An intrinsic nontriviality of graphs

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We say that a graph is intrinsically nontrivial if every spatial embedding of the graph contains a nontrivial spatial subgraph. We prove that an intrinsically nontrivial graph is intrinsically linked, namely every spatial embedding of the graph contains a nonsplittable 2–component link. We also show that there exists a graph such that every spatial embedding of the graph contains either a nonsplittable 3–component link or an irreducible spatial handcuff graph whose constituent 2–component link is split.

57M15; 57M25

Dedicated to Professor Akio Kawauchi on his 60th *birthday*

1 Introduction

Throughout this paper we work in the piecewise linear category. Let f be an embedding of a finite graph G into the 3–sphere $\mathbb{S}^3$. Then f (or $f(G)$) is called a *spatial embedding* of G or simply a *spatial graph*. We call a subgraph γ of G which is homeomorphic to the circle a *cycle*. If G is homeomorphic to the disjoint union of cycles, then f is an *n–component link* (or *knot* if $n=1$). Two spatial embeddings f and g of G are said to be *ambient isotopic* if there exists an orientation-preserving self homeomorphism Φ on $\mathbb{S}^3$ such that $\Phi\circ f=g$. A graph G is said to be *planar* if there exists an embedding of G into the 2–sphere, and a spatial embedding f of a planar graph G is said to be *trivial* if it is ambient isotopic to an embedding of G into a 2–sphere in $\mathbb{S}^3$. A spatial embedding f of G is said to be *split* if there exists a 2–sphere S in $\mathbb{S}^3$ such that $S\cap f(G)=\varnothing$ and each connected component of $\mathbb{S}^3\setminus S$ has intersection with $f(G)$, and otherwise f is said to be *nonsplittable*.

A graph G is said to be *intrinsically linked* if every spatial embedding f of G contains a nonsplittable 2–component link. Conway and Gordon [2] and Sachs [15] showed that K_6 is intrinsically linked, where K_n denotes the *complete graph* on n vertices. Conway and Gordon also showed that K_7 is *intrinsically knotted*, namely every spatial embedding f of G contains a nontrivial knot. For a positive integer n, Flapan, Foisy, Naimi and Pommersheim [4] showed that there exists an *intrinsically n–linked* graph G,

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namely every spatial embedding f of G contains a nonsplittable n–component link (see also Flapan–Naimi–Pommersheim [3] and Bowlin–Foisy [1] for the case of $n = 3$). Note that these results paid attention to only constituent knots and links of spatial graphs. Our purpose in this paper is to generalize the notion of intrinsically n–linkedness by paying attention to not only constituent knots and links but also spatial subgraphs which do not need to be knots and links, and to give graphs which have such a generalized property.

We say that a graph G is *intrinsically nontrivial* if for every spatial embedding f of G there exists a planar subgraph F of G such that $f|_F$ is not trivial. It is clear that intrinsically n–linked graphs are intrinsically nontrivial. Note that F depends on f and does not need to have the same topological type uniformly. In this situation, pioneering work has been done by Foisy [5]. Let P and P' be graphs in the *Petersen family*, which is a family of seven graphs obtained from K_6 by $\nabla - Y$ or $Y - \nabla$ *exchanges* (see Sachs [15]); all intrinsically linked graphs have a minor in this family by Robertson–Seymour–Thomas [14]. Let $P *_4 P'$ be the graph which consists of P and P' connected by four disjoint edges e_1, e_2, e_3 and e_4 as illustrated in Figure 1, which shows the case of $P = P' = K_6$. Then he showed that $P *_4 P'$ is *intrinsically knotted or* 3*–linked*, namely every spatial embedding of $P *_4 P'$ contains either a nontrivial knot or a nonsplittable 3–component link. He also showed that $K_6 *_4 K_6$ is neither intrinsically knotted nor intrinsically 3–linked.

