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A Toda bracket in the stable homotopy groups of spheres

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Let p be a prime number greater than five. In the p–local stable homotopy groups of spheres, H Toda and J Lin, respectively, constructed the elements

$$\gamma_s \in \pi_{2sp^3-2p^2-2p-2s+1}(S),$$
$$\omega_{m,n} \in \pi_{2p^{n+1}-2p^n+2p^{m+1}-2p^m+2p-6}(S)$$

of order p. In this paper, we show the nontriviality of the Toda bracket $\langle \gamma_s, p, \omega_{m,n} \rangle$ in the stable homotopy groups of spheres, where $n \geqslant m+2 > 6$, $3 \leqslant s < p$.

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1 Introduction

We are interested in the problem of detecting nontrivial elements in the stable homotopy groups of spheres. So far, several methods have been found to determine the stable homotopy groups of spheres. For example we have the classical Adams spectral sequence (ASS) [1] based on the Eilenberg–MacLane spectrum $K\mathbb{Z}_p$, whose E_2–term is $\mathrm{Ext}_A^{s,t}(\mathbb{Z}_p, \mathbb{Z}_p)$ and Adams differential given by

$$\widetilde{d}_r \colon E_r^{s,t} \longrightarrow E_r^{s+r,t+r-1},$$

where A denotes the mod p Steenrod algebra. We also have the Adams–Novikov spectral sequence (ANSS) (see Miller, Ravenel and Wilson [7] and Ravenel [8]) based on the Brown–Peterson spectrum BP.

Throughout the paper, we fix a prime $p \geqslant 7$, and put $q = 2(p-1)$. From Liulevicius [6], $\mathrm{Ext}_A^{1,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ has $\mathbb{Z}_p$–basis consisting of $a_0 \in \mathrm{Ext}_A^{1,1}(\mathbb{Z}_p, \mathbb{Z}_p)$, $h_i \in \mathrm{Ext}_A^{1,p^i q}(\mathbb{Z}_p, \mathbb{Z}_p)$ for all $i \geqslant 0$ and $\mathrm{Ext}_A^{2,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ has $\mathbb{Z}_p$–basis consisting of α_2, a_0^2, $a_0 h_i$ $(i > 0)$, g_i $(i \geqslant 0)$, k_i $(i \geqslant 0)$, b_i $(i \geqslant 0)$ and $h_i h_j$ $(j \geqslant i+2,\ i \geqslant 0)$ whose internal degrees are $2q+1$, 2, $p^i q+1$, $p^{i+1}q+2p^i q$, $2p^{i+1}q+p^i q$, $p^{i+1}q$ and $p^i q + p^j q$, respectively.

Let M be the Moore spectrum modulo the prime p given by the cofibration

$$S \xrightarrow{p} S \xrightarrow{i} M \xrightarrow{j} \Sigma S,$$

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where S is the sphere spectrum localized at the prime p. Let $\alpha\colon \Sigma^q M \to M$ be the Adams map and $V(1)$ be its cofibre given by the cofibration

$$\Sigma^q M \xrightarrow{\alpha} M \xrightarrow{i'} V(1) \xrightarrow{j'} \Sigma^{q+1} M.$$

This spectrum $V(1)$ is the known Toda–Smith spectrum. Let $V(2)$ be the cofibre of the v_2–map $\beta\colon \Sigma^{(p+1)q} V(1) \to V(1)$ given by the cofibration

$$\Sigma^{(p+1)q} V(1) \xrightarrow{\beta} V(1) \xrightarrow{\overline{i}} V(2) \xrightarrow{\overline{j}} \Sigma^{(p+1)q+1} V(1).$$

Let $\gamma\colon \Sigma^{q(p^2+p+1)} V(2) \to V(2)$ be the v_3–map.

Definition 1.1 We define, for $t \geqslant 1$, the β–element $\beta_t = j' j \beta^t i' i \in \pi_{q[tp+(t-1)]-2}(S)$ and the γ–element $\gamma_t = j j' \overline{j} \gamma^t \overline{i} i' i \in \pi_{q[tp^2+(t-1)p+(t-2)]-3}(S)$. Here the maps i, i', $\overline{i}$, β, j, j', $\overline{j}$ and γ are given above.

Theorem 1.2 *With notation as above, we have:*

(1) (Smith [9]) *For $p \geqslant 5$ and $t \geqslant 1$, $\beta_t \neq 0$ in $\pi_*(S)$.*

(2) (Toda [10]) *For $p \geqslant 7$ and $t \geqslant 1$, $\gamma_t \neq 0$ in $\pi_*(S)$.*

In [2], R Cohen constructed a certain infinite family denoted by $\zeta_k \in \pi_{q(p^{k+1}+1)-3}(S)$, $k \geqslant 1$. ζ_k is represented by

$$h_0 b_k \in \mathrm{Ext}_A^{3,q(p^{k+1}+1)}(\mathbb{Z}_p, \mathbb{Z}_p)$$

in the ASS.

Using the method of ANSS, C-N Lee [3] proved that $\beta_1^{p-1}\zeta_k$ is nontrivial for all k, ie,

$$b_0^{p-1} h_0 b_k$$

is a permanent cycle in the ASS and converges nontrivially to

$$\beta_1^{p-1}\zeta_k.$$

This result gives another infinite family of homotopy elements in the stable homotopy groups of spheres.

In [4], J Lin constructed a new nontrivial element, called $\omega_{m,n}$, in $\pi_{q(p^n+p^m+1)-4}(S)$ of order p, which is represented by

$$h_0(h_m b_{n-1} - h_n b_{m-1}) \in \mathrm{Ext}_A^{4,q(p^n+p^m+1)}(\mathbb{Z}_p, \mathbb{Z}_p)$$

in the ASS. On the way to proving the main result, he detected a new family in the stable homotopy groups of M and gave the following theorem.

Theorem 1.3 [4] *Let $p \geqslant 7$, $n \geqslant m+2 \geqslant 4$ and $h_n \in \mathrm{Ext}_A^{1,p^n q}(\mathbb{Z}_p, \mathbb{Z}_p)$. Then*

$$(i)_*(h_0 h_n h_m) \in \mathrm{Ext}_A^{3,q(p^n+p^m+1)}(H^*M, \mathbb{Z}_p)$$

is a permanent cycle in the ASS and converges to a nontrivial element

$$\xi_{m,n} \in \pi_{q(p^n+p^m+1)-3}(M).$$

In [5], X Liu obtained the following theorem.

