The homology of the stable nonorientable mapping class group

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Combining results of Wahl, Galatius–Madsen–Tillmann–Weiss and Korkmaz, one can identify the homotopy type of the classifying space of the stable nonorientable mapping class group $\mathcal{N}_\infty$ (after plus-construction). At odd primes p, the $\mathbb{F}_p$–homology coincides with that of $Q_0(\mathbb{H}\,\mathbb{P}^\infty_+)$, but at the prime 2 the result is less clear. We identify the $\mathbb{F}_2$–homology as a Hopf algebra in terms of the homology of well-known spaces. As an application we tabulate the integral stable homology of $\mathcal{N}_\infty$ in degrees up to six.

As in the oriented case, not all of these cohomology classes have a geometric interpretation. We determine a polynomial subalgebra of $H^*(\mathcal{N}_\infty;\mathbf{F}_2)$ consisting of geometrically-defined characteristic classes.

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1 Introduction

The mapping class groups $\mathcal{N}_g$ of nonorientable surfaces are not as widely studied as their counterparts for oriented surfaces, but with Wahl's proof [16] of homological stability for these groups, one can apply the machinery of Madsen and Weiss [12] used to prove the Mumford conjecture or its more concise variant by Galatius, Madsen, Tillman and Weiss [8] to study their stable homology. Together these results show that the homology of $\mathcal{N}_\infty$ coincides with that of a component of an infinite loop space, $\Omega^\infty_0\mathbf{MTO}(2)$, which we define in Section 2.2.

Inspired by Galatius' calculation [7] of the homology of the stable oriented mapping class group Γ_∞, we calculate the mod 2 homology of the stable nonorientable mapping class group $\mathcal{N}_\infty$. The odd-primary homology is much simpler: it coincides with that of $Q_0(\mathbb{H}\mathbb{P}^\infty_+)$ and is discussed in Section 5.1. Here $Q(X) := \operatorname{colim}\Omega^n\Sigma^n X$ denotes the free infinite loop space on the space X, the subscript 0 denotes the connected component of the constant loop and the subscript $+$ denotes the addition of a disjoint basepoint. The homology $H_*(Q(X);\mathbb{F}_p)$ is completely known by Cohen, Lada and May [4] as a functor of $H_*(X;\mathbb{F}_p)$.

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We adopt the convention that all (co)homology groups are with $\mathbb{F}_2$ coefficients unless otherwise stated. Our main theorems give a determination of the $\mathbb{F}_2$–homology of $\Omega^\infty_0\mathbf{MTO}(2)$ as a Hopf algebra:

Theorem A *There is an exact sequence of Hopf algebras*

$$\mathbb{F}_2 \longrightarrow H_*(\Omega^\infty_0\mathbf{MTO}(1)) \xrightarrow{\overline{\omega}_*} H_*(Q_0(BO_{1+})) \xrightarrow{\overline{\partial}_*} H_*(Q_0(S^0)) \longrightarrow \mathbb{F}_2$$

which is split as algebras.

Theorem B *There is an exact sequence of Hopf algebras*

$$\mathbb{F}_2 \longrightarrow H_*(\Omega^\infty_0\mathbf{MTO}(2)) \xrightarrow{\omega_*} H_*(Q_0(BO_{2+})) \xrightarrow{\partial_*} H_*(\Omega^\infty_0\mathbf{MTO}(1)) \longrightarrow \mathbb{F}_2$$

which is split as algebras.

We also give formulae for the maps $\overline{\partial}_*$ and ∂_*. Although these theorems determine the required homology groups, they do not do so in a very explicit manner. Thus we tabulate the first six (co)homology groups:

Degree i	1	2	3	4	5	6
Rank $QH_i(\Omega^\infty_0\mathbf{MTO}(2);\mathbb{F}_2)$	1	2	3	3	5	6
Rank $H_i(\Omega^\infty_0\mathbf{MTO}(2);\mathbb{F}_2)$	1	3	6	12	23	45
$H_i(\Omega^\infty_0\mathbf{MTO}(2);\mathbb{Z})$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3\oplus\mathbb{Z}_{12}$	$\mathbb{Z}_2^7\oplus\mathbb{Z}$	$\mathbb{Z}_2^{16}$	$\mathbb{Z}_2^{29}$
$H^i(\Omega^\infty_0\mathbf{MTO}(2);\mathbb{Z})$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3\oplus\mathbb{Z}_{12}\oplus\mathbb{Z}$	$\mathbb{Z}_2^7$	$\mathbb{Z}_2^{16}$

The integral homology can be calculated via the Bockstein spectral sequence. Only the primes 2 and 3 appear in this range: for the prime 3 the homology coincides with that of $Q(\mathbb{H}\,\mathbb{P}^\infty_+)$ and the Bockstein spectral sequence can be calculated from the known one for $\mathbb{H}\,\mathbb{P}^\infty$, which is degenerate from the first page. For the prime 2 we compare the Bockstein spectral sequence for $Q(BO_{2+})$ with the representing elements for classes belonging to $\Omega^\infty_0\mathbf{MTO}(2)$. We find that it degenerates at the E^3 page, giving the claimed homology groups.

We also study a family of mod 2 characteristic classes of surface bundles, χ_i, defined as follows. For a surface bundle $F \longrightarrow E \xrightarrow{\pi} B$ there is an associated vector bundle V of first real cohomologies, with fibre $H^1(F_b;\mathbb{R})$ over $b \in B$. We can define

$$\chi_i(E) := w_i(V) \in H^i(B;\mathbb{F}_2)$$

the i–th Stiefel–Whitney class of V. The main theorem is that these are stably independent for the nonorientable mapping class groups, so account for one indecomposable generator of the cohomology of $B\mathcal{N}_\infty$ in each dimension.

Theorem C *In the nonorientable mapping class groups, the map*

$$\mathbb{F}_2[\chi_1, \chi_2, \chi_3, ...] \longrightarrow H^*(B\mathcal{N}_g; \mathbb{F}_2)$$

is an injection in the stable range $* \leq (g-3)/4$.

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2 Recollections

2.1 Homological stability

Write $N_{g,b}$ for the nonorientable surface consisting of the connected sum of g copies of $\mathbb{RP}^2$ with b discs removed. We call g the genus of the surface, and b the number of boundary components, which we suppress in notation if it is 0. The mapping class group of $N_{g,b}$ is the group of components of the topological group of self-diffeomorphisms that fix the boundary pointwise

$$\mathcal{N}_{g,b} := \pi_0(\mathrm{Diff}(N_{g,b}, \partial N_{g,b})).$$

As long as $b \geq 1$ there are two stabilisation maps between these groups, obtained by gluing either a $N_{1,2}$ or a $N_{0,3}$ to a boundary component and extending diffeomorphisms by the identity to the new surface

$$\alpha\colon \mathcal{N}_{g,b} \longrightarrow \mathcal{N}_{g+1,b}$$
$$\beta\colon \mathcal{N}_{g,b} \longrightarrow \mathcal{N}_{g,b+1}$$

and β has a left inverse obtained by gluing a disc over a boundary component

$$\delta\colon \mathcal{N}_{g,b+1} \longrightarrow \mathcal{N}_{g,b}.$$

We quote below the theorem of Wahl on the effect on these maps on group homology.

Theorem 2.1 (Theorem A of [16]) *For $b \geq 1$, on homology:*

(1) *α gives a surjection in degrees $* \leq g/4$ and an isomorphism in degrees $* \leq (g-3)/4$.*

(2) *β gives an isomorphism in degrees $* \leq (g-3)/4$.*

(3) *δ gives a surjection in degrees $* \leq (g-1)/4$ and an isomorphism in degrees $* \leq (g-5)/4$.*

We will say "the stable range" to indicate the range $* \leq (g-3)/4$ in which α induces an isomorphism.