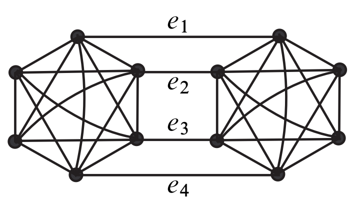

Figure 1: $K_6 *_4 K_6$

It was already known that intrinsically knotted graphs and intrinsically n–linked graphs ($n \geq 3$) are intrinsically linked by Robertson–Seymour–Thomas' characterization of intrinsically linked graphs [14]. First, we show that if a graph G is intrinsically nontrivial then every spatial embedding of G must contain a nonsplittable 2–component link as follows.

Theorem 1.1 *Intrinsically nontrivial graphs are intrinsically linked.*

Then, for an intrinsically nontrivial graph G, we are interested in the collection of nontrivial spatial graph types (except for 2–component link types) where every spatial embedding of G contains at least one spatial graph type in the collection. Foisy's above result also only paid attention to constituent knots and links of spatial graphs. We shall consider a smaller graph than $P{*_4}P'$ from the viewpoint of intrinsic nontriviality. Let $P{*_3}P'$ be the graph which is obtained from $P{*_4}P'$ by deleting e_4. Then we have the following.

Theorem 1.2 *Let P and P' be graphs in the Petersen family. Then every spatial embedding of $P{*_3}P'$ contains either a nonsplittable 3–component link or an irreducible spatial handcuff graph whose constituent 2–component link is split.*

Here a *spatial handcuff graph* is a spatial embedding f of the graph H which is illustrated in Figure 2. Note that an orientation is given to each loop, namely we regard $f(\gamma_1 \cup \gamma_2)$ as an ordered and oriented 2–component link. A spatial handcuff graph f is said to be *irreducible* if there does not exist a 2–sphere in $\mathbb{S}^3$ which intersects $f(H)$ transversely at one point. Note that an irreducible spatial handcuff graph is not trivial. We also show that $K_6{*_3}K_6$ has a spatial embedding which does not contain a nonsplittable 3–component link, and another spatial embedding which does not contain an irreducible spatial handcuff graph whose constituent 2–component link is split (Example 4.4). In particular, the former spatial embedding does not contain a nontrivial knot or a nonsplittable n–component link for $n \geq 3$. Namely Theorem 1.2 gives a new type of intrinsic nontriviality of graphs which cannot be detected by observing only constituent knots and links of its spatial embeddings.

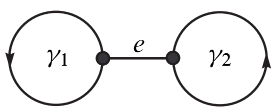

Figure 2

In the next section, we prove Theorem 1.1. In Section 3, we recall an ambient isotopy invariant of spatial handcuff graphs which was introduced by the author in [11]. In Section 4, we prove Theorem 1.2.

2 Proof of Theorem 1.1

A spatial embedding f of a graph G is said to be *free* if the fundamental group of the spatial graph complement $\pi_1(\mathbb{S}^3 \setminus f(G))$ is a free group.

We say that G is *intrinsically nonfree* if for every spatial embedding of G there exists a subgraph F of G such that $f|_F$ is not free. To prove Theorem 1.1, we show the following.

Theorem 2.1 *The following are equivalent.*

(1) *G is intrinsically nonfree.*

(2) *G is intrinsically nontrivial.*

(3) *G is intrinsically linked.*

Proof It is clear that (3) implies (2). Next we show that (2) implies (1). If G is intrinsically nontrivial, we have that for every spatial embedding f of G there exists a planar subgraph F of G such that $f|_F$ is nontrivial. Then by Scharlemann–Thompson's famous criterion [16], there exists a subgraph F' of F such that $f|_{F'}$ is not free. Since F' is also a subgraph of G, we have that G is intrinsically nonfree. Finally we show that (1) implies (3). Assume that G is not intrinsically linked. Then it follows from Robertson–Seymour–Thomas [14, (1.2)] that there exists a spatial embedding f of G such that for any cycle γ of G there exists a 2–disk D_γ in $\mathbb{S}^3$ such that $f(G)\cap D_\gamma = f(G)\cap \partial D_\gamma = f(\gamma)$. At this time, it is also known that $f|_F$ is free for any subgraph F of G [14, (3.3)] (the case that G is planar was first shown by Wu [20]). Thus we have that G is not intrinsically nonfree. □

