Large scale geometry of commutator subgroups

DANNY CALEGARI
DONGPING ZHUANG

Let $G$ be a finitely presented group, and $G'$ its commutator subgroup. Let $C$ be the Cayley graph of $G'$ with all commutators in $G$ as generators. Then $C$ is large scale simply connected. Furthermore, if $G$ is a torsion-free nonelementary word-hyperbolic group, $C$ is one-ended. Hence (in this case), the asymptotic dimension of $C$ is at least 2.

20F65, 57M07

1 Introduction

Let $G$ be a group and let $G' := [G, G]$ denote the commutator subgroup of $G$. The group $G'$ has a canonical generating set $S$, which consists precisely of the set of commutators of pairs of elements in $G$. In other words,

$$S = \{ [g, h] \text{ such that } g, h \in G \}.$$

Let $C_S(G')$ denote the Cayley graph of $G'$ with respect to the generating set $S$. This graph can be given the structure of a (path) metric space in the usual way, where edges have length 1 by fiat.

By now it is standard to expect that the large scale geometry of a Cayley graph will reveal useful information about a group. However, one usually studies finitely generated groups $G$ and the geometry of a Cayley graph $C_T(G)$ associated to a finite generating set $T$. For typical infinite groups $G$, the set of commutators $S$ will be infinite, and the Cayley graph $C_S(G')$ will not be locally compact. This is a significant complication. Nevertheless, $C_S(G')$ has several distinctive properties which invite careful study:

1. The set of commutators of a group is characteristic (ie invariant under any automorphism of $G$), and therefore the semidirect product $G' \rtimes \text{Aut}(G)$ acts on $C_S(G')$ by isometries.

2. The metric on $G'$ inherited as a subspace of $C_S(G')$ is both left- and right-invariant (unlike the typical Cayley graph, whose metric is merely left-invariant).
(3) Bounded cohomology in $G$ is reflected in the geometry of $G'$; for instance, the translation length $\tau(g)$ of an element $g \in G'$ is the stable commutator length $\text{scl}(g)$ of $g$ in $G$.

(4) Simplicial loops in $C_S(G')$ through the origin correspond to (marked) homotopy classes of maps of closed surfaces to a $K(G, 1)$.

These properties are straightforward to establish; for details, see Section 2.

This paper concerns the connectivity of $C_S(G')$ in the large for various groups $G$. Recall that a thickening $Y$ of a metric space $X$ is an isometric inclusion $X \to Y$ into a bigger metric space, such that the Hausdorff distance in $Y$ between $X$ and $Y$ is finite. A metric space $X$ is said to be large scale $k$–connected if for any thickening $Y$ of $X$ there is another thickening $Z$ of $Y$ which is $k$–connected (ie $\pi_i(Z) = 0$ for $i \leq k$; also see the definitions in Section 3). Our first main theorem, proved in Section 3, concerns the large scale connectivity of $C_S(G')$ where $G$ is finitely presented:

**Theorem A** Let $G$ be a finitely presented group. Then $C_S(G')$ is large scale simply connected.

As well as large scale connectivity, one can study connectivity at infinity. In Section 4 we specialize to word-hyperbolic $G$ and prove our second main theorem, concerning the connectivity of $G'$ at infinity:

**Theorem B** Let $G$ be a torsion-free nonelementary word-hyperbolic group. Then $C_S(G')$ is one-ended; i.e for any $r > 0$ there is an $R \geq r$ such that any two points in $C_S(G')$ at distance at least $R$ from $\text{id}$ can be joined by a path which does not come closer than distance $r$ to $\text{id}$.

Combined with a theorem of Fujiwara–Whyte [7], Theorem A and Theorem B together imply that for $G$ a torsion-free nonelementary word-hyperbolic group, $C_S(G')$ has asymptotic dimension at least 2 (see Section 5 for the definition of asymptotic dimension).

### 2 Definitions and basic properties

Throughout the rest of this paper, $G$ will denote a group, $G'$ will denote its commutator subgroup, and $S$ will denote the set of (nonzero) commutators in $G$, thought of as a generating set for $G'$. Let $C_S(G')$ denote the Cayley graph of $G'$ with respect to the generating set $S$. As a graph, $C_S(G')$ has one vertex for every element of $G'$, and two elements $g, h \in G'$ are joined by an edge if and only if $g^{-1}h \in S$. Let $d$ denote distance in $C_S(G')$ restricted to $G'$.
**Definition 2.1** Let \( g \in G' \). The *commutator length* of \( g \), denoted \( \text{cl}(g) \), is the smallest number of commutators in \( G \) whose product is equal to \( g \).

From the definition, it follows that \( \text{cl}(g) = d(\text{id}, g) \) and \( d(g, h) = \text{cl}(g^{-1}h) \) for \( g, h \in G' \).

