Small exotic 4–manifolds

ANAR AKHMEDOV

In this article, we construct the first example of a simply-connected minimal symplectic 4–manifold that is homeomorphic but not diffeomorphic to $3\mathbb{C}P^2 \# 7\mathbb{C}P^2$. We also construct the first exotic minimal symplectic $\mathbb{C}P^2 \# 5\mathbb{C}P^2$.

57N65, 57N13; 57M50

1 Introduction

Over the past several years, there has been a considerable amount of progress in the discovery of exotic smooth structures on simply-connected 4–manifolds with small Euler characteristic. In early 2004, Jongil Park [15] has constructed the first example of exotic smooth structure on $\mathbb{C}P^2 \# 7\mathbb{C}P^2$, i.e. 4–manifold homeomorphic but not diffeomorphic to $\mathbb{C}P^2 \# 7\mathbb{C}P^2$. Later that year, András Stipsicz and Zoltán Szabó used a similar technique to construct an exotic smooth structure on $\mathbb{C}P^2 \# 6\mathbb{C}P^2$ [18]. Then Fintushel and Stern [5] introduced a new technique, the double node surgery, which demonstrated that in fact $\mathbb{C}P^2 \# k\mathbb{C}P^2$, $k = 6, 7$ and 8 have infinitely many distinct smooth structures. Using the double node surgery technique [5], Park, Stipsicz and Szabó constructed infinitely many smooth structures on $\mathbb{C}P^2 \# 5\mathbb{C}P^2$ [17]. The examples in [17] are not known if symplectic. Based on similar ideas, Stipsicz and Szabó constructed the exotic smooth structures on $3\mathbb{C}P^2 \# k\mathbb{C}P^2$ for $k = 9$ [19] and Park for $k = 8$ [16]. In this article, we construct an exotic smooth structure on $3\mathbb{C}P^2 \# 7\mathbb{C}P^2$. We also construct an exotic symplectic $\mathbb{C}P^2 \# 5\mathbb{C}P^2$, the first known such symplectic example.

Our approach is different from the above constructions in the sense that we do not use any rational-blowdown surgery (Fintushel and Stern [3], Jongil [14]). Also, in contrary to the previous constructions, we use non-simply connected building blocks (Akhmedov [1], Matsumoto [11]) to produce the simply-connected examples. The main surgery technique used in our construction is the symplectic fiber sum operation (Gompf [7], McCarthy and Wolfson [12]) along the genus two surfaces. Our results can be stated as follows.

Published: 9 October 2008 DOI: 10.2140/agt.2008.8.1781
Theorem 1.1 There exist a smooth closed simply-connected minimal symplectic 4–manifold $X$ that is homeomorphic but not diffeomorphic to $3 \mathbb{C}P^2 \# 7 \overline{\mathbb{C}P^2}$.

Theorem 1.2 There exist a smooth closed simply-connected minimal symplectic 4–manifold $Y$ which is homeomorphic but not diffeomorphic to the rational surface $\mathbb{C}P^2 \# 5 \overline{\mathbb{C}P^2}$.

This article is organized as follows. The first two sections give a quick introduction to Seiberg–Witten invariants and a fiber sum operation. In Section 4, we review the symplectic building blocks for our construction. Finally, in Section 5 and Section 6, we construct minimal symplectic 4–manifolds $X$ and $Y$ homeomorphic but not diffeomorphic to $3 \mathbb{C}P^2 \# 7 \overline{\mathbb{C}P^2}$ and $\mathbb{C}P^2 \# 5 \overline{\mathbb{C}P^2}$, respectively.

Acknowledgments I would like to thank John Etnyre, Ron Stern and András Stipsicz for their interest in this work and for their encouragement. Also, I am grateful to B Doug Park for the comments on the first draft of this article, kindly pointing out some errors in the fundamental group computations and for the corrections. Finally, I wish to thank the referee for many helpful suggestions which improved the exposition of this article. This work is partially supported by NSF grant FRG-0244663.

Dedication Dedicated to Professor Ronald J Stern on the occasion of his sixtieth birthday.

2 Seiberg–Witten Invariants

In this section, we briefly recall the basics of Seiberg–Witten invariants introduced by Seiberg and Witten. Seiberg–Witten invariant of a smooth closed oriented 4–manifold $X$ with $b_2^+(X) > 1$ is an integer valued function which is defined on the set of spin$^c$ structures over $X$ (Witten [23]). For simplicity, we assume that $H_1(X, \mathbb{Z})$ has no 2–torsion. Then there is a one-to-one correspondence between the set of spin$^c$ structures over $X$ and the set of characteristic elements of $H^2(X, \mathbb{Z})$.

