A class function on the mapping class group of an orientable surface and the Meyer cocycle

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In this paper we define a \( \mathbb{Q}P^1 \)-valued class function on the mapping class group \( \mathcal{M}_{g,2} \) of a surface \( \Sigma_{g,2} \) of genus \( g \) with two boundary components. Let \( E \) be a \( \Sigma_{g,2} \)-bundle over a pair of pants \( P \). Gluing to \( E \) the product of an annulus and \( P \) along the boundaries of each fiber, we obtain a closed surface bundle over \( P \). We have another closed surface bundle by gluing to \( E \) the product of \( P \) and two disks. The sign of our class function cobounds the 2–cocycle on \( \mathcal{M}_{g,2} \) defined by the difference of the signature of these two surface bundles over \( P \).

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1 Introduction

Let \( \Sigma_{g,r} \) be a compact oriented surface of genus \( g \) with \( r \) boundary components. The mapping class group \( \mathcal{M}_{g,r} \) is \( \pi_0 \text{Diff}_+ (\Sigma_{g,r}, \partial \Sigma_{g,r}) \) where \( \text{Diff}_+ (\Sigma_{g,r}, \partial \Sigma_{g,r}) \) is the group of orientation preserving diffeomorphisms of \( \Sigma_{g,r} \) which restrict to the identity on the boundary \( \partial \Sigma_{g,r} \). We simply denote \( \Sigma_g := \Sigma_{g,0} \) and \( \mathcal{M}_g := \mathcal{M}_{g,0} \). Harer [4] proved that

\[ H^2 (\mathcal{M}_{g,r}; \mathbb{Z}) \cong \mathbb{Z}, \quad g \geq 3, \quad r \geq 0, \]

see also Korkmaz and Stipsicz [8]. Meyer [9] defined a cocycle \( \tau_g \in Z^2 (\mathcal{M}_g; \mathbb{Z}) \) \( (g \geq 0) \) called the Meyer cocycle which represents four times generator of the second cohomology class when \( g \geq 3 \). Let \( D_1, D_2, \) and \( D_3 \) be mutually disjoint disks in \( S^2 \), and \( \text{Int} D_i \) the interior of \( D_i \) for \( i = 1, 2, 3 \). We denote by \( P := S^2 - \bigcup_{i=1}^3 \text{Int} D_i \) the pair of pants, and \( \alpha, \beta, \gamma \in \pi_1 (P) \) be the homotopy classes as shown in Figure 1. We consider a \( \Sigma_{g,r} \)-bundle \( E_{g,r}^{\psi, \psi'} \) on the pair of pants \( P \) which has monodromies \( \varphi, \psi, (\psi \varphi)^{-1} \in \mathcal{M}_{g,r} \) along \( \alpha, \beta, \gamma \in \pi_1 (P) \). The diffeomorphism type of \( E_{g,r}^{\psi, \psi'} \) does not depend on the choice of representatives in the mapping classes \( \varphi \) and \( \psi \). Since \( E_{g,r}^{\psi, \psi'} \) is the oriented fiber

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bundle, it has the canonical orientation comes from that of $\Sigma_{g,r}$ and $P$. The Meyer cocycle is defined by
\[
\tau_g: \mathcal{M}_g \times \mathcal{M}_g \rightarrow \mathbb{Z},
\]
\[
(\varphi, \psi) \mapsto \text{Sign}_{E^\varphi,\psi}
\]
where $\text{Sign}_{E^\varphi,\psi}$ is the signature of the compact oriented 4–manifold $E^\varphi,\psi$. For $k > 0$, it is known as Novikov additivity that when two compact oriented $4k$–manifolds are glued by an orientation reversing diffeomorphism of their boundaries, the signature of their union is the sum of their signature. When a pants decomposition of a closed oriented 2–manifold is given, the signature of a $\Sigma_g$–bundle on the 2–manifold is the sum of the signature of the $\Sigma_g$–bundles restricted to each pair of pants. Therefore, it is important to study the Meyer cocycle to calculate the signature of compact 4–manifolds. For $g = 1, 2$ the Meyer cocycle $\tau_g$ is a coboundary, and the cobounding function of this cocycle is calculated by several authors, for instance, Meyer [9], Atiyah [1], Kasagawa [6] and Iida [5]. The Meyer cocycle is not a coboundary if genus $g \geq 3$, but the cocycle can be a coboundary when it is restricted to some subgroups. For example, on the subgroup called the hyperelliptic mapping class group, the cobounding function is calculated by Endo [2] and Morifuji [11].

Let $I$ be the unit interval $[0, 1] \subseteq \mathbb{R}$. By sewing a pair of disks onto the surface $\Sigma_{g,2}$ along the boundary, we have $\Sigma_g$. For $h \in \text{Diff}_+(\Sigma_{g,2}, \partial \Sigma_{g,2})$, if we extend $h$ by the identity on the pair of disks, we have a self-diffeomorphism of $\Sigma_g$. We denote it by $h \cup i d_{\bigcup_{i=1}^2 D^2}$. By sewing an annulus $S^1 \times I$ onto the surface $\Sigma_{g,2}$ along the boundary, we have $\Sigma_{g+1}$. In the same way, if we extend $h \in \text{Diff}_+(\Sigma_{g,2}, \partial \Sigma_{g,2})$ by the identity on the annulus, we have a self-diffeomorphism $h \cup i d_{S^1 \times I}$.
Define the induced homomorphism on the mapping class group by
\[ \theta: \mathcal{M}_{g,2} \to \mathcal{M}_g \]
\[ [h] \mapsto [h \cup i d_{1,2}] \]
and
\[ \eta: \mathcal{M}_{g,2} \to \mathcal{M}_{g+1,0} \]
\[ [h] \mapsto [h \cup i d_{S^1 \times I}] \].

Harer [3; 4] shows that \( \theta \) and \( \eta \) induce an isomorphism on the second homology classes when genus \( g \geq 5 \), so that \( \tau_g = \eta^* \tau_{g+1} - \theta^* \tau_g \) is a coboundary. Powell [12] proved that the first homology group \( H_1(\mathcal{M}_{g,r}; \mathbb{Z}) \) is trivial for \( g \geq 3 \) and \( r \geq 0 \), so by the universal coefficient theorem, it follows that the cobounding function of \( \tau_g \) is unique.