**Lemma 2.2** The group \( G' \rtimes \text{Aut}(G) \) acts on \( C_S(G') \) by isometries.

**Proof** \( \text{Aut}(G) \) acts as permutations of \( S \), and therefore the natural action on \( G \) extends to \( C_S(G') \). Further, \( G' \) acts on \( C_S(G') \) by left multiplication. \( \square \)

**Lemma 2.3** The metric on \( C_S(G') \) restricted to \( G' \) is left- and right-invariant.

**Proof** Since the inverse of a commutator is a commutator, we have \( \text{cl}(g^{-1}h) = \text{cl}(h^{-1}g) \). Since the conjugate of a commutator by any element is a commutator, we have \( \text{cl}(h^{-1}g) = \text{cl}(gh^{-1}) \). This completes the proof. \( \square \)

**Definition 2.4** Given a metric space \( X \) and an isometry \( h \) of \( X \), the *translation length* of \( h \) on \( X \), denoted \( \tau(h) \), is defined by the formula

\[
\tau(h) = \lim_{n \to \infty} \frac{d(p, h^n(p))}{n}
\]

where \( p \in X \) is arbitrary.

By the triangle inequality, the limit does not depend on the choice of \( p \).

For \( g \in G' \) acting on \( C_S(G') \) by left multiplication, we can take \( p = \text{id} \). Then \( d(\text{id}, g^n(\text{id})) = \text{cl}(g^n) \).

**Definition 2.5** Let \( G \) be a group, and \( g \in G' \). The *stable commutator length* of \( g \) is the limit

\[
\text{scl}(g) = \lim_{n \to \infty} \frac{\text{cl}(g^n)}{n}
\]

Hence we have the following:

**Lemma 2.6** Let \( g \in G' \) act on \( C_S(G') \) by left multiplication. There is an equality \( \tau(g) = \text{scl}(g) \).

**Proof** This is immediate from the definitions. \( \square \)
Stable commutator length is related to two-dimensional (bounded) cohomology. For an introduction to stable commutator length, see the book by the first author [3]; for an introduction to bounded cohomology, see Gromov [8].

If $X$ is a metric space, and $g$ is an isometry of $X$, one can obtain lower bounds on $\tau(g)$ by constructing a Lipschitz function on $X$ which grows linearly on the orbit of a point under powers of $g$. One important class of Lipschitz functions on $C_S(G')$ are quasimorphisms:

**Definition 2.7** Let $G$ be a group. A function $\phi: G \to \mathbb{R}$ is a quasimorphism if there is a least positive real number $D(\phi)$ called the defect, such that for all $g, h \in G$ there is an inequality

$$|\phi(g) + \phi(h) - \phi(gh)| \leq D(\phi).$$

From the defining property of a quasimorphism, $|\phi(id)| \leq D(\phi)$ and therefore by repeated application of the triangle inequality, one can estimate

$$|\phi([f, g, h]) - \phi(f)| \leq 7D(\phi)$$

for any $f, g, h \in G$. In other words:

**Lemma 2.8** Let $G$ be a group, and let $\phi: G \to \mathbb{R}$ be a quasimorphism with defect $D(\phi)$. Then $\phi$ restricted to $G'$ is $7D(\phi)$–Lipschitz in the metric inherited from $C_S(G')$.

Word-hyperbolic groups admit a rich family of quasimorphisms. We will exploit this fact in Section 4.

# 3 Large scale simple connectivity

The following definitions are taken from Gromov [10, pages 23–24].

**Definition 3.1** A thickening $Y$ of a metric space $X$ is an isometric inclusion $X \to Y$ with the property that there is a constant $C$ so that every point in $Y$ is within distance $C$ of some point in $X$.

**Definition 3.2** A metric space $X$ is large scale $k$–connected if for every thickening $X \subset Y$ there is a thickening $Y \subset Z$ which is $k$–connected in the usual sense; i.e. $Z$ is path-connected, and $\pi_i(Z) = 0$ for $i \leq k$. 
For $G$ a finitely generated group with generating set $T$, Gromov outlines a proof [10, 1.2] that the Cayley graph $C_T(G)$ is large scale 1–connected if and only if $G$ is finitely presented, and $C_T(G)$ is large scale $k$–connected if and only if there exists a proper simplicial action of $G$ on a $(k+1)$–dimensional $k$–connected simplicial complex $X$ with compact quotient $X/G$.

For $T$ an infinite generating set, large scale simple connectivity is equivalent to the assertion that $G$ admits a presentation $G = \langle T \mid R \rangle$ where all elements in $R$ have uniformly bounded length as words in $T$; i.e. all relations in $G$ are consequences of relations of bounded length.