Theorem 1.4 [5] *Let $p \geqslant 7$, $0 \leqslant s < p-3$. Then there exists the third Greek letter element*

$$\widetilde{\gamma}_{s+3} \in \mathrm{Ext}_A^{s+3,q[(s+3)p^2+(s+2)p+(s+1)]+s}(\mathbb{Z}_p, \mathbb{Z}_p),$$

and $\widetilde{\gamma}_{s+3}$ converges to the γ–element

$$\gamma_{s+3} \in \pi_{q[(s+3)p^2+(s+2)p+(s+1)]-3}(S)$$

in the ASS.

In this paper, I will use the new family of homotopy elements in $\pi_*(M)$ in [4] to detect a $\xi_{m,n}$–related family of filtration $s+6$ in the stable homotopy groups of spheres.

Theorem 1.5 *Let $p \geqslant 7$, $n \geqslant m+2 > 6$ and $0 \leqslant s < p-3$. Then the product*

$$h_0 h_n h_m \widetilde{\gamma}_{s+3} \in \mathrm{Ext}_A^{s+6,t(s)}(\mathbb{Z}_p, \mathbb{Z}_p)$$

is a permanent cycle in the ASS and converges to a nontrivial family of homotopy elements

$$j j' \overline{j} \gamma^{s+3} \overline{i} i' \xi_{m,n} \in \pi_{t(s)-s-6}(S),$$

where $t(s) = q[p^n + p^m + (s+3)p^2 + (s+2)p + (s+2)] + s$.

As the referee told me, in fact I show the nontriviality of Toda bracket $\langle \gamma_{s+3}, p, \omega_{m,n} \rangle$ in the stable homotopy groups of spheres and give the following theorem.

Theorem 1.6 *Let $p \geqslant 7$, $n \geqslant m+2 > 6$ and $0 \leqslant s < p-3$. Then the Toda bracket*

$$\langle \gamma_{s+3}, p, \omega_{m,n} \rangle \subset \pi_{t(s)-s-6}(S)$$

is essential. Here, $t(s) = q\big[p^n + p^m + (s+3)p^2 + (s+2)p + (s+2)\big] + s$.

The May spectral sequence (MSS) and the ASS play very important roles in the proofs of the main results. The proof of our theorem is completely elementary.

The paper is arranged as follows: after giving some propositions on the MSS in Section 2, we will make use of the MSS to obtain two low-dimensional Ext groups in Section 3. Section 4 is devoted to showing Theorem 1.5.

2 The May spectral sequence (MSS)

For computing the stable homotopy groups of spheres with the ASS, we must compute the E_2–term of the ASS, $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_p,\mathbb{Z}_p)$. The most successful method for computing $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_p,\mathbb{Z}_p)$ is the MSS.

From [8], there is a May spectral sequence (MSS) $\{E_r^{s,t,*}, d_r\}$ which converges to $\mathrm{Ext}_A^{s,t}(\mathbb{Z}_p,\mathbb{Z}_p)$ with E_1–term

(2.1) $$E_1^{*,*,*} = E(h_{m,i} \mid m>0, i\geqslant 0)\otimes P(b_{m,i} \mid m>0, i\geqslant 0)\otimes P(a_n \mid n\geqslant 0),$$

where E is the exterior algebra, P is the polynomial algebra, and

$$h_{m,i}\in E_1^{1,2(p^m-1)p^i,2m-1},\ b_{m,i}\in E_1^{2,2(p^m-1)p^{i+1},p(2m-1)},\ a_n\in E_1^{1,2p^n-1,2n+1}.$$

One has

$$d_r\colon E_r^{s,t,u}\to E_r^{s+1,t,u-r}$$

and if $x\in E_r^{s,t,*}$ and $y\in E_r^{s',t',*}$, then

$$d_r(x\cdot y) = d_r(x)\cdot y + (-1)^s x\cdot d_r(y).$$

There exists a graded commutativity in the MSS:

$$x\cdot y = (-1)^{ss'+tt'} y\cdot x$$

for $x, y = h_{m,i}, b_{m,i}$ or a_n. The first May differential d_1 is given by

$$\begin{cases} d_1(h_{i,j}) &= \sum_{0<k<i} h_{i-k,k+j}h_{k,j},\\ d_1(a_i) &= \sum_{0\leqslant k<i} h_{i-k,k}a_k,\\ d_1(b_{i,j}) &= 0.\end{cases}$$

For each element $x\in E_1^{s,t,*}$, we define $\dim x = s$, $\deg x = t$. Then we have

(2.2) $$\begin{cases} \dim h_{i,j} = \dim a_i = 1,\\ \dim b_{i,j} = 2,\\ \deg a_0 = 1,\\ \deg h_{i,j} = q(p^{i+j-1}+\cdots+p^j),\\ \deg b_{i,j} = q(p^{i+j}+\cdots+p^{j+1}),\\ \deg a_i = q(p^{i-1}+\cdots+p+1)+1,\end{cases}$$

where $i\geqslant 1,\ j\geqslant 0$.

By the knowledge on the p–adic expression in number theory, we see that for each integer $t \geqslant 0$, it can be expressed uniquely as

$$t = q(c_n p^n + c_{n-1} p^{n-1} + \cdots + c_1 p + c_0) + e,$$

where $0 \leqslant c_i < p$ $(0 \leqslant i < n)$, $p > c_n > 0$, $0 \leqslant e < q$.

Theorem 2.3 [5] *With notation as above, let s_1 be a positive integer with $0 < s_1 < p$. If $s_1 < c_j$ for some $0 \leqslant j \leqslant n$, then in the MSS, we have that the $\mathbb{Z}_p$–module*

$$E_1^{s_1,t,*} = 0.$$

Let s_2 and t' be two arbitrary positive integers. Suppose

$$t' = q(c'_n p^n + c'_{n-1} p^{n-1} + \cdots + c'_1 p + c'_0) + e',$$

where $0 \leqslant c'_i < p$ $(0 \leqslant i < n)$, $p > c'_n > 0$, $0 \leqslant e' < q$. Suppose a generator of $E_1^{s_2,t',*}$ is of the form $h = x_1 x_2 \cdots x_{s_3} \in E_1^{s_2,t',*}$, where x_i is one of a_k, $h_{r,j}$ or $b_{u,z}$, $0 \leqslant k \leqslant n+1$, $0 \leqslant r + j \leqslant n+1$, $0 \leqslant u + z \leqslant n$, $r > 0$, $j \geqslant 0$, $u > 0$, $z \geqslant 0$. By (2.2) we can assume

$$\deg x_i = q(c_{i,n} p^n + \cdots + c_{i,1} p + c_{i,0}) + e_i,$$

where $c_{i,j} = 0$ or 1, $e_i = 1$ if $x_i = a_{k_i}$, or $e_i = 0$. Then we have

$$\deg h = \sum_{i=1}^{s_3} \deg x_i = q\left(\left(\sum_{i=1}^{s_3} c_{i,n}\right) p^n + \cdots + \left(\sum_{i=1}^{s_3} c_{i,1}\right) p + \sum_{i=1}^{s_3} c_{i,0}\right) + \sum_{i=1}^{s_3} e_i.$$

Denote

$$\sum_{i=1}^{s_3} c_{i,j} \quad \text{and} \quad \sum_{i=1}^{s_3} e_i$$

by $\overline{c}_j$ and $\overline{e}$, $0 \leqslant j \leqslant n$, respectively.