In this set up, we can view the Seiberg–Witten invariant as an integer valued function

$$\text{SW}_X: \{k \in H^2(X, \mathbb{Z}) | k \equiv w_2(TX) \pmod{2}\} \rightarrow \mathbb{Z}.$$ 

The Seiberg–Witten invariant $\text{SW}_X$ is a diffeomorphism invariant. We call $\beta$ a basic class of $X$ if $\text{SW}_X(\beta) \neq 0$. It is a fundamental fact that the set of basic classes is finite. Also, if $\beta$ is a basic class, then so is $-\beta$ with

$$\text{SW}_X(-\beta) = (-1)^{(e+\sigma)(X)/4} \text{SW}_X(\beta).$$
where $e(X)$ is the Euler characteristic and $\sigma(X)$ is the signature of $X$.

**Theorem 2.1** (Taubes [20]) Suppose that $(X, \omega)$ is a closed symplectic 4–manifold with $b_2^+(X) > 1$ and the canonical class $K_X$. Then $SW_X(\pm K_X) = \pm 1$.

### 3 Fiber Sum

**Definition 3.1** Let $X$ and $Y$ be closed, oriented, smooth 4–manifolds each containing a smoothly embedded surface $\Sigma$ of genus $g \geq 1$. Assume $\Sigma$ represents a homology class of infinite order and has self-intersection zero in $X$ and $Y$, so that there exist a tubular neighborhood, say $\nu \Sigma \cong \Sigma \times D^2$, in both $X$ and $Y$. Using an orientation-reversing and fiber-preserving diffeomorphism $\psi: S^1 \times \Sigma \rightarrow S^1 \times \Sigma$, we can glue $X \setminus \nu \Sigma$ and $Y \setminus \nu \Sigma$ along the boundary $\partial(\nu \Sigma) \cong \Sigma \times S^1$. This new oriented smooth 4–manifold $X \#_\psi Y$ is called a generalized fiber sum of $X$ and $Y$ along $\Sigma$, determined by $\psi$.

**Definition 3.2** Let $e(X)$ and $\sigma(X)$ denote the Euler characteristic and the signature of a closed oriented smooth 4–manifold $X$, respectively. We define

$$c_1^2(X) := 2e(X) + 3\sigma(X), \quad \chi_h(X) := \frac{e(X) + \sigma(X)}{4}.$$ 

In the case that $X$ is a complex surface, then $c_1^2(X)$ and $\chi_h(X)$ are the self-intersection of the first Chern class $c_1(X)$ and the holomorphic Euler characteristic, respectively.

**Lemma 3.3** Let $X$ and $Y$ be closed, oriented, smooth 4–manifolds containing an embedded surface $\Sigma$ of self-intersection 0. Then

$$c_1^2(X \#_\psi Y) = c_1^2(X) + c_1^2(Y) + 8(g - 1),$$

$$\chi_h(X \#_\psi Y) = \chi_h(X) + \chi_h(Y) + (g - 1),$$

where $g$ is the genus of the surface $\Sigma$.

**Proof** The above simply follows from the well-known formulas

$$e(X \#_\psi Y) = e(X) + e(Y) - 2e(\Sigma), \quad \sigma(X \#_\psi Y) = \sigma(X) + \sigma(Y). \quad \Box$$

If $X$, $Y$ are symplectic manifolds and $\Sigma$ is an embedded symplectic submanifold in $X$ and $Y$, then according to theorem of Gompf [7] $X \#_\psi Y$ admits a symplectic structure.

We will use the following theorem of M Usher [21] to show that the symplectic manifolds constructed in Section 5 and Section 6 are minimal. Here we slightly abuse the above notation for the fiber sum.
Theorem 3.4 (Usher [21], Minimality of Symplectic Sums) Let $X = X_1 \#_{F_1} F_2 X_2$ be symplectic fiber sum of manifolds $X_1$ and $X_2$.

(i) If either $X_1 \setminus F_1$ or $X_2 \setminus F_2$ contains an embedded symplectic sphere of square $-1$, then $X$ is not minimal.

(ii) If one of the summands $X_i$ (say $X_1$) admits the structure of an $S^2$–bundle over a surface of genus $g$ such that $F_i$ is a section of this fiber bundle, then $X$ is minimal if and only if $X_2$ is minimal.

(iii) In all other cases, $X$ is minimal.

4 Building blocks

The building blocks for our construction will be as follows.

(i) The manifold $T^2 \times S^2 \# 4\mathbb{CP}^2$ equipped with the genus two Lefschetz fibration of Matsumoto [11].

(ii) The symplectic manifolds $X_K$ and $Y_K$ [1]. For the convenience of the reader, we recall the construction in [1].