To show that $C_S(G)$ is large scale 1–connected, it suffices to show that there is a constant $K$ so that for every simplicial loop in $C_S(G)$ there are a sequence of loops $\gamma = \gamma_0, \gamma_1, \ldots, \gamma_n$ where $\gamma_0$ is the trivial loop, and each $\gamma_i$ is obtained from $\gamma_{i-1}$ by cutting out a subpath $\sigma_{i-1} \subset \gamma_{i-1}$ and replacing it by a subpath $\sigma_i \subset \gamma_i$ with the same endpoints, so that $|\sigma_{i-1}| + |\sigma_i| \leq K$.

More generally, we call the operation of cutting out a subpath $\sigma$ and replacing it by a subpath $\sigma'$ with the same endpoints where $|\sigma| + |\sigma'| \leq K$ a $K$–move.

**Definition 3.3** Two loops $\gamma$ and $\gamma'$ are $K$–equivalent if there is a finite sequence of $K$–moves which begins at $\gamma$, and ends at $\gamma'$.

$K$–equivalence is (as the name suggests) an equivalence relation. The statement that $C_S(G')$ is large scale 1–connected is equivalent to the statement that there is a constant $K$ such that every two loops in $C_S(G')$ are $K$–equivalent.

First we establish large scale simple connectivity in the case of a free group.

**Lemma 3.4** Let $F$ be a finitely generated free group. Then $C_S(F')$ is large scale simply connected.

**Proof** Let $\gamma$ be a loop in $C_S(F')$. After acting on $\gamma$ by left translation, we may assume that $\gamma$ passes through id, so we may think of $\gamma$ as a simplicial path in $C_S(F')$ which starts and ends at id. If $s_i \in S$ corresponds to the $i$–th segment of $\gamma$, we obtain an expression

$$s_1 s_2 \cdots s_n = \text{id}$$

in $F$, where each $s_i$ is a commutator. For each $i$, let $a_i, b_i \in F$ be elements with $[a_i, b_i] = s_i$ (note that $a_i, b_i$ with this property are not necessarily unique). Let $\Sigma$ be a surface of genus $n$, and let $\alpha_i, \beta_i$ for $i \leq n$ be a standard basis for $\pi_1(\Sigma)$; see Figure 1.
Let $X$ be a wedge of circles corresponding to free generators for $F$, so that $\pi_1(X) = F$. We can construct a basepoint preserving map $f: \Sigma \to X$ with $f_\ast(a_i) = a_i$ and $f_\ast(b_i) = b_i$ for each $i$. Since $X$ is a $K(F, 1)$, the homotopy class of $f$ is uniquely determined by the $a_i, b_i$. Informally, we could say that loops in $C_S(F')$ correspond to based homotopy classes of maps of marked oriented surfaces into $X$ (up to the ambiguity indicated above).

Let $\phi$ be a (basepoint preserving) self-homeomorphism of $\Sigma$. The map $f \circ \phi: \Sigma \to X$ determines a new loop in $C_S(F')$ (also passing through id) which we denote $\phi_\ast(\gamma)$ (despite the notation, this image does not depend only on $\gamma$, but on the choice of elements $a_i, b_i$ as above).

**Sublemma 3.5** There is a universal constant $K$ independent of $\gamma$ or of $\phi$ (or even of $F$) so that after composing $\phi$ by an inner automorphism of $\pi_1(\Sigma)$ if necessary, $\gamma$ and $\phi_\ast(\gamma)$ as above are $K$–equivalent.

**Proof** Suppose we can express $\phi$ as a product of (basepoint preserving) automorphisms

$$\phi = \phi_m \circ \phi_{m-1} \circ \cdots \circ \phi_1$$

such that if $\alpha_i^j, \beta_i^j$ denote the images of $\alpha_i, \beta_i$ under $\phi_j \circ \phi_{j-1} \circ \cdots \circ \phi_1$, then $\phi_{j+1}$ fixes all but $K$ consecutive pairs $\alpha_i^j, \beta_i^j$ up to (basepoint preserving) homotopy. Let $s_i^j = [f_\ast \alpha_i^j, f_\ast \beta_i^j]$, and let $\gamma_j$ be the loop in $C_S(F')$ corresponding to the identity $s_1^j s_2^j \cdots s_n^j = \text{id}$ in $F$.