Theorem 2.4 *With notation as above. Suppose that there exists some $0 < j \leqslant n$ such that $\overline{c}_j = s_3$.*

(1) *If there also exist two integers i_1 and i_2 such that $0 \leqslant i_1 < i_2 < j$ and $s_3 \geqslant \overline{c}_{i_1} > \overline{c}_{i_2}$, then h cannot exist.*

(2) *If there also exists an integer i such that $0 \leqslant i < j$ and $s_3 \geqslant \overline{e} > \overline{c}_i$, h cannot exist.*

(3) *If there also exist two integers i'_1 and i'_2 such that $j < i'_1 < i'_2 \leqslant n$ and $s_3 \geqslant \overline{c}_{i'_2} > \overline{c}_{i'_1}$, then h cannot exist.*

Proof By (2.2), we easily get the desired result. □

3 Application of the MSS to two Ext groups

In this section, we make use of the MSS to determine two Ext groups which will be used in the proof of Theorem 1.5.

Lemma 3.1 *Let $p \geqslant 7$, $n \geqslant m+2 > 6$, $0 \leqslant s < p-3$ and $r \geqslant 1$. Then the May E_1–term satisfies*

$$E_1^{s+6-r,t(s)+1-r,*} = \begin{cases} 0 & r \geqslant 2; \\ 0 & r = 1 \text{ and } s < p-4; \\ M & r = 1 \text{ and } s = p-4. \end{cases}$$

Here $t(s) = q\big[p^n + p^m + (s+3)p^2 + (s+2)p + (s+2)\big] + s$ and M is the $\mathbb{Z}_p$–module generated by nine elements

$$\mathbf{g}i \in E_1^{p+1,t(p-4),(2n+1)p-2n-9} \qquad (1 \leqslant i \leqslant 8)$$
$$\mathbf{g}9 \in E_1^{p+1,t(p-4),(2m+1)p-2m-9},$$

where

$$\begin{aligned}
\mathbf{g}1 &= a_n^{p-4} h_{n,0} h_{4,0} h_{n-2,2} h_{n-3,3} h_{1,m}, \\
\mathbf{g}2 &= a_n^{p-4} h_{n,0} h_{m+1,0} h_{n-2,2} h_{1,3} h_{n-m,m}, \\
\mathbf{g}3 &= a_n^{p-4} h_{n,0} h_{4,0} h_{n-2,2} h_{m-2,3} h_{n-m,m}, \\
\mathbf{g}4 &= a_n^{p-4} h_{n,0} h_{m+1,0} h_{2,2} h_{n-3,3} h_{n-m,m}, \\
\mathbf{g}5 &= a_n^{p-4} h_{n,0} h_{4,0} h_{m-1,2} h_{n-3,3} h_{n-m,m}, \\
\mathbf{g}6 &= a_n^{p-5} a_{m+1} h_{n,0} h_{4,0} h_{n-2,2} h_{n-3,3} h_{n-m,m}, \\
\mathbf{g}7 &= a_n^{p-5} a_4 h_{n,0} h_{m+1,0} h_{n-2,2} h_{n-3,3} h_{n-m,m}, \\
\mathbf{g}8 &= a_n^{p-4} h_{m+1,0} h_{4,0} h_{n-2,2} h_{n-3,3} h_{n-m,m}, \\
\mathbf{g}9 &= a_m^{p-4} h_{m,0} h_{4,0} h_{m-2,2} h_{m-3,3} h_{1,n}.
\end{aligned}$$

Proof When $r \geqslant s+4$, we can easily show that in the MSS $E_1^{s+6-r,t(s)+1-r,*} = 0$. Thus in the rest of the proof, we assume that $1 \leqslant r < s+4$. Consider $h = x_1 x_2 \cdots x_l \in E_1^{s+6-r,t(s)-r+1,*}$ in the MSS, where x_i is one of a_k, $h_{r,j}$ or $b_{u,z}$, $0 \leqslant k \leqslant n+1$, $0 \leqslant r+j \leqslant n+1$, $0 \leqslant u+z \leqslant n$, $r > 0$, $j \geqslant 0$, $u > 0$, $z \geqslant 0$. Assume that

$$\deg x_i = q(c_{i,n} p^n + c_{i,n-1} p^{n-1} + \cdots + c_{i,1} p + c_{i,0}) + e_i,$$

where $c_{i,j} = 0$ or 1, $e_i = 1$ if $x_i = a_{k_i}$, or $e_i = 0$. It follows that

$$\dim h = \sum_{i=1}^{l} \dim x_i = s+6-r,$$

$$\begin{aligned}\deg h = \sum_{i=1}^{l} \deg x_i &= q\Big[\Big(\sum_{i=1}^{l} c_{i,n}\Big)p^n + \cdots + \Big(\sum_{i=1}^{l} c_{i,1}\Big)p + \Big(\sum_{i=1}^{l} c_{i,0}\Big)\Big] + \Big(\sum_{i=1}^{l} e_i\Big)\\ &= q\big[p^n + p^m + (s+3)p^2 + (s+2)p + (s+2)\big] + (s+1-r).\end{aligned}$$

Note that $\dim h_{i,j} = \dim a_i = 1$, $\dim b_{i,j} = 2$, $1 \leqslant r < s+4$ and $0 \leqslant s < p-3$. From $\dim h = \sum_{i=1}^{l} \dim x_i = s+6-r$ we have $l \leqslant s+6-r < p+3-r \leqslant p+2$.

We claim that $s+1-r \geqslant 0$. Otherwise, we would have $\sum_{i=1}^{l} e_i \leqslant l \leqslant p+1$. On the other hand, by $1 \leqslant r < s+4$ and $p \geqslant 7$, we would have $\sum_{i=1}^{l} e_i = q + (s-r+1) > 2p-2-3 \geqslant p+2$ which contradicts $\sum_{i=1}^{l} e_i \leqslant l \leqslant p+1$. The claim is proved.