4.1 Matsumoto fibration

First, recall that the manifold $Z = T^2 \times S^2 \# 4\mathbb{CP}^2$ can be described as the double branched cover of $S^2 \times T^2$ where the branch set $B_{2,2}$ is the union of two disjoint copies of $S^2 \times \{pt\}$ and two disjoint copies of $\{pt\} \times T^2$. The branch cover has 4 singular points, corresponding to the number of the intersections points of the horizontal lines and the vertical tori in the branch set $B_{2,2}$. After desingularizing the above singular manifold, one obtains $T^2 \times S^2 \# 4\mathbb{CP}^2$. The vertical fibration of $S^2 \times T^2$ pulls back to give a fibration of $T^2 \times S^2 \# 4\mathbb{CP}^2$ over $S^2$. A generic fiber of the vertical fibration is the double cover of $T^2$, branched over 2 points. Thus a generic fiber will be a genus two surface. According to Matsumoto [11], this fibration can be perturbed to be a Lefschetz fibration over $S^2$ with the global monodromy $(\beta_1 \beta_2 \beta_3 \beta_4)^2 = 1$, where the curves $\beta_1$, $\beta_2$, $\beta_3$ and $\beta_4$ are shown in Figure 1.

Let us denote the regular fiber by $\Sigma'_2$ and the images of standard generators of the fundamental group of $\Sigma'_2$ as $a_1$, $b_1$, $a_2$ and $b_2$. Using the homotopy exact sequence for a Lefschetz fibration,

$$\pi_1(\Sigma'_2) \longrightarrow \pi_1(Z) \longrightarrow \pi_1(S^2)$$
we have the following identification of the fundamental group of $Z$ [13]:

$$\pi_1(Z) = \pi_1(\Sigma'_2)/\langle \beta_1, \beta_2, \beta_3, \beta_4 \rangle.$$  

(1) $\beta_1 = b_1 b_2$,  
(2) $\beta_2 = a_1 b_1 a_1^{-1} b_1^{-1} = a_2 b_2 a_2^{-1} b_2^{-1}$,  
(3) $\beta_3 = b_2 a_2 b_2^{-1} a_1$,  
(4) $\beta_4 = b_2 a_2 a_1 b_1$.

Hence $\pi_1(Z) = \langle a_1, b_1, a_2, b_2 \mid b_1 b_2 = [a_1, b_1] = [a_2, b_2] = a_1 a_2 = 1 \rangle$.

Note that the fundamental group of $T^2 \times S^2 \# 4\overline{\mathbb{CP}^2}$ is $\mathbb{Z} \oplus \mathbb{Z}$, generated by two of these standard generators (say $a_1$ and $b_1$). The other two generators $a_2$ and $b_2$ are the inverses of $a_1$ and $b_1$ in the fundamental group. Also, the fundamental group of the complement of $\nu \Sigma'_2$ is $\mathbb{Z} \oplus \mathbb{Z}$. It is generated by $a_1$ and $b_1$. The normal circle $\lambda' = pt \times \partial D^2$ to $\Sigma'_2$ can be deformed using one of the exceptional spheres, thus is trivial in $\pi_1(T^2 \times S^2 \# 4\overline{\mathbb{CP}^2} \setminus \nu \Sigma'_2) = \mathbb{Z} \oplus \mathbb{Z}$.

**Lemma 4.1** $c_1^2(Z) = -4$, $\sigma(Z) = -4$ and $\chi_h(Z) = 0$.

**Proof** We have $c_1^2(Z) = c_1^2(T^2 \times S^2) - 4 = -4$, $\sigma(Z) = \sigma(T^2 \times S^2) - 4 = -4$ and $\chi_h(Z) = \chi_h(T^2 \times S^2) = 0$. \hfill $\Box$

Note that this Lefschetz fibration can be given a symplectic structure. This means that $Z$ admits a symplectic structure such that the regular fibers are symplectic submanifolds. We consider such a symplectic structure on $Z$.
4.2 Symplectic 4–manifolds cohomology equivalent to \( S^2 \times S^2 \)

Our second building block will be \( X_K \), the symplectic cohomology \( S^2 \times S^2 \) [1], or the symplectic manifold \( Y_K \), an intermediate building block in that construction [1], (see also Fintushel and Stern [4]). For the sake of completeness, the details of this construction are included below. We refer the reader to [1] for more details and for the generalization of these symplectic building blocks.

Let \( K \) be a fibered knot of genus one (ie, the trefoil or the figure eight knot) in \( S^3 \) and \( m \) be a meridional circle to \( K \). We perform 0–framed surgery on \( K \) and denote the resulting 3–manifold by \( M_K \). Since \( K \) is fibered and has genus one, it follows the 3–manifold \( M_K \) is a torus bundle over \( S^1 \), hence the 4–manifold \( M_K \times S^1 \) is a torus bundle over a torus. Furthermore, \( M_K \times S^1 \) admits a symplectic structure, and both the torus fiber and the torus section \( T_m = m \times S^1 = m \times x \) are symplectically embedded and have a self-intersection zero. The first homology of \( M_K \times S^1 \) is generated by the standard first homology generators \( m \) and \( x \) of the torus section. On the other hand, the classes of circles \( \gamma_1 \) and \( \gamma_2 \) of the fiber \( F \), coming from the Seifert surface, are trivial in homology. In addition, \( M_K \times S^1 \) is minimal symplectic, ie, it does not contain symplectic \(-1\) sphere.

