\bar{t}_{i,j}.\bar{y}^{-1} = z^{e(y)}(\bar{t}_{y(i)}y(j)).
\]

So, \(\bar{x}.\bar{t}_{i,j}.\bar{x}^{-1} = \bar{t}_{i_1}.z^{e(y)}(\bar{t}_{y(i)}y(j)).z\bar{t}_{i_1}\). By induction hypothesis one more time,

\[
\bar{x}.\bar{t}_{i,j}.\bar{x}^{-1} = z^{e(y)+1}\bar{t}_{t_{y(i)}y(i)j} = z^{e(x)}(\bar{t}_{x(i)}x(j)).
\]

The group \(\tilde{G}_n\) has another presentation in terms of generators and relations. Take \(\{z\} \cup \{\bar{t}_{i,j}\}_{i \neq j}\) where \(0 \leq i, j \leq n - 1\) as the set of generators with the following relations:

1. \(z' \cdot z' = \bar{t}' \quad z'\bar{t}_{i,j} = \bar{t}'_{i,j}z' \quad \bar{t}_{i,j} = z'\bar{t}_{i,j} \quad \bar{t}_{i,j} = z' \quad \text{for any } i, j\) \hspace{1cm} (2–1)
2. \(\bar{t}_{i,j}\bar{t}_{k,l} = z^{e(y)}(\bar{t}_{y(i)}y(j)).z\bar{t}_{i_1}\) \hspace{1cm} for any \(i, j, k, l\) if \(\{i, j\} \cap \{k, l\} = \emptyset\) \hspace{1cm} (2–2)
3. \(\bar{t}_{i,j}\bar{t}_{j,k} = z^{e(y)}(\bar{t}_{y(i)}y(j)).z\bar{t}_{i,k} \quad \text{for any } i, j, k\) \hspace{1cm} (2–3)

**Proof** Let \(G_n\) the group with \(z\) and \(\bar{t}_i\) as generators and \(G'_n\) the other one. Define \(\phi: \tilde{G}_n \rightarrow \tilde{G}'_n\) given on generators by \(\phi(\bar{t}_i) = \bar{t}_{i,i+1}'\), \(\phi(z) = z'\). For \(i < j\), let \(\phi(\bar{t}_{i,j}) = \bar{t}_{i,j}'\). By definition, (2–1) is verified. Lemma 2.4 gives equations (2–2) and (2–3). So the map \(\phi\) extends to a group isomorphism. \(\square\)
In what follows, we drop the prime exponent and only refer to $\tilde{t}_{i,j}$ and $z$ ($\tilde{t}_i$ means $\tilde{t}_{i,i+1}$).

3 The chain complex

Let $G$ be a grid presentation with complexity $n$ of the link $\tilde{L}$. Let $\Lambda$ denote the ring $\mathbb{Z}[U_{O_1}, \ldots, U_{O_n}]$. We define $\widehat{C}^- (G)$ to be the free $\Lambda$–module with generating set $\mathcal{S}_n$ quotiented by the submodule generated by $\{z+1\}$ i.e.

$$\widehat{C}^- (G) = \Lambda[\mathcal{S}_n]/ < z + 1 > .$$

Considered as module, $\widehat{C}^- (G)$ coincides with the free $\Lambda$–module with generating set $\mathcal{S}_n$. But we can also consider the structure of algebra of $\widehat{C}^- (G)$ over $\Lambda$. In this case, one can think of $\widehat{C}^- (G)$ as the group algebra of $\mathcal{S}_n$ over $\Lambda$ where the product is twisted by a non-trivial 2–cocycle (see Section 4).

We endow the set of generators with a Maslov grading $M$ and an Alexander filtration $A$ given by $M(\tilde{x}) = M(x)$ and $A(\tilde{x}) = A(x)$.

Let $\tilde{x}$ be an element of $\mathcal{S}_n$ and let $\text{Rect}(\tilde{x})$ be the set of rectangles starting at $\tilde{x}$: by definition it is the set $\{\tilde{t}_{i,j}\}_{0 \leq i < j \leq n-1}$. If we consider the set $\text{Rect}(\tilde{x}, \tilde{y})$ of rectangles connecting $x$ to $y$ (where $y = x, t_{i,j}$) as in [5], either it is the empty set, or it consists of two rectangles. We interpret the rectangle $\tilde{t}_{i,j}$ in the oriented torus $T$ as the rectangle whose bottom left corner belongs to the $i$th vertical circle. So in the case where $\text{Rect}(x, y) = \{r_1, r_2\}$ the two corresponding rectangles are $\tilde{t}_{i,j}$ and $\tilde{t}_{j,i}$. Let $r$ be the rectangle of $\text{Rect}(x, y)$ corresponding to $\tilde{r}$. A rectangle $\tilde{r} \in \text{Rect}(\tilde{x})$ is said to be empty if the corresponding rectangle $r \in \text{Rect}(x, y)$ is empty. The set of empty rectangles starting at $\tilde{x}$ is denoted $\text{Rect}^e(\tilde{x})$.