For each $j$, let $\text{supp}_{j+1}$ denote the support of $\phi_{j+1}$; ie the set of indices $i$ such that $\phi_{j+1}(\alpha_i^j) \neq \alpha_i^j$ or $\phi_{j+1}(\beta_i^j) \neq \beta_i^j$. By hypothesis, $\text{supp}_{j+1}$ consists of at most $K$ indices for each $j$.
Because it is just the marking on $\Sigma$ which has been changed and not the map $f$, if $k \leq i \leq k + K - 1$ is a maximal consecutive string of indices in $\text{supp}_j$, then there is an equality of products

$$s_j^k s_{k+1}^j \cdots s_{k+K-1}^j = s_{k}^{j+1} s_{k+1}^{j+1} \cdots s_{k+K-1}^{j+1}$$

as elements of $F$. This can be seen geometrically as follows. The expression on the left is the image under $f_\star$ of an element represented by a certain embedded based loop in $\Sigma$, while the expression on the right is its image under $f_\star \circ \phi_{j+1}$. The automorphism $\phi_{j+1}$ is represented by a homeomorphism of $\Sigma$ whose support is contained in regions bounded by such loops. Hence the expressions are equal. It follows that $\gamma^j$ and $\gamma^{j+1}$ are $2K$–equivalent.

So to prove the sublemma it suffices to show that any automorphism of $S$ can be expressed (up to inner automorphism) as a product of automorphisms $\phi_i$ with the property above.

The hypothesis that we may compose $\phi$ by an inner automorphism means that we need only consider the image of $\phi$ in the mapping class group of $\Sigma$. It is well-known since Dehn [5] that the mapping class group of a closed oriented surface $\Sigma$ of genus $g$ is generated by twists in a finite standard set of curves, each of which intersects at most two of the $\alpha_i, \beta_i$ essentially; see Figure 2.

![Figure 2: A standard set of $3g - 1$ simple curves, in yellow. Dehn twists in these curves generate the mapping class group of $\Sigma$.](image)

So write $\phi = \tau_1 \tau_2 \cdots \tau_m$ where the $\tau_i$ are all standard generators. Now define

$$\phi_j = \tau_1 \tau_2 \cdots \tau_{j-1} \tau_j \tau_{j-1}^{-1} \cdots \tau_1^{-1}.$$ 

We have

$$\phi_j \phi_{j-1} \cdots \phi_1 = \tau_1 \tau_2 \cdots \tau_j.$$
Moreover, each $\phi_j$ is a Dehn twist in a curve which is the image of a standard curve under $\phi_j^{-1} \cdots \phi_1$, and therefore intersects $\alpha_i^{-1}$ and $\beta_i^{-1}$ essentially for at most 2 (consecutive) indices $i$. This completes the proof of the sublemma (and shows, in fact, that we can take $K = 4$).

We now complete the proof of the lemma. As observed by Stallings (in eg [14]), a nontrivial map $f: \Sigma \to X$ from a closed, oriented surface to a wedge of circles factors (up to homotopy) through a pinch in the following sense. Make $f$ transverse to some edge $e$ of $X$, and look at the preimage $\Gamma$ of a regular value of $f$ in $e$. After homotoping inessential loops of $\Gamma$ off $e$, we may assume that for some edge $e$ and some regular value, the preimage $\Gamma$ contains an embedded essential loop $\delta$.

There are two cases to consider. In the first case, $\delta$ is nonseparating. In this case, let $\phi$ be an automorphism which takes $\alpha_1$ to the free homotopy class of $\delta$. Then $\gamma$ and $\phi_* (\gamma)$ are $K$–equivalent by the sublemma. However, since $f(\delta)$ is homotopically trivial in $X$, there is an identity $[\phi_* \alpha_1, \phi_* \beta_1] = \text{id}$ and therefore $\phi_* (\gamma)$ has length 1 shorter than $\gamma$.

In the second case, $\phi$ is separating, and we can let $\phi$ be an automorphism which takes the free homotopy class of $[\alpha_1, \beta_1] \cdots [\alpha_j, \beta_j]$ to $\delta$. Again, by the sublemma, $\gamma$ and $\phi_* (\gamma)$ are $K$–equivalent. But now $\phi_* (\gamma)$ contains a subarc of length $j$ with both endpoints at id, so we may write it as a product of two loops at id, each of length shorter than that of $\gamma$.

By induction, $\gamma$ is $K$–equivalent to the trivial loop, and we are done.

We are now in a position to prove our first main theorem.

**Theorem A** Let $G$ be a finitely presented group. Then $C_S (G')$ is large scale simply connected.

**Proof** Let $W$ be a smooth 4–manifold (with boundary) satisfying $\pi_1 (W) = G$. If $G = (T \mid R)$ is a finite presentation, we can build $W$ as a handlebody, with one 0–handle, one 1–handle for every generator in $T$, and one 2–handle for every relation in $R$. If $r_i \in R$ is a relation, let $D_i$ be the cocore of the corresponding 2–handle, so that $D_i$ is a properly embedded disk in $W$. Let $V \subset W$ be the union of the 0–handle and the 1–handles. Topologically, $V$ is homotopy equivalent to a wedge of circles. By the definition of cocores, the complement of $\bigcup D_i$ in $W$ deformation retracts to $V$. See eg Kirby [12, Chapter 1] for an introduction to handle decompositions of 4–manifolds.
Given $\gamma$ a loop in $C_S(G')$, translate it by left multiplication so that it passes through $id$. As before, let $\Sigma$ be a closed oriented marked surface, and $f: \Sigma \to W$ a map representing $\gamma$.