2.2 The homotopy type of the stable nonorientable mapping class group

Once we have homological stability for $\mathcal{N}_g$, the machinery of Galatius, Madsen, Tillmann and Weiss [8] identifies the stable homology of these groups.

In order to describe the result we must first introduce the spectrum $\mathbf{MTO}(2)$. Let $U_n^\perp \longrightarrow \mathrm{Gr}_2(\mathbb{R}^{n+2})$ be the n–dimensional complement to the tautological 2–plane bundle U_2. The inclusion $\mathrm{Gr}_2(\mathbb{R}^{n+2}) \longrightarrow \mathrm{Gr}_2(\mathbb{R}^{n+1+2})$ pulls back $U_{n+1}^\perp$ to $\epsilon^1 \oplus U_n^\perp$, so there is an induced map on Thom spaces

$$\mathrm{Th}(\epsilon^1 \oplus U_n^\perp) \cong S^1 \wedge \mathrm{Th}(U_n^\perp) \longrightarrow \mathrm{Th}(U_{n+1}^\perp).$$

This defines the spectrum $\mathbf{MTO}(2)$ with $(n+2)$–nd space $\mathrm{Th}(U_n^\perp)$. It is not hard to check that $\pi_0(\mathbf{MTO}(2)) \cong \mathbb{Z}$, using the cofibre sequences of spectra ((2–2) and (2–1)) in the next section.

Let $N_g \longrightarrow E \longrightarrow B$ be a smooth fibre bundle with fibre N_g a nonorientable surface of genus g. There is an embedding of E in $\mathbb{R}^{n+2} \times B$ over B, with a tubular neighbourhood homeomorphic to $N_v E$, the vertical normal bundle of E. A Pontrjagin–Thom collapse gives a map

$$S^{n+2} \wedge B_+ \longrightarrow \mathrm{Th}(N_v E)$$

and the vertical normal bundle can be classified by a map $f\colon E \longrightarrow G_2(\mathbb{R}^{n+2})$ such that $f^* U_n^\perp \cong N_v E$. Thus $f^* U_2$ is $T_v E$, the vertical tangent bundle. Composing with the classifying map we have a map

$$S^{n+2} \wedge B_+ \longrightarrow \mathrm{Th}(U_n^\perp)$$

with adjoint $B \longrightarrow \Omega^{n+2}\,\mathrm{Th}(U_n^\perp)$. Following this with the stabilisation map to $\Omega^\infty \mathbf{MTO}(2)$ defines the Madsen–Tillmann map

$$\alpha_E\colon B \longrightarrow \Omega^\infty \mathbf{MTO}(2).$$

Applying this to the universal F_g bundle over $B\mathcal{N}_g$ gives

$$\alpha_g\colon B\mathcal{N}_g \longrightarrow \Omega^\infty \mathbf{MTO}(2)$$

and similarly α_∞. The source spaces are connected and so land in a single component: it is not hard to see that it is the component of the Euler characteristic $\chi(N_g)$. However all components of an infinite loop space are homotopy equivalent, so we can translate so that the maps α_E always land in the 0 component of $\Omega^\infty\mathbf{MTO}(2)$. Together, Theorem B of [16] and Section 7 of [8] assert that the map

$$\alpha_\infty\colon B\mathcal{N}_\infty \longrightarrow \Omega^\infty_0\mathbf{MTO}(2)$$

is a homology equivalence.

One can immediately upgrade this theorem: Korkmaz [9] proves that for $g \geq 7$ the index 2 normal subgroup of $\mathcal{N}_g$ generated by Dehn twists is perfect, and that the first group homology in this range is $\mathbb{Z}/2$, so this subgroup is the maximal perfect subgroup. There is then a map

$$\alpha^+_\infty\colon B\mathcal{N}^+_\infty \longrightarrow \Omega^\infty_0\mathbf{MTO}(2)$$

from the plus-construction with respect to this maximal perfect subgroup (obtained by extending α_∞ to the plus-construction, as the map has target an infinite loop space). This is still a homology isomorphism, and $\pi_1(\Omega^\infty_0\mathbf{MTO}(2)) \cong H_1(\Omega^\infty_0\mathbf{MTO}(2)) \cong \mathbb{Z}/2$ so it is also a π_1–isomorphism, so a homotopy equivalence.

The rational cohomology of $\Omega^\infty_0\mathbf{MTO}(2)$ is not hard to determine: it is a polynomial algebra on the reductions of certain integral classes $\zeta_i \in H^{4i}(\Omega^\infty_0\mathbf{MTO}(2);\mathbb{Z})$. The author and J Ebert [6] have studied the divisibility of these classes in the integral cohomology of the stable nonorientable mapping class group: they are indivisible.

2.3 The spectra MTX(d) and tools from stable homotopy theory

There are two less well-known spectra appearing in this paper, all of the form $\mathbf{MTO}(d)$. The first is $\mathbf{MTO}(2)$ and has been introduced in the last section. The second is $\mathbf{MTO}(1)$[1] which is constructed in the same way using the unoriented Grassmannians $\mathrm{Gr}_1(\mathbb{R}^{n+1})$. A full description of these spectra is available in [8]. We collect here some of their properties.

For a spectrum $\mathbf{E} = \{E_n\}$ the associated infinite loop space is

$$\Omega^\infty\mathbf{E} := \operatorname{hocolim}\Omega^n E_n$$

where the homotopy colimit, or mapping telescope, is taken using the adjoints $E_n \longrightarrow \Omega E_{n+1}$ of the structure maps. This construction sends cofibre sequences of spectra to

[1] This spectrum is also known in homotopy theory as $\mathbb{RP}^\infty_{-1}$, though we do not use this notation.

fibre sequences of infinite loop spaces. By [8] there are cofibre sequences of spectra

$$\mathbf{MTO}(2) \xrightarrow{\omega} \Sigma^\infty BO_{2+} \xrightarrow{\partial} \mathbf{MTO}(1) \tag{2–1}$$

$$\mathbf{MTO}(1) \xrightarrow{\bar{\omega}} \Sigma^\infty BO_{1+} \xrightarrow{\bar{\partial}} \Sigma^\infty S^0 \tag{2–2}$$

and so fibre sequences of associated infinite loop spaces. It should be noted that $\pi_0(\mathbf{MTO}(1)) = 0$, but we will still write $\Omega^\infty_0\mathbf{MTO}(1)$ to remind the reader that this space is connected.

The evaluation maps $\Sigma^n\Omega^n E_n \longrightarrow E_n$ give maps on reduced homology

$$\widetilde{H}_*(\Omega^n E_n) \cong \widetilde{H}_{*+n}(\Sigma^n\Omega^n E_n) \longrightarrow \widetilde{H}_{*+n}(E_n)$$

and taking direct limits on both sides gives a map

$$\sigma_*\colon \widetilde{H}_*(\Omega^\infty\mathbf{E}) \longrightarrow H_*^{\mathrm{spec}}(\mathbf{E})$$

to the spectrum homology of $\mathbf{E}$. This is known as the *homology suspension*. There is an analogous *cohomology suspension*

$$\sigma^*\colon H^*_{\mathrm{spec}}(\mathbf{E}) \longrightarrow \widetilde{H}^*(\Omega^\infty\mathbf{E}).$$

If $\mathbf{E}$ is the suspension spectrum of a based space X, then $H_*^{\mathrm{spec}}(\mathbf{E}) \cong \widetilde{H}_*(X)$ and the inclusion

$$i_*\colon \widetilde{H}_*(X) \longrightarrow \widetilde{H}_*(Q(X)) = \widetilde{H}_*(\Omega^\infty\mathbf{E})$$

gives a right inverse for σ_*. Similarly for the cohomology suspension.