3 An invariant of spatial handcuff graphs

In this section we give the definition of an invariant of spatial handcuff graphs which can detect an irreducible one whose constituent 2–component link is split. Let $L = J_1 \cup J_2$ be an ordered and oriented 2–component link. Let D be an oriented 2–disk and x_1, x_2 disjoint arcs in ∂D, where ∂D has the orientation induced by the one of D, and each arc has an orientation induced by the one of ∂D. We assume that D is embedded in $\mathbb{S}^3$ so that $D \cap L = x_1 \cup x_2$ and $x_i \subset J_i$ with opposite orientations for each i. Then we call a knot $K_D = (L \cup \partial D) \setminus (\mathrm{int}\, x_1 \cup \mathrm{int}\, x_2)$ a *D–sum* of L. For a spatial handcuff graph f, we denote $f(\gamma_1 \cup \gamma_2)$ by L_f and consider a D–sum of L_f so that $f(e) \subset D$, $f(e)\cap \partial D = f(e)\cap L_f = \{p_1, p_2\}$ and $p_i \in \mathrm{int}\, x_i$ $(i = 1, 2)$. We call such a D–sum of L_f a *D–sum of L_f with respect to f* and denote it by $K_D(f)$. Though $K_D(f)$ is not uniquely determined up to ambient isotopy, the author showed in [11, Remark 3.4 (1)] that the modulo $\mathrm{lk}(L_f)$ reduction of $a_2(K_D(f))$ is an ambient isotopy invariant of f, where lk denotes the *linking number* in $\mathbb{S}^3$ and a_2 denotes the second coefficient of the *Conway polynomial*. Then we define

$$n(f, D) ;= a_2(K_D(f)) - a_2(f(\gamma_1)) - a_2(f(\gamma_2))$$

and denote the modulo $\mathrm{lk}(L_f)$ reduction of $n(f,D)$ by $\overline{n}(f)$. It is clear that $\overline{n}(f)$ is also an ambient isotopy invariant of f. In particular, $\overline{n}(f)$ is a uniquely determined integer if $\mathrm{lk}(L_f)=0$. In this case we denote $\overline{n}(f)$ by $n(f)$ simply. Then we have the following.

Lemma 3.1 *Let f be a spatial handcuff graph. If f is not irreducible, then for every choice of D, $n(f,D)=0$. In particular, if $\mathrm{lk}(L_f)=0$, then f is irreducible if $n(f)\neq 0$.*

Proof If f is not irreducible, then L_f is split and any D–sum of L_f with respect to f is the connected sum of $f(\gamma_1)$ and $f(\gamma_2)$. Recall that a_2 is additive under the connected sum of knots [7]. Thus we have that

$$\begin{aligned} n(f,D) &= a_2(K_D(f)) - a_2(f(\gamma_1)) - a_2(f(\gamma_2)) \\ &= a_2(f(\gamma_1)\sharp f(\gamma_2)) - a_2(f(\gamma_1)) - a_2(f(\gamma_2)) \\ &= a_2(f(\gamma_1)) + a_2(f(\gamma_2)) - a_2(f(\gamma_1)) - a_2(f(\gamma_2)) \\ &= 0. \end{aligned}$$

Therefore we have the result. □

For integers r and s, let $f_{r,s}$ be the spatial handcuff graph as illustrated in Figure 3, where the rectangles represented by r and s stand for $|r|$ full twists and $|s|$ full twists

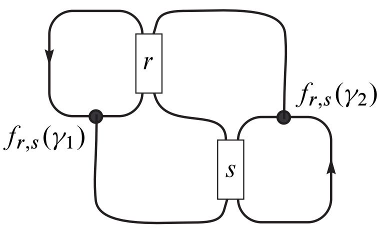

Figure 3

as illustrated in Figure 4, respectively. Note that the constituent 2–component link $L_{f_{r,s}}$ is trivial. Then we have the following.

Lemma 3.2 $n(f_{r,s})=2rs$.

