Algebraic & Geometric Topology 9 (2009) 429–454

Circular thin position for knots in S^3

FABIOLA MANJARREZ-GUTIÉRREZ

A regular circle-valued Morse function on the knot complement $C_K = S^3 \setminus K$ is a function $f\colon C_K \to S^1$ which separates critical points and which behaves nicely in a neighborhood of the knot. Such a function induces a handle decomposition on the knot exterior $E(K) = S^3 \setminus N(K)$, with the property that every regular level surface contains a Seifert surface for the knot. We rearrange the handles in such a way that the regular surfaces are as "simple" as possible. To make this precise the concept of *circular width for* $E(K)$ is introduced. When $E(K)$ is endowed with a handle decomposition which realizes the circular width we will say that the knot K is in *circular thin position*. We use this to show that many knots have more than one nonisotopic incompressible Seifert surface. We also analyze the behavior of the circular width under some knot operations.

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1 Introduction

Let M be a smooth 3–manifold. Classical Morse theory deals with a real-valued function $f\colon M \to \mathbb{R}$. This function corresponds to a handle decomposition of M namely $M = b_0 \cup N_1 \cup T_1 \cup \cdots \cup N_r \cup T_r \cup b_3$, where b_0 is a collection of 0–handles, N_i is a collection of 1–handles, T_i is a collection of 2–handles and b_3 is a collection of 3–handles. In [15] Scharlemann and Thompson introduce the concept of *thin position* for 3–manifolds; the idea is to build the manifold as described before, with a sequence of 1–handles and 2–handles chosen to keep the boundaries of the intermediate steps as simple as possible.

The Morse theory of circle-valued maps $f\colon M \to S^1$, as in the real case, relates the topology of a manifold M to the critical points of f. Morse–Novikov theory was introduced by Novikov [10] to study these functions. See Ranicki [12] for a survey of these topics.

Recently there has been work on circle-valued Morse theory on the complement of knots and links in S^3. In [16], Pajitnov, Rudolph and Weber introduced the concept of the *Morse–Novikov* number of a link $L \subset S^3$. The *Morse–Novikov number* of a

Published: 6 March 2009 DOI: 10.2140/agt.2009.9.429

link, denoted by $\mathrm{MN}(L)$, is the least possible number of critical points of a regular circle-valued Morse mapping $f\colon C_L \to S^1$. They proved that the Morse–Novikov number is subadditive with respect to the connected sum of knots; ie $\mathrm{MN}(K_1 \sharp K_2) \leq \mathrm{MN}(K_1) + \mathrm{MN}(K_2)$.

In [6], Goda pointed out that there is a handle decomposition which corresponds to a circle-valued Morse map, which he calls a Heegaard splitting for sutured manifolds.

We can consider more general circle-valued Morse functions on knot complements which correspond to handle decompositions that do not necessarily arise from a Heegaard splitting.

Analogous to Scharlemann and Thompson, we describe a process to reorder the handles of a handle decomposition of a knot exterior in such a way that the regular level surfaces are as simple as possible, giving rise to the definition of the *circular width of the knot exterior* and the *circular thin position of the knot exterior*. Similarly to regular thin position, circular thin position guarantees that all the level surfaces are either incompressible or weakly incompressible. Hence when the knot exterior is in circular thin position we obtain a nice sequence of Seifert surfaces which are alternately incompressible and weakly incompressible.

In general we expect to see several such level surfaces. However there are some special cases. Recall that a fibered knot $K \subset S^3$ is a knot with a Seifert surface R whose knot complement can be fibered over S^1 with fiber R.

In our context a fibered knot is a knot whose knot exterior has a circular thin position with one and only one incompressible level surface and no weakly incompressible level surface. This is the unique circular thin position for a fibered knot (see Burde and Zieschang [1] or Whitten [18]) and as expected, circular thin position yields no additional Seifert surfaces for the knot. The circular width of a fibered knot is defined to be zero.

