# On the homotopy type of the Deligne–Mumford compactification

JOHANNES EBERT
JEFFREY GIANSIRACUSA

An old theorem of Charney and Lee says that the classifying space of the category of stable nodal topological surfaces and isotopy classes of degenerations has the same rational homology as the Deligne–Mumford compactification. We give an integral refinement: the classifying space of the Charney–Lee category actually has the same homotopy type as the moduli stack of stable curves, and the étale homotopy type of the moduli stack is equivalent to the profinite completion of the classifying space of the Charney–Lee category.

32G15; 30F60, 14A20, 14D22

## 1 Introduction

The purpose of this note is to give an integral refinement of a relatively old theorem of Charney and Lee [6] giving a model for the rational homology of the Deligne–Mumford compactification of the moduli space of curves in terms of a category made of mapping class groups.

Let $\mathcal{M}_{g,n}$ denote the moduli stack of proper smooth algebraic curves of genus $g$ with $n$ ordered marked points, and let $\overline{\mathcal{M}}_{g,n}$ denote the moduli stack of stable curves (the Deligne–Mumford compactification of $\mathcal{M}_{g,n}$). They are both smooth Deligne–Mumford algebraic stacks defined over $\operatorname{spec}\mathbb{Z}$. These algebraic stacks have associated complex analytic stacks (orbifolds), $\mathcal{M}^{an}_{g,n}$ and $\overline{\mathcal{M}}^{an}_{g,n}$. It is well known that the coarse moduli space of $\mathcal{M}^{an}_{g,n}$ has the same rational homology as the classifying space of the mapping class group $\mathcal{MCG}_{g,n}$ of a surface of genus $g$ with $n$ marked points.

Charney and Lee defined a category $\mathcal{CL}_{g,n}$ in which:

- objects are stable nodal surfaces of genus $g$ with $n$ ordered distinct marked points in the smooth part,
- morphisms are isotopy classes of orientation-preserving diffeomorphisms and degenerations (a degeneration is a map which collapses some circles to nodes and is a diffeomorphism on the complement of these circles) that respect the marked points.

*Published: 5 November 2008* DOI: 10.2140/agt.2008.8.2049

The mapping class group $\mathcal{MCG}_{g,n}$ sits inside $\mathcal{CL}_{g,n}$ as the automorphism group of a smooth surface; automorphism groups of other objects are mapping class groups of singular surfaces appearing in the boundary of the Deligne–Mumford compactification. Note that the moduli stack $\overline{\mathcal{M}}_{g,n}$ and the category $\mathcal{CL}_{g,n}$ both have stratifications by "dual graphs".

Charney and Lee proved [6, Theorem 6.1.1] that (for $n=0$) the classifying space of $\mathcal{CL}_g$ has the same rational homology as the coarse moduli space of $\mathcal{M}_g^{\mathrm{an}}$. The moduli stack and the coarse moduli space have the same rational homology, but integrally they differ! The mod $p$ homology of the open moduli *stack* has been computed in the Harer–Ivanov stable range by Galatius [14] (using the theorem of Madsen and Weiss [18]); it contains much more than just reductions of nontorsion classes. The mod $p$ homology of the Deligne–Mumford compactified stack has been studied by Galatius and Eliashberg [15] and the authors [10], but it remains largely unknown.

An analytic stack (or more generally a topological stack) $\mathfrak{X}$ has a homotopy type which can be defined by choosing a covering $X \to \mathfrak{X}$ by a space $X$ and then taking the geometric realization of the simplicial space which in degree $n$ is the $(n+1)$–fold fiber-product $X \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} X$. That is, take the classifying space of a topological groupoid presenting the orbifold; see Moerdijk [19], Noohi [20], Ebert–Giansiracusa [10] and Ebert [9] for more details. The integral singular homology and fundamental group of the analytic stack agree with those of the homotopy type. As an example of homotopy types, it is well know that the homotopy type of the stack $\mathcal{M}_{g,n}^{\mathrm{an}}$ is $B\mathcal{MCG}_{g,n}$

We prove the following integral refinement of Charney and Lee's theorem.

**Theorem 1.1** *The classifying space of $\mathcal{CL}_{g,n}$ is homotopy equivalent to the homotopy type of the stack $\overline{\mathcal{M}}_{g,n}^{\mathrm{an}}$, so in particular, $H^*(\overline{\mathcal{M}}_{g,n}^{\mathrm{an}};\mathbb{Z}) \cong H^*(B\mathcal{CL}_{g,n};\mathbb{Z})$. Furthermore, this homotopy equivalence is compatible with the stratifications of $\mathcal{CL}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}^{\mathrm{an}}$*

By Artin–Mazur [1], Oda [21] and Frediani–Neumann [12], a Deligne–Mumford algebraic stack has an étale homotopy type (living in the category of pro-objects in the homotopy category of simplicial sets). By the Comparison Theorem of étale homotopy theory in Friedlander [13, Theorem 8.4], the étale homotopy type of a stack over $\overline{\mathbb{Q}}$ is weakly equivalent to the Artin–Mazur profinite completion of the homotopy type of the associated analytic stack. Let $\mathcal{M}_{g,n} \otimes \overline{\mathbb{Q}}$ denote the extension of scalars of $\mathcal{M}_{g,n}$ to $\overline{\mathbb{Q}}$ (ie the restriction of the moduli functor to schemes over $\overline{\mathbb{Q}}$). As explained in [21], the étale homotopy type of $\mathcal{M}_{g,n} \otimes \overline{\mathbb{Q}}$ is the Artin–Mazur profinite completion $(B\mathcal{MCG}_{g,n})^\wedge$. Similarly, Frediani–Neumann [12] describes the étale homotopy type of the moduli stack of curves with an action of a finite group $G \subset \mathcal{MCG}_{g,n}$. In this vein, the Comparison Theorem plus Theorem 1.1 yields:

**Corollary 1.2** *The étale homotopy type of $\overline{\mathcal{M}}_{g,n} \otimes \overline{\mathbb{Q}}$ is weakly equivalent to the Artin–Mazur profinite completion $(B\mathcal{CL}_{g,n})^\wedge$, and this equivalence respects the respective stratifications.*

(Recall that a weak equivalence of pro-objects is a morphism of pro-objects which induces an isomorphism on their homotopy pro-groups.)

