Small exotic 4–manifolds

ANAR AKHMEDOV

In this article, we construct the first example of a simply-connected minimal symplectic 4–manifold that is homeomorphic but not diffeomorphic to $3\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$. We also construct the first exotic minimal *symplectic* $\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2$.

57N65, 57N13; 57M50

1 Introduction

Over the past several years, there has been a considerable amount of progress in the discovery of exotic smooth structures on simply-connected 4–manifolds with small Euler characteristic. In early 2004, Jongil Park [15] has constructed the first example of exotic smooth structure on $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$, ie 4–manifold homeomorphic but not diffeomophic to $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$. Later that year, András Stipsicz and Zoltán Szabó used a similar technique to construct an exotic smooth structure on $\mathbb{CP}^2\#6\overline{\mathbb{CP}}^2$ [18]. Then Fintushel and Stern [5] introduced a new technique, the double node surgery, which demonstrated that in fact $\mathbb{CP}^2\#k\,\overline{\mathbb{CP}}^2$, $k=6$, 7 and 8 have infinitely many distinct smooth structures. Using the double node surgery technique [5], Park, Stipsicz and Szabó constructed infinitely many smooth structures on $\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2$ [17]. The examples in [17] are not known if symplectic. Based on similar ideas, Stipsicz and Szabó constructed the exotic smooth structures on $3\,\mathbb{CP}^2\#k\,\overline{\mathbb{CP}}^2$ for $k=9$ [19] and Park for $k=8$ [16]. In this article, we construct an exotic smooth structure on $3\,\mathbb{CP}^2\#7\,\overline{\mathbb{CP}}^2$. We also construct an exotic *symplectic* $\mathbb{CP}^2\#5\,\overline{\mathbb{CP}}^2$, the first known such symplectic example.

Our approach is different from the above constructions in the sense that we do not use any rational-blowdown surgery (Fintushel and Stern [3], Jongil [14]). Also, in contrary to the previous constructions, we use non-simply connected building blocks (Akhmedov [1], Matsumoto [11]) to produce the simply-connected examples. The main surgery technique used in our construction is the symplectic fiber sum operation (Gompf [7], McCarthy and Wolfson [12]) along the genus two surfaces. Our results can be stated as follows.

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Theorem 1.1 *There exist a smooth closed simply-connected minimal symplectic 4–manifold X that is homeomorphic but not diffeomorphic to $3\,\mathbb{CP}^2\#7\,\overline{\mathbb{CP}}^2$.*

Theorem 1.2 *There exist a smooth closed simply-connected minimal symplectic 4–manifold Y which is homeomorphic but not diffeomorphic to the rational surface $\mathbb{CP}^2\#5\,\overline{\mathbb{CP}}^2$.*

This article is organized as follows. The first two sections give a quick introduction to Seiberg–Witten invariants and a fiber sum operation. In Section 4, we review the symplectic building blocks for our construction. Finally, in Section 5 and Section 6, we construct minimal symplectic 4–manifolds X and Y homeomorphic but not diffeomorphic to $3\,\mathbb{CP}^2\#7\,\overline{\mathbb{CP}}^2$ and $\mathbb{CP}^2\#5\,\overline{\mathbb{CP}}^2$, respectively.

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Dedication Dedicated to Professor Ronald J Stern on the occasion of his sixtieth birthday.

2 Seiberg–Witten Invariants

In this section, we briefly recall the basics of Seiberg–Witten invariants introduced by Seiberg and Witten. Seiberg–Witten invariant of a smooth closed oriented 4–manifold X with $b_2^+(X) > 1$ is an integer valued function which is defined on the set of spin^c structures over X (Witten [23]). For simplicity, we assume that $H_1(X,\mathbf{Z})$ has no 2–torsion. Then there is a one-to-one correspondence between the set of spin^c structures over X and the set of characteristic elements of $H^2(X,\mathbf{Z})$.

In this set up, we can view the Seiberg–Witten invariant as an integer valued function

$$\mathrm{SW}_X\colon \{k \in H^2(X,\mathbf{Z}) | k \equiv w_2(TX) \pmod 2\} \longrightarrow \mathbf{Z}.$$

The Seiberg–Witten invariant SW_X is a diffeomorphism invariant. We call β a *basic class* of X if $\mathrm{SW}_X(\beta) \neq 0$. It is a fundamental fact that the set of basic classes is finite. Also, if β is a basic class, then so is $-\beta$ with

$$\mathrm{SW}_X(-\beta) = (-1)^{(e+\sigma)(X)/4}\,\mathrm{SW}_X(\beta)$$

where $e(X)$ is the Euler characteristic and $\sigma(X)$ is the signature of X.