In this paper we define a \( QP^1 \)–valued class function \( m \) on the mapping class group \( \mathcal{M}_{g,2} \) in an explicit way by using information of the first homology group of a mapping torus of \( [h] \in \mathcal{M}_{g,2} \). For \( [p : q] \in QP^1 \), we define the sign of \( [p : q] \) by \( \text{sign}([p : q]) := \text{sign}(pq) \). We prove that the sign of the function \( m \) cobounds the cocycle \( \tau_g = \eta^* \tau_{g+1} - \theta^* \tau_g \). In particular, it turns out that the cocycle \( \tau_g \) is coboundary for any \( g \geq 0 \).

This function makes a little bit easy to evaluate the Meyer cocycle on the subgroups consists of mapping classes that fix a curve on the surface. For example, consider the case \( g = 1, 2 \). We denote by \( \phi_1 \) and \( \phi_2 \) the cobounding functions of \( \tau_1 \) and \( \tau_2 \). Since \( H_1(\mathcal{M}_{g,2}; \mathbb{Q}) = 0 \), the equation \( \eta^* \tau_{g+1} = \theta^* \tau_g + \delta m \) means \( \eta^* \phi_{g+1} = \theta^* \phi_g + m \) for \( g = 1, 2 \). In particular, the function \( \phi_1 \) is described explicitly in Meyer [9]. Therefore, our function \( m \) helps to describe the cobounding function of the Meyer cocycle for genus 2 and 3 on the subgroup.

In Section 2, we construct a class function \( m \), prove some properties of this function, and calculate the image of the function. In Section 3, we prove that the sign of this function cobounds the difference \( \tau_g = \eta^* \tau_{g+1} - \theta^* \tau_g \). By the definition of the Meyer cocycle \( \tau_g, \tau_g(\varphi, \psi) \) is just the difference \( \text{Sign} E_{g+1}^{\eta, \psi} - \text{Sign} E_g^{\theta, \psi} \), so that we calculate the difference by using the sign of the function \( m \). Moreover we compute the other differences of signature \( \text{Sign} (E_{g,2}^{\varphi, \psi}) \) and \( \text{Sign} (E_{g,2}^{\eta, \psi}) \) by the function \( m \).

## 2 Class function \( m: \mathcal{M}_{g,2} \to QP^1 \)

In this section we define the class function on the mapping class group \( \mathcal{M}_{g,2} \) stated in the introduction and describe some properties of the function including the nontriviality.
For \([p : q], [r : s] \in \mathbb{P}^1\), we define an addition in \(\mathbb{P}^1\) by
\[
[p : q] + [r : s] = \begin{cases} 
(pr : ps + qr), & \text{if } [p : q] \neq [0 : 1] \text{ or } [r : s] \neq [0 : 1] \\
[0 : 1], & \text{if } [p : q] = [r : s] = [0 : 1].
\end{cases}
\]
The projective line \(\mathbb{P}^1\) forms an additive monoid under this operation with \([1 : 0]\) the zero element.

In this section, all (co)homology groups are with \(\mathbb{Q}\) coefficients.

### 2.1 Construction of the class function

Before constructing the function, we prepare a fact about homology groups of compact 3–manifolds. Let \(Y\) be a compact oriented connected 3–manifold with boundary \(\partial Y\) and \(i : \partial Y \hookrightarrow Y\) the inclusion map. Consider the commutative diagram
\[
\begin{array}{ccc}
H^1(Y) & \xrightarrow{i^*} & H^1(\partial Y) & \xrightarrow{\delta^*} & H^2(Y, \partial Y) \\
\downarrow \cap[Y] & & \downarrow \cap[\partial Y] & & \downarrow \cap[Y] \\
H_2(Y, \partial Y) & \xrightarrow{\partial_*} & H_1(\partial Y) & \xrightarrow{i_*} & H_1(Y),
\end{array}
\]
where the upper and lower rows are the exact sequences of a pair \((Y, \partial Y)\), and the vertical maps are the cap products with the (relative) fundamental classes of \(Y\) and \(\partial Y\). By the diagram and Poincaré Duality, it follows that the image of \(i^*\) is just its own annihilator with respect to the cup product of \(H^1(\partial Y)\)
\[
\text{Im } i^* = \text{Ann } (\text{Im } i^*).
\]
In particular, we have
\[
\dim \ker i_* = \dim \text{Im } i^* = \frac{1}{2} \dim H_1(\partial Y).
\]
We define the mapping torus of \(\varphi = [h] \in \mathcal{M}_{g,r}\) by
\[
X^\varphi := \Sigma_{g,r} \times I / \sim, \quad (x, 1) \sim (h(x), 0),
\]
and \(\pi : X^\varphi \to I / \partial I = S^1\) by the projection \(\pi([x, t]) = [t]\), where \([x, t] \in X^\varphi\) is the equivalent class of \((x, t) \in \Sigma_{g,r} \times I\). and \([t] \in I / \partial I = S^1\) the equivalent class of \(t \in I\).

The diffeomorphism type of the mapping torus \(X^\varphi\) does not depend on the choice of the representative \(h\). We fix the orientation on \(X^\varphi\) given by the product orientation on \(\Sigma_{g,r} \times I\). Let \(i_\varphi : \partial X^\varphi \hookrightarrow X^\varphi\) be the inclusion map. In this subsection we denote \(\Sigma := \Sigma_{g,2}\), and if we fix \(\varphi \in \mathcal{M}_{g,2}\), then we write simply \(X := X^\varphi\) and \(i := i_\varphi\). Let \(S_1\) and \(S_2\) be the two boundary components of \(\Sigma\), and \([S_k]\) \((k = 1, 2)\) the image
under the inclusion homomorphism $H_1(S_k) \to H_1(\Sigma)$ of the fundamental homology class.

We consider $\Sigma$ as a subspace of $X$ by the embedding $i: \Sigma \hookrightarrow X$ by $x \mapsto [x,0]$. We choose points $p_1 \in S_1$, $p_2 \in S_2$, and $p \in S^1$, and orientation-preserving homeomorphisms $i_1: S^1 \to S_1$ and $i_2: S^1 \to S_2$. We define singular chains $f_k: I \to (S_1 \sqcup S_2) \times S^1 = \partial X$ ($k = 1, 2, 3, 4$) by

$$f_1(t) = (i_1(t), p), \quad f_2(t) = (i_2(t), p), \quad f_3(t) = (p_1, t) \quad \text{and} \quad f_4(t) = (p_2, t)$$ respectively.