The original Charney–Lee proof could probably easily be adapted to handle surfaces with marked points and to show that the rational homology equivalence is compatible with the stratifications. However, our proof is significantly more direct than theirs, while also giving the integral refinement. Our proof is based on existence of a particularly nice atlas, first constructed by Bers [3; 2; 4; 5], which is well-adapted to the combinatorial structure of the stratification of $\overline{\mathcal{M}}_{g,n}$. Roughly speaking, the Bers atlas generalizes the Teichmüller space in the same way that the Charney–Lee category generalizes the mapping class group.

**Acknowledgements** The first author was supported by a postdoctoral grant of the German Academic Exchange Service (DAAD). He enjoyed the hospitality of the Mathematical Institute of the University of Oxford. The second author thanks C-F Bödigheimer for an invitation to visit Universität Bonn in April 2006, where this project was begun. The project was completed during the second author's stay at the IHES during 2006/2007, and he thanks both institutions for their hospitality.

## 2 The Charney–Lee category

Before proceeding with the principal content of this note, we collect here some remarks about the Charney–Lee category. We will not need either of these remarks, so we only sketch them briefly, but the reader might nevertheless find these comments illuminating.

Firstly, there is a topological version $\mathcal{CL}^{\mathrm{top}}_{g,n}$; is has the same objects as $\mathcal{CL}_{g,n}$, while the space of morphisms $\mathcal{CL}^{\mathrm{top}}_{g,n}(S,T)$ is the space of degeneration maps $S \to T$ (ie maps which collapse circles to nodes and which are orientation-preserving diffeomorphisms outside these collapsed circles). The topology of the morphism spaces is the Whitney $C^\infty$–topology. We can clearly identify $\pi_0(\mathcal{CL}^{\mathrm{top}}_{g,n})$ with $\mathcal{CL}_{g,n}$. Moreover, the obvious functor $\mathcal{CL}^{\mathrm{top}}_{g,n} \to \mathcal{CL}_{g,n}$ is a homotopy equivalence of categories; in other words, the components of the morphism spaces in $\mathcal{CL}^{\mathrm{top}}_{g,n}$ are all contractible. This is a generalization of the well-known theorem [7; 8; 16] that the diffeomorphism groups of oriented smooth surfaces with boundary (with negative Euler number) have contractible components. The space of degenerations $\mathcal{CL}^{\mathrm{top}}_{g,n}(S,T)$ fibers over the space of unparametrized 1–dimensional submanifolds in $S$ by taking the union of all curves which are collapsed. This map is a Serre fibration and the fibers are homeomorphic to diffeomorphism groups of surfaces with negative Euler characteristic, hence the components of the fibers are contractible. It follows from [11; 16] that the components of the base are also contractible.

A second remark is that $\mathcal{CL}_{g,n}$ can be described as an orbit category. The *orbit category* of $\mathcal{MCG}_{g,n}$ is the category whose objects are orbits $\mathcal{MCG}_{g,n}/H$ and whose morphisms are the $\mathcal{MCG}_{g,n}$–equivariant maps. The category $\mathcal{CL}_{g,n}$ is equivalent to the full subcategory of the orbit category containing precisely those orbits for which the isotropy subgroup $H$ is a free abelian group generated by a collection of disjoint Dehn twists. To see this, fix a smooth surface $S$ of genus $g$ with $n$ marked points, and for each object $T \in \mathcal{CL}_{g,n}$ choose a degeneration $p(T)\colon S \to T$. The Dehn twists on $S$ around the inverse images of the nodes of $T$ determine a free abelian subgroup of $\mathcal{MCG}_{g,n}$ and hence an orbit $O(T)$. Given a degeneration $\alpha\colon T \to T'$, there exists $\widetilde{\alpha} \in \mathcal{MCG}_{g,n}$ such that $\alpha \circ p(T) = p(T') \circ \widetilde{\alpha}$ — this $\widetilde{\alpha}$ is only unique up to certain Dehn twists, but it induces a well-defined morphism $O(T) \to O(T')$ in the orbit category.

## 3 The Bers atlas for $\bar{\mathcal{M}}^{\mathrm{an}}_{g,n}$

Bers [3; 2; 4; 5] has constructed an atlas $\mathfrak{D}$, which we shall call the *Bers atlas*, for the differentiable stack $\bar{\mathcal{M}}^{\mathrm{an}}_{g,n}$. (To avoid notational clutter we leave $g$ and $n$ implicit). This atlas is an extension of the atlas for the uncompactified moduli stack $\mathcal{M}^{\mathrm{an}}_{g,n}$ given by Teichmüller space.