Since $G$ is finitely presented, $H_2(G; \mathbb{Z})$ is finitely generated. Choose finitely many closed oriented surfaces $S_1, \ldots, S_r$ in $W$ which generate $H_2(G; \mathbb{Z})$. Let $K'$ be the supremum of the genus of the $S_i$. We can choose a basepoint on each $S_i$, and maps to $W$ which are basepoint preserving. By tubing $\Sigma$ repeatedly to copies of the $S_i$ with either orientation, we obtain a new surface and map $f': \Sigma' \to W$ representing a loop $\gamma'$ such that $f'(\Sigma')$ is null-homologous in $W$, and $\gamma'$ is $K'$--equivalent to $\gamma$ (note that $K'$ depends on $G$ but not on $\gamma$).

Put $f'$ in general position with respect to the $D_i$ by a homotopy. Since $f'(\Sigma')$ is null-homologous, for each proper disk $D_i$, the signed intersection number vanishes: $D_i \cap f'(\Sigma') = 0$. Hence $f'(\Sigma) \cap D_i = P_i$ is a finite, even number of points which can be partitioned into two sets of equal size corresponding to the local intersection number of $f'(\Sigma')$ with $D_i$ at $p \in P_i$.

Let $p, q \in P_i$ have opposite signs, and let $\mu$ be an embedded path in $D_i$ from $f'(p)$ to $f'(q)$. Identifying $p$ and $q$ implicitly with their preimages in $\Sigma'$, let $\alpha$ and $\beta$ be arcs in $\Sigma'$ from the basepoint to $(f')^{-1} p$ and $(f')^{-1} q$. Since $\mu$ is contractible, there is a neighborhood of $\mu$ in $D_i$ on which the normal bundle is trivializable. Hence, since $f'(\Sigma')$ and $D_i$ are transverse, we can find a neighborhood $U$ of $\mu$ in $W$ disjoint from the other $D_j$, and coordinates on $U$ satisfying:

1. $D_i \cap U$ is the plane $(x, y, 0, 0)$.
2. $\mu \cap U$ is the interval $(t, 0, 0, 0)$ for $t \in [0, 1]$.
3. $f'(\Sigma') \cap U$ is the union of the planes $(0, 0, z, w)$ and $(1, 0, z, w)$.

Let $A$ be the annulus consisting of points $(t, 0, \cos(\theta), \sin(\theta))$ where $t \in [0, 1]$. Then $A$ is disjoint from $D_i$ and all the other $D_j$, and we can tube $f'(\Sigma')$ with $A$ to reduce the number of intersection points of $f'(\Sigma')$ with $D_i$, at the cost of raising the genus by 1. Technically, we remove the disks $(f')^{-1}(0, 0, s \cos(\theta), s \sin(\theta))$ and $(f')^{-1}(1, 0, s \cos(\theta), s \sin(\theta))$ for $s \in [0, 1]$ from $\Sigma'$, and sew in a new annulus which we map homeomorphically to $A$. The result is $f'': \Sigma'' \to W$ with two fewer intersection points with $D_i$. This has the effect of adding a new (trivial) edge to the start of $\gamma'$, which is the commutator of the elements represented by the core of $A$ and the loop $f'(\alpha) \ast \mu \ast f'(\beta)$. Let $\gamma''$ denote this resulting loop, and observe that $\gamma''$ is 1--equivalent to $\gamma'$. After finitely many operations of this kind, we obtain $f'''$: $\Sigma''' \to W$ corresponding to a loop $\gamma'''$ which is max($1, K'$)--equivalent to $\gamma$, such that $f'''(\Sigma''')$ is disjoint from $\bigcup_i D_i$.
After composing with a deformation retraction, we may assume \( f' \) maps \( \Sigma'' \) into \( V \). Let \( F = \pi_1(V) \), and let \( \rho: F \to G \) be the homomorphism induced by the inclusion \( V \to W \). There is a loop \( \gamma^F \) in \( C_S(F') \) corresponding to \( f'' \) such that \( \rho_*(\gamma^F) = \gamma''' \) under the obvious simplicial map \( \rho_*: C_S(F') \to C_S(G') \). By Lemma 3.4, the loop \( \gamma^F \) is \( K \)-equivalent to a trivial loop in \( C_S(F') \). Pushing forward the sequence of intermediate loops by \( \rho_* \) shows that \( \gamma''' \) is \( K \)-equivalent to a trivial loop in \( C_S(G') \). Since \( \gamma \) was arbitrary, we are done.

\[ \square \]

**Remark 3.6** A similar, though perhaps more combinatorial argument could be made working directly with \( 2 \)-complexes in place of \( 4 \)-manifolds.