Theorem 2.1 (Taubes [20]) *Suppose that (X,ω) is a closed symplectic 4–manifold with $b_2^+(X)>1$ and the canonical class K_X. Then $SW_X(\pm K_X)=\pm 1$.*

3 Fiber Sum

Definition 3.1 Let X and Y be closed, oriented, smooth 4–manifolds each containing a smoothly embedded surface Σ of genus $g\geq 1$. Assume Σ represents a homology class of infinite order and has self-intersection zero in X and Y, so that there exist a tubular neighborhood, say $\nu\Sigma\cong\Sigma\times D^2$, in both X and Y. Using an orientation-reversing and fiber-preserving diffeomorphism $\psi\colon S^1\times\Sigma\longrightarrow S^1\times\Sigma$, we can glue $X\setminus\nu\Sigma$ and $Y\setminus\nu\Sigma$ along the boundary $\partial(\nu\Sigma)\cong\Sigma\times S^1$. This new oriented smooth 4–manifold $X\#_\psi Y$ is called a *generalized fiber sum* of X and Y along Σ, determined by ψ.

Definition 3.2 Let $e(X)$ and $\sigma(X)$ denote the Euler characteristic and the signature of a closed oriented smooth 4–manifold X, respectively. We define

$$c_1^2(X):=2e(X)+3\sigma(X),\quad \chi_h(X):=\frac{e(X)+\sigma(X)}{4}.$$

In the case that X is a complex surface, then $c_1^2(X)$ and $\chi_h(X)$ are the self-intersection of the first Chern class $c_1(X)$ and the holomorphic Euler characteristic, respectively.

Lemma 3.3 *Let X and Y be closed, oriented, smooth 4–manifolds containing an embedded surface Σ of self-intersection 0. Then*

$$c_1^2(X\#_\psi Y)=c_1^2(X)+c_1^2(Y)+8(g-1),$$
$$\chi_h(X\#_\psi Y)=\chi_h(X)+\chi_h(Y)+(g-1),$$

where g is the genus of the surface Σ.

Proof The above simply follows from the well-known formulas

$$e(X\#_\psi Y)=e(X)+e(Y)-2e(\Sigma),\quad \sigma(X\#_\psi Y)=\sigma(X)+\sigma(Y).$$ □

If X, Y are symplectic manifolds and Σ is an embedded symplectic submanifold in X and Y, then according to theorem of Gompf [7] $X\#_\psi Y$ admits a symplectic structure.

We will use the following theorem of M Usher [21] to show that the symplectic manifolds constructed in Section 5 and Section 6 are minimal. Here we slightly abuse the above notation for the fiber sum.

Theorem 3.4 (Usher [21], Minimality of Symplectic Sums) *Let $X = X_1 \#_{F_1=F_2} X_2$ be symplectic fiber sum of manifolds X_1 and X_2.*

(i) *If either $X_1 \backslash F_1$ or $X_2 \backslash F_2$ contains an embedded symplectic sphere of square -1, then X is not minimal.*

(ii) *If one of the summands X_i (say X_1) admits the structure of an S^2–bundle over a surface of genus g such that F_i is a section of this fiber bundle, then X is minimal if and only if X_2 is minimal.*

(iii) *In all other cases, X is minimal.*

4 Building blocks

The building blocks for our construction will be as follows.

(i) The manifold $T^2 \times S^2 \# 4\overline{\mathbb{CP}}^2$ equipped with the genus two Lefschetz fibration of Matsumoto [11].

(ii) The symplectic manifolds X_K and Y_K [1]. For the convenience of the reader, we recall the construction in [1].

4.1 Matsumoto fibration

First, recall that the manifold $Z = T^2 \times S^2 \# 4\overline{\mathbb{CP}}^2$ can be described as the double branched cover of $S^2 \times T^2$ where the branch set $B_{2,2}$ is the union of two disjoint copies of $S^2 \times \{\mathrm{pt}\}$ and two disjoint copies of $\{\mathrm{pt}\} \times T^2$. The branch cover has 4 singular points, corresponding to the number of the intersections points of the horizontal lines and the vertical tori in the branch set $B_{2,2}$. After desingularizing the above singular manifold, one obtains $T^2 \times S^2 \# 4\overline{\mathbb{CP}}^2$. The vertical fibration of $S^2 \times T^2$ pulls back to give a fibration of $T^2 \times S^2 \# 4\overline{\mathbb{CP}}^2$ over S^2. A generic fiber of the vertical fibration is the double cover of T^2, branched over 2 points. Thus a generic fiber will be a genus two surface. According to Matsumoto [11], this fibration can be perturbed to be a Lefschetz fibration over S^2 with the global monodromy $(\beta_1\beta_2\beta_3\beta_4)^2 = 1$, where the curves β_1, β_2, β_3 and β_4 are shown in Figure 1.