Using $0 \leqslant s+3,\ s+2,\ s+1-r < p$ and the knowledge on p–adic expression in number theory, we have

$$\textbf{(3.2)}\qquad \begin{cases} \sum_{i=1}^{l} e_i = s+1-r+\lambda_{-1}q, & \lambda_{-1} \geqslant 0;\\ \sum_{i=1}^{l} c_{i,0} + \lambda_{-1} = s+2+\lambda_0 p, & \lambda_0 \geqslant 0;\\ \sum_{i=1}^{l} c_{i,1} + \lambda_0 = s+2+\lambda_1 p, & \lambda_1 \geqslant 0;\\ \sum_{i=1}^{l} c_{i,2} + \lambda_1 = s+3+\lambda_2 p, & \lambda_2 \geqslant 0;\\ \sum_{i=1}^{l} c_{i,3} + \lambda_2 = 0+\lambda_3 p, & \lambda_3 \geqslant 0;\\ \sum_{i=1}^{l} c_{i,4} + \lambda_3 = 0+\lambda_4 p, & \lambda_4 \geqslant 0;\\ \qquad\vdots & \quad\vdots\\ \sum_{i=1}^{l} c_{i,m-1} + \lambda_{m-2} = 0+\lambda_{m-1} p, & \lambda_{m-1} \geqslant 0;\\ \sum_{i=1}^{l} c_{i,m} + \lambda_{m-1} = 1+\lambda_m p, & \lambda_m \geqslant 0;\\ \sum_{i=1}^{l} c_{i,m+1} + \lambda_m = 0+\lambda_{m+1} p, & \lambda_{m+1} \geqslant 0;\\ \qquad\vdots & \quad\vdots\\ \sum_{i=1}^{l} c_{i,n-1} + \lambda_{n-2} = 0+\lambda_{n-1} p, & \lambda_{n-1} \geqslant 0;\\ \sum_{i=1}^{l} c_{i,n} + \lambda_{n-1} = 1. \end{cases}$$

From $e_i = 0$ or 1, $c_{i,j} = 0$ or 1, and $l \leqslant p+1$, we easily have that

$$(\lambda_{-1}, \lambda_0, \lambda_1, \lambda_2) = (0,0,0,0).$$

Consider the fifth equality of (3.2) $\sum_{i=1}^{l} c_{i,3} = 0 + \lambda_3 p$. By $c_{i,3} = 0$ or 1, and $l \leqslant p+1$, we get that $\lambda_3 = 0$ or 1.

Case 1 $\lambda_3 = 0$. We claim $\lambda_4 = 0$. If $\lambda_4 = 1$, we would have $\sum_{i=1}^{l} c_{i,2} = s+3$, $\sum_{i=1}^{l} c_{i,3} = 0$, $\sum_{i=1}^{l} c_{i,4} = p$. From $\sum_{i=1}^{l} c_{i,2} = s+3$ and (2.2), there would be $s+3$ factors among h such that $\deg x_i = q$(higher terms on $p+p^2+$ lower terms on $p)+\delta_i$, where δ_i may equal 0 or 1. Similarly, from $\sum_{i=1}^{l} c_{i,4} = p$, there would be p factors among h such that $\deg x_i = q$(higher terms on $p+p^4+$ lower terms on $p)+\delta_i$. Thus, by $l \leqslant p+1$ and (2.2), there would be at least $p+s+3-(p+1) = s+2$ factors in h such that $\deg x_i = q$(higher terms on $p+p^4+p^3+p^2+$ lower terms on $p)+\delta_i$. Thus we would have $\sum_{i=1}^{l} c_{i,3} \geqslant s+2$ which contradicts $\sum_{i=1}^{l} c_{i,3} = 0$. The claim is proved.

By induction on j, we have

$$\lambda_j = 0 \quad (4 \leqslant j \leqslant n-1).$$

Then we have the following:

Subcase 1.1 If there are two factors $h_{1,n}$ and $h_{1,m}$ in h, then up to sign $h = h_{1,n} h_{1,m} \widetilde{h}$ with $\widetilde{h} \in E_1^{s+4-r,q[(s+3)p^2+(s+2)p+(s+2)]+(s+1-r),*}$.

When $r = 1$, by an argument similar to that used in the proof of Theorem 1.1 of [5], $E_1^{s+3,q[(s+3)p^2+(s+2)p+(s+2)]+s,*} = 0$.

When $r \geqslant 2$, $E_1^{s+4-r,q[(s+3)p^2+(s+2)p+(s+2)]+(s+1-r),*} = 0$ by Theorem 2.3 From the above discussion we have there cannot exist two factors $h_{1,n}$ and $h_{1,m}$ in h.

Similarly, we can show the following.

Subcase 1.2 There cannot exist two factors $h_{1,n}$ and $b_{1,m-1}$ in h.

Subcase 1.3 There cannot exist two factors $b_{1,n-1}$ and $h_{1,m}$ in h.

Subcase 1.4 There cannot exist two factors $b_{1,n-1}$ and $b_{1,m-1}$ in h.

Case 2 $\lambda_3 = 1$. When $r \geqslant 3$, it is easy to see that λ_3 is impossible to equal 1. Thus in the rest of the proof, we assume $r \leqslant 2$. From the sixth equality of (3.2), $\sum_{i=1}^{l} c_{i,4} + 1 = \lambda_4 p$, and $0 \leqslant \sum_{i=1}^{l} c_{i,4} \leqslant l \leqslant p+1$, we can deduce that

$$\lambda_4 = 1.$$

By induction on j, we have

$$\lambda_j = 1 \quad (4 \leqslant j \leqslant m-1).$$

Now consider the $(m+2)$–nd equality of (3.2), $\sum_{i=1}^{l} c_{i,m} + 1 = 1 + \lambda_m p$. Noting that $0 \leqslant \sum_{i=1}^{l} c_{i,m} \leqslant l \leqslant p+1$, we have that $\lambda_m = 0$ or 1.

Subcase 2.1 $\lambda_m = 1$. Consider the $(m+3)$–rd equality of (3.2), $\sum_{i=1}^{l} c_{i,m+1} + 1 = \lambda_{m+1} p$. Using $0 \leqslant \sum_{i=1}^{l} c_{i,m+1} \leqslant l \leqslant p+1$, we can have that

$$\lambda_{m+1} = 1.$$

By induction on j, we can show

$$\lambda_j = 1 \quad (m+1 \leqslant j \leqslant n-1).$$

Thus we have

$$(\lambda_{-1}, \lambda_0, \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_m, \lambda_{m+1}, \ldots, \lambda_{n-1}) = (0,0,0,0,1,\ldots,1,1,\ldots,1).$$

From the fifth equality of (3.2), $\sum_{i=1}^{l} c_{i,3} = p$, using $c_{i,3} = 0$ or 1, we have that $l \geqslant p$. Note that $l \leqslant s+5$. Thus $s \geqslant p-5$. By $0 \leqslant s < p-3$, we have that s may equal $p-5$ or $p-4$.