In words, Theorem A says that for \( G \) a finitely presented group, all relations amongst the commutators of \( G \) are consequences of relations involving only boundedly many commutators.

The next example shows that the size of this bound depends on \( G \):

**Example 3.7** Let \( \Sigma \) be a closed surface of genus \( g \), and \( G = \pi_1(\Sigma) \). If \( \gamma \) is a loop in \( C_S(G) \) through the origin, and \( f: \Sigma' \to \Sigma \) is a corresponding map of a closed surface, then the homology class of \( \Sigma' \) is trivial unless the genus of \( \Sigma' \) is at least as big as that of \( \Sigma \). Hence the loop in \( C_S(G) \) of length \( g \) corresponding to the relation in the “standard” presentation of \( \pi_1(\Sigma) \) is not \( K \)-equivalent to the trivial loop whenever \( K < g \).

In light of Theorem A, it is natural to ask the following question:

**Question 3.8** Let \( G \) be a finitely presented group. Is \( C_S(G') \) large scale \( k \)-connected for all \( k \)?

**Remark 3.9** Laurent Bartholdi has pointed out that for \( F \) a finitely generated free group, there is a confluent, Noetherian rewriting system for \( F' \), with rules of bounded length, which puts every word in \( F' \) over generators \( S \) into normal form (with respect to a “standard” free generating set for \( F' \)). By results of Groves [11] this should imply that \( C_S(F') \) is large scale \( k \)-connected for all \( k \), but we have not verified this implication carefully. In any case, it gives another more algebraic proof of Lemma 3.4.

### 4 Word-hyperbolic groups

In this section we specialize to the class of word-hyperbolic groups. See Gromov [9] for more details.
**Definition 4.1** A path metric space \( X \) is \( \delta \)-hyperbolic for some \( \delta \geq 0 \) if for every geodesic triangle \( abc \), and every point \( p \) on the edge \( ab \), there is \( q \in ac \cup bc \) with \( d_X(p, q) \leq \delta \). In other words, the \( \delta \) neighborhood of any two sides of a geodesic triangle contains the third side.

**Definition 4.2** A group \( G \) is word-hyperbolic if there is a finite generating set \( T \) for \( G \) such that \( C_T(G) \) is \( \delta \)-hyperbolic as a path metric space, for some \( \delta \).

**Example 4.3** Finitely generated free groups are word-hyperbolic. The fundamental group of a closed surface with negative Euler characteristic is word-hyperbolic. Discrete cocompact groups of isometries of hyperbolic \( n \)-space are word-hyperbolic.

To rule out some trivial examples, one makes the following:

**Definition 4.4** A word-hyperbolic group is *elementary* if it has a cyclic subgroup of finite index, and *nonelementary* otherwise.

The main theorem we prove in this section concerns the geometry of \( C_S(G') \) at infinity, where \( G \) is a nonelementary word-hyperbolic group. For the sake of brevity we restrict attention to torsion-free \( G \), though this restriction is not logically necessary; see Remark 4.9.

**Theorem B** Let \( G \) be a torsion-free nonelementary word-hyperbolic group. Then \( C_S(G') \) is one-ended; i.e., for any \( r > 0 \) there is an \( R \geq r \) such that any two points in \( C_S(G') \) at distance at least \( R \) from \( \text{id} \) can be joined by a path which does not come closer than distance \( r \) to \( \text{id} \).

We will estimate distance to \( \text{id} \) in \( C_S(G') \) using quasimorphisms, as indicated in Section 2. Hyperbolic groups admit a rich family of quasimorphisms. Of particular interest to us are the *Epstein–Fujiwara counting quasimorphisms*, introduced in [6], generalizing a construction due to Brooks [2] for free groups.

