About the macroscopic dimension of certain PSC–Manifolds

Dmitry Bolotov

In this note we give a partial answer to Gromov’s question about macroscopic dimension filling of a closed spin PSC–Manifold’s universal covering.

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1 Introduction

The following definition was given by M Gromov in [3].

Definition 1.1 Let $V$ be a metric space. We say that $\dim_v V \leq k$ if there exists a $k$–dimensional polyhedron $P$ and a proper uniformly co-bounded map $\phi: V \to P$ such that $\operatorname{Diam}(\phi^{-1}(p)) \leq \varepsilon$ for all $p \in P$. A metric space $V$ has the macroscopic $\dim_{mc} V \leq k$ if $\dim_v V \leq k$ for some possibly large $\varepsilon < \infty$. If $k$ is as minimal as possible, we say that $\dim_{mc} V = k$.

Gromov also posed the following conjecture.

Conjecture C1 Let $(M^n, g)$ be a closed Riemannian $n$–manifold with torsion free fundamental group and $(\tilde{M}^n, \tilde{g})$ be the universal covering of $M^n$ with the pull-back metric. Suppose that $\dim_{mc}(\tilde{M}^n, \tilde{g}) < n$. Then $\dim_{mc}(\tilde{M}^n, \tilde{g}) < n - 1$.

Remark 1.2 In fact the macroscopic dimension $\dim_{mc}(\tilde{M}^n, \tilde{g})$ of the universal covering $\tilde{M}^n$ of $M$ does not depend on a particular choice of a Riemannian metric $g$ on $M$, since the Riemannian manifolds $(\tilde{M}^n, \tilde{g})$ and $(\tilde{M}^n, \tilde{g}')$ are quasi-isometric for any two metrics $g$ and $g'$ on $M$.

This conjecture is true for $n = 3$ (see Bolotov [1]). In [2] the author shown that it fails for $n > 3$.

Actually this question arose in M Gromov’s works in connection with the study of PSC–manifolds, ie manifolds admitting a Positive Scalar Curvature metric.

The following is Gromov’s PSC–conjecture.
Conjecture C2 Let \((M^n, g)\) be a closed Riemannian PSC–manifold with torsion free fundamental group, and let \((\tilde{M}^n, \tilde{g})\) be the universal covering of \(M^n\) with the pull-back metric, then \(\dim_{mc}(\tilde{M}^n, \tilde{g}) < n - 1\).

Let us also recall the Gromov–Lawson–Rosenberg conjecture.

Conjecture 1.3 Let \(M^n\) be a closed spin manifold, \(\pi = \pi_1 M^n\), and let \(f: M^n \to B\pi\) be a classifying map. Then \(M^n\) admits a PSC–metric if and only if

\[
A \circ f_*([M^n]_{KO}) = 0
\]

in \(KO_\pi(C^*_r(\pi))\), where \([M^n]_{KO} \in KO_\pi(M^n)\) is the corresponding fundamental class in \(KO\)–theory, \(C^*_r(\pi)\) is the reduced \(C^*\)–algebra of the group \(\pi\), and

\[
A: KO_\pi(B\pi) \to KO_\pi(C^*_r(\pi))
\]

is the assembly homomorphism of homology theories.

Remark 1.4 \(f_*[M^n]_{KO}\) depends only on the bordism class \([M^n, f] \in \Omega^n_{Spin}(B\pi)\) (see Hitchin [5], Gromov–Lawson [4]).

The following important theorem is proved by J Rosenberg.

Theorem 1.5 (Rosenberg [6]) Let \(M^n\) be a spin manifold, \(\pi = \pi_1 M^n\), and \(f: M^n \to B\pi\) be a classifying map. If \(M^n\) is a PSC–manifold then \(A \circ f_*[M^n]_{KO} = 0\).

Recall that the Strong Novikov Conjecture asserts the following.

Conjecture 1.6 The assembly map \(A: KO_\pi(B\pi) \to KO_\pi(C^*_r(\pi))\) is a monomorphism.

In this paper we prove the following theorem.

Main Theorem Let \(M^n\) be a closed spin PSC–manifold and \(\pi = \pi_1 M^n\). Suppose that \(cd\pi \leq n - 1\) and that the Strong Novikov Conjecture holds for \(\pi\). Then Conjecture C2 is true for \(M^n\) as well.