(i) When $s = p-4$, $h = x_1 x_2 \cdots x_l \in E_1^{p+2-r, t(p-4)+1-r, *}$. From the first equality of (3.2), $\sum_{i=1}^{l} e_i = p-3-r$ and (2.2), there exist $(p-3-r)$ factors among h such that

$$\deg x_i = q(\text{higher terms on } p) + 1.$$

Similarly, from $\sum_{i=1}^{l} c_{i,n-1} = p-1$, there exist $(p-1)$ factors among h such that

$$\deg x_i = q(p^{n-1} + \text{ lower terms on } p) + \delta_i,$$

where δ_i may equal 0 or 1. Noting $l \leqslant p+1$ and (2.2), we have that there exist at least $(p-3-r)+(p-1)-(p+1) = p-5-r$ factors in h such that

$$\deg x_i = q(p^{n-1} + \cdots + p + 1) + 1,$$

ie, there exist at least $(p-5-r)$ a_n's among h. By the graded commutativity of $E_1^{*,*,*}$, we can let $h = a_n^{p-5-r} x_{p-4-r} \cdots x_l$. Then $h' = x_{p-4-r} \cdots x_l \in E_1^{7,t'',*}$, where $t'' = t(p-4) - (p-5-r)\deg a_n$. From $\sum_{i=1}^{l} c_{i,3} = p$, we have $l \geqslant p$. Recall that $l \leqslant p+1$. Thus $p \leqslant l \leqslant p+1$.

If $l = p$, then

$$h = a_n^{p-5-r} x_{p-4-r} \cdots x_p \in E_1^{p+2-r, t(p-4)+1-r, *}$$

and

$$h' = x_{p-4-r} \cdots x_p \in E_1^{7,t'',*}.$$

Note that $\sum_{i=p-4-r}^{p} c_{i,3}=r+5$, $\sum_{i=p-4-r}^{p} c_{i,4}=r+4$ and $\sum_{i=p-4-r}^{p} c_{i,m}=r+5$. By Theorem 2.4, we have $E_1^{7,t'',*}=0$. Thus in this case h cannot exist.

If $l=p+1$, then we can see that in this case r cannot equal 2 by $\dim h=p+2-r$ and $\dim x_i=1$ or 2. Thus in this case r must equal 1. Then $h=a_n^{p-6}x_{p-5}\cdots x_{p+1}\in E_1^{p+1,t(p-4),*}$ and $h'=x_{p-5}\cdots x_{p+1}\in E_1^{7,t'',*}$. It is easy to get that $\dim x_i=1$ for $p-5\leqslant i\leqslant p+1$, ie,

$$h'\in E(h_{m,i}\mid m>0, i\geqslant 0)\otimes P(a_n\mid n\geqslant 0),$$

and there exist at least five factors in h' such that

$\deg x_i = q(\text{higher terms on } p+p^m+p^{m-1}+\cdots+p^4+p^3+\text{lower terms on } p)+\delta_i$

by $\sum_{i=p-5}^{p+1} c_{i,3}=6$, $\sum_{i=p-5}^{p+1} c_{i,m}=6$ and (2.2), where δ_i may equal 0 or 1. We can divide the seven factors of h' into the following three disjoint classes, using $\sum_{i=p-5}^{p+1} c_{i,3}=6$, $\sum_{i=p-5}^{p+1} c_{i,4}=5,\dots,\sum_{i=p-5}^{p+1} c_{i,m-1}=5$, $\sum_{i=p-5}^{p+1} c_{i,m}=6$ and (2.2):

$$\begin{cases} S_1=\{x\mid \deg x=q(\text{higher terms on } p+p^m+\cdots+p^3+\text{lower terms on } p)+\delta\};\\ |S_1|=5;\\ S_2=\{x\mid \deg x=q(\text{higher terms on } p+p^m)\};\\ |S_2|=1;\\ S_3=\{x\mid \deg x=q(p^3+\text{lower terms on } p)+\delta\};\\ |S_3|=1.\end{cases}$$

Here δ may equal 0 or 1. Now we list the deg's of the seven factors in the following table.

$\deg x_i$	higher terms		$p^m q$		$p^{m-1}q$	$\cdots$	$p^3 q$		lower terms
$\deg x_{p-5}$	higher terms	+	$p^m q$	+	$p^{m-1}q$	$+\cdots+$	$p^3 q$	+	lower terms
$\deg x_{p-4}$	higher terms	+	$p^m q$	+	$p^{m-1}q$	$+\cdots+$	$p^3 q$	+	lower terms
$\deg x_{p-3}$	higher terms	+	$p^m q$	+	$p^{m-1}q$	$+\cdots+$	$p^3 q$	+	lower terms
$\deg x_{p-2}$	higher terms	+	$p^m q$	+	$p^{m-1}q$	$+\cdots+$	$p^3 q$	+	lower terms
$\deg x_{p-1}$	higher terms	+	$p^m q$	+	$p^{m-1}q$	$+\cdots+$	$p^3 q$	+	lower terms
$\deg x_p$	higher terms	+	$p^m q$						
$\deg x_{p+1}$							$p^3 q$	+	lower terms

Similarly, from $\sum_{i=p-5}^{p+1} c_{i,m}=6$, $\sum_{i=p-5}^{p+1} c_{i,m+1}=5,\cdots,\sum_{i=p-5}^{p+1} c_{i,n-1}=5$ and (2.2), we know that there exist at least four factors in h' such that

$$\deg x_i=q(p^{n-1}+\cdots+p^m+\text{lower terms on } p)+\delta_i,$$

where δ_i may equal 0 or 1. Then there exist two probabilities which are listed in the following two tables.