Proof Let K_+, K_- and K_0 be two oriented knots and an oriented 2–component link which are identical except inside the depicted regions as illustrated in Figure 5.

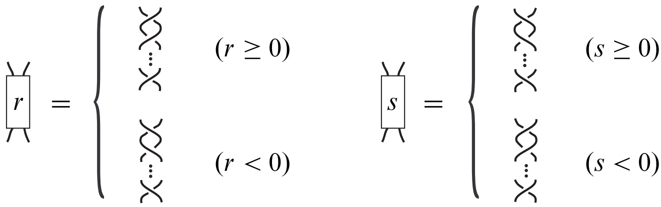

Figure 4

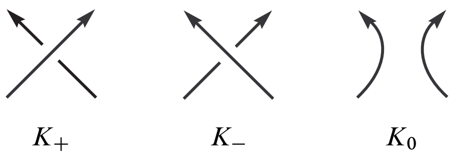

Figure 5

Then it is well known that $a_2(K_+) - a_2(K_-) = \mathrm{lk}(K_0)$ [6]. We consider the D–sum of L_f with respect to $f_{r,s}$ and the skein tree started from $K_D(f_{r,s})$ as illustrated in Figure 6, where $\varepsilon = \pm 1$ is the usual sign of marked crossing. Note that $\varepsilon = 1$ if $r > 0$ and -1 if $r < 0$. Then

$$a_2(K_D(f_{r,s})) - a_2(J) = \varepsilon \mathrm{lk}(M_1) = \varepsilon(r - \varepsilon + s), \tag{3-1}$$

$$a_2(K_D(f_{r-\varepsilon,s})) - a_2(J) = \varepsilon \mathrm{lk}(M_2) = \varepsilon(r - \varepsilon - s). \tag{3-2}$$

Thus by (3-1) and (3-2),

$$a_2(K_D(f_{r,s})) - a_2(K_D(f_{r-\varepsilon,s})) = 2\varepsilon s.$$

Hence we have that

$$\begin{aligned} a_2(K_D(f_{r,s})) &= a_2(K_D(f_{r-\varepsilon,s})) + 2\varepsilon s \\ &= a_2(K_D(f_{r-2\varepsilon,s})) + 2\varepsilon s + 2\varepsilon s \\ &\vdots \\ &= a_2(K_D(f_{r-|r|\varepsilon,s})) + 2\varepsilon |r| s \\ &= a_2(K_D(f_{0,s})) + 2rs. \end{aligned} \tag{3-3}$$

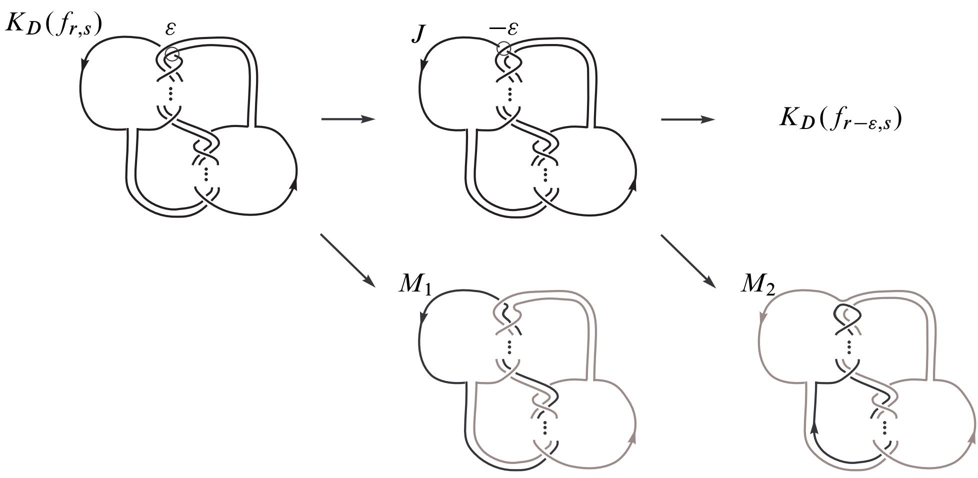

Figure 6

It is easy to see that $f_{0,s}$ is trivial, namely $a_2(K_D(f_{0,s})) = 0$. Therefore by (3-3),

$$n(f_{r,s}) = a_2(K_D(f_{r,s})) - a_2(f_{r,s}(\gamma_1)) - a_2(f_{r,s}(\gamma_2)) = 2rs.$$