For a proper smooth fibre bundle $p\colon E \longrightarrow X$ with fibre F (in other words, F is a compact smooth manifold without boundary), there is a stable map $\mathrm{trf}_p\colon \Sigma^\infty X_+ \longrightarrow \Sigma^\infty E_+$ the *Becker–Gottlieb transfer*. There is a simple description in the case that the base X is compact: then there is an embedding of E into $X \times \mathbb{R}^n$ over X, for some n. Choose a tubular neighbourhood U of E in $X \times \mathbb{R}^n$, which is diffeomorphic to the normal bundle of E in $X \times \mathbb{R}^n$. Restricted to a fibre F_x embedded in $\mathbb{R}^n$ this bundle is simply the normal bundle, so the direct sum with the vertical tangent bundle $T_v E$ is trivial. Thus we write $-T_v E$ for the bundle diffeomorphic to U. A Pontrjagin–Thom collapse map gives the *pretransfer*

$$\mathrm{prt}\colon \Sigma^n(X_+) \longrightarrow U^+ \cong \mathrm{Th}(-T_v E).$$

The inclusion map $-T_v E \longrightarrow T_v E \oplus -T_v E$ is proper, so on one-point compactification gives the *inclusion*

$$\mathrm{inc}\colon \mathrm{Th}(-T_v E) \longrightarrow \Sigma^n(E_+).$$

The Becker–Gottlieb transfer is the composition inc $\circ$ prt: $\Sigma^n(X_+) \longrightarrow \Sigma^n(E_+)$, or its extension to suspension spectra. If the base X is not compact but admits an exhaustion by compact subspaces, then one has a similar definition. In this case one only obtains a map on suspension spectra.

We gather below some results on the Becker–Gottlieb transfer.

Lemma 2.2 [1; 3, Equation (2.3), page 137] *Let p be a fibre bundle as above. The composition of the induced maps in cohomology $\mathrm{trf}_p^* \circ p^*$ is multiplication by $\chi(F)$, the Euler characteristic of the fibre. Furthermore, if $q\colon \widetilde{E} \longrightarrow E$ is another smooth proper fibre bundle, then $p \circ q\colon \widetilde{E} \longrightarrow X$ is also a smooth proper fibre bundle and $\mathrm{trf}_{p\circ q}$ and $\mathrm{trf}_p \circ \mathrm{trf}_q$ are homotopic.*

If E is oriented then the effect of the transfer on cohomology is given by

$$\mathrm{trf}_p^*(x) = p_!(x \cdot e(T_v E)) \tag{2–3}$$

where $p_!$ is the Gysin or fibre-integration map corresponding to p, and $e(T_v E)$ is the Euler class of the vertical tangent bundle.

2.4 Universal definition of characteristic classes

If $c \in H^i(BO_2; R)$ is a characteristic class of unoriented 2–plane bundles we can define a characteristic class of unoriented surface bundles: let $E \xrightarrow{\pi} B$ be a surface bundle then $\nu_c(E) := \mathrm{trf}_\pi^*(c(T_v E)) \in H^i(B; R)$ is a characteristic class, by the naturality properties of the transfer. Classes that arise in this manner have a good interpretation in terms of the spectrum $\mathbf{MTO}(2)$.

Definition 2.3 Define a cohomology class $\overline{\nu}_c$ in $\Omega_0^\infty\mathbf{MTO}(2)$ by the image of c under

$$H^*(BO_2; R) \xrightarrow{\sigma^*} H^*(Q_0(BO_{2+}); R) \xrightarrow{\omega^*} H^*(\Omega_0^\infty\mathbf{MTO}(2); R).$$

Theorem 2.4 *If $\alpha_E\colon B \longrightarrow \Omega_0^\infty\mathbf{MTO}(2)$ is the Madsen–Tillmann map for a surface bundle $E \xrightarrow{\pi} B$ then $\alpha_E^*(\overline{\nu}_c) = \nu_c(E)$. Thus the $\overline{\nu}_c$ are universal characteristic classes, in the sense that they do not depend on the genus of the fibre of π.*

Proof Let $T\colon E \to BO_2$ classify $T_v E$, and $\eta\colon \mathrm{Th}(-T_v E) \longrightarrow \mathbf{MTO}(2)$ be given by the maps classifying each vertical normal bundle. Then the diagram below commutes.

$$\begin{array}{ccc}
\Sigma^\infty E_+ & \xrightarrow{\Sigma^\infty T} & \Sigma^\infty BO_{2+} \\
{\scriptstyle \mathrm{inc}}\uparrow & & \uparrow{\scriptstyle \omega} \\
\mathrm{Th}(-T_v E) & \xrightarrow{\eta} & \mathbf{MTO}(2) \\
{\scriptstyle \mathrm{prt}}\uparrow & \overset{\alpha^\sharp}{\nearrow} & \\
\Sigma^\infty B_+ & &
\end{array}$$

By definition $\nu_c = \mathrm{trf}^*(c(T_v E))$, and so it is also given by $(\alpha^\sharp)^*\omega^*(c)$. We wish to identify ν_c via maps of spaces. Note that the adjoint of $\alpha^\sharp$ is the map α_E. The following diagram commutes by the naturality of the cohomology suspension.

$$\begin{array}{ccc}
H^*_{\mathrm{spec}}(\Sigma^\infty B_+) & = & H^*(B) \\
{\scriptstyle (\alpha^\sharp)^*}\uparrow & & \uparrow{\scriptstyle \alpha_E^*} \\
H^*_{\mathrm{spec}}(\mathbf{MTO}(2)) & \xrightarrow{\sigma^*} & H^*(\Omega^\infty \mathbf{MTO}(2)) \\
{\scriptstyle \omega}\uparrow & & \uparrow{\scriptstyle \Omega^\infty \omega^*} \\
H^*_{\mathrm{spec}}(\Sigma^\infty BO_{2+}) & \xrightarrow{\sigma^*} & H^*(Q(BO_{2+}))
\end{array}$$

Thus $\nu_c = \alpha_E^*(\Omega^\infty\omega)^*\sigma^*(c) = \alpha_E^*(\overline{\nu}_c)$. □

Example 2.5 The powers of the first Pontrjagin class $p_1^i \in H^{4i}(BO_2;\mathbb{Z})$ give the characteristic classes ζ_i defined by Wahl [16].

Example 2.6 The Stiefel–Whitney classes $w_1^i w_2^j \in H^{i+2j}(BO_2;\mathbb{F}_2)$ define characteristic classes $\mu_{i,j}$. Even the classes $\mu_{i,0}$ are difficult to analyse. They satisfy relations $\mu_{i,0}^2 = \mu_{2i,0}$, as the cohomology suspension σ^* commutes with Steenrod squares, but the algebraic dependence between the odd classes $\mu_{2i+1,0}$ is not known.

Example 2.7 The same result is true in the oriented case, with BO_2 replaced by BSO_2. Here the powers of the first Chern class $c_1^i \in H^{2i}(BSO_2;\mathbb{Z})$ define integral characteristic classes κ_i which in the oriented mapping class groups give the Miller–Morita–Mumford classes.

2.5 Hopf algebras

We will make much use of the structure of Hopf algebras over a field $\mathbb{F}$. Let us fix notation, which coincides with that of the classic reference on Hopf algebras of Milnor

and Moore [14]. Recall that an *abelian* Hopf algebra is one that is commutative and cocommutative. All Hopf algebras appearing in this paper are homology groups of infinite loop spaces and as such are abelian. We write $\psi\colon X \longrightarrow X \otimes X$ for the coproduct of a coalgebra X, and $\psi_Y\colon Y \longrightarrow X \otimes Y$ for the coaction of the coalgebra X on a right comodule Y.