We form a twisted fiber sum of two copies of the manifold \( M_K \times S^1 \), we identify the fiber \( F \) of one fibration to the section \( T_m \) of other. Let \( Y_K \) denote the mentioned twisted fiber sum \( Y_K = M_K \times S^1 \#_{F=T_m} M_K \times S^1 \). It follows from Gompf’s theorem [7] that \( Y_K \) is symplectic and by Usher’s Theorem 3.4 that \( Y_K \) is minimal symplectic.

Let \( T_1 \) be the section of the first copy of \( M_K \times S^1 \) and \( T_2 \) be the fiber in the second copy. Then the genus two surface \( \Sigma_2 = T_1 \# T_2 \) symplectically embeds into \( Y_K \) and has self-intersection zero. Let \( X_K \) be a symplectic 4–manifold constructed as follows: Take two copies of \( Y_K \) and form the fiber sum along the genus two surface \( \Sigma_2 \) using the special glueing diffeomorphism \( \phi \), the vertical involution of \( \Sigma_2 \) with two fixed points. Thus \( X_K := Y_K \# \phi Y_K \). Let \( m \), \( x \), \( \gamma_1 \) and \( \gamma_2 \) denote the generators of \( \pi_1(\Sigma_2) \) under the inclusion. The diffeomorphism \( \phi: T_1 \# T_2 \to T_1 \# T_2 \) of \( \Sigma_2 \) maps on the generators as follows: \( \phi_*(m') = \gamma_1 \), \( \phi_*(x') = \gamma_2 \), \( \phi_*(\gamma_1') = m \) and \( \phi_*(\gamma_2') = x \). In [1] we show that the manifold \( X_K \) has first Betti number zero and has the integral cohomology of \( S^2 \times S^2 \). Furthermore, \( H_2(X_K, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \), where the basis for the second homology are the classes of \( \Sigma_2 = S \) and the new genus two surface \( T \) resulting from the last fiber sum operation (two punctured genus one surfaces glues to form a genus two surface). Also, \( S^2 = T^2 = 0 \) and \( S \cdot T = 1 \). Furthermore, \( c_1^2(X_K) = 8 \), \( \sigma(X_K) = 0 \) and \( \chi_b(X_K) = 1 \). Since \( Y_K \) is minimal symplectic, it follows from Theorem 3.4 that \( X_K \) is minimal symplectic as well.
4.2.1 Fundamental Group of $M_K \times S^1$ We will assume that $K$ is the trefoil knot. Let $a$, $b$ and $c$ denote the Wirtinger generators of the trefoil. The knot group of the trefoil has the following presentations: $\pi_1(K) = \langle a, b , c \mid ab = bc, ca = ab \rangle = \langle a, b \mid aba = bab \rangle = \langle u, v \mid u^2 = v^3 \rangle$ where $u = bab$ and $v = ab$. The homotopy classes of the meridian and the longitude of the trefoil are given as follows: $m = uv^{-1} = b$ and $l = u^2( uv^{-1} )^{-6} = ab^2ab^{-4}$ (Burde and Zieschang [2]). Also, the homotopy classes of $\gamma_1$ and $\gamma_2$ are given as follows: $\gamma_1 = a^{-1}b$ and $\gamma_2 = b^{-1}aba^{-1}$. Notice that the fundamental group of $M_K$, 0–surgery on the trefoil, is obtained from the knot group of the trefoil by adjoining the relation $l = u^2( uv^{-1} )^{-6} = ab^2ab^{-4} = 1$. Thus, we have $\pi_1(M_K) = \langle u, v \mid u^2 = v^3, \quad u^2( uv^{-1} )^{-6} = 1 \rangle = \langle a, b \mid aba = bab, \quad ab^2a = b^4 \rangle$ and $\pi_1(M_K \times S^1) = \langle a, b, x \mid aba = bab, \quad ab^2a = b^4, \quad [x, a] = [x, b] = 1 \rangle$.

4.2.2 Fundamental Group of $Y_K$ The next step is to take two copies of the manifold $M_K \times S^1$ and perform the fiber sum along symplectic tori. In the first copy of $M_K \times S^1$, we take a tubular neighborhood of the torus section $T_m$, remove it from $M_K \times S^1$ and denote the resulting manifold by $C_S$. In the second copy, we remove a tubular neighborhood of the fiber $F$ and denote it by $C_F$. Notice that $C_S = M_K \times S^1 \setminus vT_m = (M_K \setminus v(m)) \times S^1$. We have $\pi_1(C_S) = \pi_1(K) \oplus \langle x \rangle$ where $x$ is the generator corresponding to the $S^1$ copy. Also using the above computation, we easily derive: $H_1(C_S) = H_1(M_K) = \langle m \rangle \oplus \langle x \rangle$.