Remark 1.7 Clearly, for the proof of this result it is sufficient to show that the classifying map \(f: M^n \to B\pi\) can be deformed into the \((n - 2)\)–skeleton of \(B\pi\). In this case the covering map \(\tilde{f}: \tilde{M} \to \tilde{B}\pi^{(n-2)}\) would yield the result. Also notice that the Main Theorem is nontrivial only in the case \(cd\pi = n - 1\).
2 Proof of Main Theorem

Proof Let $M$ be an oriented, closed, $n$–dimensional, spin manifold with torsion free fundamental group $\pi$. Notice that the Main Theorem is trivial for $n = 2$. Moreover, since $\text{cd}\pi \neq 2$ for $n = 3$, we will assume that $n \geq 4$. Suppose also that $\text{cd} \pi = n - 1$ and $[M] = 0 \in \Omega^\text{Spin}_n(*)$.

Consider the following composition of maps:

$$M \xrightarrow{f} B\pi \xrightarrow{p} B\pi/B\pi^{(n-2)},$$

where $p$ is the factor-map to the factor-space $B\pi/B\pi^{(n-2)}$ of $B\pi$ by its $(n-2)$–skeleton.

Since $\text{cd} \pi = n - 1$, we can assume that $\dim B\pi = n - 1$ and $B\pi/B\pi^{(n-2)}$ is homeomorphic to a bouquet of $(n-1)$–dimensional spheres.

We can also assume that $M$ is endowed with cellular decomposition having only one cell in each dimensions 0 and $n$, and that $f$ is a cellular map. Let $f^{(n-2)}$ be the restriction of $f$ to the $(n-2)$–skeleton of $M$. The first obstruction class $[c^{n-1}_f]$ for the extension of $f^{(n-2)}$ to the $(n-1)$–skeleton belongs to the group $H^{n-1}(M, \pi_{n-2}(B\pi^{(n-2)}))$.

Notice that by Hurewicz’s theorem

$$\pi_{n-2}(B\pi^{(n-2)}) \cong H_{n-2}(B\pi^{(n-2)}, \mathbb{Z}[\pi])$$

is a free $\mathbb{Z}[\pi]$–module. Hence by Poincaré duality with twisted coefficients

$$H^k(M, \oplus_i \mathbb{Z}[\pi]) \cong H^k_c(\hat{M}, \oplus_i \mathbb{Z}) \cong H_{n-k}(\hat{M}, \oplus_i \mathbb{Z}).$$  \hfill (*)

Since $H_1(\hat{M}, \oplus_i \mathbb{Z}) = 0$, we can extend $f^{(n-2)}$ to a map $f^{(n-1)}: M^{(n-1)} \to B\pi^{(n-2)}$ changing (if necessary) $f^{(n-2)}$ on the $(n-2)$–skeleton, but not changing $f^{(n-2)}$ on the $(n-3)$–skeleton.

Since $B\pi$ is a $K(\pi, 1)$–space, we can also extend $f^{(n-1)}$ to the map $\hat{f}: M \to B\pi$. In the sequel we will denote this map $\hat{f}$ by $f$.

Notice that the first obstruction $c^n_f \in C^n(M, \pi_{n-1}(B\pi^{(n-2)}))$ for an extension of $f^{(n-1)}$ to all of $M$ can be represented as a composition of $\mathbb{Z}[\pi]$–module homomorphisms:

$$c^n_f: C_n(M, \mathbb{Z}[\pi]) \cong \pi_n(M, M^{(n-1)}) \xrightarrow{\partial} \pi_{n-1}(M^{(n-1)}) \xrightarrow{\partial} \pi_{n-1}(B\pi^{(n-2)}).$$
Consider the following commutative diagram:

\[
\begin{array}{ccc}
\pi_n(M, M^{(n-1)}) & \xrightarrow{f_*} & \pi_n(B\pi, B\pi^{(n-2)}) \\
\downarrow\partial & & \downarrow\partial \\
\pi_{n-1}(M^{(n-1)}) & \xrightarrow{f_{*(n-1)}} & \pi_{n-1}(B\pi^{(n-2)})
\end{array}
\]

Notice that \(\pi_n(B\pi, B\pi^{(n-2)}) \xrightarrow{\partial} \pi_{n-1}(B\pi^{(n-2)})\) is an isomorphism. This can easily be seen from the exact sequence of the pair \((B\pi, B\pi^{(n-2)})\) since \(\pi_i(B\pi) = 0\) for \(i \geq 2\).