For a coalgebra A, a left A–comodule B and a right A–comodule C, the *cotensor product* $B\square_A C$ is defined as the kernel of

$$B \otimes C \xrightarrow{\psi_B \otimes \mathrm{id}_C - \mathrm{id}_B \otimes \psi_C} B \otimes A \otimes C.$$

In general $B\square_A C$ is just a vector space, but if A is cocommutative then it is again an A–comodule.

The Hopf algebra *kernel* and *cokernel* of a map $f\colon A \longrightarrow B$ of Hopf algebras are $A\backslash\backslash f := A\square_B \mathbb{F}$ and $B/\!/f := B \otimes_A \mathbb{F}$ respectively. When A and B are abelian, kernels and cokernels are again (abelian) Hopf algebras. We will write PA and QA for the vector spaces of primitive and indecomposable elements respectively.

We will make use of two corollaries of Borel's structure theorem for finite-type abelian Hopf algebras over a perfect field [14, Theorem 7.11].

Lemma 2.8 *Firstly, a Hopf subalgebra of a polynomial algebra is again polynomial. Secondly, if $f\colon A \longrightarrow B$ is an injective homomorphism of Hopf algebras, then B is a free A–module. Dually, if f is surjective then A is a free B–comodule.*

2.6 The Eilenberg–Moore spectral sequence

In this section all homology is with coefficients in some field $\mathbb{F}$. For a fibration sequence $F \longrightarrow E \xrightarrow{\pi} B$ of connected spaces, the Eilenberg–Moore spectral sequence is

$$E^2_{p,q} = \mathrm{Cotor}^{H_*(B)}_{p,q}(H_*(E), \mathbb{F}) \Rightarrow H_{p+q}(F)$$

where the subscript on Cotor denotes the degree q part of the p–th derived functor of $-\square_{H_*(B)}\mathbb{F}$. The $H_*(B)$–comodule structure on $H_*(E)$ is induced by the map π_*. The usual requirement for convergence is that B be simply connected, but we will use the exotic convergence of [5]: base spaces will have $\mathbb{Z}/2$ as fundamental group, which always acts nilpotently on any finitely-generated $\mathbb{F}_2$–homology group.

When the fibration is one of connected components of infinite loop spaces,

$$\Omega^\infty_0\mathbf{F} \longrightarrow \Omega^\infty_0\mathbf{E} \xrightarrow{\pi} \Omega^\infty_0\mathbf{B}$$

coming from a cofibre sequence of finite-type spectra, all the $\mathbb{F}$–homology groups are finite-type abelian Hopf algebras over $\mathbb{F}$. Suppose that some condition guaranteeing the convergence of the Eilenberg–Moore spectral sequence holds. If $\pi_*\colon H_*(\Omega^\infty_0\mathbf{E};\mathbb{F}) \longrightarrow H_*(\Omega^\infty_0\mathbf{B};\mathbb{F})$ is surjective then Lemma 2.8 implies that $H_*(\Omega^\infty_0\mathbf{E};\mathbb{F})$ is a free $H_*(\Omega^\infty_0\mathbf{B};\mathbb{F})$–comodule. Thus the E^2–term of the Eilenberg–Moore spectral sequence is

$$\begin{aligned}\operatorname{Cotor}^{H_*(\Omega^\infty_0\mathbf{B};\mathbb{F})}_{p,q}(H_*(\Omega^\infty_0\mathbf{E};\mathbb{F}),\mathbb{F}) &= H_*(\Omega^\infty_0\mathbf{E};\mathbb{F})\square_{H_*(\Omega^\infty_0\mathbf{B};\mathbb{F})}\mathbb{F}\\ &= H_*(\Omega^\infty_0\mathbf{E};\mathbb{F})\backslash\!\backslash\pi_*\end{aligned}$$

concentrated in the line $p=0$, so the spectral sequence collapses. Thus

$$\mathbb{F} \longrightarrow H_*(\Omega^\infty_0\mathbf{F};\mathbb{F}) \longrightarrow H_*(\Omega^\infty_0\mathbf{E};\mathbb{F}) \xrightarrow{\pi_*} H_*(\Omega^\infty_0\mathbf{B};\mathbb{F}) \longrightarrow \mathbb{F}$$

is a short exact sequence of Hopf algebras.

2.7 Araki–Kudo operations and the homology of $Q(Y_+)$

An infinite loop space X has a rich structure on its $\mathbb{F}_2$–homology. Firstly the H-space structure gives the homology the structure of a Hopf algebra, with the Pontrjagin product denoted $*$. Secondly there are Araki–Kudo operations [10]: these are a family of homomorphisms

$$Q^s\colon H_i(X;\mathbb{F}_2) \longrightarrow H_{i+s}(X;\mathbb{F}_2)$$

that are natural for maps of infinite loop spaces. These have formal properties similar to the Steenrod squares: a Cartan formula and Adém relations. The reference is Cohen, Lada and May [4], and we will use many properties of these operations. We recall some elementary notions. The sequence $I=(s_1,\dots,s_k)$ is called *admissible* if $s_i \le 2s_{i+1}$ for all i. The *excess* of such a sequence is the integer $e(I) := s_1 - \sum_{i=2}^k s_i$. The *length* of I is $l(I) := k$. The *degree* of I is $d(I) := \sum_i s_i$. We write $Q^I = Q^{s_1}\cdots Q^{s_k}$. We write $\mathcal{R}$ for the algebra of all operations, and call it the *Dyer–Lashof algebra*.

For a space Y, the $\mathbb{F}_2$–homology of $Q(Y_+)$ has a (non functorial) description as follows. Let $\mathcal{B}$ be a homogenous basis for $H_*(Y;\mathbb{F}_2)$. Then the $\mathbb{F}_2$–homology of $Q(Y_+)$ is the polynomial algebra on the set

$$\{Q^I(x) \mid x\in\mathcal{B},\ I \text{ admissible},\ e(I) > |x|\}.$$

If the space Y is connected, $\pi_0(Q(Y_+)) \cong \mathbb{Z}$ and we write $[n]$ for the image of $n\in\pi_0$ in $H_0(Q(Y_+))$. The $\mathbb{F}_2$–homology of the 0 component $Q_0(Y_+)$ is then the polynomial algebra on the set

$$\{Q^I(x) * [-2^{l(I)}] \mid x\in\mathcal{B},\ I \text{ admissible},\ e(I) > |x|, |Q^I(x)| > 0\}.$$

3 Mod 2 homology of Ω_0^∞MTO(1)

It is clear that the map

$$\overline{\partial}\colon Q_0(BO_{1+}) \longrightarrow Q_0(S^0)$$

is the transfer for the double covering $EO_1 \longrightarrow BO_1$, $t\colon Q(BO_{1+}) \longrightarrow Q(S^0)$. The effect of this map on homology has been calculated by Mann, Miller and Miller [13]:

Lemma 3.1 *Let $e_i \in H_i(BO_1;\mathbb{F}_2)$ be the unique nontrivial class. Then $t_*(e_i) = Q^i([1])$.*

Corollary 3.2 *Thus the map $\overline{\partial}\colon Q_0(BO_{1+}) \longrightarrow Q_0(S^0)$ on homology is*

$$\overline{\partial}_*(e_i * [-1]) = Q^i([1]) * [-2]$$

so is surjective, as these elements generate $H_*(Q_0(S^0))$ over the Dyer–Lashof algebra.

Proof of Theorem A The Eilenberg–Moore spectral sequence for the fibration

$$\Omega_0^\infty\mathbf{MTO}(1) \longrightarrow Q_0(BO_{1+}) \xrightarrow{\overline{\partial}} Q_0(S^0)$$

converges to $H_*(\Omega_0^\infty\mathbf{MTO}(1))$ as the base space has fundamental group $\mathbb{Z}/2$. By Corollary 3.2, $\overline{\partial}_*$ is surjective, so by the discussion in Section 2.6 we obtain the short exact sequence of Hopf algebras.