This completes the proof. □

By Lemma 3.1 and Lemma 3.2, we have that if $r, s \neq 0$ then $f_{r,s}$ is irreducible. Since $L_{f_{r,s}}$ is trivial, $\{f_{r,s}\}_{r,s\neq 0}$ is a family of *minimally knotted* spatial handcuff graphs, namely each $f_{r,s}$ is nontrivial and any of whose spatial proper subgraphs is trivial.

4 Proof of Theorem 1.2

Let P_4 be the oriented graph consists of four edges e_1, e_2 e_3, e_4 and two loops e_5, e_6 as illustrated in Figure 7. We denote the cycles e_5, $e_1 \cup e_2$, $e_3 \cup e_4$ and e_6 of P_4 by c_1, c_2, c_3 and c_4, respectively, and the subgraph $c_1 \cup e_i \cup e_j \cup c_4$ $(i = 1, 2,\ j = 3, 4)$ by H_{ij}. Note that H_{ij} is homeomorphic to the graph H illustrated in Figure 2. Let f be a spatial embedding of P_4 with $\mathrm{lk}(f(c_1 \cup c_4)) = 0$. Then we define $\xi(f) \in \mathbb{Z}$ by

$$\begin{aligned}\xi(f) &= \sum_{i,j} (-1)^{i+j} n(f|_{H_{ij}}) \\ &= n(f|_{H_{13}}) - n(f|_{H_{14}}) - n(f|_{H_{23}}) + n(f|_{H_{24}}).\end{aligned}$$

Proposition 4.1 *$\xi(f)$ is a Delta equivalence invariant of f.*

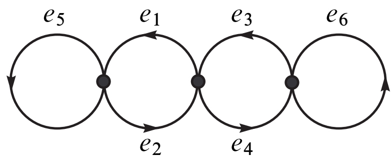

Figure 7

Here, a *Delta equivalence* is an equivalence relation on spatial graphs which is generated by *Delta moves* and ambient isotopies, where a Delta move is a local move on a spatial graph as illustrated in Figure 8 [8; 10]. It is shown in [9] that a Delta equivalence

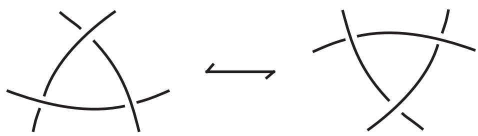

Figure 8

coincides with a *(spatial graph-)homology*, which is an equivalence relation on spatial graphs introduced in [18]. Note that a Delta move preserves the linking number of each of the constituent 2–component links. We show the following lemma needed to prove Proposition 4.1.

Lemma 4.2 *Let g be a spatial handcuff graph with $\mathrm{lk}(L_g) = 0$ and h a spatial handcuff graph obtained from g by a single Delta move. Then we have the following.*

(1) *If either $g(\gamma_1)$ or $g(\gamma_2)$ does not appear in the Delta move as the strings, then $n(g) = n(h)$.*

(2) *If both $g(\gamma_1)$ and $g(\gamma_2)$ appear and $g(e)$ does not appear in the Delta move as the strings, then $n(g) - n(h) = \pm 1$.*

(3) *If all of $g(\gamma_1)$, $g(\gamma_2)$ and $g(e)$ appear in the Delta move as the strings, then $n(g) - n(h) = \pm 2$ or 0.*

Proof (1) If all of the three strings in such an intended Delta move belong to the same spatial edge (such a move is called a *self Delta move*), we have the result because it is known that $n(g)$ is invariant under a self Delta move [11, Theorem 2.1]. Next we consider the case that at least one of the three strings in the Delta move belong to $g(e)$. If $g(\gamma_j)$ does not appear in the Delta move as the strings ($j = 1$ or 2), then by applying the deformation on $g(H)$ as illustrated in Figure 9 repeatedly, we can see that such a Delta move may be realized by self Delta moves on $g(\gamma_i)$ $(i \neq j)$. Therefore we have that $n(g) = n(h)$.