Recall that \(n \geq 4\) and

\[
\pi_n(B\pi, B\pi^{(n-2)}) \cong \pi_n(\widetilde{B}\pi, \widetilde{B}\pi^{(n-2)}) \cong \pi_n(\widetilde{B}\pi / \widetilde{B}\pi^{(n-2)})
\]

is a free \(\mathbb{Z}_2[\pi]\)-module.

Using Poincaré duality for the \(\mathbb{Z}[\pi]\)-module \(\Lambda = \pi_n(\widetilde{B}\pi / \widetilde{B}\pi^{(n-2)})\) it is not hard to verify that

\[
H^n(M, \Lambda) \cong \Lambda \otimes \mathbb{Z}[\pi] \mathbb{Z} \cong \oplus_1 \mathbb{Z}_2.
\]

Consider the following commutative diagram:

\[
\begin{array}{ccc}
\pi_n(M, M^{(n-1)}) & \xrightarrow{f_*} & \pi_n(B\pi, B\pi^{(n-2)}) \\
\downarrow\otimes\mathbb{Z} & & \downarrow\otimes\mathbb{Z} \\
\pi_n(M, M^{(n-1)}) \otimes \mathbb{Z} & \xrightarrow{f_*} & \pi_n(B\pi, B\pi^{(n-2)}) \otimes \mathbb{Z}
\end{array}
\]

Clearly, \(\pi_n(M, M^{(n-1)}) \otimes \mathbb{Z} \cong \pi_n(M/M^{(n-1)}) \cong \pi_n(S^n)\) and

\[
\pi_n(B\pi, B\pi^{(n-2)}) \otimes \mathbb{Z} \cong \pi_n(B\pi / B\pi^{(n-2)}) \cong \pi_n(S^n).\]

We conclude that \([c_f] = (f_* \circ \otimes \mathbb{Z})(c)\), where \(c\) is a generator of free module \(\pi_n(M, M^{(n-1)})\), and \([c_f]\) is represented by the map (as an element of the homotopy group \(\pi_n(S^{n-1})\)):

\[
M/M^{(n-1)} \cong S^n \xrightarrow{f} B\pi / B\pi^{(n-2)} \cong S^{n-1}.\]
Consider the following commutative diagram:

\[
\begin{array}{ccc}
M & \xrightarrow{q} & S^n & \xrightarrow{id} & S^n \\
\downarrow f & & \downarrow \bar{f} & & \downarrow h \\
\B_\pi & \xrightarrow{p} & S^{n-1} & \xrightarrow{p_k} & S^{n-1}
\end{array}
\]  (***)

Suppose that for some \( k \) the map \( p_k \circ \bar{f} \) is not null-homotopic, where \( p_k \) is the projection on the \( k \)-factor of the bouquet. The map

\[
p_k \circ p \circ f : M \to S^{n-1}
\]

induces a composition of homomorphisms

\[
p_k^* \circ p^* \circ f^* : KO_n(M) \to KO_n(S^{n-1}).
\]

Clearly, if

\[
p_k^* \circ p^* \circ f^*[M]_{KO} = (h \circ q)^*[M]_{KO} \neq 0,
\]

then \( f^*[M]_{KO} \neq 0 \) as well.

**Lemma 2.1** Let \( \Omega^n_{Spin}(S^{n-1}) \) be the \( n \)th bordism group of \( S^{n-1} \). Then \( [(M, h \circ q)] = [(S^n, h)] \) in \( \Omega^n_{Spin}(S^{n-1}) \).