deg x_i	$p^{n-1}q$	$\cdots$	$p^{m+1}q$	p^mq	$p^{m-1}q$	$\cdots$	p^3q	lower terms
deg x_{p-5}	$p^{n-1}q$	$+\cdots+$	$p^{m+1}q$	$+\ p^mq$	$+\ p^{m-1}q$	$+\cdots+$	p^3q	$+$ lower terms
deg x_{p-4}	$p^{n-1}q$	$+\cdots+$	$p^{m+1}q$	$+\ p^mq$	$+\ p^{m-1}q$	$+\cdots+$	p^3q	$+$ lower terms
deg x_{p-3}	$p^{n-1}q$	$+\cdots+$	$p^{m+1}q$	$+\ p^mq$	$+\ p^{m-1}q$	$+\cdots+$	p^3q	$+$ lower terms
deg x_{p-2}	$p^{n-1}q$	$+\cdots+$	$p^{m+1}q$	$+\ p^mq$	$+\ p^{m-1}q$	$+\cdots+$	p^3q	$+$ lower terms
deg x_{p-1}	$p^{n-1}q$	$+\cdots+$	$p^{m+1}q$	$+\ p^mq$	$+\ p^{m-1}q$	$+\cdots+$	p^3q	$+$ lower terms
deg x_p				p^mq				
deg x_{p+1}							p^3q	$+$ lower terms

Table 1

It is easy to see that in Table 1 the sixth factor is $h_{1,m}$.

deg x_i	$p^{n-1}q$	$\cdots$	$p^{m+1}q$	p^mq	$p^{m-1}q$	$\cdots$	p^3q	lower terms
deg x_{p-5}	$p^{n-1}q$	$+\cdots+$	$p^{m+1}q$	$+\ p^mq$	$+\ p^{m-1}q$	$+\cdots+$	p^3q	$+$ lower terms
deg x_{p-4}	$p^{n-1}q$	$+\cdots+$	$p^{m+1}q$	$+\ p^mq$	$+\ p^{m-1}q$	$+\cdots+$	p^3q	$+$ lower terms
deg x_{p-3}	$p^{n-1}q$	$+\cdots+$	$p^{m+1}q$	$+\ p^mq$	$+\ p^{m-1}q$	$+\cdots+$	p^3q	$+$ lower terms
deg x_{p-2}	$p^{n-1}q$	$+\cdots+$	$p^{m+1}q$	$+\ p^mq$	$+\ p^{m-1}q$	$+\cdots+$	p^3q	$+$ lower terms
deg x_{p-1}				p^mq	$+\ p^{m-1}q$	$+\cdots+$	p^3q	$+$ lower terms
deg x_p	$p^{n-1}q$	$+\cdots+$	$p^{m+1}q$	$+\ p^mq$				
deg x_{p+1}							p^3q	$+$ lower terms

Table 2

It is easy to see that in Table 2 the sixth factor is $h_{n-m,m}$.

Consider Table 1. By $h_{i,j}^2=0$, $\sum_{i=p-5}^{p+1} e_i=2$, $\sum_{i=p-5}^{p+1} c_{i,0}=4$, $\sum_{i=p-5}^{p+1} c_{i,1}=4$, $\sum_{i=p-5}^{p+1} c_{i,2}=5$ and (2.2), we can get that the deg's of the seven factors of h' must be the following.

deg x_i	$p^{n-1}q$	$\cdots$	p^mq	$p^{m-1}q$	$\ldots$	p^3q	p^2q	pq	q	1
deg x_{p-5}	$p^{n-1}q$	$+\cdots+$	p^mq	$+\ p^{m-1}q$	$+\cdots+$	p^3q	$+\ p^2q$	$+\ pq$	$+\ q$	$+\ 1$
deg x_{p-4}	$p^{n-1}q$	$+\cdots+$	p^mq	$+\ p^{m-1}q$	$+\cdots+$	p^3q	$+\ p^2q$	$+\ pq$	$+\ q$	$+\ 1$
deg x_{p-3}	$p^{n-1}q$	$+\cdots+$	p^mq	$+\ p^{m-1}q$	$+\cdots+$	p^3q	$+\ p^2q$	$+\ pq$	$+\ q$	
deg x_{p-2}	$p^{n-1}q$	$+\cdots+$	p^mq	$+\ p^{m-1}q$	$+\cdots+$	p^3q	$+\ p^2q$			
deg x_{p-1}	$p^{n-1}q$	$+\cdots+$	p^mq	$+\ p^{m-1}q$	$+\cdots+$	p^3q				
deg x_p			p^mq							
deg x_{p+1}						p^3q	$+\ p^2q$	$+\ pq$	$+\ q$	

Thus by (2.2), we have that in this case $h'=a_n^2h_{n,0}h_{n-2,2}h_{n-3,3}h_{1,m}h_{4,0}$. Then up to sign $h=a_n^{p-4}h_{n,0}h_{4,0}h_{n-2,2}h_{n-3,3}h_{1,m}\in E_1^{p+1,t(p-4),(2n+1)p-2n-9}$, denoted by $\mathbf{g}1$.

Now consider Table 2. Similarly, we also get $E_1^{p+1,t(p-4),(2n+1)p-2n-9}$ has seven generators $\mathbf{g}2,\dots,\mathbf{g}8$.

From the above discussion, we get that in this case $E_1^{p+1,t(p-4),*}$ has the eight generators $\mathbf{g}i$ $(1\leqslant i\leqslant 8)$.

(ii) When $s=p-5$, $h=x_1x_2\cdots x_l\in E_1^{p+1-r,t(p-5)+1-r,*}$. From the equality $\sum_{i=1}^{l}c_{i,3}=p$ of (3.2), we have that $l\geqslant p$. Note that $\dim x_i=1$ or 2. Since $\dim h=\sum_{i=1}^{l}\dim x_i=p+1-r$, it follows that $l\leqslant p+1-r$. Thus we have

$$p\leqslant l\leqslant p+1-r.$$

It is easy to see that in this case r is impossible to equal 2. We only consider the case $r=1$. Thus we have that $l=p$ and $h=x_1x_2\cdots x_p\in E_1^{p,t(p-5),*}$. By Theorem 2.4, we know that in this case, it is impossible for h to exist.

Subcase 2.2 $\lambda_m=0$. By an argument similar to that used in the proof of $\lambda_4=0$ in Case 1, we also get that in this case

$$\lambda_{m+1}=0.$$

By induction on j, we have that

$$\lambda_j=0\quad(m+1\leqslant j\leqslant n-1).$$

Thus we have

$$(\lambda_{-1},\dots,\lambda_2,\lambda_3,\dots,\lambda_{m-1},\lambda_m,\lambda_{m+1},\dots,\lambda_{n-1})=(0,\dots,0,1,\dots,1,0,0,\dots,0).$$

By (2.2), it is easy to see that in this case there exists a factor $h_{1,n}$ or $b_{1,n-1}$ in h. By the graded commutativity of $E_1^{*,*,*}$, we can denote the $h_{1,n}$ or $b_{1,n-1}$ by x_l. Then $h=h''h_{1,n}$ or $h=h''b_{1,n-1}$, where $h''=x_1\cdots x_{l-1}$.