The Bers atlas is defined as follows. Let $S$ be a fixed stable nodal topological surface of genus $g$ with $n$ marked points. An *$S$–marked Riemann surface* is a stable nodal Riemann surface $F$ with $n$ marked points lying in the smooth part, together with a degeneration $F \to S$ which respects the marked points. Two $S$–marked Riemann surfaces $f\colon F \to S$ and $f'\colon F' \to S$ are defined to be equivalent if there exists a biholomorphic map $g\colon F \xrightarrow{\cong} F'$ (respecting the marked points) such that the diagram

$$\begin{tikzcd} F \arrow[rr, "g"] \arrow[dr, "f"'] & & F' \arrow[dl, "f'"] \\ & S & \end{tikzcd}$$

commutes up to a homotopy that is constant on the marked points.

Let $\mathfrak{D}(S)$ denote the set of all equivalence classes of $S$–marked Riemann surfaces. In [2; 5] Bers defined a topology on $\mathfrak{D}(S)$ making it into a *contractible* manifold, and such that when $S$ is smooth then $\mathfrak{D}(S)$ is the usual Teichmüller space of $S$.

In fact, the Fenchel–Nielsen coordinates give a homeomorphism between $\mathfrak{D}(S)$ and an open ball as follows. Let $N$ denote the set of nodes of $S$ and choose a complete cutsystem $C$ on $S$ (ie a collection of disjoint simple closed curves in the smooth part of $S$ such that the complement of $C \sqcup N$ is a disjoint union of pairs of pants. Given a point $[f\colon F \to S] \in \mathfrak{D}(S)$, there is a unique compatible hyperbolic metric on $F$. The free homotopy class of each curve of $f^{-1}(C)$ has a minimal geodesic length and a twist; these numbers determine a point in $(\mathbb{R}_+ \times \mathbb{R})^C \cong \mathbb{H}^C$. For a node $n \in N$, if $f^{-1}(n)$ is a simple closed curve then this free homotopy class has a length and a twist in $\mathbb{R}_+ \times \mathbb{R}/\mathbb{Z} \cong \mathbb{C}^\times$, and the coordinates converge to the origin as the inverse image of $n$ in $F$ collapses to a node. Hence the Fenchel–Nielsen coordinates give a map

$$\mathfrak{D}(S) \to \mathbb{H}^C \times \mathbb{C}^N,$$

which one can show is a homeomorphism. A particularly nice exposition for smooth surfaces can be found in Hubbard's book [17, p 320 ff].

Bers also endowed $\mathfrak{D}(S)$ with the structure of a complex manifold which embeds as a bounded domain in $\mathbb{C}^{3g-3+n}$, generalizing the Maskit coordinates, but we shall not need this fact.

The Bers atlas is given by

$$\coprod_S \mathfrak{D}(S) \to \overline{\mathcal{M}}^{\mathrm{an}}_{g,n},$$

where the disjoint union runs over each diffeomorphism class of stable nodal surfaces $S$ having genus $g$ and $n$ marked points; the map to the moduli space is given informally by forgetting the markings, sending a marked Riemann surface $[F \to S]$ to $F$. More precisely, there is a tautological family over $\mathfrak{D}(S)$ whose fiber over $[F \to S]$ is $F$, and the map to $\overline{\mathcal{M}}^{\mathrm{an}}_{g,n}$ is given by classifying this tautological family.

**Theorem 3.1** *The morphism $\coprod_S \mathfrak{D}(S) \to \overline{\mathcal{M}}^{\mathrm{an}}_{g,n}$ defines a proper étale atlas for $\overline{\mathcal{M}}^{\mathrm{an}}_{g,n}$ as a differentiable (or even complex analytic) stack.*

**Proof** This is essentially contained in the work of Bers; it follows from Theorems 6 and 7 announced in [2]. □

Put differently, the representable submersion $\coprod_S \mathfrak{D}(S) \to \overline{\mathcal{M}}^{\mathrm{an}}_{g,n}$ determines a Lie groupoid which is proper and étale (ie an orbifold groupoid, though not always effective),

$$\mathfrak{D} := \left[\left(\coprod_S \mathfrak{D}(S)\right) \times_{\overline{\mathcal{M}}^{\mathrm{an}}_{g,n}} \left(\coprod_S \mathfrak{D}(S)\right) \rightrightarrows \left(\coprod_S \mathfrak{D}(S)\right)\right].$$

We call this the *Bers groupoid*. An object of this groupoid is the equivalence class of an $S$–marked Riemann surface $F$ for some $S$; a morphism $[F \to S] \to [F' \to T]$ is a biholomorphic map $g\colon F \xrightarrow{\cong} F'$ respecting the marked points but completely ignoring the maps to $S$ and $T$. We call this the Bers groupoid and denote it $\mathfrak{D}$. Since it is a presentation of the stack $\overline{\mathcal{M}}^{\mathrm{an}}_{g,n}$, its classifying space is a model for the homotopy type of $\overline{\mathcal{M}}^{\mathrm{an}}_{g,n}$. In particular, $B\mathfrak{D}$ has the same integral (co)homology as the stack $\overline{\mathcal{M}}^{\mathrm{an}}_{g,n}$.