We define an almost fibered knot to be a knot whose complement possesses a circular thin position in which there is one and only one weakly incompressible Seifert surface S and one and only one incompressible Seifert surface F.

Goda [5] showed that all nonfibered knots up to ten crossings are handle number one knots. In our context these knots have a circular thin position with one incompressible Seifert surface F of minimal genus and a weakly incompressible Seifert surface S with $\mathrm{genus}(S) = \mathrm{genus}(F) + 1$. Thus, all nonfibered knots up to ten crossings are examples of almost fibered knots. Goda's examples also illustrate that almost fibered knots do not have a unique circular thin position. He describes knots with two nonisotopic minimal genus Seifert surfaces which can be used to find two different circular thin positions, both giving the structure of an almost fibered knot.

Given all these concepts and definitions we prove the main theorem:

Theorem 3.6 *Let $K \subset S^3$. At least one of the following holds:*

(1) *K is fibered.*

(2) *K is almost fibered.*

(3) *K contains a closed essential surface in its complement. Moreover this closed essential surface is in the complement of an incompressible Seifert surface for the knot.*

(4) *K has at least two nonisotopic incompressible Seifert surfaces.*

We also study the behavior of circular width under two natural operations on knot exteriors. Given two knots K_1 and K_2 in S^3, we can take their connected sum, or we can glue their exteriors together along their common boundary ensuring that preferred longitudes match.

In both cases, we find upper bounds for the circular width of the resulting manifold, which depend on the circular width of the original knot exteriors.

It is natural to ask for an example of a knot which is neither fibered nor almost fibered. The candidate we propose is the connected sum of two almost fibered knots. As we will see in Section 4 this connected sum inherits a circular structure from the knot summands. It seems to be hard to prove that this is indeed a circular thin position for the connected sum.

In Section 2 we review definitions concerning surfaces, circle-valued Morse functions and Heegaard splittings.

We study circle-valued Morse functions on knots in Section 3. We define and introduce the terminology of circular handle decomposition, circular width and almost fibered knots. We prove Theorem 3.6.

Section 4 is about the behavior of circular width under two knots operations: connected sum of knots and boundary sum of knot exteriors. Using these two operations we construct new manifolds, in one case the exterior of the connected sum of two knots and in the other case a closed manifold. In both cases there is a natural circular decomposition inherited by these manifolds, so we can prove that the circular width of the manifolds is bounded above by an n–tuple which depends on the circular width of the original knot exteriors.

Acknowledgements I want to thank my advisor Professor Abigail Thompson for her guidance, encouragement and for so many helpful conversations.

This work is part of my doctoral dissertation at University of California, Davis and was supported by a UC Mexus–CONACyT fellowship.

2 Preliminaries

2.1 Knots and surfaces

This section is devoted to definitions related to knots and Seifert surfaces, as well as to properties of Seifert surfaces under two operations on knots. The definitions and operations are mostly classical; see Rolfsen [13] for more details.

Let K be a knot in S^3. The knot complement will be denoted by $C_K = S^3 \setminus K$. An open tubular neighborhood of K in will be denoted by $N(K)$ and the exterior of the knot K by $E(K) = S^3 \setminus N(K)$.

A Seifert surface R' for a knot K is an oriented compact 2–submanifold of S^3 with no closed components such that $\partial R' = K$. The intersection of R' with $E(K)$, $R = R' \cap E(K)$, is also called a Seifert surface for K.

Since R is two sided we can specify a $+$ side and a $-$ side of R. We say that a disk D, such that $\partial D \subset R$, lies on the $+$ side (resp. in the $-$ side) of R if the collar of its boundary lies on the $+$ side (resp. in the $-$ side) of R.

Definition 2.1 We say that R is *compressible* if there is a 2–disk $D \subset E(K)$ such that $D \cap \operatorname{int}(R) = \partial D$ does not bound a disk in R. If R is not compressible, it is said to be *incompressible*. D is a compressing disk for S.