We endow $\widehat{C}^- (G)$ with a differential $\tilde{\partial}^-$ given on elements of $\mathcal{S}_n$ by:

$$\tilde{\partial}^- \tilde{x} = \sum_{\tilde{r} \in \text{Rect}^e(\tilde{x})} U_{O_{O_1}}(\tilde{r}) \ldots U_{O_{O_n}}(\tilde{r}) \tilde{x} \tilde{r}$$

where $O_k(\tilde{r})$ is the number of times $O_k$ appears in the interior of $r$.

**Proposition 3.1** The differential $\tilde{\partial}^-$ drops the Maslov degree by one and respect the Alexander filtration.

**Proof** It is a straightforward consequence of calculus done in [5].

_Algberaic & Geometric Topology, Volume 8 (2008)_
Figure 3: Rectangles. Black dots represent \( x \) and white dots \( y \). The two hatched regions correspond to rectangles \( \tau_{0,2} \in \text{Rect}(\bar{x}) \) and \( \tau_{2,0} \in \text{Rect}(\bar{x}) \). The rectangle \( \tau_{0,2} \) is an empty rectangle while \( \tau_{2,0} \) is not.

**Proposition 3.2** The endomorphism \( \tilde{\partial}^\sim \) of \( \tilde{C}^\sim (G) \) is a differential, i.e.

\[
\tilde{\partial}^\sim \circ \tilde{\partial}^\sim = 0.
\]

**Proof** Let \( \bar{x} = s(x) \in \tilde{C}_n \), viewed as a generator of \( \tilde{C}^\sim (G) \). Then

\[
\tilde{\partial}^\sim \circ \tilde{\partial}^\sim (\bar{x}) = \sum_{\tilde{r}_2 \in \text{Rect}^\sim(\bar{x}, \bar{r}_1)} \sum_{\bar{r}_1 \in \text{Rect}^\circ(\bar{x})} U_{\bar{O}_1} \tilde{O}_1(\tilde{r}_1) + U_{\bar{O}_2} \tilde{O}_2(\tilde{r}_2) + \ldots + U_{\bar{O}_n} \tilde{O}_n(\tilde{r}_2). \]

There are different cases which are illustrated by Figure 4.

**Cases 1,2** The rectangles corresponding to \( \bar{\tau}_{i,j} \) and \( \bar{\tau}_{k,l} \) give the elements \( \bar{z}_1 = \bar{x} \cdot \bar{\tau}_{k,l} \cdot \bar{\tau}_{i,j} \) and \( \bar{z}_2 = \bar{x} \cdot \bar{\tau}_{i,j} \cdot \bar{\tau}_{k,l} \). By equation (2–2) contribution to \( \tilde{\partial}^\sim \circ \tilde{\partial}^\sim (\bar{x}) \) is null.

**Case 3** Supports of the rectangles have a common edge. The two corresponding elements are \( \bar{z}_1 = \bar{x} \cdot \bar{\tau}_{i,j} \cdot \bar{\tau}_{k,l} \) and \( \bar{z}_2 = \bar{x} \cdot \bar{\tau}_{i,j} \cdot \bar{\tau}_{k,l} \). By equation (2–3), \( \bar{z}_1 = z \bar{z}_2 \) and so the contribution is null. Other cases work in a similar way.

**Case 4** The vertical annulus is of width 1 and corresponds to \( \bar{z}_1 = U_{\bar{O}_m} \bar{x} \cdot \bar{\tau}_{i,j} \cdot \bar{\tau}_l \) (it is a consequence of the condition on rectangles to be empty).

To this vertical annulus corresponds the horizontal annulus of height 1 which contains \( O_m \). This horizontal annulus contributes for \( U_{\bar{O}_m} \bar{x} \cdot \bar{\tau}_{i,k} \cdot \bar{\tau}_{k,l} \) for a pair \( k < l \in \{0, \ldots, n-1\} \). So, the contribution of each vertical annulus is canceled by the corresponding horizontal annulus. The global contribution to \( \tilde{\partial}^\sim \circ \tilde{\partial}^\sim (\bar{x}) \) is null. □
4 Sign assignment induced by the complex

In this section we prove that the chain complex \( \tilde{C}^- (G) \) coincides with the chain complex \( C^- (G) \) over \( \mathbb{Z} \) after a choice of a sign assignment.