Finally, $H_*(Q_0(S^0))$ is a free algebra so the sequence splits as algebras. □

Remark 3.3 If I is some sequence, Q^I is an Araki–Kudo operation and one can iteratively apply Adém relations to write it as a linear combination of Q^J with J admissible. We write $Q^I = \sum_J \lambda^I_J Q^J$ where the λ are the coefficients needed to express Q^I as admissible monomials. We adopt the convention that we sum over all J, and that $\lambda^I_J = 0$ if J is not admissible. Adém relations decrease excess, so $\lambda^I_J = 0$ if $e(J) > e(I)$ also.

For a sequence I and an integer i such that I is admissible, $e(I) > i$ and (I,i) is not admissible, there is an element $v^{I,i}$ in $QH_*(Q_0(BO_{1+}))$ given by

$$v^{I,i} := Q^I(e_i) * [-2^{1+l(I)}] + \sum_{J,j} \lambda^{I,i}_{J,j} Q^J(e_j) * [-2^{1+l(J)}]$$

where the sum is over all sequences (J,j). This clearly lies in the kernel of $Q(\overline{\partial}_*)$. We adopt the convention that $v^{I,i}$ is also defined for (I,i) admissible, and is 0. This is the natural extension, as $Q^I Q^i$ is already written in terms of admissibles.

Theorem 3.4 *The Hopf algebra $H_*(\Omega^\infty_0\mathbf{MTO}(1);\mathbb{F}_2)$ is polynomial on a set*

$$\{V^{I,i} \mid I \text{ admissible},\ e(I) > i,\ (I,i) \text{ not admissible}\}$$

where $\overline{\omega}_*(V^{I,i}) = v^{I,i}$ modulo decomposables.

Proof It is now clear that $H_*(\Omega^\infty_0\mathbf{MTO}(1))$ is a polynomial algebra, as for Hopf algebras over $\mathbb{F}_2$ a Hopf subalgebra of a polynomial algebra is again polynomial. Furthermore as the sequence splits as algebras there is an exact sequence of indecomposable quotients

$$0 \longrightarrow QH_*(\Omega^\infty_0\mathbf{MTO}(1)) \xrightarrow{Q(\overline{\omega}_*)} QH_*(Q_0(BO_{1+})) \xrightarrow{Q(\overline{\partial}_*)} QH_*(Q_0(S^0)) \longrightarrow 0.$$

The elements $v^{I,i}$ described earlier are indecomposable and lie in the kernel of $Q(\overline{\partial}_*)$. Thus there are polynomial generators $V^{I,i}$ such that $Q(\overline{\omega}_*)(V^{I,i}) = v^{I,i}$. The theorem follows from the following Lemma. □

Lemma 3.5 *The $v^{I,i}$ form an additive basis of* $\mathrm{Ker}(Q(\overline{\partial}_*))$.

Proof Let $v = \sum \mu_{I,i} Q^I(e_i) * [-2^{1+l(I)}]$ be an element in the kernel of $Q(\overline{\partial}_*)$. Then

$$\begin{aligned} 0 = \overline{\partial}_*(v) &= \sum_{I,i} \mu_{I,i} Q^I Q^i([1]) * [-2^{1+l(I)}] \\ &= \sum_{I,i} \mu_{I,i} \sum_{J,j} \lambda^{I,i}_{J,j} Q^J Q^j([1]) * [-2^{1+l(J)}] \\ &= \sum_{J,j} \left(\sum_{I,i} \mu_{I,i} \lambda^{I,i}_{J,j} \right) Q^J Q^j([1]) * [-2^{1+l(J)}]. \end{aligned}$$

so $\sum_{I,i} \mu_{I,i} \lambda^{I,i}_{J,j} = 0$ for each (J, j) admissible and of positive excess. Now consider

$$\begin{aligned} v - \sum_{I,i} \mu_{I,i} v^{I,i} &= \sum_{I,i} \mu_{I,i} \left(\sum_{J,j} \lambda^{I,i}_{J,j} Q^J(e_j) \right) * [-2^{1+l(J)}] \\ &= \sum_{J,j} \left(\sum_{I,i} \mu_{I,i} \lambda^{I,i}_{J,j} \right) Q^J(e_j) * [-2^{1+l(J)}] \\ &= 0. \end{aligned}$$

So the $v^{I,i}$ span the kernel of the map $Q(\overline{\partial}_*)$.

Now let $\sum_{I,i}\mu_{I,i}v^{I,i}=0$, so

$$\begin{aligned}\sum\mu_{I,i}Q^I(e_i)*[-2^{1+l(I)}] &= \sum_{I,i}\mu_{I,i}\sum_{J,j}\lambda^{I,i}_{J,j}Q^J(e_j)*[-2^{1+l(J)}]\\ &= \sum_{J,j}\Big(\sum_{I,i}\mu_{I,i}\lambda^{I,i}_{J,j}\Big)Q^J(e_j)*[-2^{1+l(J)}].\end{aligned}$$

If (I,i) is not admissible then $Q^I(e_i)*[-2^{1+l(I)}]$ does not appear on the right hand side, so $\mu_{I,i}=0$. So all the coefficients are 0 and the $v^{I,i}$ are linearly independent. □

It will also be important later to understand the action of the Araki–Kudo operations on the polynomial generators $v^{I,i}$ and $V^{I,i}$.

Theorem 3.6 *If (l,I) is an admissible sequence then in $QH_*(Q_0(BO_{2+}))$*

$$Q^l(v^{I,i})=v^{(l,I),i}+\sum_{j,J,J'}\lambda^{I,i}_{J,j}\lambda^{l,J}_{J'}v^{J',j}$$

and the same formula holds for $V^{I,i}$ in $QH_*(\Omega^\infty_0\mathbf{MTO}(1))$. Furthermore, if $\lambda^{I,i}_{J,j}\neq 0$ then $j>i$.

Proof Firstly we analyse the effect of Q^l on $v^{I,i}*[2^{1+l(I)}]$.

$$\begin{aligned}Q^l(v^{I,i}*[2^{1+l(I)}]) &= Q^lQ^I(e_i)+\sum_{J,j}\lambda^{I,i}_{J,j}Q^lQ^J(e_j)\\ &= Q^{(l,I)}(e_i)+\sum_{J,j}\lambda^{I,i}_{J,j}\sum_{J'}\lambda^{l,J}_{J'}Q^{J'}(e_j).\end{aligned}$$

Note that this element is in the kernel of $\overline{\partial}_*$, as $v^{I,i}$ is and $\overline{\partial}_*$ commutes with Araki–Kudo operations and the Pontrjagin product. Thus

$$\begin{aligned}0 &= Q^{(l,I)}Q^i([1])+\sum_{J',J,j}\lambda^{I,i}_{J,j}\lambda^{l,J}_{J'}Q^{J'}Q^j([1])\\ &= Q^{(l,I,i)}([1])+\sum_{J'',J',J,j}\lambda^{I,i}_{J,j}\lambda^{l,J}_{J'}\lambda^{J',j}_{J''}Q^{J''}([1])\end{aligned}$$

and the terms in the summation are all admissible. Thus this is the unique way to express $Q^{(l,I,i)}$ in terms of admissible operations, so

$$\lambda^{l,I,i}_{J''}=\sum_{J',J,j}\lambda^{I,i}_{J,j}\lambda^{l,J}_{J'}\lambda^{J',j}_{J''}.$$

Secondly we analyse the expression for $v^{(l,I),i} * [2^{2+l(I)}]$.

