Classification of string links 
up to self delta-moves and concordance

AKIRA YASUHARA

For an $n$–component string link, the Milnor’s concordance invariant is defined for each sequence $I = i_1i_2\cdots i_m$ ($i_j \in \{1, \ldots , n\}$). Let $r(I)$ denote the maximum number of times that any index appears. We show that two string links are equivalent up to self $\Delta$–moves and concordance if and only if their Milnor invariants coincide for all sequences $I$ with $r(I) \leq 2$.

1 Introduction

For an $n$–component link $L$, the Milnor $\bar{\mu}$–invariant $\bar{\mu}_L(I)$ is defined for each sequence $I = i_1i_2\cdots i_m$ ($i_j \in \{1, \ldots , n\}$); see Milnor [13; 14]. Let $r(I)$ denote the maximum number of times that any index appears. For example, $r(1123) = 2$ and $r(1231223) = 3$. It is known that if $r(I) = 1$, then $\bar{\mu}_L(I)$ is a link-homotopy invariant [13], where link-homotopy is an equivalence relation on links generated by self crossing changes. Similarly, for a string link $L$, the Milnor $\mu$–invariant $\mu_L(I)$ is defined; see Habegger and Lin [7]. Milnor $\mu$–invariants give a link-homotopy classification for string links.

Theorem 1.1 [7] Two $n$–component string links $L$ and $L'$ are link-homotopic if and only if $\bar{\mu}_L(I) = \bar{\mu}_{L'}(I)$ for any $I$ with $r(I) = 1$.

Theorem 1.1 implies the following.

Theorem 1.2 [13; 7] A link $L$ in $S^3$ is link-homotopic to the trivial link if and only if $\bar{\mu}_L(I) = 0$ for any $I$ with $r(I) = 1$.

Although Milnor invariants for sequences $I$ with $r(I) \geq 2$ are not necessarily link-homotopy invariants, they are generalized link-homotopy invariants. In fact, T Fleming and the author [4, Theorem 1.1] showed that $\bar{\mu}$–invariants for sequences $I$ with $r(I) \leq k$ are self $C_k$–equivalence invariants of links in $S^3$, where the self $C_k$–equivalence is an equivalence relation on (string) links generated by self $C_k$–moves, and a $C_k$–move is
a local move on links defined by Habiro [9; 10]. This statement holds for \( \mu \)-invariants of string links as well. The proof is the same as the one of [4, Theorem 1.1] except for using Proposition 3.1 instead of [14, Theorem 7]. The link-homotopy coincides with the self \( C_1 \)-equivalence. The self \( C_2 \)-equivalence coincides with the self \( \Delta \)-equivalence, which is an equivalence relation generated by \( \Delta \)-moves. A \( \Delta \)-move is a local move as illustrated in Figure 1; see Murakami and Nakanishi [15]. The \( \Delta \)-move is called a self \( \Delta \)-move if all strands in Figure 1 belong to the same component of a (string) link; see Shibuya [19].

![Figure 1](image)

A self \( \Delta \)-equivalence classification of 2-component links was given by Y Nakanishi and Y Ohyama [16] and for 2-component string links by Fleming and the author [3]. It is still open for (string) links with at least 3 components.

The following result is a generalization of Theorem 1.2.

**Theorem 1.3** [23, Corollary 1.5] A link \( L \) is self \( \Delta \)-equivalent to the trivial link if and only if \( \bar{\mu}_L(I) = 0 \) for any \( I \) with \( r(I) \leq 2 \).

In this paper, we generalize Theorem 1.1 and give a certain geometric characterization for string links whose \( \mu \)-invariants coincide for all sequences \( I \) with \( r(I) \leq 2 \). It is known that self \( \Delta \)-equivalence is too fine to give the characterization, i.e., there are 2-component string links such that their \( \mu \)-invariants vanish for all sequences \( I \) with \( r(I) \leq 2 \) and they are not self \( \Delta \)-equivalent to the trivial string link [3].