We now recall some facts about the Bers atlas from [2; 5]. A degeneration $\alpha\colon S \to T$ induces a map $\alpha_*\colon \mathfrak{D}(S) \to \mathfrak{D}(T)$ by change-of-marking, ie

$$[F \xrightarrow{f} S] \mapsto [F \xrightarrow{\alpha\circ f} T].$$

The induced map $\alpha_*$ is a local homeomorphism. Its image is precisely the subspace consisting of those points $[F \to T]$ for which the marking can be lifted along $\alpha$ to $S$; with appropriate Fenchel–Nielsen coordinates one easily sees that this is the complement of a collection of complex coordinate hyperplanes. The map $\alpha_*$ only depends on the isotopy class of $\alpha$ because of the equivalence relation on degenerations $F \to S$ used in defining the space $\mathfrak{D}(S)$. In particular, there is a properly discontinuous action of the mapping class group $\mathcal{MCG}(S)$ of $S$ on $\mathfrak{D}(S)$ and the quotient stack $[\mathfrak{D}(S)/\mathcal{MCG}(S)]$ is isomorphic to the image of $\mathfrak{D}(S)$ in $\overline{\mathcal{M}}^{\mathrm{an}}_{g,n}$.

# 4 The Bers groupoid and the Charney–Lee category

We shall now describe a subcategory of the Bers groupoid which is more visibly related to the Charney–Lee category. We then give a completely explicit description of the Bers groupoid in terms of this subcategory.

The spaces $\mathfrak{D}(-)$ together with the change-of-marking maps described above determine a functor $\widehat{\mathfrak{D}}\colon \mathcal{CL}_{g,n} \to \mathrm{Spaces}$, and we may form the transport category (or Grothendieck construction) $\mathcal{CL}_{g,n} \int \widehat{\mathfrak{D}}$. Concretely, an object of the transport category is a point $[f\colon F \to S]$ in $\mathfrak{D}(S)$ for some $S$. A morphism from $[f\colon F \to S]$ to $[f'\colon F' \to T]$ is represented *a priori* by a biholomorphic map $g\colon F \to F'$ together with the isotopy class of a degeneration $\alpha\colon S \to T$ such that the diagram

$$\begin{tikzcd} F \arrow[r, "g"] \arrow[d, "f"'] & F' \arrow[d, "f'"] \\ S \arrow[r, "\alpha"] & T \end{tikzcd}$$

commutes up to homotopy. However, since the Charney–Lee category possesses the right cancelation property,

$$[\alpha]\circ[\gamma]=[\beta]\circ[\gamma] \text{ implies } [\alpha]=[\beta],$$

the isomorphism $g$ uniquely determines the degeneration isotopy class $[\alpha]$. Note that not every isomorphism covers (up to homotopy) a degeneration. Thus a morphism $[F\to S]\to[F'\to T]$ can be specified simply by a biholomorphic map $g\colon F\to F'$ for which there exists degeneration isotopy class that it covers.

By comparing the definitions the following is now apparent.

**Proposition 4.1** *The topological category $\mathcal{CL}_{g,n}\int\widehat{\mathfrak{D}}$ is isomorphic to a subcategory of the Bers groupoid $\mathfrak{D}$; namely, it is the subcategory with all objects of $\mathfrak{D}$ and only those biholomorphic maps which cover (up to isotopy) a degeneration of the markings.*

**Lemma 4.2** *The inclusion $\mathcal{CL}_{g,n}\int\widehat{\mathfrak{D}}\hookrightarrow\mathfrak{D}$ induces a homotopy equivalence of classifying spaces.*

We will give the proof of Lemma 4.2 in Section 6 after some preparation in Section 5. Assuming this lemma for the moment, the proof of Theorem 1.1 is straightforward.

**Proof of Theorem 1.1** The Bers atlas is an atlas for $\overline{\mathcal{M}}^{\mathrm{an}}_{g,n}$, and so by definition of the homotopy type of a stack, the classifying space $B\mathfrak{D}$ of the Bers groupoid $\mathfrak{D}$ is the homotopy type of $\overline{\mathcal{M}}^{\mathrm{an}}_{g,n}$. Because $\widehat{\mathfrak{D}}$ takes any $S\in\mathcal{CL}_{g,n}$ to a contractible space, the forgetful functor $\mathcal{CL}_{g,n}\int\widehat{\mathfrak{D}}\to\mathcal{CL}_{g,n}$ induces a homotopy equivalence on classifying spaces. Therefore, by Lemma 4.2:

$$\mathrm{Ho}(\overline{\mathcal{M}}^{\mathrm{an}}_{g,n})\simeq B\mathfrak{D}\xleftarrow{\simeq}B\left(\mathcal{CL}_{g,n}\int\widehat{\mathfrak{D}}\right)\xrightarrow{\simeq}B\mathcal{CL}_{g,n}.$$

We postpone the discussion of compatibility with the stratifications until Section 7. □

**Remark 4.3** It is possible to show that, as abstract categories, when one formally adjoins inverses to all arrows of $\mathcal{CL}_{g,n}\int\widehat{\mathfrak{D}}$ then one obtains precisely the Bers groupoid $\mathfrak{D}$. In particular, an arrow $[f\colon F\to S]\to[f'\colon F'\to T]$ can be represented by $\alpha^{-1}\circ\beta$ for a pair of degenerations $S\xleftarrow{\alpha}F'\xrightarrow{\beta}T$, and this representation is unique up to precomposition with an element of the mapping class group of $F'$.

# 5 A lifting property of the Bers atlas

Let $X$ be a space and $\sigma\colon X \to \overline{\mathcal{M}}^{\mathrm{an}}_{g,n}$ be a map. We say that a lift $\widetilde{\sigma}\colon X \to \mathfrak{D}(S)$ of $\sigma$ is *maximal* if $\widetilde{\sigma}$ does not admit a lift to $\mathfrak{D}(S')$ for any $S'$ with a strict degeneration $S' \to S$. Clearly, if $\sigma$ admits a lift to some $\mathfrak{D}(T)$ then it lifts further to a maximal lift.