Let $e_k \in H_1(\partial X)$ be the homology class of $f_k$ ($k = 1, 2, 3, 4$). Then the set \{e_1, e_2, e_3, e_4\} forms a basis for $H_1(\partial X)$, and the intersection number

$$e_i \cdot e_j = \begin{cases} 1 & \text{if } j = i + 2, \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, 2$ and $j = 3, 4$. Now we describe the kernel of the homomorphism $i_*: H_1(\partial X) \to H_1(X)$. Since $e_1$ and $e_2$ lie in the kernel of $(\pi|_{\partial X})_*$ and $\pi_*(e_3) = \pi_*(e_4) = [S^1] \in H_1(S^1)$, we have

$$\text{Ker } i_* \subset \text{Ker } (\pi_* i_*) = \mathbb{Q}e_1 \oplus \mathbb{Q}e_2 \oplus \mathbb{Q}(e_3 - e_4).$$

By the definition of the map $f_k$, $(i \circ f_k)_*[S^1] = i_*[S_k]$, and so $i_*(e_1 + e_2) = i_*([S_1] + [S_2]) \in H_1(X)$. Since $S_1 \sqcup S_2$ is the boundary of $\Sigma$, we have $[S_1] + [S_2] = 0 \in H_1(\Sigma)$. Hence

$$\mathbb{Q}(e_1 + e_2) \subset \text{Ker } i_*.$$

As we saw at the beginning of this subsection, $\dim \text{Ker } i_* = \frac{1}{2} \dim H_1(\partial X) = 2$. It follows that $\text{Ker } i_* = \mathbb{Q}(e_1 + e_2) \oplus \mathbb{Q}(p(e_3 - e_4) + qe_1)$ for some $p, q \in \mathbb{Q}$. Now we can define a class function.

**Definition 2.1** For $\varphi \in \mathcal{M}_{g,2}$, we take $p, q \in \mathbb{Q}$ such that $\text{Ker } i_{\varphi} = \mathbb{Q}(e_1 + e_2) \oplus \mathbb{Q}(p(e_3 - e_4) + qe_1)$.

We define $m: \mathcal{M}_{g,2} \to \mathbb{Q}P^1$ by $m(\varphi) = [p:q]$.

**Lemma 2.2** For $\varphi, \psi \in \mathcal{M}_{g,2}$,

$$m(\psi \varphi \psi^{-1}) = m(\varphi).$$

**Proof** Define $\Psi: X^\varphi \to X^{\psi \varphi \psi^{-1}}$ by $\Psi(x,t) = (\psi(x), t)$. Then $\Psi$ maps $e_i$ as defined in $H_1(X^\varphi)$ to the corresponding $e_i$ as defined in $H_1(X^{\psi \varphi \psi^{-1}})$, and the
following diagram commutes

\[
\begin{array}{ccc}
H_1(\partial X^\varphi) & \xrightarrow{i_{\varphi}^*} & H_1(X^\varphi) \\
\downarrow \Psi_* & & \downarrow \Psi_* \\
H_1(\partial X^{\varphi\varphi^{-1}}) & \xrightarrow{i_{\varphi\varphi^{-1}}^*} & H_1(X^{\varphi\varphi^{-1}}).
\end{array}
\]

As we see from the diagram, \(\Psi_*\) gives the natural isomorphism between the kernels \(\text{Ker} (H_1(\partial X^\varphi) \to H_1(X^\varphi))\) and \(\text{Ker} (H_1(\partial X^{\varphi\varphi^{-1}}) \to H_1(X^{\varphi\varphi^{-1}}))\). Hence we have \(m(\varphi\varphi^{-1}) = m(\varphi)\).

2.2 Some properties and the nontriviality of the class function

By the Serre spectral sequence of the \(\Sigma\)–bundle \(\pi: X \to S^1\), we have the exact sequence

\[
0 \to \text{Coker} (\varphi_* - 1) \xrightarrow{i_*} H_1(X) \xrightarrow{\pi_*} H_1(S^1) \to 0,
\]

where \(\text{Coker} (\varphi_* - 1)\) is the cokernel of the homomorphism \(\varphi_* - 1: H_1(\Sigma) \to H_1(\Sigma)\).

Then we have a unique homomorphism \(j_{\varphi}: \mathbb{Q}e_1 \oplus \mathbb{Q}e_2 \oplus \mathbb{Q}(e_3 - e_4) \to \text{Coker} (\varphi_* - 1)\) such that the diagram with exact rows

\[
\begin{array}{ccc}
0 & \xrightarrow{\omega} & \mathbb{Q}e_1 \oplus \mathbb{Q}e_2 \oplus \mathbb{Q}(e_3 - e_4) \\
\downarrow j_{\varphi} & & \downarrow i_* \\
0 & \xrightarrow{i_*} & H_1(X) \xrightarrow{\pi_*} H_1(S^1) \to 0
\end{array}
\]

commutes. By the diagram, we have

\[
\text{Ker} i_* = \text{Ker} j_{\varphi} \quad \text{and} \quad j_{\varphi}(e_1) = -j_{\varphi}(e_2) = [S_1] \in \text{Coker} (\varphi_* - 1).
\]

Now we introduce a cochain \(\omega_I \in C^1(\mathcal{M}_{g,2}; H_1(\Sigma))\) defined by Kawazumi [7]. On the fiber \(\Sigma = \pi^{-1}(0) \subset X\), pick a path \(l\) such that \(l(0) \in S_2\) and \(l(1) \in S_1\). Define \(\omega_I\) by

\[
\omega_I(\varphi) := [\varphi(l) - l] \in H_1(\Sigma).
\]

Then we have the following lemma.

**Lemma 2.3**

\[
j_{\varphi}(e_3 - e_4) = [\omega_I(\varphi)] \in \text{Coker} (\varphi_* - 1).
\]
Apply
At the beginning of this section, we defined the commutative monoid structure on $\mathbb{Q}P^1$. Hence, $i_*(e_3 - e_4) = \iota_*(\lfloor \varphi(l) - l \rfloor) \in H_1(X)$

Since $\iota_*$ is injective, the lemma follows.

From the lemma, we see the homology class $[\omega_l(\varphi)] \in \text{Coker}(\varphi_* - 1)$ is independent of the choice of the path $l$. If $\omega_l(\varphi) = 0$, then $j_\varphi(e_3 - e_4) = 0$.

Remark 2.4 If there exists a path $l$ from a point in $S_2$ to a point in $S_1$ which has no common point with the support of a representative of $\varphi \in \mathcal{M}_{g,2}$, then $m(\varphi) = [1 : 0]$. In particular, $m(id) = [1 : 0]$, the zero element of the monoid $\mathbb{Q}P^1$.