Since Milnor invariants are concordance invariants by Casson [1] and concordance does not imply self \( \Delta \)-equivalence by Nakanishi and Shibuya [17] and Nakanishi, Shibuya and Yasuhara [18], an equivalence relation generated by concordance and self \( \Delta \)-moves is looser than the self \( \Delta \)-equivalence and preserves Milnor invariants for all sequences \( I \) with \( r(I) \leq 2 \). We define the equivalence relation as follows. Two (string) links \( L \) and \( L' \) are self-\( \Delta \) concordant if there is a sequence \( L = L_1, \ldots, L_m = L' \) of (string) links such that for each \( i (\in \{1, \ldots, m-1\} \), \( L_i \) and \( L_{i+1} \) are either concordant or self \( \Delta \)-equivalent.

*Algebraic & Geometric Topology, Volume 9 (2009)*
The following is the main result of this paper.

**Theorem 1.4** Two $n$–component string links $L$ and $L'$ are self–$\Delta$ concordant if and only if $\mu_L(I) = \mu_{L'}(I)$ for any $I$ with $r(I) \leq 2$.

For an $n$–component string link $L$, let $L(k)$ be a $kn$–component string link obtained from $L$ by replacing each component of $L$ with $k$ zero framed parallels of it. By combining Theorem 1.4 and Proposition 3.2, we have the following corollary.

**Corollary 1.5** Two string links $L$ and $L'$ are self–$\Delta$ concordant if and only if $L(2)$ and $L'(2)$ are link-homotopic.

Let $\mathcal{SL}(n)$ be the set of $n$–component string links, and let $\mathcal{SL}(n)/(s\Delta + c)$ (resp. $\mathcal{SL}(n)/C_m$) be the set of self–$\Delta$ concordance classes (resp. the set of $C_m$–equivalence classes). Habiro showed that $\mathcal{SL}(n)/C_m$ is a nilpotent group [10, Theorem 5.4]. Since $C_{2n}$–equivalence for $n$–component (string) links implies self $\Delta$–equivalence [4, Lemma 1.2], we have that $\mathcal{SL}(n)/(s\Delta + c)$ is a nilpotent group. Moreover, since the first nonvanishing $\mu$–invariants are additive under the stacking product (for example see Cochran [2] and Habegger and Masbaum [8]), by Theorem 1.4, we have the following proposition.

**Proposition 1.6** The quotient $\mathcal{SL}(n)/(s\Delta + c)$ forms a torsion–free nilpotent group under the stacking product.

**Acknowledgments** The author would like to thank Professor Nathan Habegger and Professor Uwe Kaiser for many useful discussions which inspired him to consider the equivalence relation generated by self $\Delta$–moves and concordance. He is also very grateful to Professor Jonathan Hillman for helpful comments. He is partially supported by a Grant-in-Aid for Scientific Research (C) (#20540065) of the Japan Society for the Promotion of Science.

## 2 String links and Milnor invariants

In this section, we summarize the definitions of string links and Milnor invariants of links and string links.

A string link is a generalization of a pure braid defined by Habegger and Lin [7].
2.1 String links

Let $D$ be the unit disk in the plane and let $I = [0, 1]$ be the unit interval. Choose $n$ points $p_1, \ldots, p_n$ in the interior of $D$ so that $p_1, \ldots, p_n$ lie in order on the $x$–axis, see Figure 2. An $n$–component string link $L = K_1 \cup \cdots \cup K_n$ in $D \times I$ is a disjoint union of oriented arcs $K_1, \ldots, K_n$ such that each $K_i$ runs from $(p_i, 0)$ to $(p_i, 1)$ ($i = 1, \ldots, n$). A string link $K_1 \cup \cdots \cup K_n$ with $K_i = \{ p_i \} \times I$ ($i = 1, \ldots, n$) is called the $n$–component trivial string link and denoted by $I_n$. For a string link $L$ in $D \times I$, the closure $\text{cl}(L)$ of $L$ is a link in $S^3$ obtained from $L$ by identifying points of $\partial(D \times I)$ with their images under the projection $D \times I \to D$. It is easy to see that every link is the closure of some string link.