(i) If $x_l=h_{1,n}$, then $h''=x_1\cdots x_{l-1}\in E_1^{s+5-r,t(s)-p^nq+1-r,*}$. In this case we have $\sum_{i=1}^{l-1}c_{i,3}=p$. Thus $l\geqslant p+1$. Note that $l\leqslant p+2-r$. It is easy to see that in this case r is impossible to equal 2. Thus we only consider the case $r=1$. Then $l=p+1$. By $\dim x_i=2$ or 1, we have that $\dim h=\dim x_1x_2\cdots x_{p+1}=s+5\geqslant p+1$, then $s\geqslant p-4$. Note that $0\leqslant s<p-3$. It follows that $s=p-4$. Then we have that

$$h''=x_1\cdots x_p\in E_1^{p,t(p-4)-p^nq,*}.$$

By an argument similar to that used in the proof of Subcase 2.1 (i), we get that up to sign $h=a_m^{p-4}h_{m,0}h_{4,0}h_{m-2,2}h_{m-3,3}h_{1,n}\in E_1^{p+1,t(p-4),(2m+1)p-2m-9}$, denoted by $\mathbf{g}9$.

(ii) If $x_l = b_{1,n-1}$, then $h'' = x_1 \cdots x_{l-1} \in E_1^{s+4-r,t(s)-p^n q+1-r,*}$. In this case we have $\sum_{i=1}^{l-1} c_{i,3} = p$. Thus $l-1 \geqslant p$, and then

$$\dim h'' = \sum_{i=1}^{l-1} \dim x_i \geqslant p.$$

On the other hand, we also have

$$\dim h'' = s+4-r < p+1-r \leqslant p$$

by $0 \leqslant s < p-3$. This yields a contradiction. Thus in this case, it is impossible for h to exist.

Combining Cases 1 and 2, we complete the proof of the lemma. □

Lemma 3.3 *Let $p \geqslant 7$, $n \geqslant m+2 > 6$, $0 \leqslant s < p-3$. Then the May E_2–term*

$$E_2^{s+5,t(s),*} = 0.$$

Here, $t(s) = q[p^n + p^m + (s+3)p^2 + (s+2)p + (s+2)] + s$.

Proof When $0 \leqslant s < p-4$, from Lemma 3.1 we know that in the MSS, $E_1^{s+5,t(s),*} = 0$. Then we have

$$E_2^{s+5,t(s),*} = 0.$$

Now we consider the case $s = p-4$. From Lemma 3.1 we have that

$$E_1^{p+1,t(p-4),*} = \mathbb{Z}_p\{\mathbf{g}1, \mathbf{g}2, \cdots, \mathbf{g}9\}.$$

By the first May differential and graded commutativity of $E_1^{*,*,*}$, we can easily get that

$$\begin{aligned}
d_1(\mathbf{g}1) &= -a_n^{p-4} h_{n,0} h_{4,0} h_{n-2,2} h_{1,3} h_{n-4,4} h_{1,m} + \cdots \neq 0,\\
d_1(\mathbf{g}2) &= a_n^{p-4} h_{n,0} h_{m+1,0} h_{n-2,2} h_{1,3} h_{1,m} h_{n-m-1,m+1} + \cdots \neq 0,\\
d_1(\mathbf{g}3) &= a_n^{p-4} h_{n,0} h_{4,0} h_{n-2,2} h_{m-2,3} h_{1,m} h_{n-m-1,m+1} + \cdots \neq 0,\\
d_1(\mathbf{g}4) &= a_n^{p-4} h_{n,0} h_{m+1,0} h_{2,2} h_{n-3,3} h_{1,m} h_{n-m-1,m+1} + \cdots \neq 0,\\
d_1(\mathbf{g}5) &= a_n^{p-4} h_{n,0} h_{4,0} h_{m-1,2} h_{n-3,3} h_{1,m} h_{n-m-1,m+1} + \cdots \neq 0,\\
d_1(\mathbf{g}6) &= a_n^{p-5} a_{m+1} h_{n,0} h_{4,0} h_{n-2,2} h_{n-3,3} h_{1,m} h_{n-m-1,m+1} + \cdots \neq 0,\\
d_1(\mathbf{g}7) &= a_n^{p-5} a_4 h_{n,0} h_{m+1,0} h_{n-2,2} h_{n-3,3} h_{1,m} h_{n-m-1,m+1} + \cdots \neq 0,\\
d_1(\mathbf{g}8) &= a_n^{p-4} h_{m+1,0} h_{4,0} h_{n-2,2} h_{n-3,3} h_{1,m} h_{n-m-1,m+1} + \cdots \neq 0,\\
d_1(\mathbf{g}9) &= -a_m^{p-4} h_{m,0} h_{4,0} h_{m-2,2} h_{1,3} h_{m-4,4} h_{1,n} + \cdots \neq 0.
\end{aligned}$$

It is easy to check that $a_n^{p-4}h_{n,0}h_{4,0}h_{n-2,2}h_{1,3}h_{n-4,4}h_{1,m}$ only appears in $d_1(\mathbf{g}1)$, doesn't appear in $d_1(\mathbf{g}i)$ $(i \geqslant 2)$. Similarly, we can show that the eight elements

$$a_n^{p-4}h_{n,0}h_{m+1,0}h_{n-2,2}h_{1,3}h_{1,m}h_{n-m-1,m+1}$$
$$a_n^{p-4}h_{n,0}h_{4,0}h_{n-2,2}h_{m-2,3}h_{1,m}h_{n-m-1,m+1}$$
$$\vdots$$
$$a_n^{p-4}h_{m+1,0}h_{4,0}h_{n-2,2}h_{n-3,3}h_{1,m}h_{n-m-1,m+1}$$
$$a_m^{p-4}h_{m,0}h_{4,0}h_{m-2,2}h_{1,3}h_{m-4,4}h_{1,n}$$

only appear in $d_1(\mathbf{g}2)$, $d_1(\mathbf{g}3)$, $\cdots$, $d_1(\mathbf{g}8)$ and $d_1(\mathbf{g}9)$ respectively. It follows that $d_1(\mathbf{g}1)$, $d_1(\mathbf{g}2)$, $\cdots$, $d_1(\mathbf{g}9)$ are linearly independent. Consequently, we have $E_2^{p+1,t(p-4),*} = 0$. □

Theorem 3.4 *Let $p \geqslant 7$, $n \geqslant m+2 > 6$, $0 \leqslant s < p-3$. Then the product*

$$h_0h_nh_m\widetilde{\gamma}_{s+3} \neq 0 \in \mathrm{Ext}_A^{s+6,t(s)}(\mathbb{Z}_p,\mathbb{Z}_p),$$

where $t(s) = q[p^n + p^m + (s+3)p^2 + (s+2)p + (s+2)] + s$.