To compute the fundamental group of the $C_F$, we will use the following observation: $vF$ is the preimage of the small disk on $T_{m'} = m' \times y$. The elements $y$ and $m' = d$ of the $\pi_1(C_F)$ do not commute anymore, but $y$ still commutes with generators $\gamma_1'$ and $\gamma_2'$. The fundamental group and the first homology of the $C_F$ will be isomorphic to the following: $\pi_1(C_F) = \langle d, y, \gamma_1', \gamma_2' \mid [y, \gamma_1'] = [y, \gamma_2'] = [y_1', y_2'] = 1, \quad d \gamma_1'^{-d} = \gamma_1' \gamma_2', \quad d \gamma_2'^{-d} = (y_1')^{-1} \rangle$ and $H_1(C_F) = \langle d \rangle \oplus \langle y \rangle$.

We use the Van Kampen’s Theorem to compute the fundamental group of $Y_K$.

$$\pi_1(Y_K) = \pi_1(C_F) *_{\pi_1(K)} \pi_1(C_F)$$

$$= \langle d, y, \gamma_1', \gamma_2' \mid [y, \gamma_1'] = [y, \gamma_2'] = [y_1', y_2'] = 1, \quad d \gamma_1'^{-d} = \gamma_1' \gamma_2', \quad d \gamma_2'^{-d} = 1 \rangle$$

$$= \langle a, b, x, \gamma_1', \gamma_2', d, y \mid aba = bab, \quad [x, a] = [x, b] = 1 \rangle$$

$$= \langle a, b, x, \gamma_1', \gamma_2', d, y \mid aba = bab, \quad [x, a] = [x, b] = 1 \rangle$$

$$= \langle a, b, x, \gamma_1', \gamma_2' \mid \gamma_1' = 1, \quad d \gamma_1'^{-d} = \gamma_1' \gamma_2', \quad d \gamma_2'^{-d} = (y_1')^{-1}, \quad \gamma_1' = x, \gamma_2' = b, \quad [y_1', y_2'] = [d, y] \rangle.$$
homomorphism maps the standard generators of $\pi_1(\Sigma_2)$ to $a^{-1}b$, $b^{-1}aba^{-1}$, $d$ and $y$ in $\pi_1(Y_K)$.

**Lemma 4.2** ([1]) There are nonnegative integers $m$ and $n$ such that

\[
\pi_1(Y_K \setminus \nu\Sigma_2) = \langle a, b, x, d, y; g_1, \ldots, g_m \mid aba = bab, \n\]
\[
[y, x] = [y, b] = 1, \quad dxd^{-1} = xb, \quad dbd^{-1} = x^{-1}, \n\]
\[
ab^2ab^{-4} = [d, y], \quad r_1 = \cdots = r_n = 1, \quad r_{n+1} = 1, \n\]

where the generators $g_1, \ldots, g_m$ and relators $r_1, \ldots, r_n$ all lie in the normal subgroup $N$ generated by the element $[x, b]$ and the relator $r_{n+1}$ is a word in $x, a$ and elements of $N$. Moreover, if we add an extra relation $[x, b] = 1$, then the relation $r_{n+1} = 1$ simplifies to $[x, a] = 1$.

**Proof** This follows from Van Kampen’s Theorem. Note that $[x, b]$ is a meridian of $\Sigma_2$ in $Y_K$. Hence setting $[x, b] = 1$ should turn $\pi_1(Y_K \setminus \nu\Sigma_2)$ into $\pi_1(Y_K)$. Also note that $[x, a]$ is the boundary of a punctured section in $C_S \setminus \nu(\text{fiber})$ and is no longer trivial in $\pi_1(Y_K \setminus \nu\Sigma_2)$. By setting $[x, b] = 1$, the relation $r_{n+1} = 1$ is to turn into $[x, a] = 1$.

It remains to check that the relations in $\pi_1(Y_K)$ other than $[x, a] = [x, b] = 1$ remain the same in $\pi_1(Y_K \setminus \nu\Sigma_2)$. By choosing a suitable point $\theta \in S^1$ away from the image of the fiber that forms part of $\Sigma_2$, we obtain an embedding of the knot complement $(S^3 \setminus \nu K) \times \{\theta\} \hookrightarrow C_S \setminus \nu(\text{fiber})$. This means that $aba = bab$ holds in $\pi_1(Y_K \setminus \nu\Sigma_2)$. Since $[\Sigma_2]^2 = 0$, there exists a parallel copy of $\Sigma_2$ outside $\nu\Sigma_2$, wherein the identity $ab^2ab^{-4} = [d, y]$ still holds. The other remaining relations in $\pi_1(Y_K)$ are coming from the monodromy of the torus bundle over a torus. Since these relations will now describe the monodromy of the punctured torus bundle over a punctured torus, they hold true in $\pi_1(Y_K \setminus \nu\Sigma_2)$. \hfill \Box