Milnor defined in [13; 14] a family of invariants of oriented, ordered links in $S^3$, known as Milnor $\mu$–invariants.

2.2 Milnor invariants of links

Given an $n$–component link $L = K_1 \cup \cdots \cup K_n$ in $S^3$, denote by $G$ the fundamental group of $S^3 \setminus L$, and by $G_q$ the $q$–th subgroup of the lower central series of $G$. We have a presentation of $G/G_q$ with $n$ generators, given by a meridian $\alpha_i$ of each component $K_i$. So, for each $j \in \{1, \ldots, n\}$, the longitude $l_j$ of the $j$–th component of $L$ is expressed modulo $G_q$ as a word in the $\alpha_i$’s (abusing notation, we still denote this word by $l_j$). The Magnus expansion $E(l_j)$ of $l_j$ is the formal power series in noncommuting variables $X_1, \ldots, X_n$ obtained by substituting $1 + X_i$ for $\alpha_i$ and $1 - X_i + X_i^2 - X_i^3 + \cdots$ for $\alpha_i^{-1}$, $i = 1, \ldots, n$. Let $I = i_1 i_2 \cdots i_k$ ($k \leq q$) be a sequence in $\{1, \ldots, n\}$. Denote by $\mu_I(L)$ the coefficient of $X_{i_1} \cdots X_{i_k}$ in the Magnus expansion $E(l_j)$. Milnor $\bar{\mu}$–invariant $\bar{\mu}(L)$ is the residue class of $\mu_I(L)$ modulo the greatest common divisor of all $\mu_I(J)$ such that $J$ is obtained from $I$ by removing at least one index, and permuting the remaining indices cyclically.

In [7], Habegger and Lin define the Milnor invariants of string links. We also refer the reader to Habegger and Masbaum [8].

2.3 Milnor invariants of string links

In the unit disk $D$, we chose a point $e \in \partial D$ and loops $\alpha_1, \ldots, \alpha_n$ as illustrated in Figure 2. For an $n$–component string link $L = K_1 \cup \cdots \cup K_n$ in $D \times I$ with $\partial K_j = \{(p_j, 0), (p_j, 1)\}$ ($j = 1, \ldots, n$), set $Y = (D \times I) \setminus L$, $Y_0 = (D \times \{0\}) \setminus L$, and $Y_1 = (D \times \{1\}) \setminus L$. We may assume that each $\pi_1(Y_i)$ ($i \in \{0, 1\}$) with base point $(e, 1)$ is the free group $F(n)$ on generators $\alpha_1, \ldots, \alpha_n$. We denote the image of $\alpha_j$ in the lower central series quotient $F(n)/F(n)_q$ again by $\alpha_j$. By Stallings’ theorem [22], the inclusions
Classification of string links up to self delta-moves and concordance

Figure 2

\[ i_t: Y_t \to Y \] induce isomorphisms \((i_t)_*: \pi_1(Y_t)/\pi_1(Y_t)_q \to \pi_1(Y)/\pi_1(Y)_q \) for any positive integer \(q\). Hence the induced map \((i_t)_1^{-1} \circ (i_0)_*\) is an automorphism of \( F(n)/F(n)_q \) and sends each \( \alpha_j \) to a conjugate \( l_j\alpha_j l_j^{-1} \) of \( \alpha_j \), where \( l_j \) is the longitude of \( K_j \) defined as follows. Let \( \gamma_j \) be a zero framed parallel of \( K_j \) such that the endpoints \((c_j, t) \in D \times \{t\} \) lie on the \( x \)-axis in \( \mathbb{R}^2 \times \{t\} \). The longitude \( l_j \in F(n)/F(n)_q \) is an element represented by the union of the arc \( \gamma_j \) and the segments \( e \times I, \ c_j e \times \{0, 1\} \) under \((i_1)_1^{-1}\). The coefficient \( \mu_L(i_1i_2\cdots i_{k-1}j) \) \((k \leq q)\) of \( X_{i_1}\cdots X_{i_{k-1}} \) in the Magnus expansion \( E(l_j) \) is well-defined invariant of \( L \), and it is called a Milnor \( \mu \)-invariant of \( L \).