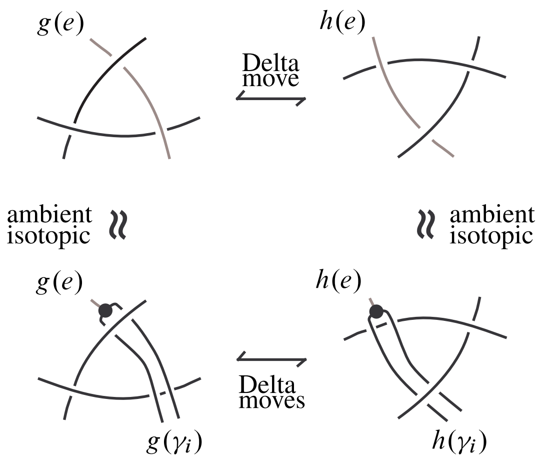

Figure 9

(2) If h is obtained from g by such an intended single Delta move, then we may consider $K_D(g)$ and $K_D(h)$ by the same 2–disk D so that $K_D(h)$ is obtained from $K_D(g)$ by a single Delta move. Then by the result of Okada [13] that if a knot K_1 is obtained from a knot K_2 by a single delta move then $a_2(K_1) - a_2(K_2) = \pm 1$, we have that $a_2(K_D(g)) - a_2(K_D(h)) = \pm 1$. On the other hand, since a Delta move is a 3–*component Brunnian local move* [19, Section 2, Examples (2)], we have that $g(\gamma_i)$ and $h(\gamma_i)$ are ambient isotopic $(i = 1, 2)$. Hence we have that $n(g) - n(h) = a_2(K_D(g)) - a_2(K_D(h)) = \pm 1$.

(3) If h is obtained from g by such an intended single Delta move, then we may consider $K_D(g)$ and $K_{D'}(h)$ so that $K_{D'}(h)$ is obtained from $K_D(g)$ by twice Delta moves as illustrated in Figure 10. Thus by Okada's result as we said in the proof of (2), we have that $a_2(K_D(g)) - a_2(K_{D'}(h)) = \pm 2$ or 0. Note that $g(\gamma_i)$ and $h(\gamma_i)$ are ambient isotopic $(i = 1, 2)$ by Brunnian property of the Delta move. Hence we have that $n(g) - n(h) = a_2(K_D(g)) - a_2(K_{D'}(h)) = \pm 2$ or 0. □

Proof of Proposition 4.1 Let f' be a spatial handcuff graph which is obtained from f by a single Delta move. It is sufficient to show that $\xi(f) = \xi(f')$. If either $f(e_5)$ or $f(e_6)$ does not appear in the Delta move as the strings, then by Lemma 4.2 (1)

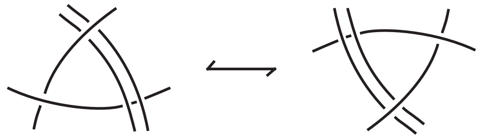

Figure 10

and Brunnian property of the Delta move, we have that $n(f|_{H_{ij}}) = n(f'|_{H_{ij}})$ for any $i = 1,2$ and $j = 3,4$. Hence $\xi(f) = \xi(f')$. If none of $f(e_1)$, $f(e_2)$, $f(e_3)$ or $f(e_4)$ appears in the Delta move as the strings, then by Lemma 4.2 (2) we have that $n(f|_{H_{ij}}) - n(f'|_{H_{ij}}) = \pm 1$ $(i = 1,2,\ j = 3,4)$. Note that if two oriented knots differ by a single Delta move then the variation of a_2 of them is determined only by the order of strings in the Delta move and their orientations arising from following the knot along the orientation (cf [12; 21, Theorem 6]). Hence we have that $n(f|_{H_{ij}}) - n(f'|_{H_{ij}}) = 1$ $(i = 1,2,\ j = 3,4)$ or $n(f|_{H_{ij}}) - n(f'|_{H_{ij}}) = -1$ $(i = 1,2,\ j = 3,4)$. Then

$$\xi(f) - \xi(f') = \sum_{i,j} (-1)^{i+j} \left(n(f|_{H_{ij}}) - n(f'|_{H_{ij}}) \right) = (\pm 1) \sum_{i,j} (-1)^{i+j} = 0.$$