We say that R is *strongly compressible* if there are two compressing disks D_1 lying on the $+$ side of R and D_2 lying on the $-$ side of R with ∂D_1 and ∂D_2 disjoint essential closed curves in R. Otherwise we say that R is *weakly incompressible*.

Definition 2.2 The *connected sum of two knots* K_1 and K_2, denoted by $K_1 \sharp K_2$, is constructed by removing a short segment from each K_i and joining each free end of K_1 to a different end of K_2 to form a new knot. This operation is well-defined up to orientation. There is a 2–sphere Σ that intersects $K_1 \sharp K_2$ in two points. Σ is called a *separating* sphere. See Figure 1.

Given Seifert surfaces S_1 and S_2 for K_1 and K_2, respectively, one may construct a Seifert surface for the knot $K_1 \sharp K_2$ by taking a boundary connected sum of S_1 and S_2, denoted by $S_1 \sharp_\partial S_2$. Figure 2 shows the simplest case for a boundary connected sum of Seifert surfaces.

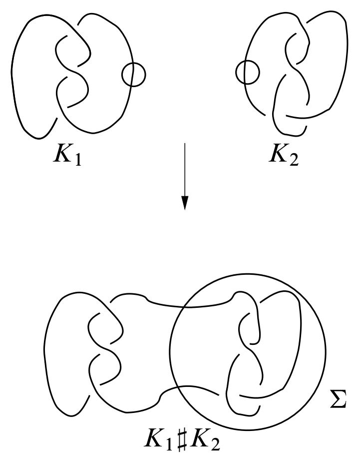

Figure 1: Connected sum of a trefoil and a figure–8 knot

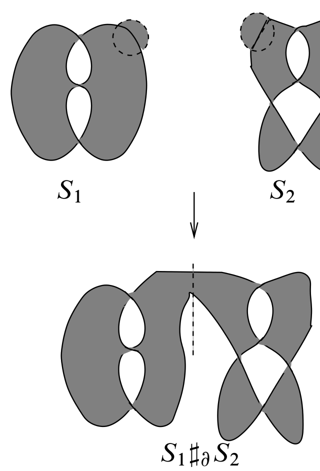

Figure 2: A boundary connected sum of two Seifert surfaces

Now let us consider another operation on the exterior $E(K_1)$ and $E(K_2)$ of two knots K_1 and K_2 in S^3. Let S_1 and S_2 be Seifert surfaces for K_1 and K_2, respectively.

Definition 2.3 Let M be the orientable closed 3–manifold obtained from $E(K_1)$ and $E(K_2)$ by identifying their boundaries via a homeomorphism h: $\partial E(K_1) \to \partial E(K_2)$ such that $h(l_1) = l_2$, where $l_i = \partial S_i$ for $i = 1, 2$. Denote this manifold by $M = E(K_1) \cup_\partial E(K_2)$ and call it the *boundary sum* of the knot exteriors $E(K_1)$ and $E(K_2)$. Under this identification S_1 and S_2 are glued together along their boundaries via h, obtaining a closed embedded surface in M. Denote this surface by $F = S_1 \cup_\partial S_2$ and call it a *boundary sum* of Seifert surfaces S_1 and S_2.

The following lemma analyzes the behavior of incompressible Seifert surfaces under the homeomorphism h used to obtain the manifold $M = E(K_1) \cup_\partial E(K_2)$.

Lemma 2.4 *If S_1 and S_2 are incompressible Seifert surfaces in $E(K_1)$ and $E(K_2)$, respectively, then $F = S_1 \cup_\partial S_2$ is incompressible in M.*

Proof Let c be the image of l_i in M and let T be the image of $\partial E(K_i)$ in M.

Suppose F is compressible. Then we can choose a nontrivial compressing disk $D \subset M$ with $\partial D \subset F$ such that D intersects T in a minimal number of arcs. $D \cap T \neq \varnothing$, otherwise $D \subset E(K_i)$, contradicting the fact that S_i is incompressible (for $i = 1, 2$).

Since F is a two sided surface in M, D lies on one side of F, say the + side.