3 Proof of Theorem 1.4

By an argument similar to that in the proof of [14, Theorem 7], we have the following proposition.

**Proposition 3.1** (cf [14, Theorem 7]) Let \( L_j' \) \((j = 1, 2)\) be an \( l \)-component string link obtained from an \( n \)-component string link \( L_j \) by replacing each component of \( L_j \) with zero framed parallels of it. Suppose that the \( i \)-th components of \( L_1' \) and \( L_2' \) correspond to the \( h(i) \)-th components of \( L_1 \) and \( L_2 \) respectively. For a sequence \( i_1i_2\cdots i_m \) of integers in \( \{1, 2, \ldots, l\} \), \( \mu_{L_1'}(I) = \mu_{L_2'}(I) \) for any subsequence \( I \) of \( i_1i_2\cdots i_m \) if and only if \( \mu_{L_1}(J) = \mu_{L_2}(J) \) for any subsequence \( J \) of \( h(i_1)h(i_2)\cdots h(i_m) \).

**Remark** It is shown that for a link \( L' \) in \( S^3 \) obtained from a link \( L \) by taking zero framed parallels of the components of \( L \), if the \( i \)-th component of \( L' \) corresponds to the \( h(i) \)-th component of \( L \), then \( \bar{\mu}_{L'}(i_1i_2\cdots i_m) = \bar{\mu}_{L}(h(i_1)h(i_2)\cdots h(i_m)) \) [14, Theorem 7]. Although this looks stronger than the proposition above, it holds for the residue class \( \bar{\mu} \). It does not hold for \( \mu \)-invariant of string links. In fact, there is the
following example: Let $L$ be the 2–component string link illustrated in Figure 3 and $L’$ the 3–component string link illustrated in Figure 3, which is obtained from $L$ by taking two zero framed parallels of the first component. Note that $h(1) = h(2) = 1$ and $h(3) = 2$. Then the Magnus expansion of the 2nd longitude of $L$ is $1 + X_1$ and the expansion of the 3rd longitude of $L’$ is $1 + X_2 + X_1X_2$. Hence we have $\mu_L(112) \neq \mu_{L’}(123)$.

**Proof** It is enough to consider the special case where $L_j’ (j = 1, 2)$ is an $(n + 1)$–component string link obtained from an $n$–component string link $L_j$ by replacing the $n$–th component of $L_j$ with two parallels of it. We may assume that the two parallels are contained in a tubular neighborhood $N_j$ of the $n$–th component of $L_j$. Then there is the natural homomorphism from $\pi_1(E_j)(\cong \pi_1(E_j \setminus N_j))$ to $\pi_1(E_j’)$, where $E_j$ and $E_j’$ are the complements of $L_j$ and $L_j’$ respectively. The $i$–th longitudes $l_{ji}$ (i = 1, . . . , n) of $L_j$ map to the $i$–th longitudes $l_{ji}’$ of $L_j’$, the $i$–th meridians $\alpha_{ji}$ (i = 1, . . . , n − 1) of $L_j$ map to the $i$–th meridians $\alpha_{ji}’$ of $L_j’$, and the $n$–th meridian $\alpha_{jn}$ maps to $\alpha_{jn}’\alpha_{jn+1}’$. Note that $l_{jn+1}’$ is equal to $l_{jn}’$. The Magnus expansion $M(l_{ji}’) of l_{ji}’ can be obtained from the expansion

$$M(l_{ji}h(i)) = 1 + \sum \mu_{L_j}(h_1 \cdots h_s h(i))X_{h_1} \cdots X_{h_s}$$

by substituting $M(\alpha_{jn}’\alpha_{jn+1}’) - 1 = X_n + X_{n+1} + X_nX_{n+1}$ for $X_n$.