The goal of the present section is to prove the following result.

**Lemma 5.1** *Suppose $X$ is simply connected and $X \to \overline{\mathcal{M}}^{\mathrm{an}}_{g,n}$ admits maximal lifts $\sigma_1\colon X \to \mathfrak{D}(S)$ and $\sigma_2\colon X \to \mathfrak{D}(T)$. Then there exists a diffeomorphism (unique up to isotopy) $\alpha\colon S \cong T$ with $\alpha_*\sigma_1 = \sigma_2$.*

An equivalent formulation of the above lemma is that for any pair of stable surfaces $S$ and $T$, there exists a stable surface $R$ degenerating onto $S$ and $T$ such that the map from $\mathfrak{D}(R)$ to any component of the universal cover of

$$\mathfrak{D}(S) \times_{\overline{\mathcal{M}}^{\mathrm{an}}_{g,n}} \mathfrak{D}(T)$$

is a homeomorphism. However, we do not know a more direct proof of this fact.

The main tool for the proof of Lemma 5.1 is a sheaf of sets $\mathcal{Z}$ on the differentiable stack $\overline{\mathcal{M}}^{\mathrm{an}}_{g,n}$. This sheaf encodes the continuity property of markings on the fibers in a marked family of stable Riemann surfaces. The idea of the sheaf is as follows. Given a family $E \to X$ of stable Riemann surfaces, an element of $\mathcal{Z}(X)$ should be thought of as the isotopy class of a continuous subfamily $C \subset E$ that restricts in each fiber to either a node or a simple closed curve that does not meet the nodes and marked points and does not retract to a node. If $X \to \mathfrak{D}(S)$ is a maximal lift then one can reconstruct the homeomorphism type of $S$ from the sections of $\mathcal{Z}$ over $X$ that restrict to a node in some fiber: each node in each fiber determines a node of $S$ and maximality of the lift ensures that $S$ has no superfluous nodes. This will show that maximal lifts are essentially unique.

We construct the sheaf precisely by defining its stalks over the Bers groupoid, topologizing its étale space, and then showing it descends to a sheaf on the stack $\overline{\mathcal{M}}^{\mathrm{an}}_{g,n}$. The stalk $\mathcal{Z}_{[S]}$ at a point $[S] \in \overline{\mathcal{M}}^{\mathrm{an}}_{g,n}$ is defined to be the union of the set of nodes of $S$ with the set of isotopy classes of unoriented simple closed curves in $S \setminus \{\text{nodes and marked points}\}$ which bound neither a disc nor a once-punctured disc. A degeneration $\alpha\colon T \to S$ induces an injective map $\alpha^*\colon \mathcal{Z}_{[S]} \hookrightarrow \mathcal{Z}_{[T]}$ by taking preimages of curves and nodes. Thus over $\mathfrak{D}(S)$ the markings canonically identify $\mathcal{Z}_{[S]}$ with a subset of each stalk.

As a set, the étale space $\mathcal{Z}_{\mathrm{et}}$ of $\mathcal{Z}$ is the disjoint union of the stalks; we topologize it as follows. Given a point $[\alpha\colon F \to S] \in \mathfrak{D}(S)$, there exists a neighborhood $U$ of this point which lifts along the local homeomorphism $\alpha_*\colon \mathfrak{D}(F) \to \mathfrak{D}(S)$, and so the markings identify $\mathcal{Z}_{[S]}$ with a subset of each stalk over $U$. The topology is determined by the condition that a section over $U$ is continuous at $[F \to S]$ if and only if it is locally constant with respect to these identifications.

Next we claim that $\mathcal{Z}$ is a sheaf on the stack $\overline{\mathcal{M}}^{\mathrm{an}}_{g,n}$. To justify this, we need to argue that $\mathcal{Z}$ satisfies the appropriate descent conditions. More precisely, let $d_0, d_1\colon \mathrm{Mor}(\mathfrak{D}) \to \mathrm{Obj}(\mathfrak{D})$ be source and target maps and let

$$d_0, d_1, d_2\colon \mathrm{Mor}(\mathfrak{D}) \times_{\mathrm{Obj}(\mathfrak{D})} \mathrm{Mor}(\mathfrak{D}) \to \mathrm{Mor}(\mathfrak{D})$$

be the three simplicial structure maps in the nerve of $\mathfrak{D}$ (they are left projection, composition, and right projection respectively). A descent datum for $\mathcal{Z}$ is an isomorphism $f\colon d_0^*\mathcal{Z} \to d_1^*\mathcal{Z}$ which makes the hexagon of sheafs and isomorphisms on $\mathrm{Mor}(\mathfrak{D}) \times_{\mathrm{Obj}(\mathfrak{D})} \mathrm{Mor}(\mathfrak{D})$ commutative (the equalities are induced from simplicial identities):

$$\begin{tikzcd}
 & d_0^*d_0^*\mathcal{Z} \arrow[r, "d_0^* f"] \arrow[dl, equal] & d_0^*d_1^*\mathcal{Z} \arrow[dr, equal] & \\
d_1^*d_0^*\mathcal{Z} \arrow[dr, "d_1^* f"] & & & d_2^*d_0^*\mathcal{Z} \arrow[dl, "d_2^* f"] \\
 & d_1^*d_1^*\mathcal{Z} \arrow[r, equal] & d_2^*d_1^*\mathcal{Z} &
\end{tikzcd}$$

There is an obvious bijection of étale spaces $d_0^*\mathcal{Z}_{\mathrm{et}} \cong d_1^*\mathcal{Z}_{\mathrm{et}}$ and the topology is designed so that this is a homeomorphism. The commutativity of the above diagram is also clear. Thus $\mathcal{Z}$ is a sheaf of sets on $\overline{\mathcal{M}}^{\mathrm{an}}_{g,n}$.