Let us denote the regular fiber by Σ_2' and the images of standard generators of the fundamental group of Σ_2' as a_1, b_1, a_2 and b_2. Using the homotopy exact sequence for a Lefschetz fibration,

$$\pi_1(\Sigma_2') \longrightarrow \pi_1(Z) \longrightarrow \pi_1(S^2)$$

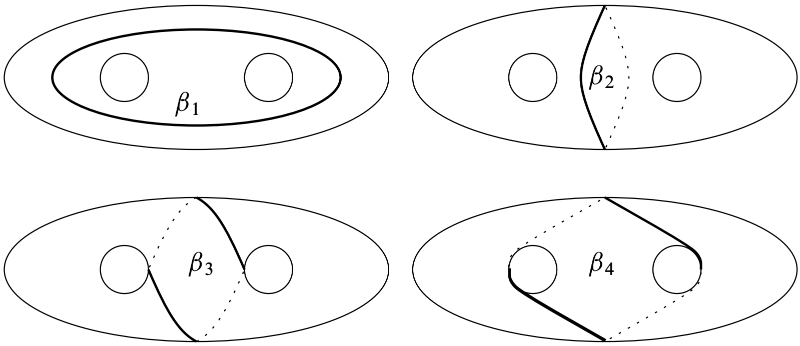

Figure 1: Dehn Twists for Matsumoto Fibration

we have the following identification of the fundamental group of Z [13]:

$$\pi_1(Z) = \pi_1(\Sigma_2')/\langle \beta_1, \beta_2, \beta_3, \beta_4 \rangle.$$

(1) $$\beta_1 = b_1 b_2,$$

(2) $$\beta_2 = a_1 b_1 {a_1}^{-1} {b_1}^{-1} = a_2 b_2 {a_2}^{-1} {b_2}^{-1},$$

(3) $$\beta_3 = b_2 a_2 {b_2}^{-1} a_1,$$

(4) $$\beta_4 = b_2 a_2 a_1 b_1.$$

Hence $\pi_1(Z) = \langle a_1, b_1, a_2, b_2 \mid b_1 b_2 = [a_1, b_1] = [a_2, b_2] = a_1 a_2 = 1 \rangle$.

Note that the fundamental group of $T^2 \times S^2 \# 4\overline{\mathbb{CP}}^2$ is $\mathbf{Z} \oplus \mathbf{Z}$, generated by two of these standard generators (say a_1 and b_1). The other two generators a_2 and b_2 are the inverses of a_1 and b_1 in the fundamental group. Also, the fundamental group of the complement of $\nu\Sigma_2'$ is $\mathbf{Z} \oplus \mathbf{Z}$. It is generated by a_1 and b_1. The normal circle $\lambda' = pt \times \partial D^2$ to Σ_2' can be deformed using one of the exceptional spheres, thus is trivial in $\pi_1(T^2 \times S^2 \# 4\overline{\mathbb{CP}}^2 \setminus \nu\Sigma_2') = \mathbf{Z} \oplus \mathbf{Z}$.

Lemma 4.1 ${c_1}^2(Z) = -4$, $\sigma(Z) = -4$ *and* $\chi_h(Z) = 0$.

Proof We have ${c_1}^2(Z) = {c_1}^2(T^2 \times S^2) - 4 = -4$, $\sigma(Z) = \sigma(T^2 \times S^2) - 4 = -4$ and $\chi_h(Z) = \chi_h(T^2 \times S^2) = 0$. □

Note that this Lefschetz fibration can be given a symplectic structure. This means that Z admits a symplectic structure such that the regular fibers are symplectic submanifolds. We consider such a symplectic structure on Z.

4.2 Symplectic 4–manifolds cohomology equivalent to $S^2 \times S^2$

Our second building block will be X_K, the symplectic cohomology $S^2 \times S^2$ [1], or the symplectic manifold Y_K, an intermediate building block in that construction [1], (see also Fintushel and Stern [4]). For the sake of completeness, the details of this construction are included below. We refer the reader to [1] for more details and for the generalization of these symplectic building blocks.

Let K be a fibered knot of genus one (ie, the trefoil or the figure eight knot) in S^3 and m be a meridional circle to K. We perform 0–framed surgery on K and denote the resulting 3–manifold by M_K. Since K is fibered and has genus one, it follows the 3–manifold M_K is a torus bundle over S^1; hence the 4–manifold $M_K \times S^1$ is a torus bundle over a torus. Furthermore, $M_K \times S^1$ admits a symplectic structure, and both the torus fiber and the torus section $T_m = m \times S^1 = m \times x$ are symplectically embedded and have a self-intersection zero. The first homology of $M_K \times S^1$ is generated by the standard first homology generators m and x of the torus section. On the other hand, the classes of circles γ_1 and γ_2 of the fiber F, coming from the Seifert surface, are trivial in homology. In addition, $M_K \times S^1$ is minimal symplectic, ie, it does not contain symplectic -1 sphere.