Hence, if $\mu_{L_1}(J) = \mu_{L_2}(J)$ for any subsequence $J$ of $h(i_1) \cdots h(i_m)$, then $\mu_{L_1’}(I) = \mu_{L_2’}(I)$ for any subsequence of $i_1 \cdots i_m$. Recall that $h(i) = i$ (i = 1, . . . , n − 1) and $h(n) = h(n + 1) = n$.

On the other hand, suppose that there is a subsequence $Jh$ of $h(i_1) \cdots h(i_m)$ such that $\mu_{L_1}(Jh) \neq \mu_{L_2}(Jh)$. We may assume that the length of $J$ is minimal among all such subsequences, ie, for any subsequence $J’(\neq J)$ of $J$, $\mu_{L_1}(J’h) = \mu_{L_2}(J’h)$. Let $k_1 \cdots k_r k$ be a subsequence of $i_1 \cdots i_m$ with $h(k_1) \cdots h(k_r) = J$ and $h(k) = h$. Note that $\mu_{L_1’}(k_1 \cdots k_r k)$ might not be equal to $\mu_{L_1}(Jh)$ (if $k_1 \cdots k_r$ contains the
pattern \( n(n + 1) \). Since the Magnus expansion \( M(l_j') \) can be obtained from \( M(l_j) \) by substituting \( X_n + X_{n+1} + X_nX_{n+1} \) for \( X_n \), there is a set \( S_J \), possibly \( S_J = \{ J \} \), of subsequences of \( J \) such that

\[
\mu_{L_j'}(k_1 \cdots k_r k) = \sum_{J' \in S_J} \mu_{L_j}(J'h) \quad (j = 1, 2).
\]

The minimality of \( J \) implies

\[
\mu_{L_1'}(k_1 \cdots k_r k) - \mu_{L_2'}(k_1 \cdots k_r k) = \mu_{L_1}(Jh) - \mu_{L_2}(Jh) \neq 0.
\]

This completes the proof.

By combining Theorem 1.1 and Proposition 3.1, we have the following characterization for string links whose \( \mu \)-invariants coincide for all sequences \( I \) with \( r(I) \leq k \).

**Proposition 3.2** (cf [2, Proposition 9.3]) Let \( L_1 \) and \( L_2 \) be \( n \)-component string links and \( k \) a natural number. Let \( L_j(k) \) be a \( kn \)-component string link obtained from \( L_j \) by replacing each component of \( L_j \) with \( k \) zero framed parallels of it \( (j = 1, 2) \). Then the following are equivalent:

1. \( \mu_{L_1}(J) = \mu_{L_2}(J) \) for any \( J \) with \( r(J) \leq k \).
2. \( \mu_{L_1(k)}(I) = \mu_{L_2(k)}(I) \) for any \( I \) with \( r(I) = 1 \).
3. \( L_1(k) \) and \( L_2(k) \) are link-homotopic.

**Proof** Theorem 1.1 implies “(2) \( \Leftrightarrow \) (3)”. We only need to show “(1) \( \Leftrightarrow \) (2)”.

Suppose that the \( i \)-th components of \( L_1(k) \) and \( L_2(k) \) correspond to the \( h(i) \)-th components of \( L_1 \) and \( L_2 \) respectively. For a sequence \( I = i_1 i_2 \cdots i_m \) of integers in \( \{1, 2, \ldots, nk\} \), let \( h(I) \) denote \( h(i_1) h(i_2) \cdots h(i_m) \).

(1) \( \Rightarrow \) (2) Let \( I \) be a sequence of integers in \( \{1, 2, \ldots, nk\} \) with \( r(I) = 1 \). Since \( r(h(I)) \leq k \), for any subsequence \( J \) of \( h(I) \), we have \( r(J) \leq k \), and hence \( \mu_{L_1}(J) = \mu_{L_2}(J) \). By Proposition 3.1, we have \( \mu_{L_1(k)}(I) = \mu_{L_2(k)}(I) \).