The following property follows immediately from the definition of the topology on the étale space of $\mathcal{Z}$.

**Lemma 5.2** *Let $U \subset \mathfrak{D}(S)$ be a neighborhood of the origin $[S \to S]$. There is a canonical bijection $\mathcal{Z}(U) \cong \mathcal{Z}_{[S]}$ induced in one direction by restriction to the stalk over $[S \to S]$ and in the other direction by using the markings to identify $\mathcal{Z}_{[S]}$ with a subset of each stalk.*

A section of $\mathcal{Z}$ over a base $X$ is said to be *nodal* if it restricts to a node in some stalk. Lemma 5.2 implies that the nodal sections over $\mathfrak{D}(S)$ are precisely those which restrict to nodes at the origin $[S \to S]$.

**Lemma 5.3** *Given a point $[F \xrightarrow{\alpha} S] \in \mathfrak{D}(S)$, the marking $\alpha$ collapses to nodes precisely those curves in $F$ which are the restrictions of nodal sections over $\mathfrak{D}(S)$.*

**Proof** By Lemma 5.2, a curve in $F$ is the preimage of a node in $S$ if and only if it is the restriction of a nodal section over $\mathfrak{D}(S)$. □

Let $T$ be a stable nodal surface and let $\alpha_*\colon T \to S$ be a degeneration which collapses a single curve in $T$ to a node $n \in S$. The node $n$ determines a nodal section over $\mathfrak{D}(S)$, and the maximal subset over which this section is *not* nodal is precisely the image of the change-of-marking $\alpha_*\colon \mathfrak{D}(T) \to \mathfrak{D}(S)$.

**Lemma 5.4** *Suppose $X$ is simply connected and $\widetilde{\sigma}\colon X \to \mathfrak{D}(S)$ is a maximal lift. Then the map $\widetilde{\sigma}^*\colon \mathcal{Z}(\mathfrak{D}(S)) \hookrightarrow \mathcal{Z}(X)$ restricts to a bijection between nodal sections.*

**Proof** Clearly every nodal section over $X$ is the pullback of a nodal section over $\mathfrak{D}(S)$. Conversely, if there exists a nodal section over $\mathfrak{D}(S)$ which pulls backs to a non-nodal section over $X$ then $X$ lies in the image of a change-of-marking $\alpha_*\colon \mathfrak{D}(T) \to \mathfrak{D}(S)$ for some $T$ with strictly fewer nodes than $S$. Since $X$ is simply connected and the change-of-marking maps are local homeomorphisms, $X$ lifts further, which contradicts the maximality hypothesis. □

**Proof of Lemma 5.1** Choose a point $x \in X$ and let $F_x$ denote the fiber over $x$. Consider the commutative diagram

$$\begin{tikzcd} & \mathcal{Z}(X) \arrow[d, hook] \\ \mathcal{Z}(\mathfrak{D}(S)) \arrow[ur, hook, "\sigma_1^*"] \arrow[r, hook] & \mathcal{Z}_{[F_x]} \end{tikzcd}$$

where the vertical and horizontal arrows are induced by restriction to the stalk at $x$ (which is identified with the stalk at $\sigma_1(x)$. By Lemma 5.4, a curve in $F_x$ is the restriction of a nodal section over $X$ if and only if it is the restriction of a nodal section over $\mathfrak{D}(S)$. By Lemma 5.3, $S$ is topologically obtained from $F_x$ by collapsing those curves which are the restrictions of nodal sections over $\mathfrak{D}(S)$ (equivalently, nodal sections over $X$). By the same reasoning, $T$ is topologically obtained from $F_x$ by collapsing the same set of curves. Hence $S$ and $T$ are abstractly homeomorphic. Finally, since the image of $\mathfrak{D}(S)$ in $\overline{\mathcal{M}}^{\mathrm{an}}_{g,n}$ is isomorphic to the quotient stack $[\mathfrak{D}(S)/\mathcal{MCG}_{g,n}(S)]$, it follows that any two lifts to $\mathfrak{D}(S)$ are related by a unique change of marking. □

# 6 Proof of Lemma 4.2

We shall now prove Lemma 4.2. It will follow from Lemma 5.1 together with Waldhausen's Theorem A' [22, p 165], which is a simplicial version of Quillen's Theorem A. We first recall Waldhausen's Theorem A'. Suppose $F\colon A_\bullet \to B_\bullet$ is a functor of simplicial categories. Given an object $\sigma \in \operatorname{Obj} B_n$ of simplicial degree $n$, the simplicial fiber category $(F/\sigma)_\bullet$ is given in degree $k$ by

$$(F/\sigma)_k \colon= \coprod_{u\colon \underline{k}\to\underline{n}} F_k/u^*\sigma,$$

where the disjoint union is taken over all monotone maps from $\{0,\dots k\}$ to $\{0\dots n\}$. The theorem states that if each of these simplicial fiber categories has contractible classifying space then $F$ is a homotopy equivalence of classifying spaces.