**4.2.3 Fundamental Group of $X_K$** Finally, we carry out the computations of the fundamental group and the first homology of $X_K$. Suppose that $e$, $f$, $z$, $s$ and $t$ are the generators of the fundamental group in the second copy of $Y_K$ corresponding to the generators $a$, $b$, $x$, $d$ and $y$ as in above discussion. Our gluing map $\phi$ maps the generators of $\pi_1(\Sigma_2)$ as follows:

\[
\phi_*(a^{-1}b) = s, \quad \phi_*(b^{-1}aba^{-1}) = t, \quad \phi_*(d) = e^{-1}f, \quad \phi_*(y) = f^{-1}efe^{-1}. \n\]
By Van Kampen’s Theorem and Lemma 4.2, we have
\[
\pi_1(X_K) = \langle a, b, x, d, y; e, f, z, s, t; g_1, \ldots, g_m; h_1, \ldots, h_m \mid \begin{align*}
aba &= bab, \ [y, x] = [y, b] = 1, \\
dxd^{-1} &= xb, \ dbd^{-1} = x^{-1}, \ ab^2ab^{-4} = [d, y], \\
r_1 &= \cdots = r_{n+1} = 1, \ r'_1 = \cdots = r'_{n+1} = 1, \\
efe &= fef, \ [t, z] = [t, f] = 1, \\
szs^{-1} &= zf, \ sfs^{-1} = z^{-1}, \ ef^2ef^{-4} = [s, t], \\
d &= e^{-1}f, \ y = f^{-1}efe^{-1}, \ a^{-1}b = s, \ b^{-1}aba^{-1} = t, \\
[x, b] &= [z, f] \rangle,
\]
where the elements \( g_i, h_i \ (i = 1, \ldots, m) \) and \( r_j, r'_j \ (j = 1, \ldots, n + 1) \) all are in the normal subgroup generated by \([x, b] = [z, f])

Notice that it follows from our gluing that the images of standard generators of the fundamental group of \( \Sigma_2 \) are \( a^{-1}b, b^{-1}aba^{-1}, d \) and \( y \) in \( \pi_1(X_K) \). By abelianizing \( \pi_1(X_K) \), we easily see that \( H_1(X_K, \mathbb{Z}) = 0 \).

5 Construction of an exotic \( 3 \mathbb{C}P^2 \# 7 \overline{\mathbb{C}P}^2 \)

In this section, we construct a simply-connected minimal symplectic 4–manifold \( X \) homeomorphic but not diffeomorphic to \( 3 \mathbb{C}P^2 \# 7 \overline{\mathbb{C}P}^2 \). Using Seiberg–Witten invariants, we will distinguish \( X \) from \( 3 \mathbb{C}P^2 \# 7 \overline{\mathbb{C}P}^2 \).

Our manifold \( X \) will be the symplectic fiber sum of \( X_K \) and \( Z = T^2 \times S^2 \# 4\overline{\mathbb{C}P}^2 \) along the genus two surfaces \( \Sigma_2 \) and \( \Sigma'_2 \). Recall that \( a^{-1}b, b^{-1}aba^{-1}, d, y \) and \( \lambda = \{pt\} \times S^1 = [x, b][z, f]^{-1} \) generate the inclusion-induced image of \( \pi_1(\Sigma_2 \times S^1) \) inside \( \pi_1(X_K \setminus \nu \Sigma_2) \). Let \( a_1, b_1, a_2, b_2 \) and \( \lambda' = 1 \) be generators of \( \pi_1(Z \setminus \nu \Sigma'_2) \) as in Section 4.1. We choose the gluing diffeomorphism \( \psi: \Sigma_2 \times S^1 \to \Sigma'_2 \times S^1 \) that maps the fundamental group generators as follows:

\[
\psi_*(a^{-1}b) = a_2, \ \psi_*(b^{-1}aba^{-1}) = b_2, \ \psi_*(d) = a_1, \ \psi_*(y) = b_1, \ \psi_*(\lambda) = \lambda'.
\]

\( \lambda \) and \( \lambda' \) above denote the meridians of \( \Sigma \) and \( \Sigma'_2 \) in \( X_K \) and \( Z \), respectively.

It follows from Gompf’s theorem [7] that \( X = X_K \# \psi(T^2 \times S^2 \# 4\overline{\mathbb{C}P}^2) \) is symplectic.