(2) \( \Rightarrow \) (1) Let \( J \) be a sequence of integers in \( \{1, 2, \ldots, n\} \) with \( r(J) \leq k \). Then there is a sequence \( I' \) of integers in \( \{1, 2, \ldots, nk\} \) with \( r(I') = 1 \) and \( h(I') = J \). Since any subsequence \( I \) of \( I' \) satisfies \( r(I) = 1 \), \( \mu_{L_1(k)}(I) = \mu_{L_2(k)}(I) \). By Proposition 3.1, we have \( \mu_{L_1}(J) = \mu_{L_2}(J) \).
By using Proposition 3.2, we have the following proposition.

**Proposition 3.3** Let $L$ and $L'$ be $n$–component string links and $k$ a natural number. Then $\mu_L(I) = \mu_{L'}(I)$ for any $I$ with $r(I) \leq k$ if and only if $\mu_{L * T}(I) = 0$ for any $I$ with $r(I) \leq k$, where $*$ is the stacking product and $\overline{L}$ is the horizontal mirror image of $L'$ with the orientation reversed.

Note that, for a string link $L$, $\overline{L}$ is the inverse of $L$ under the concordance, ie, both $\overline{L} \ast L$ and $L \ast \overline{L}$ are concordant to a trivial string link.

**Proof** By Proposition 3.2, $\mu_L(I) = \mu_{L'}(I)$ (resp. $\mu_{L * T}(I) = 0$) for any $I$ with $r(I) \leq k$ if and only if $L(k)$ and $L'(k)$ (resp. $(L * T')(k)$ and the $k$–component trivial string link $1_{kn}$) are link-homotopic. Hence it is enough to show that $L(k)$ and $L'(k)$ are link-homotopic if and only if $(L * \overline{L}')(k)$ and $1_{kn}$ are link-homotopic. Note that $(L \ast \overline{L})(k) = L(k) \ast \overline{L}(k)$.

If $L(k)$ and $L'(k)$ are link-homotopic, then $L(k) \ast \overline{L}(k)$ is link-homotopic to $L'(k) \ast \overline{L}(k)$, which is concordant to $1_{kn}$. Since concordance of string links implies link-homotopy [5; 6]\(^1\), $L(k) \ast \overline{L}(k)$ is link-homotopic to $1_{kn}$.

If $L(k) \ast \overline{L}(k)$ is link-homotopic to $1_{kn}$, then $L(k) \ast \overline{L}(k) \ast L'(k)$ is link-homotopic to $L'(k)$. Since $L(k) \ast \overline{L}(k) \ast L'(k)$ is concordant to $L(k)$, $L(k)$ is link-homotopic to $L'(k)$. This completes the proof. \(\Box\)

Two $n$–component string links $L$ and $L'$ are **weak self $\Delta$–equivalent** if the closure $\text{cl}(L \ast \overline{L})$ is self $\Delta$–equivalent to the trivial link.

T Shibuya defined weak self $\Delta$–equivalence for links in $S^3$ [21], and showed that two links in $S^3$ are weak self $\Delta$–equivalent if and only if they are self–$\Delta$ concordant [20]. (In [21] and [20], the self–$\Delta$ concordance is called the $\Delta$–**cobordism**.) Here we give the same result for string links.

**Proposition 3.4** (cf [20, Theorem]) Two string links $L$ and $L'$ are weak self $\Delta$–equivalent if and only if they are self–$\Delta$ concordant.

Before proving the proposition above, we need some preparation.

Let $L = K_1 \cup \cdots \cup K_n$ be an $n$–component (string) link and $b$ a band attaching a single component $K_i$ with coherent orientation, ie, $b \cap L = b \cap K_i \subset \partial b$ consists

\(^1\) In [5; 6], it was shown that concordance of links in $S^3$ implies link-homotopy. It still holds for string links. It also follows from Theorem 1.1 since Milnor invariants are concordance invariants.
of two arcs whose orientations from $K_i$ are opposite to those from $\partial b$. Then $L' = (L \cup \partial b) \setminus \text{int}(b \cap K_i)$, which is a union of an $n$–component (string) link and a knot, is said to be obtained from $L$ by fusion (along a band $b$), and conversely $L$ is said to be obtained from $L'$ by fusion [12].