$$\begin{aligned}
v^{(l,I),i} * [2^{2+l(I)}] &= Q^{(l,I)}(e_i) + \sum_{J''',j'} \lambda^{l,I,i}_{J''',j'} Q^{J'''}(e_{j'}) \\
&= Q^{(l,I)}(e_i) + \sum_{J''',j'} \left(\sum_{J',J,j} \lambda^{I,i}_{J,j} \lambda^{l,J}_{J'} \lambda^{J',j}_{J''',j'} \right) Q^{J'''}(e_{j'}) \\
&= Q^{(l,I)}(e_i) + \sum_{J',J,j} \lambda^{I,i}_{J,j} \lambda^{l,J}_{J'} \left(\sum_{J''',j'} \lambda^{J',j}_{J''',j'} Q^{J'''}(e_{j'}) \right) \\
&= Q^{(l,I)}(e_i) + \sum_{J',J,j} \lambda^{I,i}_{J,j} \lambda^{l,J}_{J'} \left(v^{J',j} * [2^{1+l(J')}] + Q^{J'}(e_j) \right) \\
&= Q^{(l,I)}(e_i) + \sum_{J,j} \lambda^{I,i}_{J,j} Q^{(l,J)}(e_j) \\
&\qquad + \sum_{J',J,j} \lambda^{I,i}_{J,j} \lambda^{l,J}_{J'} v^{J',j} * [2^{1+l(J')}] \\
&= Q^{l}(v^{I,i} * [2^{1+l(I)}]) + \sum_{J',J,j} \lambda^{I,i}_{J,j} \lambda^{l,J}_{J'} v^{J',j} * [2^{1+l(J')}].
\end{aligned}$$

Then $Q^l(v^{I,i} * [2^{1+l(I)}]) = Q^l(v^{I,i}) * [2^{2+l(I)}]$ modulo decomposables, so the result follows after translating back to the 0 component, as $l(J') = l(I) + 1$.

Finally, as (I,i) is not admissible but I is, to write $Q^{(I,i)}$ as admissibles an Adém relation must be applied involving the last term Q^i. Applying an Adém relation to $Q^a Q^b$ gives monomials $Q^A Q^B$ with B strictly larger than b, so if $\lambda^{I,i}_{J,j} \neq 0$ then $j > i$. □

4 The map ∂ on homology

We wish to study the composition

$$Q(BO_{2+}) \xrightarrow{\partial} \Omega^\infty \mathbf{MTO}(1) \xrightarrow{\overline{\omega}} Q(BO_{1+})$$

and as ∂ is the pretransfer for the circle bundle $S^1 \longrightarrow BO_1 \longrightarrow BO_2$, it then follows that $\overline{\omega} \circ \partial$ is the transfer for this bundle.

Remark 4.1 In general there are fibre bundles $S^{n-1} \longrightarrow \mathbb{S}(\gamma_n) \xrightarrow{\pi} BO_n$ and a map $f\colon \mathbb{S}(\gamma_n) \longrightarrow BO_{n-1}$ classifying the vertical tangent bundle. Then the composition

$$Q(BO_{n+}) \xrightarrow{\mathrm{trf}_\pi} Q(\mathbb{S}(\gamma_n)_+) \xrightarrow{f} Q(BO_{n-1+})$$

coincides with the composition

$$Q(BO_{n+}) \xrightarrow{\partial} \Omega^\infty \mathbf{MTO}(n-1) \xrightarrow{\omega} Q(BO_{n-1+})$$

of maps occurring in the analogs of the fibrations (2–1) and (2–2).

To evaluate this transfer we will use the technique of Brumfiel and Madsen [3] of reducing it to a transfer for a finite-sheeted cover. Thus we must find a nondegenerate vector field on the fibre that is equivariant for the action of the structure group. We can not do this directly as S^1 does not admit any O_2–invariant nondegenerate vector fields, so instead we consider the pullback bundle via $d\colon BO_1 \times BO_1 \longrightarrow BO_2$, the map classifying $\gamma_1 \times \gamma_1$,

$$S^1 \longrightarrow \mathbb{S}(\gamma_1 \times \gamma_1) \longrightarrow BO_1 \times BO_1.$$

There is a nondegenerate $O_1 \times O_1$–invariant vector field on S^1. It has 4 singular points split into two orbits of two each, having opposite indices. Thus the singular locus $\Sigma \subset \mathbb{S}(\gamma_1 \times \gamma_1)$ is simply $EO_1 \times BO_1 \coprod BO_1 \times EO_1$. We can thus compute the transfer for this bundle in terms of the transfer map t of Section 3.

Proposition 4.2 *The composition $BO_1 \times BO_1 \xrightarrow{d} BO_2 \xrightarrow{T} Q(BO_{1+})$ is homotopic to*

$$\begin{aligned} BO_1 \times BO_1 &\xrightarrow{\Delta} BO_1 \times BO_1 \times BO_1 \times BO_1 \\ &\xrightarrow{(t\wedge\mathrm{id})\times(\mathrm{id}\wedge t)} Q_2(EO_1 \times BO_{1+}) \times Q_2(BO_1 \times EO_{1+}) \\ &\xrightarrow{\mathrm{id}\times\chi} Q_2(BO_{1+}) \times Q_{-2}(BO_{1+}) \xrightarrow{\mu} Q_0(BO_{1+}). \end{aligned}$$

Proof This is an elementary application of Theorem 2.10 of [3]. We are aware that there has been a correction to this paper in [11], but our indices are very simple and there is no difference. □

We can now effectively identify the map ∂_*. The map d_* is surjective and we proved in the last section that $\bar{\omega}_*$ is injective, so the map $T_* \circ d_*$ determines ∂_*.

Theorem 4.3 *The composition*

$$H_*(BO_1 \times BO_1) \xrightarrow{d_*} H_*(BO_2) \longrightarrow H_*(Q_1(BO_{2+})) \xrightarrow{\bar{\omega}_*\partial_*} H_*(Q_0(BO_{1+}))$$

is given by

$$e_i \otimes e_j \mapsto \sum_{a=0}^{i}\sum_{b=0}^{j}\sum_{s=0}^{b}\sum_{t=0}^{a}\binom{b-s}{s}\binom{a-t}{t}Q^{i-a+s}(e_{b-s}) * \chi\left(Q^{j-b+t}(e_{a-t})\right)$$

which modulo decomposables is

$$\sum_{s=0}^{j}\binom{j-s}{s}Q^{i+s}(e_{j-s}) * [-2] + \sum_{t=0}^{i}\binom{i-t}{t}Q^{j+t}(e_{i-t}) * [-2].$$

Proof Applying the description of the composition from the previous proposition gives

$$e_i \otimes e_j \mapsto \sum_{a=0}^{i}\sum_{b=0}^{j}\left(Q^{i-a}([1]) \wedge e_b\right) * \chi\left(e_a \wedge Q^{j-b}([1])\right).$$

We then use the following formulae from [4] for evaluating the smash product on homology:

$$(a * b) \wedge c = \sum (a \wedge c') * (b \wedge c'') \text{ when } \Delta_*(c) = \sum c' \otimes c''$$
$$Q^i(x) \wedge y = \sum Q^{i+t}(x \wedge Sq^t_*(y)).$$

So $$Q^{i-a}([1]) \wedge e_b = \sum_{s=0}^{b} Q^{i-a+s}([1] \wedge Sq^s_*(e_b))$$

which can be expressed as

$$\sum_{s=0}^{b}\binom{b-s}{s}Q^{i-a+s}(e_{b-s}),$$

as $Sq^s_*(e_b) = \binom{b-s}{s}e_{b-s}$. This gives the first expression. The expression modulo decomposables follows immediately, using the fact that $\chi(e_i) = e_i * [-2]$ modulo decomposables. □

Corollary 4.4 *The formula in Theorem 4.3 for the indecomposable part of the image of $e_i \otimes e_j$ lies in the kernel of $Q(\overline{\partial}_*)$ and so can be expressed as a linear combination of the $v^{I,i}$. This expression is simply*

$$\sum_{s=0}^{j}\binom{j-s}{s}v^{i+s,j-s} + \sum_{t=0}^{i}\binom{i-t}{t}v^{j+t,i-t}.$$

Thus $e_0 \otimes e_i$ maps to $v^{i,0}$ modulo decomposables.