Define the subgroups $\mathcal{I}' := \text{Ker}(\mathcal{M}_{g,2} \to \text{Aut}(H_1(\Sigma_{g,2}; \mathbb{Z})))$ and $\mathcal{I} := \ker(\mathcal{M}_{g,2} \to \text{Aut}(H_1(\Sigma_{g,2}, \partial \Sigma_{g,2}; \mathbb{Z})))$. For $\varphi \in \mathcal{I}'$, $m(\varphi) = [p : q]$ means $p(\varphi(l) - l) + qe_1 = 0 \in H_1(\Sigma_{g,2}; \mathbb{Z})$. This shows that $m$ is homomorphic on $\mathcal{I}'$. For $\varphi \in \mathcal{I}$, $\omega(\varphi) = 0 \in H_1(\Sigma_{g,2}; \mathbb{Z})$. This shows that $m(\varphi) = [1 : 0]$ for all $\varphi \in \mathcal{I}$.

Remark 2.5 The restriction of $m$ on $\mathcal{I}$ is trivial, and the restriction of $m$ on $\mathcal{I}'$ is a nontrivial monoid homomorphism.

At the beginning of this section, we defined the commutative monoid structure on $\mathbb{Q}P^1$. So integral multiples of $m(\varphi)$ are well-defined.

Proposition 2.6 If $\varphi \in \mathcal{M}_{g,2}$ and $k \in \mathbb{Z}$, then

$$m(\varphi^k) = km(\varphi).$$

Proof The proposition is trivial for $k = 0$ and $k = 1$. Assume $k \geq 2$.

Let $m(\varphi) = [p : q]$. By the definition of $j_\varphi$, $pj_\varphi(e_3 - e_4) = -q[S_1] \in \text{Coker}(\varphi_* - 1)$. Hence, there exists $v \in H_1(\Sigma)$ such that

$$p[\varphi(l) - l] = -q[S_1] + (\varphi_* - 1)v \in H_1(\Sigma).$$

Apply $\varphi^i$ ($i = 0, 1, \ldots, k - 1$) to the both sides of the equation and sum over $i$. Then

$$\sum_{i=0}^{k-1} p[\varphi^{i+1}(l) - \varphi^i(l)] = \sum_{i=0}^{k-1} \{-q[S_1] + (\varphi_*^{i+1}(v) - \varphi_*^i(v))\},$$

that is

$$p[\varphi^k(l) - l] = -kq[S_1] + (\varphi_*^k - 1)v.$$
Hence, $m(\varphi^k) = [p : kq] = km(\varphi)$ for $k \geq 0$.

By applying $\varphi^{-1}$ to the equation $p[\varphi(l) - l] = -q[S_1] + (\varphi_* - 1)v$, we have
\[
p[\varphi^{-1}(l) - l] = q[S_1] + (\varphi_*^{-1} - 1)v \in H_1(\Sigma).
\]
Hence, $m(\varphi^{-1}) = [p : -q] = -m(\varphi)$. Since $m(\varphi^{-k}) = -m(\varphi^k) = -km(\varphi)$ for $k > 0$, the proposition follows for the case $k < 0$.

Now we compute the image of the function $m$. In particular, we see that $m$ is nontrivial.

**Proposition 2.7** For $g \geq 1$, $m$ is surjective. For $g = 0$, $\text{Im} \ (m) = [1 : \mathbf{Z}]$.

**Proof** Suppose $g \geq 1$. We choose oriented simple closed curves $\alpha$, $\alpha'$, and $\beta$ and paths $l$ and $l'$ as shown in Figure 2. We denote the Dehn twists along a simple closed curve $C \subset \Sigma$ by $t_C$, and the homology class of $C$ by $[C]$. Then $[\alpha] + [\alpha'] + [\beta] = 0 \in H_1(\Sigma)$ since they bound a 2–chain. For $p \in \mathbf{Z}$, if we denote $\varphi := t_{\alpha}^p t_{\alpha'}^{-1} t_{\beta}^{-1}$, then
\[
j_\varphi((p + 1)(e_3 - e_4)) = \omega_{l}(\varphi) + p\omega_{l'}(\varphi)
\]
\[
= [(t_{\alpha}^p t_{\alpha'}^{-1} t_{\beta}^{-1})(l) - l] + p((t_{\alpha}^p t_{\alpha'}^{-1} t_{\beta}^{-1})(l') - l')
\]
\[
= p([\alpha] + [\alpha'] + [\beta]) + [\beta] = [\beta] = [S_1].
\]
Hence, $j_\varphi((p + 1)(e_3 - e_4) - e_1) = 0$, so that
\[
m(\varphi) = [p + 1 : -1].
\]
By Proposition 2.6, we have
\[
m(\varphi^{-q}) = -q[p + 1 : -1] = \begin{cases} [p + 1 : q], & \text{if } p \neq -1 \\ [0 : 1], & \text{if } p = -1. \end{cases} \quad (q \in \mathbf{Z})
\]
Since \( p \) and \( q \) can run over all integers, we see \( m \) is surjective for \( g \geq 1 \).
For \( g = 0 \), \( \mathcal{M}_{0, 2} \) is the infinite cyclic group generated by \( t_{\beta} \). Since \( m(t_{\beta}^{-1}) = [1 : q] \), we have \( \text{Im}(m) = [1 : \mathbb{Z}] \).

3 The difference of two Meyer cocycles \( \eta^* \tau_{g+1} \) and \( \theta^* \tau_g \)

In this section (co)homology groups are with \( \mathbb{Z} \) coefficient unless specified.