Proof Since it is known that $h_{1,i}$ and $a_3^s h_{3,0}h_{2,1}h_{1,2} \in E_1^{*,*,*}$ are permanent cycles in the MSS and converge nontrivially to

$$h_i, \quad \widetilde{\gamma}_{s+3} \in \mathrm{Ext}_A^{*,*}(\mathbb{Z}_p,\mathbb{Z}_p)$$

for $i \geqslant 0$ respectively (see Theorem 1.4),

$$h_{1,0}h_{1,n}h_{1,m}a_3^{s+3}h_{3,0}h_{2,1}h_{1,2} \in E_1^{s+6,t(s),*}$$

is a permanent cycle in the MSS and converges to $h_0h_nh_m\widetilde{\gamma}_{s+3} \in \mathrm{Ext}_A^{s+6,t(s)}(\mathbb{Z}_p,\mathbb{Z}_p)$.

Case 1 When $0 \leqslant s < p-4$, from Lemmas 3.1 and 3.3 we know that in the MSS

$$E_r^{s+5,t(s),*} = 0 \quad (r \geqslant 1).$$

Thus the permanent cycle $h_{1,0}h_{1,n}h_{1,m}a_3^s h_{3,0}h_{2,1}h_{1,2} \in E_r^{s+6,t(s),*}$ does not bound and converges to

$$h_0h_nh_m\widetilde{\gamma}_{s+3} \in \mathrm{Ext}_A^{s+6,t(s)}(\mathbb{Z}_p,\mathbb{Z}_p)$$

nontrivially in the MSS, ie, $h_0h_nh_m\widetilde{\gamma}_{s+3} \neq 0 \in \mathrm{Ext}_A^{s+6,t(s)}(\mathbb{Z}_p,\mathbb{Z}_p)$.

Case 2 When $s = p-4$, from Lemma 3.3 we have that $E_2^{p+1,t(p-4),*} = 0$. Thus

$$E_r^{p+1,t(p-4),*} = 0 \quad (r \geqslant 2).$$

Meanwhile, in the MSS the third degree of $h_{1,0}h_{1,n}h_{1,m}a_3^{p-4}h_{3,0}h_{2,1}h_{1,2}$ is $7p-16$ and none of the third degrees of the generators $\mathbf{g}i$ $(1 \leqslant i \leqslant 9)$ is $7p-15$. Thus no element hits $h_{1,0}h_{1,n}h_{1,m}a_3^{p-4}h_{3,0}h_{2,1}h_{1,2}$ under the May differential d_1. From the above discussion, $h_{1,0}h_{1,n}h_{1,m}a_3^{p-4}h_{3,0}h_{2,1}h_{1,2} \in E_r^{p+1,t(p-4),*}$ does not bound in the MSS and converges nontrivially to $h_0h_nh_m\widetilde{\gamma}_{p-1} \in \mathrm{Ext}_A^{p+2,t(p-4)}(\mathbb{Z}_p,\mathbb{Z}_p)$. Thus $h_0h_nh_m\widetilde{\gamma}_{p-1} \neq 0$.

From Cases 1 and 2, Theorem 3.4 follows. □

Theorem 3.5 *Let $p \geqslant 7$, $n \geqslant m+2 > 6$, $0 \leqslant s < p-3$, $2 \leqslant r \leqslant s+6$. Then we have*

$$\mathrm{Ext}_A^{s+6-r,t(s)-r+1}(\mathbb{Z}_p,\mathbb{Z}_p) = 0,$$

where $t(s) = q[p^n + p^m + (s+3)p^2 + (s+2)p + (s+2)] + s$.

Proof From Lemma 3.1, we have that in this case $E_1^{s+6-r,t(s)+1-r,*} = 0$. By the MSS, we easily have the desired result. □

4 Proof of Theorem 1.5

From Theorem 1.3, $i_*(h_0h_nh_m) \in \mathrm{Ext}_A^{3,q(p^n+p^m+1)}(H^*M,\mathbb{Z}_p)$ is a permanent cycle in the ASS and converges to a nontrivial element $\xi_{m,n} \in \pi_{q(p^n+p^m+1)-3}(M)$.

Consider the composition of maps $\varphi = jj'\overline{j}\gamma^{s+3}\overline{i}i'\xi_{m,n}$. Since $\xi_{m,n}$ is represented by $i_*(h_0h_nh_m) \in \mathrm{Ext}_A^{3,q(p^n+p^m+1)}(H^*M,\mathbb{Z}_p)$ in the ASS, then the above φ is represented in the ASS by $\overline{c} = (jj'\overline{j}\gamma^{s+3}\overline{i}i'i)_*(h_0h_nh_m)$.

From Theorem 1.4 and the knowledge of Yoneda products we know that the composition

$$\mathrm{Ext}_A^{0,0}(\mathbb{Z}_p,\mathbb{Z}_p) \xrightarrow{(\overline{i}i'i)_*} \mathrm{Ext}_A^{0,0}(H^*V(2),\mathbb{Z}_p) \xrightarrow{(jj'\overline{j})_*(\gamma_*)^{s+3}} \mathrm{Ext}_A^{s+3,q[(s+3)p^2+(s+2)p+(s+1)]+s}(\mathbb{Z}_p,\mathbb{Z}_p)$$

is a multiplication up to nonzero scalar by

$$\widetilde{\gamma}_{s+3} \in \mathrm{Ext}_A^{s+3,q[(s+3)p^2+(s+2)p+(s+1)]+s}(\mathbb{Z}_p,\mathbb{Z}_p).$$

Hence, $\overline{f}$ is represented up to nonzero scalar by

$$\overline{c} = \widetilde{\gamma}_{s+3}h_0h_nh_m \neq 0 \in \mathrm{Ext}_A^{s+6,t(s)}(\mathbb{Z}_p,\mathbb{Z}_p)$$

in the ASS (cf Theorem 3.4).

Moreover, from Theorem 3.5, we know that $\widetilde{\gamma}_{s+3}h_0h_nh_m$ cannot be hit by any differential in the ASS. Consequently, the corresponding homotopy element φ is nontrivial. This shows Theorem 1.5. □

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