We form a twisted fiber sum of two copies of the manifold $M_K \times S^1$, we identify the fiber F of one fibration to the section T_m of other. Let Y_K denote the mentioned twisted fiber sum $Y_K = M_K \times S^1 \#_{F=T_m} M_K \times S^1$. It follows from Gompf's theorem [7] that Y_K is symplectic and by Usher's Theorem 3.4 that Y_K is minimal symplectic.

Let T_1 be the section of the first copy of $M_K \times S^1$ and T_2 be the fiber in the second copy. Then the genus two surface $\Sigma_2 = T_1 \# T_2$ symplectically embeds into Y_K and has self-intersection zero. Let X_K be a symplectic 4–manifold constructed as follows: Take two copies of Y_K and form the fiber sum along the genus two surface Σ_2 using the special gluing diffeomorphism ϕ, the vertical involution of Σ_2 with two fixed points. Thus $X_K := Y_K \#_\phi Y_K$. Let m, x, γ_1 and γ_2 denote the generators of $\pi_1(\Sigma_2)$ under the inclusion. The diffeomorphism ϕ: $T_1 \# T_2 \longrightarrow T_1 \# T_2$ of Σ_2 maps on the generators as follows: $\phi_*(m') = \gamma_1$, $\phi_*(x') = \gamma_2$, $\phi_*({\gamma_1}') = m$ and $\phi_*({\gamma_2}') = x$. In [1] we show that the manifold X_K has first Betti number zero and has the integral cohomology of $S^2 \times S^2$. Furthermore, $H_2(X_K, \mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z}$, where the basis for the second homology are the classes of $\Sigma_2 = S$ and the new genus two surface T resulting from the last fiber sum operation (two punctured genus one surfaces glues to form a genus two surface). Also, $S^2 = T^2 = 0$ and $S \cdot T = 1$. Furthermore, ${c_1}^2(X_K) = 8$, $\sigma(X_K) = 0$ and $\chi_h(X_K) = 1$. Since Y_K is minimal symplectic, it follows from Theorem 3.4 that X_K is minimal symplectic as well.

4.2.1 Fundamental Group of $M_K \times S^1$ We will assume that K is the trefoil knot. Let a, b and c denote the Wirtinger generators of the trefoil. The knot group of the trefoil has the following presentations: $\pi_1(K) = \langle a,b,c \mid ab = bc,\ ca = ab\rangle = \langle a,b \mid aba = bab\rangle = \langle u,v \mid u^2 = v^3\rangle$ where $u = bab$ and $v = ab$. The homotopy classes of the meridian and the longitude of the trefoil are given as follows: $m = uv^{-1} = b$ and $l = u^2(uv^{-1})^{-6} = ab^2ab^{-4}$ (Burde and Zieschang [2]). Also, the homotopy classes of γ_1 and γ_2 are given as follows: $\gamma_1 = a^{-1}b$ and $\gamma_2 = b^{-1}aba^{-1}$. Notice that the fundamental group of M_K, 0–surgery on the trefoil, is obtained from the knot group of the trefoil by adjoining the relation $l = u^2(uv^{-1})^{-6} = ab^2ab^{-4} = 1$. Thus, we have $\pi_1(M_K) = \langle u,v \mid u^2 = v^3,\ u^2(uv^{-1})^{-6} = 1\rangle = \langle a,b \mid aba = bab,\ ab^2a = b^4\rangle$ and $\pi_1(M_K \times S^1) = \langle a,b,x \mid aba = bab,\ ab^2a = b^4,\ [x,a] = [x,b] = 1\rangle$.

4.2.2 Fundamental Group of Y_K The next step is to take two copies of the manifold $M_K \times S^1$ and perform the fiber sum along symplectic tori. In the first copy of $M_K \times S^1$, we take a tubular neighborhood of the torus section T_m, remove it from $M_K \times S^1$ and denote the resulting manifold by C_S. In the second copy, we remove a tubular neighborhood of the fiber F and denote it by C_F. Notice that $C_S = M_K \times S^1 \setminus \nu T_m = (M_K \setminus \nu(m)) \times S^1$. We have $\pi_1(C_S) = \pi_1(K) \oplus < x >$ where x is the generator corresponding to the S^1 copy. Also using the above computation, we easily derive: $H_1(C_S) = H_1(M_K) = \langle m\rangle \oplus \langle x\rangle$.