Let $\alpha \in D \cap T$ be an outermost arc in D. Then there is an arc $\alpha' \subset D$ such that α and α' share endpoints and bound a disk in D. If α' is trivial in S_i, then it can be pushed across T reducing $|D \cap T|$. Hence α' must be essential in S_i.

Suppose α' is in S_1. We can cap off the edge α in a neighborhood of T, creating a compressing disk D' for S_1. Since S_1 is incompressible, $\partial D'$ bounds a disk in S_1. Using this disk we can push α' across T, decreasing $|D \cap T|$, which is a contradiction. Therefore F is incompressible. □

2.2 Circle-valued Morse functions for knots

We will assume basic definitions and results from real-valued Morse theory; for details, see Matsumoto [8] and Milnor [9].

The following has been adapted from Pajitnov's book *Circle-valued Morse theory* [11].

Let M be a smooth compact 3–manifold. Let f be a smooth function from M to the one dimensional sphere S^1. The Morse theory of circle-valued maps $f\colon M \to S^1$, as in the real case, relates the topology of a manifold M to the critical points of f. Morse–Novikov theory was introduced by Novikov [10] to study these functions. The motivation came from a problem in hydrodynamics.

For a point $x \in M$ choose a neighborhood V of $f(x)$ in S^1 diffeomorphic to an open interval of $\mathbb{R}$, and let $U = f^{-1}(V)$. The map $f|U$ is then identified with a smooth map from U to R. Thus all the local notions of critical points, nondegeneracy, index, etc. are defined in the same way as for the real-valued case.

Definition 2.5 A smooth map $f\colon M \to S^1$ is called a *Morse map*, if all its critical points are nondegenerate. For a Morse map $f\colon M \to S^1$ we denote by $S(f)$ the set of all critical points of f, and by $S_k(f)$ the set of all critical points of index k.

If M is compact, the set $S(f)$ is finite; in this case we denote by $m(f)$ the cardinality of $S(f)$ and by $m_k(f)$ the cardinality of $S_k(f)$.

We turn our attention to circle-valued Morse theory for the complement of a knot K in S^3.

A *circle-valued Morse function* on $C_k = S^3 \setminus K$ is a function $f\colon C_k \to S^1$ which has only nondegenerate critical points.

Let K be an oriented knot in S^3. The manifold C_K is not compact and to develop a reasonable Morse theory it is natural to impose a restriction on the behavior of the Morse map in a neighborhood of K. This restriction will allow f to have a finite set of critical points. We require the circle-valued Morse map f to behave "nicely" in a neighborhood of K.

Definition 2.6 Let K be a knot in S^3. A Morse map $f\colon C_K \to S^1$ is said to be *regular* if K has a neighborhood framed as $S^1 \times D^2$ such that $K = S^1 \times \{0\}$ and the restriction $f|_{S^1\times(D^2-\{0\})}\colon S^1 \times (D^2 - \{0\}) \to S^1$ is given by $(x, y) \to y/|y|$.

The set of critical points of a regular Morse map f is finite.

From now on we will be considering *regular circle-valued Morse functions* on knot complements. For simplicity we will just refer to them as *circle-valued Morse functions*.

Recall that a knot $K \subset S^3$ is fibered if there is fibration $\phi\colon C_K \to S^1$ "behaving nicely" in a neighborhood of K. This fibration is unique; see Burde and Zieschang [1] or Whitten [18]. So if we consider a Morse map $f\colon C_K \to S^1$ with minimum number of critical points (which is zero), then K is a fibered knot, and f is homotopic to ϕ.

If K is not fibered then any Morse map $f\colon C_K \to S^1$ will necessarily have critical points. It is natural to expect to find nice relationships between circle-valued Morse functions on knot complements and the topology of the knot complement, just as in the real-valued case. We will discuss this relationship in Section 3.

2.3 Heegaard splittings

Heegaard splittings were first introduced by Poul Heegaard in his dissertation in 1898. He proved that a closed connected orientable compact 3–manifold contains a surface which decomposes the 3–manifold into two *handlebodies*.