Let \( g \geq 0 \) be a positive integer. In the introduction, we defined the homomorphisms \( \eta: \mathcal{M}_{g, 2} \to \mathcal{M}_{g+1, 0} \) and \( \theta: \mathcal{M}_{g, 2} \to \mathcal{M}_g \) to be the induced maps by sewing a pair of disks and by sewing an annulus onto the surface \( \Sigma_{g, 2} \) along their boundaries respectively. We denote the Meyer cocycle on the mapping class group of genus \( g \) closed orientable surface \( \mathcal{M}_g \) by \( \tau_g \in \mathbb{Z}^2(\mathcal{M}_g) \) and define \( \bar{\tau}_g \in \mathbb{Z}^2(\mathcal{M}_{g, 2}) \) to be the difference between the Meyer cocycles

\[
\bar{\tau}_g := \eta^* \tau_{g+1} - \theta^* \tau_g.
\]

Let \( P := S^2 - \bigsqcup_{i=1}^3 D^2 \). In this section, we prove the main theorem and calculate the changes of signature associated with sewing a pair of trivial disk bundles \( P \times \bigsqcup_{i=1}^3 D^2 \) and sewing a trivial annulus bundles \( P \times (S^1 \times I) \) onto \( \Sigma_{g, 2} \)–bundle on the pair of pants \( P \) along their boundaries. To state the main theorem, we define the sign of \([p : q] \in \mathbb{Q}P^1\) by

\[
\text{sign}([p : q]) := \text{sign}(pq) = \begin{cases} 1 & \text{if } pq > 0, \\ 0 & \text{if } pq = 0, \\ -1 & \text{if } pq < 0. \end{cases}
\]

**Theorem 3.1** For \( \varphi, \psi \in \mathcal{M}_{g, 2} \), we define

\[
\tilde{\phi}_g(\varphi) := \text{sign}(m(\varphi)).
\]

Then \( \tilde{\phi}_g \) cobounds the difference \( \bar{\tau}_g \) between the Meyer cocycles \( \eta^* \tau_{g+1} \) and \( \theta^* \tau_g \)

\[
\bar{\tau}_g(\varphi, \psi) = \delta \tilde{\phi}_g(\varphi, \psi) \\
= \text{sign}(m(\varphi)) + \text{sign}(m(\psi)) + \text{sign}(m((\varphi \psi)^{-1})).
\]

**Remark 3.2** Let \( k \) be an integer. By **Lemma 2.2** and **Proposition 2.6**, \( \tilde{\phi}_g \) has the properties

\[
\tilde{\phi}_g(\varphi \psi \psi^{-1}) = \tilde{\phi}_g(\varphi) \quad \text{and} \\
\tilde{\phi}_g(\varphi^k) = \text{sign}(k)\tilde{\phi}_g(\varphi)
\]
for any $g \geq 0$.

### 3.1 Proof of Main Theorem

In this subsection we prove Theorem 3.1.

In the introduction, we defined compact oriented 4–manifold $E^g_r$ as a $\Sigma_{g,r}$–bundle on the pair of pants $P$ which has monodromies $\varphi$, $\psi$, and $(\psi \varphi)^{-1} \in \mathcal{M}_{g,r}$ along $\alpha$, $\beta$, and $\gamma \in \pi_1(P)$ respectively, and in Section 2.1, we defined compact oriented 3–manifold $X^g_r$ by the mapping torus of $\Sigma_{g,r} \times I/\sim$ where $(x, 1) \sim (h(x), 0)$ for $\varphi = [h] \in \mathcal{M}_{g,r}$.

Gluing to $E^g_{g,2}$ the trivial annulus bundle on $P$ along the boundaries of each fiber, we obtain

$$E^g_{g+1} = E^g_{g,2} \cup (-S^1 \times I \times P).$$

Similarly, glue to $X^g_{g,2}$ the trivial annulus bundle on $S^1$. Then we have

$$X^g_{g+1} = X^g_{g,2} \cup (-S^1 \times I \times S^1).$$

Define

$$G: \partial D^2 \times I \to \{1\} \times S^1 \times I.$$

$$(x, t) \mapsto (1, x, \frac{1+t}{2}).$$

By the map $G$, we can glue $D^2 \times I$ to $I \times S^1 \times I$ as shown in Figure 3.

![Figure 3: Gluing map $G$](image)

Glue $D^2 \times I \times P$ to $I \times E^g_{g+1}$ with the gluing map $G \times id: \partial D^2 \times I \times P \to \{1\} \times S^1 \times I \times P$. In the same way, glue $D^2 \times I \times S^1$...
to \( I \times X_{g+1}^\eta(\varphi) = (I \times X_{g+1}) \cup (-I \times S^1 \times I \times S^1) \) with \( G \times \text{id}_{S^1} \partial D^2 \times I \times S^1 \to \{1\} \times S^1 \times I \times S^1 \). Namely, we construct two manifolds

\[
\tilde{E}^{\varphi, \psi} := (I \times E_{g+1}^{\eta(\varphi), \eta(\psi)}) \cup_{G \times \text{id}_{D^P}} (D^2 \times I \times P)
\]

and

\[
\tilde{X}^{\varphi} := (I \times X_{g+1}^{\eta(\varphi)}) \cup_{G \times \text{id}_{S^1}} (D^2 \times I \times S^1).
\]

Fix the orientations of these manifolds induced from the product orientations of \( I \times E_{g+1}^{\eta(\varphi), \eta(\psi)} \) and \( I \times X_{g+1}^{\eta(\varphi)} \). To prove main theorem, it suffices to prove Lemma 3.3 and Lemma 3.4 below.

**Lemma 3.3**

\[(\eta^* \tau_{g+1} - \theta^* \tau_g)(\varphi, \psi) = \text{Sign} \tilde{X}^{\varphi} + \text{Sign} \tilde{X}^{\psi} + \text{Sign} \tilde{X}(\varphi \psi)^{-1} \text{ for } \varphi, \psi \in M_{g, 2}, \ g \geq 0.\]

**Lemma 3.4**

\[\text{Sign} \tilde{X}^{\varphi} = \text{sign} (m(\varphi)) \text{ for } \varphi \in M_{g, 2}, \ g \geq 0.\]

**Proof of Lemma 3.3**  Note that

\[\tilde{X}^{\varphi} = \tilde{E}^{\varphi, \psi} |_{\partial D_1}.\]

Then we can see

\[
\partial \tilde{E}^{\varphi, \psi} = (\tilde{E}^{\varphi, \psi} |_{\partial D_1} \cup \tilde{E}^{\varphi, \psi} |_{\partial D_2} \cup \tilde{E}^{\varphi, \psi} |_{\partial D_3}) \cup E_{g}^{\theta(\varphi), \theta(\psi)} \cup -E_{g+1}^{\eta(\varphi), \eta(\psi)}
\]

\[
= (\tilde{X}^{\varphi} \cup \tilde{X}^{\psi} \cup \tilde{X}(\varphi \psi)^{-1}) \cup E_{g}^{\theta(\varphi), \theta(\psi)} \cup -E_{g+1}^{\eta(\varphi), \eta(\psi)}.
\]

Since the Signature is a bordism invariant (for example, see Milnor and Stasheff [10, Lemma 17.3]), we have \( \text{Sign} \partial \tilde{E}^{\varphi, \psi} = 0 \). By Novikov Additivity, we see that

\[\text{Sign} (E_{g+1}^{\eta(\varphi), \eta(\psi)}) - \text{Sign} (E_{g}^{\theta(\varphi), \theta(\psi)}) = \text{Sign} \tilde{X}^{\varphi} + \text{Sign} \tilde{X}^{\psi} + \text{Sign} \tilde{X}(\varphi \psi)^{-1}.\]

Notice that \( \tilde{X}(\varphi \psi)^{-1} \) is diffeomorphic to \( \tilde{X}(\varphi)^{-1} \), so that \( \text{Sign} \tilde{X}(\varphi \psi)^{-1} = \text{Sign} \tilde{X}(\varphi)^{-1} \).