To compute the fundamental group of the C_F, we will use the following observation: νF is the preimage of the small disk on $T_{m'} = m' \times y$. The elements y and $m' = d$ of the $\pi_1(C_F)$ do not commute anymore, but y still commutes with generators γ_1' and γ_2'. The fundamental group and the first homology of the C_F will be isomorphic to the following: $\pi_1(C_F) = \langle d, y, \gamma_1', \gamma_2' \mid [y,\gamma_1'] = [y,\gamma_2'] = [\gamma_1',\gamma_2'] = 1,\ d\gamma_1' d^{-1} = \gamma_1'\gamma_2',\ d\gamma_2' d^{-1} = (\gamma_1')^{-1}\rangle$ and $H_1(C_F) = \langle d\rangle \oplus \langle y\rangle$.

We use the Van Kampen's Theorem to compute the fundamental group of Y_K.

$$\begin{aligned}
\pi_1(Y_K) = \pi_1(C_F) *_{\pi_1(T^3)} \pi_1(C_F)
&= \langle d, y, \gamma_1', \gamma_2' \mid [y,\gamma_1'] = [y,\gamma_2'] = [\gamma_1',\gamma_2'] = 1,\ d\gamma_1' d^{-1} = \gamma_1'\gamma_2',\ d\gamma_2' d^{-1} \\
&\quad = ({\gamma_1}')^{-1}\rangle_{\langle \gamma_1' = x,\ \gamma_2' = b,\ \lambda' = \lambda\rangle} \langle a,b,x \mid aba = bab,\ [x,a] = [x,b] = 1\rangle \\
&= \langle a,b,x,\gamma_1',\gamma_2',d,y \mid aba = bab, [x,a] = [x,b][y,\gamma_1'] = [y,\gamma_2'] \\
&\quad = [\gamma_1',\gamma_2'] = 1,\ d\gamma_1' d^{-1} = \gamma_1'\gamma_2',\ d\gamma_2' d^{-1} = ({\gamma_1}')^{-1},\ \gamma_1' = x, \\
&\quad \gamma_2' = b, [\gamma_1',\gamma_2'] = [d,y]\rangle.
\end{aligned}$$

Inside Y_K, we can find a genus 2 symplectic submanifold Σ_2 which is the internal sum of a punctured fiber in C_S and a punctured section in C_F. The inclusion-induced homomorphism maps the standard generators of $\pi_1(\Sigma_2)$ to $a^{-1}b$, $b^{-1}aba^{-1}$, d and y in $\pi_1(Y_K)$.

Lemma 4.2 ([1]) *There are nonnegative integers m and n such that*

$$\begin{aligned}\pi_1(Y_K \setminus \nu\Sigma_2) = \langle a, b, x, d, y;\, g_1, \dots, g_m \mid\ & aba = bab,\\ & [y, x] = [y, b] = 1,\ dxd^{-1} = xb,\ dbd^{-1} = x^{-1},\\ & ab^2ab^{-4} = [d, y],\ r_1 = \cdots = r_n = 1,\ r_{n+1} = 1\rangle,\end{aligned}$$

where the generators $g_1, \dots, g_m$ and relators $r_1, \dots, r_n$ all lie in the normal subgroup N generated by the element $[x, b]$ and the relator r_{n+1} is a word in x, a and elements of N. Moreover, if we add an extra relation $[x, b] = 1$, then the relation $r_{n+1} = 1$ simplifies to $[x, a] = 1$.

Proof This follows from Van Kampen's Theorem. Note that $[x, b]$ is a meridian of Σ_2 in Y_K. Hence setting $[x, b] = 1$ should turn $\pi_1(Y_K \setminus \nu\Sigma_2)$ into $\pi_1(Y_K)$. Also note that $[x, a]$ is the boundary of a punctured section in $C_S \setminus \nu(\text{fiber})$ and is no longer trivial in $\pi_1(Y_K \setminus \nu\Sigma_2)$. By setting $[x, b] = 1$, the relation $r_{n+1} = 1$ is to turn into $[x, a] = 1$.