**Definition 4.1** A sign assignment is a function \( S : \text{Rect}^\circ \rightarrow \{ \pm 1 \} \) such that

- (Sq) for any distincts \( r_1, r_2, r'_1, r'_2 \in \text{Rect}^\circ \) such that \( r_1 \ast r_2 = r'_1 \ast r'_2 \) we have
  \[
  S(r_1).S(r_2) = -S(r'_1).S(r'_2).
  \]

- (V) if \( r_1, r_2 \in \text{Rect}^\circ \) are such that \( r_1 \ast r_2 \) is a vertical annulus then
  \[
  S(r_1).S(r_2) = -1.
  \]

- (H) if \( r_1, r_2 \in \text{Rect}^\circ \) are such that \( r_1 \ast r_2 \) is a horizontal annulus then
  \[
  S(r_1).S(r_2) = +1.
  \]
Let \( s: \mathfrak{S}_n \to \widetilde{\mathfrak{S}}_n \) be a section of the map \( p \) that is \( p \circ s = \text{id}_{\mathfrak{S}_n} \).

\[
1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{i} \widetilde{\mathfrak{S}}_n \xrightarrow{p_s} \mathfrak{S}_n \longrightarrow 1
\]

To define the sign assignment we need the 2–cocycle \( c \in C^2(\mathfrak{S}_n, \mathbb{Z}/2\mathbb{Z}) \) associated to the map \( s \) given by

\[
s(x).s(y) = (i \circ c(x, y))s(x, y).
\]

(4–1) The cohomological class of \( c \) measures how \( s \) fails to be a group morphism. In particular, it is non-trivial \((n \geq 4)\) since \( \mathfrak{S}_n \) is a non-trivial central extension of \( \mathfrak{S}_n \) by \( \mathbb{Z}/2\mathbb{Z} \).

We say that a rectangle \( r \) is horizontally torn if given the coordinates \((i_{hl}, j_{hl})\) of its bottom left corner and \((i_{tr}, j_{tr})\) of its top right corner then \( i_{hl} > i_{tr} \). Otherwise, \( r \) is said to be not horizontally torn.

**Lemma 4.2** The complex \((\mathcal{C}^+, \widetilde{\mathcal{D}}^-)\) induces a sign assignment in the sense of Definition 4.1: for all \((x, y) \in \mathfrak{S}^2_n\) and all \( r \in \text{Rect}^\circ(x, y)\)

\[
S(r) = \varepsilon(r).c(x^{-1}, y, x)
\]

where \( \varepsilon(r) = +1 \) if \( r \) is a rectangle not horizontally torn and \( \varepsilon(r) = -1 \) otherwise.

**Remark** The sign assignment in the sense of Definition 4.1 is unique up to a 1–coboundary: if \( S_1 \) and \( S_2 \) are two sign assignments then there exists an application \( f: \mathfrak{S}_n \to \{\pm 1\} \) such that for all rectangles \( r \in \text{Rect}^\circ(x, y)\), \( S_1(r) = f(x).f(y).S_2(r) \).

It is a consequence of the fact that the central extension corresponds to a 2–cohomological class in \( H^2(\mathfrak{S}_n, \mathbb{Z}/2\mathbb{Z}) \) (compare with [5, Theorem 4.2]). Here, we construct explicitly a map \( s: \mathfrak{S}_n \to \widetilde{\mathfrak{S}}_n \) such that \( p \circ s = \text{id} \) which means making a choice of a representative of this class, another choice must differ by a 1–coboundary.

**Proof** Since \( c \) is 2–cocycle we have \( \delta c = 1 \) ie for all \((x, y, z) \in \mathfrak{S}^3_n\)

\[
\delta c(x, y, z) = c(y, z).c(x, y, z).c(x, y, z).c(x, y) = 1.
\]

By definition we have \( c(x, 1) = c(1, x) = 1 \) and \( c(\tau_{i,j}, \tau_{i,j}) = -1 \). Let's prove that \( S \) satisfy properties (Sq), (V) et (H).

(Sq) Let any four distincts rectangles \( S r_1, r_2, r'_1, r'_2 \in \text{Rect}^\circ \) such that \( r_1 \ast r_2 = r'_1 \ast r'_2 \).