**Proof of Lemma 4.2** By taking the total singular simplicial set one has an inclusion of simplicial categories

$$j\colon S_\bullet\left(\mathcal{CL}_{g,n}\int \widehat{\mathfrak{D}}\right) \to S_\bullet\mathfrak{D}.$$

We will apply Waldhausen's Theorem A' to the simplicial functor $j$. Fix an object $\phi\colon \Delta^n \to \operatorname{Obj}\mathfrak{D}$ of $S_n\mathfrak{D}$. The image of $\phi$ lands in $\mathfrak{D}(S)$ for some stable nodal surface $S$. In simplicial degree $k$ the simplicial fiber category $(j/\phi)_\bullet$ is a disjoint union of ordinary fiber categories of the form $j_k/\sigma$ for various objects $\sigma$ of simplicial degree $k$. Suppose that each of these categories is contractible. Then collapsing them to points maps the simplicial fiber category $(j/\phi)_\bullet$ by a levelwise homotopy equivalence to the standard simplicial model for the $n$–simplex given in degree $k$ by $\coprod_{u\colon \underline{k}\to\underline{n}} *$. The geometric realization of this map is thus a homotopy equivalence

$$|B(j/\phi)_\bullet| \to \Delta^n \simeq *,$$

and so Waldhausen's Theorem A' yields the result.

It thus suffices to show that each category $j_n/\sigma$ has an initial object, where $\sigma\colon \Delta^n \to \mathfrak{D}(S) \subset \operatorname{Obj}\mathfrak{D}$. Let $\overline{\sigma}\colon \Delta^n \to \overline{\mathcal{M}}^{\mathrm{an}}_{g,n}$ denote the composition of $\sigma$ with the projection of $\mathfrak{D}(S)$ down to $\overline{\mathcal{M}}^{\mathrm{an}}_{g,n}$. Explicitly, an object of the category $j_n/\sigma$ is a lift (up to a specified 2–morphism) of $\overline{\sigma}$ to some chart $\mathfrak{D}(T)$; ie a 2–commutative diagram

$$\begin{tikzcd} & \mathfrak{D}(T) \arrow[d] \\ \Delta^n \arrow[ur, "\tau"] \arrow[r, "\overline{\sigma}"'] \arrow[ur, phantom, shift left=3, "\Downarrow\theta"] & \overline{\mathcal{M}}^{\mathrm{an}}_{g,n}. \end{tikzcd}$$

A morphism $(\tau_1, \theta_1) \to (\tau_2, \theta_2)$ is an isotopy class of degenerations $\alpha\colon T_1 \to T_2$ such that the induced 2–morphism $\Phi(\alpha)$

$$\begin{tikzcd} & \mathfrak{D}(T_1) \arrow[dr] & \\ \Delta^n \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\Downarrow \Phi(\alpha)"] & & \overline{\mathcal{M}}^{\mathrm{an}}_{g,n} \\ & \mathfrak{D}(T_2) \arrow[ur] & \end{tikzcd}$$

satisfies $\theta_2 \circ \Phi(\alpha) = \theta_1$. This is equivalent to saying that $\alpha_* \circ \tau_1 = \tau_2$. Since $j_n$ is the inclusion of a subcategory into a groupoid, there is at most one arrow between any two objects of $j_n/\sigma$. We are thus reduced to showing that there is an object $(\sigma_0, \theta_0)$ which maps to all other objects of $j_n/\sigma$. Every lift of $\overline{\sigma}$ to some $\mathfrak{D}(T)$ lifts further to a maximal lift, and Lemma 5.1 says that a maximal lift of $\overline{\sigma}$ is unique up to isomorphism. A maximal lift therefore provides the desired initial object. □

# 7 Stratifications

The strata of $\overline{\mathcal{M}}^{\mathrm{an}}_{g,n}$ are indexed by *stable graphs* with $n$ external legs; equivalently the strata are indexed by diffeomorphism types of stable nodal surfaces of genus $g$ with $n$ labeled points. A stable nodal surface $T$ corresponds to an open stratum $R_T\overline{\mathcal{M}}^{\mathrm{an}}_{g,n}$ which is the locus of all stable nodal Riemann surfaces $F$ which are topologically diffeomorphic to $T$. The closure $\overline{R}_T\overline{\mathcal{M}}^{\mathrm{an}}_{g,n}$ is the locus of all Riemann surfaces $F$ for which $T$ admits a degeneration onto $F$. This stratification gives a corresponding stratification of the spaces $\mathfrak{D}(S)$, and so there are atlases:

$$\coprod_S R_T\mathfrak{D}(S) \to R_T\overline{\mathcal{M}}^{\mathrm{an}}_{g,n},$$
$$\coprod_S \overline{R}_T\mathfrak{D}(S) \to \overline{R}_T\overline{\mathcal{M}}^{\mathrm{an}}_{g,n}$$

which give rise to subgroupoids of the Bers groupoid $\mathfrak{D}$. The Fenchel–Nielsen coordinates show that $\overline{R}_T\mathfrak{D}(S)$ is homeomorphic to a proper ball in $\mathfrak{D}(S)$ of codimension equal to the number of nodes of $S$ minus the number of nodes of $T$. In particular, $\overline{R}_T\mathfrak{D}(S)$ is contractible.

The stratification of $\mathcal{CL}_{g,n}$ is as follows:

$$R_T\mathcal{CL}_{g,n} = \text{full subcategory on the object } T = \mathcal{MCG}_{g,n}(T)$$
$$\overline{R}_T\mathcal{CL}_{g,n} = \{\text{full subcategory on } S \text{ such that } T \text{ admits a degeneration onto } S\}$$

The proof of Theorem 1.1 (along with the proofs of all propositions and lemmas it employs) remains valid upon inserting $\bar{R}_T$ in front of all occurrences of the symbols $\mathcal{CL}_{g,n}$, $\overline{\mathcal{M}}^{\mathrm{an}}_{g,n}$, and $\mathfrak{D}$. Thus the homotopy equivalence $\overline{\mathcal{M}}^{\mathrm{an}}_{g,n} \simeq B\mathcal{CL}_{g,n}$ restricts to an equivalence of each closed stratum.