Fix a word-hyperbolic group \( G \) and a finite generating set \( T \). Let \( C_T(G) \) denote the Cayley graph of \( G \) with respect to \( T \). Let \( \sigma \) be an oriented simplicial path in \( C_T(G) \). A *copy* of \( \sigma \) is a translate \( g \cdot \sigma \) for some \( g \in G \). If \( \gamma \) is an oriented simplicial path in \( C_T(G) \), let \( |\gamma|_\sigma \) denote the maximal number of disjoint copies of \( \sigma \) contained in \( \gamma \). For \( g \in G \), define

\[
c_\sigma(g) = d(\text{id}, g) - \inf_{\gamma} (\text{length}(\gamma) - |\gamma|_\sigma)
\]

where the infimum is taken over all directed paths \( \gamma \) in \( C_T(G) \) from \( \text{id} \) to \( g \), and \( d(\cdot, \cdot) \) denotes distance in \( C_T(G) \).
Definition 4.5 (Epstein–Fujiwara) A counting quasimorphism on $G$ is a function of the form
\[ h_\sigma(g) := c_\sigma(g) - c_{\sigma^{-1}}(g) \]
where $\sigma^{-1}$ denotes the same simplicial path as $\sigma$ with the opposite orientation.

Since $|\gamma|_{\sigma}$ takes discrete values, the infimum is realized in the definition of $c_\sigma$. A path $\gamma$ for which
\[ c_\sigma(g) = d(\text{id}, g) - \text{length}(\gamma) + |\gamma|_{\sigma} \]
is called a realizing path for $g$. Realizing paths exist, and satisfy the following geometric property:

Lemma 4.6 (Epstein–Fujiwara [6, Proposition 2.2]) Any realizing path for $g$ is a $(K, \epsilon)$–quasigeodesic in $C_T(G)$, where
\[ K = \frac{\text{length}(\sigma)}{\text{length}(\sigma) - 1} \quad \text{and} \quad \epsilon = \frac{2 \cdot \text{length}(\sigma)}{\text{length}(\sigma) - 1}. \]

Moreover, the following holds:

Lemma 4.7 (Epstein–Fujiwara [6, Proposition 2.13]) Let $\sigma$ be a path in $C_T(G)$ of length at least 2. Then there is a constant $K(\delta)$ (where $T$ is such that $C_T(G)$ is $\delta$–hyperbolic as a metric space) such that $D(h_\sigma) \leq K(\delta)$.

Counting quasimorphisms are very versatile, as the following lemma shows:

Lemma 4.8 Let $G$ be a torsion-free, nonelementary word-hyperbolic group. Let $g_i$ be a finite collection of elements of $G$. There is a commutator $s \in G'$ and a quasimorphism $\phi$ on $G$ with the following properties:

1. $|\phi(g_i)| = 0$ for all $i$.
2. $|\phi(s^n) - n| \leq K_1$ for all $n$, where $K_1$ is a constant which depends only on $G$.
3. $D(\phi) \leq K_2$ where $K_2$ is a constant which depends only on $G$.

Proof Fix a finite generating set $T$ so that $C_T(G)$ is $\delta$–hyperbolic. There is a constant $N$ such that for any nonzero $g \in G$, the power $g^N$ fixes an axis $L_g$ [9]. Since $G$ is nonelementary, it contains quasigeodesically embedded copies of free groups, of any fixed rank. So we can find a commutator $s$ whose translation length (in $C_T(G)$) is as big as desired. In particular, given $g_1, \ldots, g_j$ we choose $s$ with $\tau(s) \gg \tau(g_i)$ for all $i$. Let $L$ be a geodesic axis for $s^N$, and let $\sigma$ be a fundamental domain for the action of $s^N$ on $L$. Since $|\sigma| = N \tau(s) \gg \tau(g_i)$, Lemma 4.6 implies...
that there are no copies of $\sigma$ or $\sigma^{-1}$ in a realizing path for any $g_i$. Hence $h_\sigma(g_i) = 0$ for all $i$. By Lemma 4.7, $D(h_\sigma) \leq K(\delta)$. It remains to estimate $h_\sigma(s^n)$.

In fact, the argument of [4] Theorem A' (which establishes explicitly an estimate that is implicit in [6]) shows that for $N$ sufficiently large (depending only on $G$ and not on $s$) no copies of $1$ are contained in any realizing path for $s^n$ with $n$ positive, and therefore $|h_\sigma(s^n) - [n/N]|$ is bounded by a constant depending only on $G$. The quasimorphism $\phi = N \cdot h_\sigma$ has the desired properties.

\[ \square \]

**Remark 4.9** The hypothesis that $G$ is torsion-free is included only to ensure that $s$ is not conjugate to $s^{-1}$. It is possible to remove this hypothesis by taking slightly more care in the definition of $s$, using the methods of the proof of Proposition 2 from [1]. We are grateful to the referee for pointing this out.