By the definition of the Meyer cocycle, we have

\[\text{Sign} (E_{g+1}^{\eta(\varphi), \eta(\psi)}) = \eta^* \tau_{g+1}(\varphi, \psi), \text{ and } \text{Sign} (E_{g}^{\theta(\varphi), \theta(\psi)}) = \theta^* \tau_g(\varphi, \psi).\]

Define \( \tilde{\phi} = \text{Sign} (\tilde{X}^{\varphi}) \); then we have \( \delta \tilde{\phi} = \eta^* \tau_{g+1} - \theta^* \tau_g. \) We get the cobounding function \( \phi \).

---

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Proof of Lemma 3.4 Write simply $X := X^\eta(\phi)$, $X' := X^\phi$, and $Y := \widetilde{Y}^\phi = (I \times X) \cup_{G \times \text{id}_{S^1}} (D^2 \times I \times S^1)$.

For $i = 0, 1$, define

$$j_i : X \to I \times X \hookrightarrow Y,$$

where $I \times X \hookrightarrow Y$ is a natural embedding. We will prove there is an exact sequence

$$H_2(X') \xrightarrow{j_{0*}} H_2(Y) \xrightarrow{\delta} H_1(X_1 \cap Y_2) \xrightarrow{\delta} H_1(Y_1) \oplus H_1(Y_2).$$

Define the submanifolds $Y_1 := I \times X'$ and $Y_2 := Y - \text{Int} Y_1 = (-I \times S^1 \times I \times S^1) \cup_{G \times S^1} (D^2 \times I \times S^1)$. Then we see that

$$Y_1 \simeq X', Y_2 \simeq S^1, Y_1 \cap Y_2 \simeq \partial X' = (S_1 \sqcup S_2) \times S^1.$$

By the Meyer–Vietoris exact sequence, we have the exact sequence

$$H_2(Y_1) \oplus H_2(Y_2) \xrightarrow{\delta} H_2(Y) \xrightarrow{\delta} H_1(Y_1 \cap Y_2) \xrightarrow{\delta} H_1(Y_1) \oplus H_1(Y_2).$$

Denote the map $H_1(\partial X') \to H_1(X') \oplus H_1(S^1)$ in the above diagram by $h$. the projection $H_1(\partial X') \to H_1(S^1)$ to the second entry of $h$ is the composite of inclusion homomorphism $H_1(\partial X') \to H_1(X')$ and $\pi_* : H_1(X') \to H_1(S^1)$. Therefore,

$$\text{Ker} (H_1(\partial X') \to H_1(X') \oplus H_1(S^1)) = \text{Ker} (H_1(\partial X') \to H_1(X')).$$

So the sequence is exact.

Next, we will construct the splitting

$$H_2(Y; \mathbb{Q}) = j_1*H_2(X'; \mathbb{Q}) \oplus \text{Ker} (H_1(\partial X'; \mathbb{Q}) \to H_1(X'; \mathbb{Q})).$$

Note that there exist $p, q \in \mathbb{Q}$ such that

$$\text{Ker} (H_1(\partial X'; \mathbb{Q}) \to H_1(X'; \mathbb{Q})) = \mathbb{Q}(e_1 + e_2) \oplus \mathbb{Q}\{p(e_3 - e_4) + qe_1\}$$

as in Section 2. To construct the splitting, we choose elements of inverse images of $e_1 + e_2$, $p(e_3 - e_4) + qe_1$ under $H_2(Y) \to H_1(\partial X')$. Define $\iota_Y : \Sigma_{g+1} \to Y$ by

$$\Sigma_{g+1} \to X \to I \times X \hookrightarrow Y,$$

$$x \mapsto (x, 0) \mapsto (0, x, 0).$$
Then we see that
\[ H_2(Y) \to H_1(Y_1 \cap Y_2) \to H_1(\partial X'). \]

so we choose \( t_\gamma * [\Sigma g + 1] \) as an element of the inverse image of \( e_1 + e_2 \).

Next, we choose an element of the inverse image of \( p(e_3 - e_4) + qe_1 \). Since \( p(e_3 - e_4) + qe_1 \in \text{Ker}(H_1(\partial X'; q) \to H_1(X'; q)) \), there exists a singular 2–chain \( s \in C_2(X'; q) \) such that
\[ \partial s = p(f_3 - f_4) + qf_1 \in B_1(X'; q). \]

For \( i = 0, 1 \), define \( s'_{0i} : I \times S^1 \to I \times S^1 \times I \times S^1 \hookrightarrow Y_2 \) by \( s'_{0i}(t, u) = (i, 0, t, u) \). Then we see that
\[ [\partial s'_{0i}] = [ji f_3 - ji f_4] \in H_1(Y_1 \cap Y_2; q). \]

Figure 4: Images of \( s'_{10} \) and \( s'_{11} \subset Y_2 \).

Define \( s'_{1i} : D^2 \to Y_2 = (-I \times S^1 \times I \times S^1) \cup_{G \times S^1} (D^2 \times I \times S^1) \subset Y \) as shown in
Figure 4 by
\[
\begin{align*}
    s'_{10}(x) &= \begin{cases} 
        (6x, 1, 0) & \in D^2 \times I \times S^1 (||x|| \leq \frac{1}{6}), \\
        (2 - 6||x||, \frac{x}{||x||}, \frac{2}{3}, 0) & \in I \times S^1 \times I \times S^1 \left( \frac{1}{6} \leq ||x|| \leq \frac{1}{3} \right), \\
        (0, \frac{x}{||x||}, 1 - ||x||, 0) & \in I \times S^1 \times I \times S^1 \left( \frac{1}{3} \leq ||x|| \leq 1 \right). 
    \end{cases} \\
    s'_{11}(x) &= \begin{cases} 
        (\frac{1}{2} x, 0, 0) & \in D^2 \times I \times S^1 (||x|| \leq \frac{1}{2}), \\
        (1, \frac{x}{||x||}, 1 - ||x||, 0) & \in I \times S^1 \times I \times S^1 \left( \frac{1}{2} \leq ||x|| \leq 1 \right). 
    \end{cases}
\end{align*}
\]

Then, we have \([\partial s'_{1i}] = [ji f_1] \in H_1(Y_1 \cap Y_2; q)\).