**Lemma 3.5** Let $L_1, L_2, L_3$ be oriented tangles such that $L_2$ is obtained from $L_1$ by a single (self) $\Delta$–move, and that $L_3$ is obtained from $L_2$ by a single fusion. Then there is an oriented tangle $L'_2$ such that $L'_2$ is obtained from $L_1$ by a single fusion, and that $L_3$ is obtained from $L'_2$ by a single (self) $\Delta$–move.

**Proof** Let $B$ be a 3–ball such that $L_1 \setminus B = L_2 \setminus B$, and that the pair of tangles $(B, L_1 \cap B)$ and $(B, L_2 \cap B)$ is a (self) $\Delta$–move. Let $b$ be a fusion band with $L_3 = (L_2 \cup \partial b) \setminus \text{int}(b \cap L_2)$. If $b$ intersects $B$, then we can move it out of $B$ by an isotopy fixing $L_2$ since $(B, L_2 \cap B)$ is a trivial tangle. Thus we may assume that $b$ is contained in $L_1 \setminus B$. Let $L'_2$ be a link obtained from $L_1$ by fusion along $b$. Then $L_3$ is obtained from $L'_2$ by a (self) $\Delta$–move, which corresponds to substituting $(B, L_2 \cap B)$ for $(B, L_1 \cap B)$. $\square$

**Proof of Proposition 3.4** If $L$ and $L'$ are self–$\Delta$ concordant, then $\mu_L(I) = \mu_{L'}(I)$ for any $I$ with $r(I) \leq 2$ [1; 4]. Proposition 3.3 and Theorem 1.3 imply that $\text{cl}(L * \overline{L'})$ is self $\Delta$–equivalent to the trivial link.

Suppose $L$ and $L'$ are weak self $\Delta$–equivalent. Since $L$ is concordant to $L * \overline{L'} * L'$, it is enough to show that $L * \overline{L'} * L'$ and $L'$ are self–$\Delta$ concordant. The split sum of $L'$ and $\text{cl}(L * \overline{L'})$ is obtained from $L * \overline{L'} * L'$ by a finite sequence of fission, and $\text{cl}(L * \overline{L'})$ is self $\Delta$–equivalent to the trivial link $O$. So $L * \overline{L'} * L'$ is obtained from the split sum of $L'$ and $O$ by a sequence of self $\Delta$–moves and fusion. By Lemma 3.5, we can freely choose to perform all fusion first, and then all self $\Delta$–moves. Hence we have that the fusion of $L'$ and $O$, which is concordant to $L'$, is self $\Delta$–equivalent to $L * \overline{L'} * L'$. $\square$

Now we are ready to prove Theorem 1.4.

**Proof of Theorem 1.4** By Theorem 1.3, $L$ and $L'$ are weak self $\Delta$–equivalent if and only if $\mu_{\text{cl}(L_1 * \overline{L'})}(I) = \mu_{L_1 * \overline{L'}}(I) = 0$ for any $I$ with $r(I) \leq 2$. Proposition 3.3 and Proposition 3.4 complete the proof. $\square$

**Remark** Although the $C_k$–move ($k \geq 3$) is not an unknotting operation, it might be reasonable to consider the following question: For two string links $L$ and $L'$ whose components are trivial, if $\mu_L(I) = \mu_{L'}(I)$ for any $I$ with $r(I) \leq k$, then are $L$ and $L'$...
equivalent up to self $C_k$–move and concordance? The question is still open, but the answer is likely negative. For example, the Hopf link with both components Whitehead doubled, which is a boundary link and thus all its Milnor invariants vanish, is neither self $C_3$–equivalent [4] nor concordant [11, Section 7.3] to the trivial link.

References


Classification of string links up to self delta-moves and concordance


Tokyo Gakugei University
Department of Mathematics, Koganei, Tokyo 184–8501, Japan

yasuhara@u-gakugei.ac.jp

Received: 6 July 2008 Revised: 26 November 2008