Proof Similarly to the proof of Lemma 3.5, consider

$$Q(\partial_*\overline{\omega}_*d_*)(e_i\otimes e_j)+\sum_{s=0}^{j}\binom{j-s}{s}v^{i+s,j-s}+\sum_{t=0}^{i}\binom{i-t}{t}v^{j+t,i-t}$$

which by the definition of the $v^{I,i}$ can be written as

$$\sum_{s=0}^{j}\binom{j-s}{s}\sum_{a,b}\lambda_{a,b}^{i+s,j-s}Q^a(e_b)*[-2]+\sum_{t=0}^{i}\binom{i-t}{t}\sum_{a,b}\lambda_{a,b}^{j+t,i-t}Q^a(e_b)*[-2]$$

so the coefficient of $Q^a(e_b)*[-2]$ is

$$\sum_{s=0}^{j}\binom{j-s}{s}\lambda_{a,b}^{i+s,j-s}+\sum_{t=0}^{i}\binom{i-t}{t}\lambda_{a,b}^{j+t,i-t}.$$

This element lies in the kernel of $\overline{\partial}_*$, and the $Q^a(e_b)$ are sent to $Q^{(a,b)}([1])*[-4]$ where (a,b) is admissible. These are linearly independent in the homology of $Q_0(S^0)$ and so all the coefficients are 0. Thus the expression above is identically 0, and the result follows. □

Proposition 4.5 *The map $Q(\partial_*)\colon QH_*(Q_0(BO_{2+}))\longrightarrow QH_*(\Omega^\infty_0\mathbf{MTO}(1))$ is surjective.*

Proof Define an increasing filtration $G^i:=\langle\, V^{a,b}\mid b\le i\,\rangle$ of $G^\infty:=\langle\, V^{a,b}\,\rangle\subset QH_*(\Omega^\infty_0\mathbf{MTO}(1))$. The previous corollary implies that G^0 is in the image of $Q(\partial_*)$. Consider the indecomposable element $V^{i-a,a}$, so $i-a>2a$, and $a<i/3$. Then

$$Q(\partial_*d_*)(e_a\otimes e_{i-a})=\sum_{s=0}^{i-a}\binom{i-a-s}{s}V^{a+s,i-a-s}+\sum_{t=0}^{a}\binom{a-t}{t}V^{i-a+t,a-t}.$$

Either $2s+a>i$, so $s>i-a-s$ and the binomial coefficient in the first sum is 0, or $2s+a\le i$. This together with $a<i/3$ implies that $a+s\le 2(i-a-s)$, so $(a+s,i-a-s)$ is admissible and $V^{a+s,i-a-s}$ is 0. Thus the first sum is 0. The second sum is

$$V^{i-a,a}+\sum_{t=1}^{a}\binom{a-t}{t}V^{i-a+t,a-t}$$

so is $V^{i-a,a}$ modulo G^{a-1}. Thus by induction along the filtration G^*, $G^\infty=\langle V^{a,b}\rangle$ is in the image of $Q(\partial_*)$. The map ∂ is an infinite loop map, so commutes with Araki–Kudo operations: thus $\mathcal{R}\cdot G^\infty$ is also in the image.

Now we introduce a new filtration $F^i := \langle\, V^{I,j} \mid j \geq i \,\rangle$ of $QH_*(\Omega^\infty_0\mathbf{MTO}(1))$. This filtration is decreasing, with F^0 the whole vector space, but it has the property that in any fixed degree n the filtration F^*_n is has finite length. In particular

$$F^n_n := \langle\, V^{I,j} \mid j \geq n,\, d(I) + j = n \,\rangle = \{0\}$$

as $d(I) > 0$. We will proceed by induction up the filtration in each degree. By Theorem 3.6, for (l, I) admissible,

$$Q^l(V^{I,i}) = V^{(l,I),i} + \sum_{j,J,J'} \lambda^{I,i}_{J,j}\lambda^{l,J}_{J'}\, V^{J',j}$$

and for each term in the sum $j > i$. In particular the sum lies in F^{i+1}, so $Q^l(V^{I,i}) = V^{(l,I),i}$ modulo F^{i+1}. Thus by iterating,

$$V^{(I,k),i} = Q^I(V^{k,i}) \text{ modulo } F^{i+1}$$

and so $F^i \subseteq \mathcal{R}\cdot G^\infty + F^{i+1}$. In degree n, $F^n_n = \{0\}$ is in the image of $Q(\partial_*)$, so by this inclusion F^{n-1}_n is too, and so on: thus F^0_n is in the image. This holds in all degrees n so F^0 is in the image, but this is the entire space $QH_*(\Omega^\infty_0\mathbf{MTO}(1))$. □

5 The homology of $\Omega^\infty_0\mathbf{MTO}(2)$

Proof of Theorem B We now study the Eilenberg–Moore spectral sequence for the fibration

$$\Omega^\infty_0\mathbf{MTO}(2) \longrightarrow Q_0(BO_{2+}) \xrightarrow{\partial} \Omega^\infty_0\mathbf{MTO}(1)$$

noting that $\pi_1(\Omega^\infty_0\mathbf{MTO}(1)) \cong \mathbb{Z}/2$ and so by the discussion in Section 2.6 the spectral sequence converges to $H_*(\Omega^\infty_0\mathbf{MTO}(2))$. By Proposition 4.5, ∂_* is surjective, so by the discussion in Section 2.6 we obtain the short exact sequence of Hopf algebras.

Finally, $H_*(\Omega^\infty_0\mathbf{MTO}(1))$ is a polynomial algebra, so free, so the sequence is split as algebras. □

Corollary 5.1 *It now follows that $H_*(\Omega^\infty_0\mathbf{MTO}(1);\mathbb{F}_2)$ is also dual to a polynomial algebra, as $H^*(\Omega^\infty_0\mathbf{MTO}(1);\mathbb{F}_2)$ injects into $H^*(Q_0(BO_{2+});\mathbb{F}_2)$, which is polynomial.*

5.1 Remarks on integral (co)homology

Write $X[\frac{1}{2}]$ for the $\mathbb{Z}[\frac{1}{2}]$–localisation in spaces or spectra. The spectrum $\mathbf{MTO}(1)[\frac{1}{2}]$ is contractible (this follows immediately from Corollary 6.3 of [16]), so

$$\mathbf{MTO}(2)[\tfrac{1}{2}] \xrightarrow{\simeq} \Sigma^\infty BO_{2+}[\tfrac{1}{2}]$$

is an equivalence. There is a zig-zag

$$BO_2 \longrightarrow BSO_3 \longleftarrow BSU_2 = \mathbb{HP}^\infty$$

of maps that induce homology equivalences with $\mathbb{Z}[\frac{1}{2}]$–coefficients, so a zig-zag of homotopy-equivalences after $\mathbb{Z}[\frac{1}{2}]$–localisation. Thus

$$\Omega^\infty\mathbf{MTO}(2)[\tfrac{1}{2}] \simeq Q(\mathbb{HP}^\infty_+)[\tfrac{1}{2}]$$

and on homology $H_*(\Omega^\infty_0\mathbf{MTO}(2);\mathbb{Z}[\frac{1}{2}]) \cong H_*(Q_0(\mathbb{HP}^\infty_+);\mathbb{Z}[\frac{1}{2}])$ as Hopf algebras. The homology of $Q_0(\mathbb{H}\,\mathbb{P}^\infty_+)$ can be completely calculated integrally by piecing together its $\mathbb{F}_p$ homologies, as its Bockstein spectral sequence is a functor of that of $\mathbb{HP}^\infty$. This determines the integral homology of $\Omega^\infty_0\mathbf{MTO}(2)$ except for its 2-torsion. By Theorem B the map $\Omega^\infty_0\mathbf{MTO}(2) \longrightarrow Q_0(BO_{2+})$ gives an injection on the E_1 pages of their Bockstein spectral sequences, but it is not clear what happens on subsequent pages.