The chain \( s'_i := ps'_{0i} + qs'_{1i} \) satisfies
\[ [\partial s'_i] = [ji(p(f_3 - f_4) + qf_1)] \in H_1(Y_1 \cap Y_2; q), \]

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so that we have \([\partial(j_i s - s_i')] = 0 \in H_1(Y_1 \cap Y_2; \mathbb{Q})\).

We see
\[
H_2(Y; \mathbb{Q}) \to H_1(Y_1 \cap Y_2; \mathbb{Q}) \to H_1(\partial X'; \mathbb{Q}),
\]
\[
[j_i s - s_i'] \mapsto \partial_*(j_i s - s_i') \mapsto p(e_3 - e_4) + q e_1
\]
so that we can choose \([j_i s - s_i']\) as an element of the inverse image of \(p(e_3 - e_4) + q e_1\).

Now we calculate the intersection form of \(H_2(Y; \mathbb{Q})\). Define the subspace \(X''_1 = j_1(X) \cup_G \text{id}_{S^1} (D^2 \times 0 \times S^1) \subset Y\). Then we see that \(X''_1\) is a deformation retract of \(Y\). Hence, every element of \(H_2(Y; \mathbb{Q})\) is represented by a cycle in \(X''_1\). Therefore, a homology class is included in the annihilator of intersection form in \(H_2(Y; \mathbb{Q})\) if it is represented by a cycle which has no common point with \(X''_1\). We see
\[
j_0(X') \cap X''_1 = \emptyset \quad \text{and} \quad \iota_Y(\Sigma_{g+1}) \cap X''_1 = \emptyset,
\]
so that the preimage of \(\mathbb{Q}(e_1 + e_2)\) and \(j_0 \ast H_2(X'; \mathbb{Q})\) are included in the annihilator of intersection form in \(H_2(Y; \mathbb{Q})\).

To describe the signature of \(Y\), it suffices to calculate the self-intersection number of \([j_i s - s_i'] = p(e_3 - e_4) + q e_1\). The cycle \(j_i s - s_i'\) satisfies
\[
\text{Im} (j_0 s) \cap (\text{Im} (j_1 s) \cup \text{Im} (s'_{01}) \cup \text{Im} (s'_{11})) = \emptyset
\]
\[
\text{Im} (s'_{00}) \cap (\text{Im} (j_1 s) \cup \text{Im} (s'_{01}) \cup \text{Im} (s'_{11})) = \emptyset
\]
\[
\text{Im} (s'_{10}) \cap (\text{Im} (j_1 s) \cup \text{Im} (s'_{11})) = \emptyset,
\]
so that
\[
(j_0 s - s_0') \cdot (j_1 s - s_1') = (j_0 s - (p s'_{00} + q s'_{10})) \cdot (j_1 s - (p s'_{01} + q s'_{11}))
\]
\[
= q s'_{10} \cdot p s'_{01}.
\]

If necessary, perturb the chain \(s'_{01}\). Then we see that \(s'_{01}\) and \(s'_{10}\) intersect only once positively. Hence, we have \(\text{Sign}(Y) = \text{sign}(p q) = \text{sign}(m(\varphi))\). \(\square\)

### 3.2 Wall’s non-additivity formula

In the introduction, we stated the Novikov additivity of Signature. Wall derives a formula from this additivity in a more general case, when two compact oriented smooth 4\(k\)–manifolds are glued along common submanifolds of their boundaries. We will give the specific case of his formula for \(k = 1\).

Let \(Z\) be a closed oriented smooth 2–manifold, \(X_-, X_0, X_+\) compact oriented smooth 3–manifolds with the boundaries \(\partial X_- = \partial X_0 = \partial X_+ = Z\), and \(Y_-, Y_+\) compact oriented smooth 4–manifolds with the boundaries \(\partial Y_- = X_- \cup_Z (-X_0)\).
\[ \partial Y_+ = X_0 \cup_Z (-X_+) \]. Here we denote by \( M \cup_B (-N) \) the union of two manifolds \( M \) and \( N \) glued by orientation reversing diffeomorphism of their common boundaries \( \partial M = \partial N = B \). Let \( Y = Y_- \cup X_0 Y_+ \) be the union of \( Y_- \) and \( Y_+ \) glued along submanifolds \( X_0 \) of their boundaries. Suppose \( Y \) is oriented by the induced orientation of \( Y_- \) and \( Y_+ \).

Write \( V = H_1(Z; \mathbb{R}) \); let \( A, B, \) and \( C \) be the kernels of the maps on first homology induce by the inclusions of \( Z \) in \( X_- \), \( X_0 \) and \( X_+ \) respectively.

We define

\[ W := \frac{B \cap (C + A)}{(B \cap C) + (B \cap A)}. \]

and a bilinear form \( \Psi \) by

\[ \Psi: W \times W \rightarrow \mathbb{R}, \quad (b, b') \mapsto b \cdot c'. \]

Here \( c' \) is an element of \( C \) such that there exists an element \( a' \in A \) such that \( a' + b' + c' = 0 \), and \( b \cdot c' \) denotes the intersection product of \( b \) and \( c' \). It is known that \( \Psi \) is independent of the choice of \( c' \) and well-defined on \( W \). Denote the signature of the bilinear form \( \Psi \) by \( \text{Sign}(V; BCA) \) and the signature of the compact oriented 4–manifold \( M \) by \( \text{Sign} M \). We are now ready to state the formula.

**Theorem 3.5** (Wall [13]) \( \text{Sign} Y = \text{Sign} Y_- + \text{Sign} Y_+ - \text{Sign}(V; BCA) \).

**3.3 The differences \( \text{Sign} E_g - \text{Sign} E_{g,2} \) and \( \text{Sign} E_{g+1} - \text{Sign} E_{g,2} \)**

In this subsection, we calculate the difference of signature associated with sewing the trivial Disk bundles onto the \( \Sigma_{g,2} \)-bundle.