To see that it restricts to a homotopy equivalence on each open stratum, one uses the fact that each open stratum $R_T\overline{\mathcal{M}}^{\mathrm{an}}_{g,n}$ is the stack quotient of a finite group acting on a product of uncompactified moduli spaces, and $R_T\mathcal{CL}_{g,n}$ is the homotopy quotient of the same finite group acting on the corresponding product of classifying spaces of mapping class groups. Thus $R_T\overline{\mathcal{M}}^{\mathrm{an}}_{g,n} \simeq B(R_T\mathcal{CL}_{g,n})$ follows from the equivalence $\mathcal{M}^{\mathrm{an}}_{g,n} \simeq B\mathcal{MCG}_{g,n}$ discussed in the introduction, since the homotopy type of the stack quotient is the homotopy quotient.

## References


[1] **M Artin**, **B Mazur**, *Etale homotopy*, Lecture Notes in Math. 100, Springer, Berlin (1969) MR0245577

[2] **L Bers**, *On spaces of Riemann surfaces with nodes*, Bull. Amer. Math. Soc. 80 (1974) 1219–1222 MR0361165

[3] **L Bers**, *Spaces of degenerating Riemann surfaces*, from: "Discontinuous groups and Riemann surfaces (Proc. Conf., Univ. Maryland, College Park, Md., 1973)", Ann. of Math. Studies 79, Princeton Univ. Press (1974) 43–55 MR0361051

[4] **L Bers**, *Deformations and moduli of Riemann surfaces with nodes and signatures*, Math. Scand. 36 (1975) 12–16 MR0374496 Collection of articles dedicated to Werner Fenchel on his 70th birthday

[5] **L Bers**, *Finite-dimensional Teichmüller spaces and generalizations*, Bull. Amer. Math. Soc. (N.S.) 5 (1981) 131–172 MR621883

[6] **R Charney**, **R Lee**, *Moduli space of stable curves from a homotopy viewpoint*, J. Differential Geom. 20 (1984) 185–235 MR772131

[7] **C J Earle**, **J Eells**, *A fibre bundle description of Teichmüller theory*, J. Differential Geometry 3 (1969) 19–43 MR0276999

[8] **C J Earle**, **A Schatz**, *Teichmüller theory for surfaces with boundary*, J. Differential Geometry 4 (1970) 169–185 MR0277000

[9] **J Ebert**, *The homotopy type of a topological stack*, In preparation

[10] **J Ebert**, **J Giansiracusa**, *Pontrjagin–Thom maps and the homology of the moduli stack of stable curves*, Submitted `arXiv:0712.0702`

[11] **D B A Epstein**, *Curves on 2–manifolds and isotopies*, Acta Math. 115 (1966) 83–107 MR0214087

[12] **P Frediani**, **F Neumann**, *Étale homotopy types of moduli stacks of algebraic curves with symmetries*, $K$–Theory 30 (2003) 315–340 MR2064243 Special issue in honor of Hyman Bass on his seventieth birthday. Part IV

[13] **E M Friedlander**, *Étale homotopy of simplicial schemes*, Annals of Math. Studies 104, Princeton University Press (1982) MR676809

[14] **S Galatius**, *Mod $p$ homology of the stable mapping class group*, Topology 43 (2004) 1105–1132 MR2079997

[15] **S Galatius**, **Y Eliashberg**, *Homotopy theory of compactified moduli spaces*, Oberwolfach Reports (2006) 761–766

[16] **A Gramain**, *Le type d'homotopie du groupe des difféomorphismes d'une surface compacte*, Ann. Sci. École Norm. Sup. (4) 6 (1973) 53–66 MR0326773

[17] **J H Hubbard**, *Teichmüller theory and applications to geometry, topology, and dynamics. Vol. 1*, Matrix Editions, Ithaca, NY (2006) MR2245223

[18] **I Madsen**, **M Weiss**, *The stable moduli space of Riemann surfaces: Mumford's conjecture*, Ann. of Math. (2) 165 (2007) 843–941 MR2335797

[19] **I Moerdijk**, *Orbifolds as groupoids: an introduction*, from: "Orbifolds in mathematics and physics (Madison, WI, 2001)", Contemp. Math. 310, Amer. Math. Soc. (2002) 205–222 MR1950948

[20] **B Noohi**, *Homotopy types of topological stacks* arXiv:0808.3799

[21] **T Oda**, *Etale homotopy type of the moduli spaces of algebraic curves*, from: "Geometric Galois actions, 1", London Math. Soc. Lecture Note Ser. 242, Cambridge Univ. Press (1997) 85–95 MR1483111

[22] **F Waldhausen**, *Algebraic $K$–theory of spaces, a manifold approach*, from: "Current trends in algebraic topology, Part 1 (London, Ont., 1981)", CMS Conf. Proc. 2, Amer. Math. Soc. (1982) 141–184 MR686115

*Mathematisches Institut, der Universität Bonn*
*Beringstrasse 1, 53115 Bonn, Germany*

*Mathematical Institute, University of Oxford*
*24–29 St. Giles', Oxford, OX1 3LB, United Kingdom*

ebert@maths.ox.ac.uk, giansira@maths.ox.ac.uk

Received: 16 July 2008