**Lemma 5.1** \( X \) is simply connected.
Proof  By Van Kampen’s theorem, we have
\[
\pi_1(X) = \frac{\pi_1(X_K \setminus v\Sigma_2) * \pi_1(Z \setminus v\Sigma'_2)}{\{a^{-1}b = a_2, b^{-1}aba^{-1} = b_2, d = a_1, y = b_1, \lambda = 1\}}.
\]

Since \(\lambda'\) is nullhomotopic in \(Z \setminus v\Sigma'_2\), the normal circle \(\lambda\) of \(\pi_1(X_K \setminus v\Sigma_2)\) becomes trivial in \(\pi_1(X)\). Also, using the relations \(b_1b_2 = [a_1, b_1] = [a_2, b_2] = b_2a_2b_2^{-1}a_1 = a_1a_2 = 1\) in \(\pi_1(Z \setminus v\Sigma'_2)\), we get the following relations in the fundamental group of \(X\): \(a^{-1}bd = [a^{-1}b, b^{-1}aba^{-1}] = [d, y] = [d, b^{-1}aba^{-1}] = yb^{-1}aba^{-1} = 1\). Note that the fundamental group of \(Z\) is an abelian group of rank two. In addition, we have the following relations in \(\pi_1(X)\) coming from the fundamental group of \(\pi_1(X_K \setminus v\Sigma_2)\):

\[
\begin{align*}
aba & = bab, 
dfe = fef, 
[y, b] = [f, t] = 1, 
dbd^{-1} & = x^{-1}, 
dxd^{-1} = xb, 
sfs^{-1} = z^{-1}, 
szs^{-1} = zf, 
a^{-1}b = s, 
b^{-1}aba^{-1} = t, 
y = f^{-1}efe^{-1} 
\end{align*}
\]

These are the relations that give rise to the following identities:

\[
\begin{align*}
yab & = ba, 
a & = bd, 
yb & = by, 
aba & = bab.
\end{align*}
\]

Next, multiply the relation (5) by \(a\) from the right and use \(aba = bab\). We have \(yaba = ba^2 \implies yabab = ba^2\). By cancelling the element \(b\), we obtain \(yab = a^2\). Finally, applying the relation (5) again, we have \(ba = a^2\). The latter implies that \(b = a\). Since \(a = bd, dbd^{-1} = x^{-1}, dxd^{-1} = xb, aba = bab\) and \(yb^{-1}aba^{-1} = 1\), we obtain \(d = y = x = b = a = 1\). Furthermore, using the relations \(a^{-1}b = s, b^{-1}aba^{-1} = t, efe = fef, e^{-1}f = d, sfs^{-1} = z^{-1}\) and \(zszs^{-1} = zf\), we similarly have \(s = t = z = f = e = 1\). Thus, we can conclude that the elements \(a, b, x, d, y, e, f, z, s\) and \(t\) are all trivial in the fundamental group of \(X\). Since we identified \(a^{-1}b\) and \(b^{-1}aba^{-1}\) with generators \(a_2\) and \(b_2\) of the group \(\pi_1(Z \setminus v\Sigma'_2) = \mathbb{Z} \oplus \mathbb{Z}\), it follows that \(a_2\) and \(b_2\) are trivial in the fundamental group of \(X\) as well. This proves that \(X\) is simply connected.

Lemma 5.2  \(c_1^2(X) = 12, \sigma(X) = -4\) and \(\chi_h(X) = 2\).

Proof  We have \(c_1^2(X) = c_1^2(X_K) + c_1^2(T^2 \times S^2 \# 4\mathbb{CP}^2) + 8, \sigma(X) = \sigma(X_K) + \sigma(T^2 \times S^2 \# 4\mathbb{CP}^2)\) and \(\chi_h(X) = \chi_h(X_K) + \chi_h(T^2 \times S^2 \# 4\mathbb{CP}^2) + 1\). Since \(c_1^2(X_K) = 8, \sigma(X_K) = 0\) and \(\chi_h(X_K) = 1\), the result follows from Lemma 3.3 and Lemma 4.1.
By Freedman’s theorem [6], Lemma 5.1 and Lemma 5.2, $X$ is homeomorphic to $3\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$. It follows from Taubes Theorem 2.1 that $SW_X(K_X) = \pm 1$. Next we apply the connected sum theorem for the Seiberg–Witten invariant and show that $SW$ function is trivial for $3\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$. Since the Seiberg–Witten invariants are diffeomorphism invariants, we conclude that $X$ is not diffeomorphic to $3\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$. Notice that case (i) of Theorem 3.4 does not apply and $X_K$ is a minimal symplectic manifold. Thus, we can conclude that $X$ is minimal. Since symplectic minimality implies irreducibility for simply-connected 4–manifolds with $b_2^+ > 1$ (Kotschick [9]), it follows that $X$ is also smoothly irreducible.

6 Construction of an exotic symplectic $\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2$

In this section, we construct a simply-connected minimal symplectic 4–manifold $Y$ homeomorphic but not diffeomorphic to $\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2$. Using Usher’s Theorem [21], we will distinguish $Y$ from $\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2$.