In the introduction, we defined \( E_{g,r}^{\phi,\psi} \) as a oriented \( \Sigma_{g,r} \)-bundle on \( P \) which has monodromies \( \phi, \psi, (\psi \phi)^{-1} \in \pi_1(P) \). If we fix \( \phi, \psi \in M_{g,2} \), we denote simply

\[ E_{g,2} := E_{g,2}^{\phi,\psi}, \quad E_g := E_{g}^{\theta(\phi), \theta(\psi)}, \quad \text{and} \quad E_{g+1} := E_{g+1}^{\eta(\phi), \eta(\psi)} (g \geq 0). \]

**Proposition 3.6** \( \text{Sign}(E_g) - \text{Sign}(E_{g,2}) = -\text{sign}(m(\phi) + m(\psi) + m((\psi \phi)^{-1})) \) for \( g \geq 0 \).

**Proof** \( E_g \) is the union of \( E_{g,2} \) and \( E_D := (D^2 \sqcup D^2) \times P \) glued along their boundaries. Using Non-additivity formula **Theorem 3.5**, we calculate \( \text{Sign}(E_g) - \text{Sign}(E_{g,2}) \).

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Define $Y_-, Y_+, X_-, X_0, X_+$, and $Z$ by

$$
Y_- := (\bigcup_{j=1}^2 D^2) \times P, \quad Y_+ := E_{g,2},
$$
$$
X_- := (\bigcup_{j=1}^2 D^2) \times \partial P, \quad X_+ := E_{g,2} \mid \partial P, \quad X_0 := (\bigcup_{j=1}^2 \partial D^2) \times P,
$$
and $Z := (\bigcup_{j=1}^2 \partial D^2) \times \partial P$, respectively.

Here, by the notation stated in Section 2.1,

$$
X_+ = E_{g,2} \mid \partial P \cong X^\psi \sqcup X^\psi \sqcup X^{(\psi \varphi)^{-1}}, \quad Z \cong \partial X^\psi \sqcup \partial X^\psi \sqcup \partial X^{(\psi \varphi)^{-1}}.
$$

Define $V$, $A$, $B$, and $C$ as stated in Section 3.1.

Since $X^\psi = X^{(\psi \varphi)^{-1}} = S^1 \times S^1$, we can choose the bases of $H_1(\partial X^\psi; \mathbb{R})$, $H_1(\partial X^\psi; \mathbb{R})$, and $H_1(\partial X^{(\psi \varphi)^{-1}}; \mathbb{R})$ as stated in Section 2.1. Denote their bases by $\{e_{11}, e_{12}, e_{13}, e_{14}\}$, $\{e_{21}, e_{22}, e_{23}, e_{24}\}$, and $\{e_{31}, e_{32}, e_{33}, e_{34}\}$ respectively.

Since $Z = \partial X^\psi \sqcup \partial X^\psi \sqcup \partial X^{(\psi \varphi)^{-1}}$, we think of $e_{ij}$ as an element of $H_1(Z; \mathbb{R})$.

Denote $m(\varphi) = [a_1 : b_1]$, $m(\psi) = [a_2 : b_2]$, and $m((\psi \varphi)^{-1}) = [a_3 : b_3]$ respectively. Then we have

$$
V = H_1(Z, \mathbb{R}) = \bigoplus_{i=1}^3 \bigoplus_{j=1}^4 \mathbb{R}e_{ij},
$$
$$
A = \mathbb{R}e_{11} \oplus \mathbb{R}e_{21} \oplus \mathbb{R}e_{31} \oplus \mathbb{R}e_{12} \oplus \mathbb{R}e_{22} \oplus \mathbb{R}e_{32},
$$
$$
B = \mathbb{R}(e_{11} - e_{21}) \oplus \mathbb{R}(e_{11} - e_{31}) \oplus \mathbb{R}(e_{12} - e_{22}) \oplus \mathbb{R}(e_{12} - e_{32}) \oplus \mathbb{R}(e_{13} + e_{23} + e_{33}) \oplus \mathbb{R}(e_{14} + e_{24} + e_{34}).
$$
$$
C = \bigoplus_{i=1}^3 \begin{cases} 
\mathbb{R}(e_{i1} + e_{i2}) \oplus \mathbb{R}(e_{i3} - e_{i4} + m_i e_{i1}) & \text{if } a_i \neq 0 \\
\mathbb{R}e_{i1} \oplus \mathbb{R}e_{i2} & \text{if } a_i = 0.
\end{cases}
$$

Here we denote $m_i := \frac{b_i}{a_i}$. Hence,

$$
B \cap A = \mathbb{R}(e_{11} - e_{21}) \oplus \mathbb{R}(e_{12} - e_{22}) \oplus \mathbb{R}(e_{11} - e_{31}) \oplus \mathbb{R}(e_{12} - e_{32}).
$$
Then, the space $W$ is generated by the element represented by

$$b := e_{13} + e_{23} + e_{33} - e_{14} - e_{24} - e_{34} \in B \cap (C + A).$$

Choose the elements

$$a := m_1e_{11} + m_2e_{21} + m_3e_{31} \in A \text{ and } c := -\sum_{i=1}^{3} (e_{13} - e_{14} + m_ie_{i1}) \in C.$$

Then we see that $a + b + c = 0$ and obtain $\Psi(b, b) = b \cdot c = m_1 + m_2 + m_3$. This shows that $\text{Sign}(V; BCA) = \text{sign}(m_1 + m_2 + m_3)$. The other cases follow in similar ways.

Hence, we obtain

$$\text{Sign}(V; BCA) = \text{sign}(m(\varphi) + m(\psi) + m((\varphi\psi)^{-1})).$$
By the non-additivity formula, we have
\[ \text{Sign}(E_g) = \text{Sign}(E_D) + \text{Sign}(E_{g,2}) - \text{Sign}(V;BCA). \]
Since \( E_D \) is a trivial bundle \((D^2 \sqcup D^2) \times P\), we have \( \text{Sign}(E_D) = 0 \).
This completes the proof of the proposition. \( \square \)

By Theorem 3.1 and Proposition 3.6, we can calculate the difference of signature
\[ \text{Sign}(E_g) - \text{Sign}(E_{g,2}). \]

**Corollary 3.7** For \( g \geq 0 \),
\[ \text{Sign}(E_{g+1}) - \text{Sign}(E_{g,2}) = \text{sign}(m(\varphi)) + \text{sign}(m(\psi)) + \text{sign}(m((\varphi\psi)^{-1})) \]
\[ - \text{sign}(m(\varphi) + m(\psi) + m((\varphi\psi)^{-1})). \]

**References**


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