Finally we consider the case that all of $f(e_5)$, $f(e_6)$ and $f(e_k)$ appear in the Delta move as the strings $(k = 1,2,3,4)$. It is sufficient to check the case of $k = 1$. Then by Lemma 4.2 (3) and noting as before the variation of a_2 of two oriented knots differing by a single Delta move, it holds that any one of $n(f|_{H_{1j}}) - n(f'|_{H_{1j}}) = 2$ $(j = 3,4)$, $n(f|_{H_{1j}}) - n(f'|_{H_{1j}}) = -2$ $(j = 3,4)$ or $n(f|_{H_{1j}}) - n(f'|_{H_{1j}}) = 0$ $(j = 3,4)$. Note that $f|_{H_{2j}}$ and $f'|_{H_{2j}}$ are ambient isotopic for any $j = 3,4$ by Brunnian property of the Delta move. Then

$$\xi(f) - \xi(f') = n(f|_{H_{13}}) - n(f'|_{H_{13}}) - \left(n(f|_{H_{14}}) - n(f'|_{H_{14}}) \right) = 0.$$

Therefore we have the desired conclusion. □

Lemma 4.3 *Let f be a spatial embedding of P_4 with $\mathrm{lk}(f(c_1 \cup c_4)) = 0$. Then we have that $\mathrm{lk}(f(c_1 \cup c_3))\mathrm{lk}(f(c_2 \cup c_4)) \neq 0$ if and only if $\xi(f) \neq 0$.*

Proof It is known that two spatial embeddings of a planar graph are Delta equivalent if and only if their corresponding constituent 2–component links have the same linking number [17]. Therefore we have that if $\mathrm{lk}(f(c_1 \cup c_3)) = s$ and $\mathrm{lk}(f(c_2 \cup c_4)) = r$, then f is Delta equivalent to the spatial embedding $h_{r,s}$ of P_4 as illustrated in Figure 11, where the rectangles represented by r and s stand for $|r|$ full twists and $|s|$ full twists as illustrated in Figure 4, respectively. Note that $h_{r,s}|_{H_{13}}$, $h_{r,s}|_{H_{14}}$ and $h_{r,s}|_{H_{23}}$ are

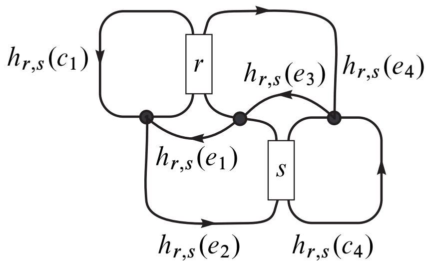

Figure 11

trivial spatial handcuff graphs. Then by combining Proposition 4.1 and Lemma 3.2,

$$\xi(f) = \xi(h_{r,s}) = n(h_{r,s}|_{H_{24}}) = 2rs.$$

This implies the result. □

Note that if $\xi(f) \neq 0$ then there must exist a subgraph H_{ij} such that $n(f|_{H_{ij}}) \neq 0$. Then by Lemma 3.1 we have that $f|_{H_{ij}}$ is irreducible.