It remains to check that the relations in $\pi_1(Y_K)$ other than $[x, a] = [x, b] = 1$ remain the same in $\pi_1(Y_K \setminus \nu\Sigma_2)$. By choosing a suitable point $\theta \in S^1$ away from the image of the fiber that forms part of Σ_2, we obtain an embedding of the knot complement $(S^3 \setminus \nu K) \times \{\theta\} \hookrightarrow C_S \setminus \nu(\text{fiber})$. This means that $aba = bab$ holds in $\pi_1(Y_K \setminus \nu\Sigma_2)$. Since $[\Sigma_2]^2 = 0$, there exists a parallel copy of Σ_2 outside $\nu\Sigma_2$, wherein the identity $ab^2ab^{-4} = [d, y]$ still holds. The other remaining relations in $\pi_1(Y_K)$ are coming from the monodromy of the torus bundle over a torus. Since these relations will now describe the monodromy of the punctured torus bundle over a punctured torus, they hold true in $\pi_1(Y_K \setminus \nu\Sigma_2)$. □

4.2.3 Fundamental Group of X_K Finally, we carry out the computations of the fundamental group and the first homology of X_K. Suppose that e, f, z, s and t are the generators of the fundamental group in the second copy of Y_K corresponding to the generators a, b, x, d and y as in above discussion. Our gluing map ϕ maps the generators of $\pi_1(\Sigma_2)$ as follows:

$$\phi_*(a^{-1}b) = s,\ \phi_*(b^{-1}aba^{-1}) = t,\ \phi_*(d) = e^{-1}f,\ \phi_*(y) = f^{-1}efe^{-1}.$$

By Van Kampen's Theorem and Lemma 4.2, we have

$$\begin{aligned}\pi_1(X_K) = \langle a, b, x, d, y;\ e, f, z, s, t;\ g_1, \dots, g_m;\ h_1, \dots, h_m \mid \\ aba = bab,\ [y, x] = [y, b] = 1, \\ dxd^{-1} = xb,\ dbd^{-1} = x^{-1},\ ab^2ab^{-4} = [d, y], \\ r_1 = \cdots = r_{n+1} = 1,\ r'_1 = \cdots = r'_{n+1} = 1, \\ efe = fef,\ [t, z] = [t, f] = 1, \\ szs^{-1} = zf,\ sfs^{-1} = z^{-1},\ ef^2ef^{-4} = [s, t], \\ d = e^{-1}f,\ y = f^{-1}efe^{-1},\ a^{-1}b = s,\ b^{-1}aba^{-1} = t, \\ [x, b] = [z, f]\rangle,\end{aligned}$$

where the elements g_i, h_i $(i = 1, \cdots, m)$ and $r_j, r_j{}'$ $(j = 1, \cdots, n+1)$ all are in the normal subgroup generated by $[x, b] = [z, f]$.

Notice that it follows from our gluing that the images of standard generators of the fundamental group of Σ_2 are $a^{-1}b$, $b^{-1}aba^{-1}$, d and y in $\pi_1(X_K)$. By abelianizing $\pi_1(X_K)$, we easily see that $H_1(X_K, \mathbf{Z}) = 0$.

5 Construction of an exotic $3\,\mathbb{CP}^2\#7\,\overline{\mathbb{CP}}^2$

In this section, we construct a simply-connected minimal symplectic 4–manifold X homeomorphic but not diffeomorphic to $3\,\mathbb{CP}^2\#7\,\overline{\mathbb{CP}}^2$. Using Seiberg–Witten invariants, we will distinguish X from $3\,\mathbb{CP}^2\#7\,\overline{\mathbb{CP}}^2$.

Our manifold X will be the symplectic fiber sum of X_K and $Z = T^2 \times S^2\,\#4\overline{\mathbb{CP}}^2$ along the genus two surfaces Σ_2 and Σ'_2. Recall that $a^{-1}b$, $b^{-1}aba^{-1}$, d, y and $\lambda = \{\mathrm{pt}\} \times S^1 = [x, b][z, f]^{-1}$ generate the inclusion-induced image of $\pi_1(\Sigma_2 \times S^1)$ inside $\pi_1(X_K \setminus \nu\Sigma_2)$. Let a_1, b_1, a_2, b_2 and $\lambda' = 1$ be generators of $\pi_1(Z \setminus \nu\Sigma'_2)$ as in Section 4.1. We choose the gluing diffeomorphism $\psi\colon \Sigma_2 \times S^1 \to \Sigma'_2 \times S^1$ that maps the fundamental group generators as follows:

$$\psi_*(a^{-1}b) = a_2,\ \psi_*(b^{-1}aba^{-1}) = b_2,\ \psi_*(d) = a_1,\ \psi_*(y) = b_1,\ \psi_*(\lambda) = \lambda'.$$

λ and λ' above denote the meridians of Σ and Σ'_2 in X_K and Z, respectively.

It follows from Gompf's theorem [7] that $X = X_K\#_\psi(T^2 \times S^2\,\#4\overline{\mathbb{CP}}^2)$ is symplectic.