Suppose \( \bar{r}_{i,j} = \tau_{i} \in \text{Rect}^\circ(\bar{x}) \) corresponds to \( r_1 \) and \( \bar{r}_{k,l} = \tau_{k,l} \in \text{Rect}^\circ(\bar{x}, \tau_{i,j}) \) corresponds to \( r_2 \). Then \( \bar{r}'_1 = \tau_{k,l} \in \text{Rect}^\circ(\bar{x}) \) corresponds to \( r'_1 \) and \( \bar{r}'_2 = \tau_{i,j} \in \text{Rect}^\circ(\bar{x}) \).
Rect\(^c\)(\(\tilde{X}, \tilde{T}_{k,l}\)) corresponds to \(r'_2\). There are several cases to verify, as for the proof of \(\tilde{\partial}^{-} \circ \tilde{\partial}^{-} = 0\) but all cases can be verified in a similar way. We verify the case \(i < j < k < l\). We calculate \(\delta c(\tau_{k,l}, \tau_{i,j}, x)\) and \(\delta c(\tau_{i,j}, \tau_{k,l}, x)\). With equalities \(c(\tau_{i,j}, \tau_{k,l}, x) = c(\tau_{k,l}, \tau_{i,j}, x)\) and \(c(\tau_{i,j}, \tau_{k,l}) = -c(\tau_{k,l}, \tau_{i,j})\) we get

\[
S(r_1)S(r_2) = -S(r'_1)S(r'_2).
\]

(V) Let \(r_1, r_2 \in \text{Rect}^c\) such that \(r_1 \ast r_2\) is a vertical annulus. Suppose that \(\tilde{r}_1 = \tilde{t}_i \in \text{Rect}^c(\tilde{X})\) corresponds to \(r_1\) and \(\tilde{r}_2 = \tilde{t}_j \in \text{Rect}^c(\tilde{X}, \tilde{T}_i)\) corresponds to \(r_2\). We calculate \(\delta c(\tau_{i,j}, \tau_{i,j}, x)\) and with equalities \(c(x, 1) = 1, c(\tau_{i,j}, \tau_{i,j}) = -1\) we get

\[
S(r_1)S(r_2) = -1.
\]

(H) Let \(r_1, r_2 \in \text{Rect}^c\) such that \(r_1 \ast r_2\) is a horizontal annulus (of height one). Suppose \(\tilde{r}_1 = \tilde{t}_{i,j} \in \text{Rect}^c(\tilde{X})\) corresponds to \(r_1\) and \(\tilde{r}_2 = \tilde{t}_{j,k} \in \text{Rect}^c(\tilde{X}, \tilde{T}_{i,j})\) corresponds to \(r_2\). We calculate \(\delta c(\tau_{i,j}, \tau_{i,j}, x)\) and with equalities \(c(x, 1) = 1, c(\tau_{i,j}, \tau_{i,j}) = -1\) we get

\[
S(r_1)S(r_2) = +1.
\]

\[\Box\]

**Proposition 4.3** The filtered chain complex \((\tilde{C}^{-}(G), \tilde{\partial}^{-})\) is filtered isomorphic to the filtered chain complex \((C^{-}(G), \partial^{-})\).

**Proof** The map \(s: \mathfrak{S}_n \to \tilde{\mathfrak{S}}_n\) extends linearly with respect to \(\mathbb{Z}[U_1, \ldots, U_n]\) uniquely to a map \(s: C^{-}(G) \to \tilde{C}^{-}(G)\) which is an isomorphism of modules. It commutes with the differentials \(ie s \circ \partial^{-} = \tilde{\partial}^{-} \circ s\) where the sign assignment \(S\) is given by equation (4–2). By definition, \(s\) respects the Alexander filtration and the Maslov grading. So \(s\) defines a filtered isomorphism between the complexes \((C^{-}(G), \partial^{-})\) and \((\tilde{C}^{-}(G), \tilde{\partial}^{-})\). \[\Box\]

A consequence of the above proposition and [5, Theorem 1.2] is the following.

**Corollary 4.4** Let \(\hat{L}\) be an oriented link with \(\ell\) components. We number the \(\bigcirc\) so that \(O_1, \ldots, O_\ell\) correspond to the different components of \(\hat{L}\). Then the filtered quasi-isomorphism type of \((\tilde{C}^{-}(G), \tilde{\partial}^{-})\) over \(\mathbb{Z}[U_{O_1}, \ldots, U_{O_\ell}]\) is an invariant of the link.

**Remark** The proof of this theorem can also be done by adapting the original proof in [5], sometimes with slightly simplified arguments.

*Algebraic & Geometric Topology, Volume 8 (2008)*
Étienne Gallais

References


Laboratoire de Mathématiques Jean Leray (LMJL), UFR Sciences et Techniques
2 rue de la Houssinière - BP 92208, 44 322 Nantes Cedex 3, France
Etienne.Gallais@univ-nantes.fr
http://www.math.sciences.univ-nantes.fr/~gallais/

Received: 4 July 2007 Revised: 30 May 2008

*Algebraic & Geometric Topology, Volume 8 (2008)*