We now give the proof of Theorem B:

**Proof** Let $g, h \in G'$ have commutator length at least $R$. Let $g = s_1s_2 \cdots s_n$ and $h = t_1t_2 \cdots t_m$ where $n, m \geq R$ are equal to the commutator lengths of $g$ and $h$ respectively, and each $s_i, t_i$ is a commutator in $G$. Let $s$ be a commutator with the properties described in Lemma 4.8 with respect to the elements $g, h$; that is, we want $s$ for which there is a quasimorphism $\phi$ with $\phi(g) = \phi(h) = 0$, with $|\phi(s^n) - n| \leq K_1$ for all $n$, and with $D(\phi) \leq K_2$. Let $N \gg R$ be very large. We build a path in $C_S(G')$ from $g$ to $h$ out of four segments, none of which come too close to id.

The first segment is

$g, gs, gs^2, gs^3, \ldots, gs^n$.

Since $s$ is a commutator, $d(gs^i, \text{id}) \geq R - i$ for any $i$. On the other hand,

$\phi(gs^i) \geq \phi(g) + \phi(s^i) - D(\phi) \geq i - K_2 - K_1$

where $K_1, K_2$ are as in Lemma 4.8 (and do not depend on $g, h, s$). From Lemma 2.8 we can estimate

$d(gs^i, \text{id}) \geq \frac{\phi(gs^i)}{7D(\phi)} \geq \frac{i - K_2 - K_1}{7K_2}$.

Hence $d(gs^i, \text{id}) \geq R/14K_2 - (K_1 + K_2)/7K_2$ for all $i$, so providing $R \gg K_1, K_2$, the path $gs^i$ never gets too close to id.

The second segment is

$gs^N = s_1s_2 \cdots s_ns^N, s_2 \cdots s_ns^N, \ldots, s^N$.
Note that consecutive elements in this segment are distance 1 apart in $C_S(G')$, by Lemma 2.3. Since $d(gs^N, \text{id}) \geq (N - K_2 - K_1)/7K_2 \gg R$ for $N$ sufficiently large, we have that for all $i$,
\[ d(s_i \cdots s_n s^N, \text{id}) \gg R. \]

The third segment is
\[ s^N, t_m s^N, t_{m-1} t_m s^N, \ldots, t_1 t_2 \cdots t_m s^N = h s^N \]
and the fourth is
\[ h s^N, h s^{N-1}, \ldots, h s, h. \]
For the same reason as above, neither of these segments gets too close to $\text{id}$. This completes the proof of the theorem, taking $r = R/14K_2 - (K_1 + K_2)/7K_2$. \hfill \Box

5 Asymptotic dimension

The main point of this section is to make the observation that $G'$ for $G$ as above is not a quasitree, and to restate this observation in terms of asymptotic dimension. We think it is worth making this restatement explicitly. The notion of asymptotic dimension was introduced by Gromov [10, page 32].

**Definition 5.1** Let $X$ be a metric space, and $X = \bigcup_i U_i$ a covering by subsets. For given $D \geq 0$, the $D$–multiplicity of the covering is at most $n$ if for any $x \in X$, the closed $D$–ball centered at $x$ intersects at most $n$ of the $U_i$.

A metric space $X$ has **asymptotic dimension at most** $n$ if for every $D \geq 0$ there is a covering $X = \bigcup_i U_i$ for which the diameters of the $U_i$ are uniformly bounded, and the $D$–multiplicity of the covering is at most $n + 1$. The least such $n$ is the asymptotic dimension of $X$, and we write
\[ \text{asdim}(X) = n. \]

If $X$ is a metric space, we say $H_1(X)$ is uniformly generated if there is a constant $L$ such that $H_1(X)$ is generated by loops of length at most $L$. It is clear that if $X$ is large scale 1–connected, then $H_1(X)$ is uniformly generated. Fujiwara–Whyte [7] prove the following theorem:

**Theorem 5.2** (Fujiwara–Whyte [7, Theorem 0.1]) Let $X$ be a geodesic metric space with $H_1(X)$ uniformly generated. $X$ has asdim$(X) = 1$ if and only if $X$ is quasi-isometric to an unbounded tree.
A group whose Cayley graph is quasi-isometric to an unbounded tree has more than one end (see e.g. Manning [13], especially Sections 2.1 and 2.2). Hence Theorem A and Theorem B together imply the following:

**Corollary 5.3** Let $G$ be a nonelementary torsion-free word-hyperbolic group. Then $\text{asdim}(C_{S}(G')) \geq 2$.

**Acknowledgments** We would like to thank Koji Fujiwara for some useful conversations. We would also like to thank the anonymous referee for a careful reading, and many useful comments. Danny Calegari was partially funded by NSF grant DMS 0707130.

**References**


Department of Mathematics, California Institute of Technology
Pasadena CA 91125, USA
dannyc@its.caltech.edu, dongping@its.caltech.edu
http://www.its.caltech.edu/~dannyc

Received: 29 July 2008 Revised: 1 October 2008