Lemma 5.1 *X is simply connected.*

Proof By Van Kampen's theorem, we have

$$\pi_1(X) = \frac{\pi_1(X_K \setminus \nu\Sigma_2) * \pi_1(Z \setminus \nu\Sigma_2')}{\langle a^{-1}b = a_2,\ b^{-1}aba^{-1} = b_2,\ d = a_1,\ y = b_1,\ \lambda = 1\rangle}.$$

Since λ' is nullhomotopic in $Z \setminus \nu\Sigma_2'$, the normal circle λ of $\pi_1(X_K \setminus \nu\Sigma_2)$ becomes trivial in $\pi_1(X)$. Also, using the relations $b_1b_2 = [a_1, b_1] = [a_2, b_2] = b_2a_2{b_2}^{-1}a_1 = a_1a_2 = 1$ in $\pi_1(Z \setminus \nu\Sigma_2')$, we get the following relations in the fundamental group of X: $a^{-1}bd = [a^{-1}b, b^{-1}aba^{-1}] = [d, y] = [d, b^{-1}aba^{-1}] = yb^{-1}aba^{-1} = 1$. Note that the fundamental group of Z is an abelian group of rank two. In addition, we have the following relations in $\pi_1(X)$ coming from the fundamental group of $\pi_1(X_K \setminus \nu\Sigma_2)$: $aba = bab$, $efe = fef$, $[y, b] = [t, f] = 1$, $dbd^{-1} = x^{-1}$, $dxd^{-1} = xb$, $sfs^{-1} = z^{-1}$, $szs^{-1} = zf$, $a^{-1}b = s$, $b^{-1}aba^{-1} = t$, $y = f^{-1}efe^{-1}$ and $e^{-1}f = d$. These set of relations give rise to the following identities:

$$yab = ba, \tag{5}$$

$$a = bd, \tag{6}$$

$$yb = by, \tag{7}$$

$$aba = bab. \tag{8}$$

Next, multiply the relation (5) by a from the right and use $aba = bab$. We have $yaba = ba^2 \implies ybab = ba^2$. By cancelling the element b, we obtain $yab = a^2$. Finally, applying the relation (5) again, we have $ba = a^2$. The latter implies that $b = a$. Since $a = bd$, $dbd^{-1} = x^{-1}$, $dxd^{-1} = xb$, $aba = bab$ and $yb^{-1}aba^{-1} = 1$, we obtain $d = y = x = b = a = 1$. Furthermore, using the relations $a^{-1}b = s$, $b^{-1}aba^{-1} = t$, $efe = fef$, $e^{-1}f = d$, $sfs^{-1} = z^{-1}$ and $szs^{-1} = zf$, we similarly have $s = t = z = f = e = 1$. Thus, we can conclude that the elements a, b, x d, y, e, f, z, s and t are all trivial in the fundamental group of X. Since we identified $a^{-1}b$ and $b^{-1}aba^{-1}$ with generators a_2 and b_2 of the group $\pi_1(Z \setminus \nu\Sigma_2') = \mathbf{Z} \oplus \mathbf{Z}$, it follows that a_2 and b_2 are trivial in the fundamental group of X as well. This proves that X is simply connected. □

Lemma 5.2 *${c_1}^2(X) = 12$, $\sigma(X) = -4$ and $\chi_h(X) = 2$.*

Proof We have ${c_1}^2(X) = {c_1}^2(X_K) + {c_1}^2(T^2 \times S^2 \# 4\overline{\mathbb{CP}}^2) + 8$, $\sigma(X) = \sigma(X_K) + \sigma(T^2 \times S^2 \# 4\overline{\mathbb{CP}}^2)$ and $\chi_h(X) = \chi_h(X_K) + \chi_h(T^2 \times S^2 \# 4\overline{\mathbb{CP}}^2) + 1$. Since ${c_1}^2(X_K) = 8$, $\sigma(X_K) = 0$ and $\chi_h(X_K) = 1$, the result follows from Lemma 3.3 and Lemma 4.1. □

By Freedman's theorem [6], Lemma 5.1 and Lemma 5.2, X is homeomorphic to $3\,\mathbb{CP}^2\#7\,\overline{\mathbb{CP}}^2$. It follows from Taubes Theorem 2.1 that $\mathrm{SW}_X(K_X) = \pm 1$. Next we apply the connected sum theorem for the Seiberg–Witten invariant and show that SW function is trivial for $3\,\mathbb{CP}^2\#7\,\overline{\mathbb{CP}}^2$. Since the Seiberg–Witten invariants are diffeomorphism invariants, we conclude that X is not diffeomorphic to $3\,\mathbb{CP}^2\#7\,\overline{\mathbb{CP}}^2$. Notice that case (i) of Theorem 3.4 does not apply and X_K is a minimal symplectic manifold. Thus, we can conclude that X is minimal. Since symplectic minimality implies irreducibility for simply-connected 4–manifolds with $b_2^+ > 1$ (Kotschick [9]), it follows that X is also smoothly irreducible.