Proof of Theorem 1.2 Let f be a spatial embedding of $P *_3 P'$. Note that every spatial embedding of a graph in the Petersen family contains a 2–component link with odd linking number [2; 19]. Hence there exists a pair of two disjoint cycles γ_1, γ_2 of P such that $\mathrm{lk}(f(\gamma_1 \cup \gamma_2)) \neq 0$, and there exists a pair of two disjoint cycles γ_1', γ_2' of P' such that $\mathrm{lk}(f(\gamma_1' \cup \gamma_2')) \neq 0$. If there exist two edges e_i and e_j $(1 \leq i < j \leq 3)$ each of which connects γ_l with γ_k' for some l and k, by Bowlin–Foisy's argument [1, Lemma 3], we have that $f(\gamma_1 \cup \gamma_2 \cup \gamma_1' \cup \gamma_2' \cup e_i \cup e_j)$ contains a nonsplittable 3–component link. If for any l and k there do not exist two edges e_i and e_j $(1 \leq i < j \leq 3)$ each of which connects γ_l with γ_k', then we may assume that e_1 connects γ_1 with γ_1', e_2 connects γ_1 with γ_2' and e_3 connects γ_2 with γ_2' without loss of generality. If $f(\gamma_2 \cup \gamma_1')$ is nonsplittable, then $f(\gamma_1 \cup \gamma_2 \cup \gamma_1')$ is a nonsplittable 3–component link. If $f(\gamma_2 \cup \gamma_1')$ is split, then let us consider the subgraph $F = \gamma_1 \cup \gamma_2 \cup \gamma_1' \cup \gamma_2' \cup e_1 \cup e_2 \cup e_3$ of $P *_3 P'$. We denote the graph obtained from F by contracting e_1, e_2 and e_3 by $F/e_1/e_2/e_3$. Note that $F/e_1/e_2/e_3$ is homeomorphic to P_4 which is illustrated in Figure 7. Let $\bar{f}$ is a spatial embedding of $F/e_1/e_2/e_3$ naturally induced from $f|_F$. Then by applying Lemma 4.3 and Lemma 3.1 to $\bar{f}$, we have that $\bar{f}(F/e_1/e_2/e_3)$ contains an irreducible spatial handcuff graph whose constituent 2–component link is split. Then it is clear that $f(F)$ also contains a spatial handcuff graph with the same spatial graph type as above. This implies that f contains an irreducible spatial handcuff graph whose constituent 2–component link is split. This completes the proof. □

Example 4.4 Let f be the spatial embedding of $K_6 *_3 K_6$ as illustrated in Figure 12. Then we can see that each of the spatial handcuff graphs contained in f is trivial or ambient isotopic to the spatial handcuff graph as illustrated in Figure 13. Namely f does not contain an irreducible spatial handcuff graph whose constituent 2–component link is split. But f contains exactly one nonsplittable 3–component link. Next, let g be the spatial embedding of $K_6 *_3 K_6$ as illustrated in Figure 12 which is obtained from f by a single crossing change. Then we can see that each of the constituent knots of g is trivial and each of the constituent n–component links of g is split for $n \geq 3$. But g contains exactly one irreducible spatial handcuff graph whose constituent 2–component link is split.

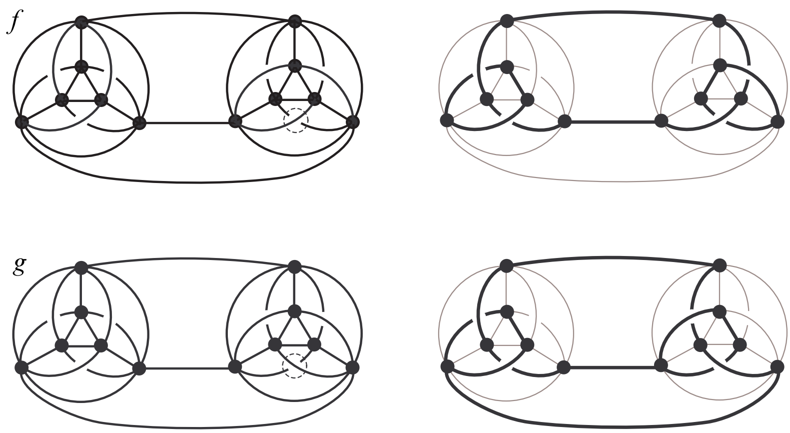

Figure 12

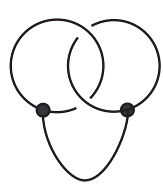

Figure 13

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