The manifold $Y$ will be the symplectic fiber sum of $Y_K$ and $T^2 \times S^2 \#4\overline{\mathbb{CP}}^2$ along the genus two surfaces $\Sigma_2$ and $\Sigma'_2$. Let us choose the gluing diffeomorphism $\varphi: \Sigma_2 \times S^1 \to \Sigma'_2 \times S^1$ that maps the generators $a^{-1}b, b^{-1}aba^{-1}, d, y$ and $\mu$ of $\pi_1(Y_K \setminus \nu \Sigma_2)$ to the generators $a_1, b_1, a_2, b_2$ and $\mu'$ of $\pi_1(Z \setminus \nu \Sigma'_2)$ according to the following rule:

$$\varphi^*(a^{-1}b) = a_2, \quad \varphi^*(b^{-1}aba^{-1}) = b_2, \quad \varphi^*(d) = a_1, \quad \varphi^*(y) = b_1, \quad \varphi^*(\mu) = \mu'.$$

Here, $\mu$ and $\mu'$ denote the meridians of $\Sigma$ and $\Sigma'_2$.

Again, by Gompf’s theorem [7], $Y = Y_K\#_q(T^2 \times S^2 \#4\overline{\mathbb{CP}}^2)$ is symplectic.

**Lemma 6.1** $Y$ is simply connected.

**Proof** By Van Kampen’s theorem, we have

$$\pi_1(Y) = \frac{\pi_1(Y_K \setminus \nu \Sigma_2) \ast \pi_1(Z \setminus \nu \Sigma'_2)}{(a^{-1}b = a_2, \ b^{-1}aba^{-1} = b_2, \ d = a_1, \ y = b_1, \ \lambda = 1)}.$$ 

The following set of relations hold in $\pi_1(Y)$.

(9) \hspace{1cm} a = bd, \\
(10) \hspace{1cm} yb = by, \\
(11) \hspace{1cm} aba = bab, \\
(12) \hspace{1cm} yab = ba.$
Using the same argument as in proof of Lemma 5.1, we have \( a = b = x = d = y = 1 \). Thus \( \pi_1(Y) = 0 \). \( \square \)

**Lemma 6.2** \( c_1^2(Y) = 4 \), \( \sigma(Y) = -4 \) and \( \chi_h(Y) = 1 \).

**Proof** We have \( c_1^2(Y) = c_1^2(Y_K) + c_1^2(T^2 \times S^2 \# 4\mathbb{CP}^2) + 8 \), \( \sigma(Y) = \sigma(Y_K) + \sigma(T^2 \times S^2 \# 4\mathbb{CP}^2) \) and \( \chi_h(Y) = \chi_h(Y_K) + \chi_h(T^2 \times S^2 \# 4\mathbb{CP}^2) + 1 \). Since \( c_1^2(Y_K) = 0 \), \( \sigma(Y_K) = 0 \) and \( \chi_h(Y_K) = 0 \), the result follows from Lemma 3.3 and Lemma 4.1. \( \square \)

By Freedman’s classification theorem [6], Lemma 6.1 and Lemma 6.2 above, \( Y \) is homeomorphic to \( \mathbb{CP}^2 \# 5 \mathbb{CP}^2 \). Notice that \( Y \) is a fiber sum of the non-minimal manifold \( Z = T^2 \times S^2 \# 4\mathbb{CP}^2 \) with the minimal manifold \( Y_K \). All 4 exceptional spheres \( E_1 \), \( E_2 \), \( E_3 \) and \( E_4 \) in \( Z \) meet with the genus two fiber \( 2T + S - E_1 - E_2 - E_3 - E_4 \). Also, any embedded symplectic \(-1\) sphere in \( T^2 \times S^2 \# 4\mathbb{CP}^2 \) is of the form \( mS \pm E_i \), thus intersect non-trivially with the fiber class \( 2T + S - E_1 - E_2 - E_3 - E_4 \). It follows from Theorem 3.4 that \( Y \) is a minimal symplectic manifold. Since symplectic minimality implies irreducibility for simply-connected 4–manifolds for \( b^+ = 1 \) [8], it follows that \( Y \) is also smoothly irreducible. We conclude that \( Y \) is not diffeomorphic to \( \mathbb{CP}^2 \# 5 \mathbb{CP}^2 \).

**Remark** Alternatively, one can apply the concept of symplectic Kodaira dimension to prove the exoticness of \( X \) and \( Y \). We refer the reader to the articles by Li and Yau [10] and Usher [22] for a detailed treatment of how the Kodaira dimension behaves under the symplectic fiber sum.

**References**


*Algebraic & Geometric Topology, Volume 8 (2008)*
Small exotic 4–manifolds


[22] M Usher, Kodaira dimension and Symplectic Sums, to appear in Commentarii Mathematici Helvetici

Anar Akhmedov

Mathematics Department, Columbia University, New York, NY 10027, USA
anar@math.columbia.edu

Received: 12 March 2007    Revised: 7 July 2008