**Proof** Since \( [M] = 0 \in \Omega^n_{Spin}(*) \), there exists an \((n+1)\)-dimensional spin manifold \( W \) with \( \partial W = M \). Let \( B \subset W \) be a small open ball and

\[
i : D^n \times I \to W \setminus B
\]

be a regular normal neighborhood of the transversal segment \( i : 0 \times I \to W \setminus B \), such that \( i(0, 0) \in M \) and \( i(0, 1) \in \partial B \cong S^n \). Define the following map:

\[
i(D^n \times I) \xrightarrow{\text{retraction}} i(D^n \times 0) \xrightarrow{\text{quotient}} i(D^n \times 0) / i(\partial D^n \times 0) \cong S^n \xrightarrow{h} S^{n-1}.
\]

We can extend it to the map \( F : W \setminus B \to S^{n-1} \) which is constant outside \( i(D^n \times I) \). Clearly, the restriction \( F|_M \) is homotopic to \( h \circ q \) in \( M \) and the restriction \( F|_{\partial B} \) is homotopic to \( h \in \partial B \).

Since \( (h \circ q)^*[M]_{KO} \) depends only on the bordism class in \( \Omega^n_{Spin}(S^{n-1}) \) (Hitchin [5]), we obtain from **Lemma 2.1** that

\[
(h \circ q)^*[M]_{KO} = h^*[S^n]_{KO}.
\]

We will now show that \( [h] \in \pi_1^x \) represents a non-zero element \( h^*[S^n]_{KO} \) in \( KO_n(S^{n-1}) \).
By assumption $h$ is not homotopic to zero. Therefore $h$ must be homotopic to the Hopf $(n-3)$–suspension $H: S^n \to S^{n-1}$ which induces a homomorphism $H_*: KO_n(S^n) \to KO_n(S^{n-1})$. But $H_*[S^n]_{KO} \neq 0$. Indeed, let $\tilde{S}^1$ be a circle with nontrivial spin structure and $pr: S^{n-1} \times S^1 \to S^n$ be the natural projection. Using framed surgery along generating circle $\tilde{S}^1$ it is easy to verify that:

$$[(S^n, H)] = [(S^{n-1} \times \tilde{S}^1, pr)] \in \Omega^\text{Spin}_n(S^{n-1}).$$

But

$$KO_n(S^{n-1}) = KO_n(\mathbb{R}^{n-1}) \oplus KO_n(*) .$$

Moreover, $pr_*[(S^{n-1} \times \tilde{S}^1)]_{KO}$ is equal to the generator of $KO_n(\mathbb{R}^{n-1}) \cong KO_{n-1}(\mathbb{R}^{n-1}) \otimes KO_1(*) \cong \mathbb{Z} \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2$ and

$$h_*[S^n]_{KO} = H_*[S^n]_{KO} = pr_*[(S^{n-1} \times \tilde{S}^1)]_{KO} \neq 0.$$ 

Thus both $(h \circ q)_* [M]_{KO}$ and $f_* [M]_{KO}$ are non-zero.

Therefore if the Strong Novikov Conjecture is true, then by Theorem 1.5, $M$ does not admit a PSC–metric.

We conclude that if $M$ is a PSC–manifold, then $[\overline{f}] = 0$. Therefore $f$ can be deformed to the $(n-2)$–skeleton of $B\pi$ and by Remark 1.7 dim$_{mc} \widetilde{M} \leq n-2$.

In the case when $[M] \neq 0 \in \Omega^\text{Spin}_n(*)$ we can consider the manifold $M \times S^1$ representing $0 \in \Omega^\text{Spin}_{n+1}(*)$. Clearly, $M \times S^1$ is a PSC–manifold whenever $M$ is a PSC–manifold.

Let us consider the following diagram:

$$
\begin{array}{ccccccc}
M \times S^1 & \xrightarrow{S} & SM & \xrightarrow{Sg} & S^{n+1} & \xrightarrow{id} & S^{n+1} \\
\downarrow f & & \downarrow Sf & & \downarrow S\bar{f} & & \downarrow Sh \\
B\pi \times S^1 & \xrightarrow{S} & SB\pi & \xrightarrow{Sp} & \forall S^n & \xrightarrow{p_k} & S^n
\end{array}
$$

where the symbol $S$ means a suspension.

For the natural cell decomposition of $M \times S^1$ the result for $M$ follows from the previous discussion of $M \times S^1$ taking into account that if $h \sim H$, then $Sh \sim SH$.

**Corollary 2.2** The counterexamples to the Conjecture C1 constructed in [2] do not admit PSC–metrics.

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References


B Verkin Institute for Low Temperature Physics, Lenina Ave 47
Kharkov 61103, Ukraine

bolotov@univer.kharkov.ua

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