In the range $* \leq 6$ only the primes 2 and 3 contribute. The Bockstein spectral sequence for $Q_0(\mathbb{H}\,\mathbb{P}^\infty_+)$ at the prime 3 collapses at the E_2 page, so the only odd primary contribution is a $\mathbb{Z}/3$ in degree 3. In this range one can deduce the Bockstein spectral sequence for $\Omega^\infty_0\mathbf{MTO}(2)$ at the prime 2, as the map above still gives an injection on the E_2 page, and it collapses at the E_3 page. The only unusual contribution is a $\mathbb{Z}/4$ in degree 3.

6 A polynomial family in the mod 2 cohomology of the stable nonorientable mapping class group

Let $F \longrightarrow E \xrightarrow{\pi} B$ be a bundle of surfaces. We can define mod 2 characteristic classes as follows. There is an associated first real cohomology bundle V, with fibre $H^1(F_b;\mathbb{R})$ over the point $b \in B$. Define

$$\chi_i(E) := w_i(V) \in H^i(B;\mathbb{F}_2)$$

the i–th Stiefel–Whitney class of this vector bundle. The main result is that in the stable nonorientable mapping class group these account for one indecomposable generator in each dimension, and are stably independent.

Theorem 6.1 *In the nonorientable mapping class groups, the map*

$$\mathbb{F}_2[\chi_1, \chi_2, \chi_3, ...] \longrightarrow H^*(B\mathcal{N}_g;\mathbb{F}_2)$$

is an injection in the stable range $* \leq (g-3)/4$.

The theorem of Korkmaz given in Section 2.2 implies that $H^1(B\mathcal{N}_g;\mathbb{F}_2)\cong\mathbb{F}_2$ for $g\geq 7$, so χ_1 is the generator. This gives another interpretation of χ_1: it is the obstruction to reducing the structure group of a bundle of nonorientable surfaces to the index 2 subgroup of the mapping class group generated by Dehn twists.

The proof of the theorem is somewhat indirect. There is a class $1\in KO^0(E)$ representing the trivial 1–dimensional bundle, and applying the Becker–Gottlieb transfer in real K-theory we obtain a virtual bundle $\mathrm{trf}^*_\pi(1)\in KO^0(B)$. An application of the Atiyah–Singer index theorem for families due to Becker and Schultz implies the following theorem, which gives a homotopy-theoretic characterisation of the virtual bundle V.

Theorem 6.2 *Suppose the surface bundle $E\xrightarrow{\pi}B$ is smooth and the fibres are compact, connected and nonorientable. Then the K-theory class $\mathrm{trf}^*_\pi(1)$ coincides with $1-V$ in $KO^0(B)$. In particular $\chi_i(E)=w_i(-\mathrm{trf}^*_\pi(1))$ in $H^i(B;\mathbb{F}_2)$.*

Proof By [2, Theorem 6.1], $\mathrm{trf}^*_\pi(1)=\sum_i(-1)^i[H^i(F_b;\mathbb{R})]$. In our situation this sum is $1-V$ as $[H^0(F_b;\mathbb{R})]$ is the constant rank 1 vector bundle, $[H^1(F_b;\mathbb{R})]$ is the vector bundle V, and the higher terms are 0 as the fibres F_b are nonorientable surfaces. □

The virtual bundle $-\mathrm{trf}^*_\pi(1)$ is classified by

$$B\xrightarrow{\mathrm{trf}}Q(E_+)\xrightarrow{\text{collapse}}Q(S^0)\xrightarrow{Q(i)}\mathbb{Z}\times BO\xrightarrow{\chi}\mathbb{Z}\times BO$$

where $Q(i)$ is the extension to the free infinite loop space of the inclusion i of S^0 to the 0 and 1 components of $\mathbb{Z}\times BO$, and χ is the inversion map on $\mathbb{Z}\times BO$. The discussion in Section 2.4 implies that the above composition is homotopic to

$$B\xrightarrow{\alpha_E}\Omega^\infty\mathbf{MTO}(2)\xrightarrow{\omega}Q(BO_{2+})\xrightarrow{\text{collapse}}Q(S^0)\xrightarrow{Q(i)}\mathbb{Z}\times BO\xrightarrow{\chi}\mathbb{Z}\times BO$$

as the collapse map $E\longrightarrow *$ can be taken to factor through the map $E\longrightarrow BO_2$ classifying the vertical tangent bundle. Define $\overline{\chi}_i\in H^i(\Omega^\infty_0\mathbf{MTO}(2);\mathbb{F}_2)$ to be the pullback of the i–th Stiefel–Whitney class w_i by the composition. We call $\overline{\chi}_i$ the *universal χ_i class*, as by the previous theorem $\alpha_E^*(\overline{\chi}_i)=\chi_i(E)\in H^i(B;\mathbb{F}_2)$ for any smooth, nonorientable surface bundle with compact fibres.

Lemma 6.3 *The composition $Q(i)\circ\mathrm{collapse}\circ\omega\colon\Omega^\infty_0\mathbf{MTO}(2)\longrightarrow BO$ is injective on $\mathbb{F}_2$–cohomology. The same is true after applying the inversion map χ, as it is a homotopy equivalence.*

Proof It is enough to show that the map is surjective on $\mathbb{F}_2$–homology, and to do this it is enough to show that the composition

$$\Omega^\infty\mathbf{MTO}(2) \xrightarrow{\omega} Q(BO_{2+}) \xrightarrow{\text{collapse}} Q(S^0) \xrightarrow{Q(i)} \mathbb{Z}\times BO$$

without restriction to 0 components is surjective on $\mathbb{F}_2$–homology. The action of the Dyer–Lashof algebra on the $\mathbb{F}_2$–homology of $\mathbb{Z}\times BO$ has been computed by Priddy [15], and it is generated over the Dyer–Lashof algebra (and Pontrjagin product) by the class $x_0 \in H_0(\mathbb{Z}\times BO;\mathbb{F}_2)$ representing the component $\{1\}\times BO$. As the composition is a map of infinite loop spaces, it is enough to show that the class x_0 is in the image.

The composition

$$\pi_0(B) \xrightarrow{\pi_0(\alpha_E)} \pi_0(\Omega^\infty\mathbf{MTO}(2)) \cong \mathbb{Z} \longrightarrow \pi_0(Q(S^0)) \cong \mathbb{Z}$$

for a surface bundle $E \xrightarrow{\pi} B$ picks out the Euler characteristic of the fibre. In particular, any bundle with fibre $\mathbb{RP}^2$ lands in the 1 component, and so the top composition is an isomorphism on π_0. In particular, the class x_0 is in the image. □

The lemma implies that

$$\mathbb{F}_2[\overline{\chi}_1, \overline{\chi}_2, \overline{\chi}_3, ...] \longrightarrow H^*(\Omega^\infty_0\mathbf{MTO}(2);\mathbb{F}_2)$$

is injective, because the Stiefel–Whitney classes are algebraically independent in $H^*(BO;\mathbb{F}_2)$. Theorem 6.1 follows by the homology stability of Section 2.1.

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