6 Construction of an exotic symplectic $\mathbb{CP}^2\#5\,\overline{\mathbb{CP}}^2$

In this section, we construct a simply-connected minimal symplectic 4–manifold Y homeomorphic but not diffeomorphic to $\mathbb{CP}^2\#5\,\overline{\mathbb{CP}}^2$. Using Usher's Theorem [21], we will distinguish Y from $\mathbb{CP}^2\#5\,\overline{\mathbb{CP}}^2$.

The manifold Y will be the symplectic fiber sum of Y_K and $T^2\times S^2\,\#4\overline{\mathbb{CP}}^2$ along the genus two surfaces Σ_2 and Σ_2'. Let us choose the gluing diffeomorphism $\varphi\colon \Sigma_2\times S^1 \to \Sigma_2'\times S^1$ that maps the generators $a^{-1}b$, $b^{-1}aba^{-1}$, d, y and μ of $\pi_1(Y_K\setminus \nu\Sigma_2)$ to the generators a_1, b_1, a_2, b_2 and μ' of $\pi_1(Z\setminus \nu\Sigma_2')$ according to the following rule:

$$\varphi_*(a^{-1}b) = a_2,\ \varphi_*(b^{-1}aba^{-1}) = b_2,\ \varphi_*(d) = a_1,\ \varphi_*(y) = b_1,\ \varphi_*(\mu) = \mu'.$$

Here, μ and μ' denote the meridians of Σ and Σ_2'.

Again, by Gompf's theorem [7], $Y = Y_K\#_\varphi(T^2\times S^2\,\#4\overline{\mathbb{CP}}^2)$ is symplectic.

Lemma 6.1 *Y is simply connected.*

Proof By Van Kampen's theorem, we have

$$\pi_1(Y) = \frac{\pi_1(Y_K\setminus \nu\Sigma_2) * \pi_1(Z\setminus \nu\Sigma_2')}{\langle a^{-1}b = a_2,\ b^{-1}aba^{-1} = b_2,\ d = a_1,\ y = b_1,\ \lambda = 1\rangle}.$$

The following set of relations hold in $\pi_1(Y)$.

(9) $$a = bd,$$

(10) $$yb = by,$$

(11) $$aba = bab,$$

(12) $$yab = ba.$$

Using the same argument as in proof of Lemma 5.1, we have $a = b = x = d = y = 1$. Thus $\pi_1(Y) = 0$. □

Lemma 6.2 $c_1{}^2(Y) = 4$, $\sigma(Y) = -4$ *and* $\chi_h(Y) = 1$.

Proof We have $c_1{}^2(Y) = c_1{}^2(Y_K) + c_1{}^2(T^2 \times S^2 \# 4\overline{\mathbb{CP}}^2) + 8$, $\sigma(Y) = \sigma(Y_K) + \sigma(T^2 \times S^2 \# 4\overline{\mathbb{CP}}^2)$ and $\chi_h(Y) = \chi_h(Y_K) + \chi_h(T^2 \times S^2 \# 4\overline{\mathbb{CP}}^2) + 1$. Since $c_1{}^2(Y_K) = 0$, $\sigma(Y_K) = 0$ and $\chi_h(Y_K) = 0$, the result follows from Lemma 3.3 and Lemma 4.1. □

By Freedman's classification theorem [6], Lemma 6.1 and Lemma 6.2 above, Y is homeomorphic to $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}}^2$. Notice that Y is a fiber sum of the non-minimal manifold $Z = T^2 \times S^2 \# 4\overline{\mathbb{CP}}^2$ with the minimal manifold Y_K. All 4 exceptional spheres E_1, E_2, E_3 and E_4 in Z meet with the genus two fiber $2T + S - E_1 - E_2 - E_3 - E_4$. Also, any embedded symplectic -1 sphere in $T^2 \times S^2 \# 4\overline{\mathbb{CP}}^2$ is of the form $mS \pm E_i$, thus intersect non-trivially with the fiber class $2T + S - E_1 - E_2 - E_3 - E_4$. It follows from Theorem 3.4 that Y is a minimal symplectic manifold. Since symplectic minimality implies irreducibility for simply-connected 4–manifolds for $b_2^+ = 1$ [8], it follows that Y is also smoothly irreducible. We conclude that Y is not diffeomorphic to $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}}^2$.

Remark *Alternatively, one can apply the concept of symplectic Kodaira dimension to prove the exoticness of X and Y. We refer the reader to the articles by Li and Yau [10] and Usher [22] for a detailed treatment of how the Kodaira dimension behaves under the symplectic fiber sum.*

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Mathematics Department, Columbia University, New York, NY 10027, USA

anar@math.columbia.edu

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