Definition 2.7 A *handlebody* is a connected compact orientable 3–manifold with boundary containing n pairwise disjoint, properly embedded 2–disks such that the manifold resulting from cutting along the disks is a 3–ball.

For manifolds with nonempty boundary, one needs the concept of compression body, introduced by Casson and Gordon [2]. It is a generalization of a handlebody. Definition 2.8, Definition 2.9 and Definition 2.10 are from [2].

Definition 2.8 A *compression body* W is a cobordism rel ∂ between surfaces $\partial_+ W$ and $\partial_- W$ such that $W \cong \partial_+ W \times I \cup$ 2–handles $\cup$ 3–handles and $\partial_- W$ has no 2–sphere components. If $\partial_- W \neq \varnothing$ and W is connected, then W is obtained from $\partial_- W \times I$ by attaching a number of 1–handles along disks on $\partial_- W \times \{1\}$ where $\partial_- W$ corresponds to $\partial_- W \times \{0\}$.

Denote by $h(W)$ the number of 1–handles attached to $\partial_- W \times I$.

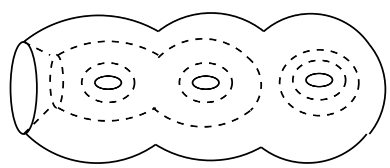

Figure 3: A compression body W with $\partial_- W$ a genus 2 surface with one boundary component and a genus 1 surface. $\partial_+ W$ is a genus 3 surface with one boundary component

Definition 2.9 A 3*–manifold triad* $(M; N, N')$ is a cobordism M rel ∂ between surfaces N and N'. Thus N and N' are disjoint surfaces in ∂M with $\partial N \cong \partial N'$ such that $\partial M = N \cup N' \cup (\partial N \times I)$.

Definition 2.10 A *Heegaard splitting* of $(M; N, N')$ is a pair of compression bodies (W, W') such that $W \cup W' = M$, $W \cap W' = \partial_+ W = \partial_+ W' (= S)$ and $\partial_- W = N$, $\partial_- W' = N'$.

S is called a Heegaard surface and $\partial S = \partial N$.

The *genus* of a Heegaard splitting is defined by the genus of the Heegaard surface.

A Heegaard splitting (W, W') is said to be *weakly reducible* if there are disks $D_1 \subset W$ and $D_2 \subset W'$ with $\partial D_i \subset S$ an essential curve for $i = 1, 2$ and such that $\partial D_1 \cap \partial D_2 = \varnothing$.

If the Heegaard splitting is not weakly reducible then it is said to be *strongly irreducible*.

The next lemma is proved in [2]; we will need it in Section 3.

Lemma 2.11 *If $\partial_- W$ or $\partial_- W'$ are compressible in (W, W') then (W, W') is weakly reducible.*

3 Thinning circle-valued Morse functions

Given a regular Morse function $f\colon C_K \to S^1$, as in the case of real-valued Morse functions, there is a correspondence between f and a handle decomposition for $E(K)$, namely

$$E(K) = \big((R \times I) \cup N_1 \cup T_1 \cup N_2 \cup T_2 \cup \cdots \cup N_k \cup T_k\big)/R \times 0 \sim R \times 1,$$

where R is a Seifert surface for K, $R \setminus K$ is a regular level surface of f, N_i is a collection of 1–handles corresponding to index 1 critical points, and T_i is a collection of 2–handles corresponding to index 2 critical points.

We will call this decomposition a *circular handle decomposition* for $E(K)$ (Figure 4).

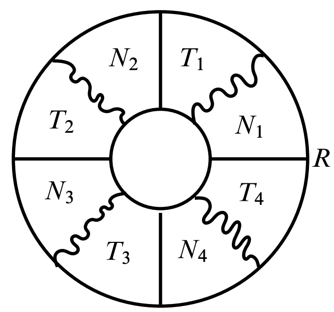

Figure 4: Circular decomposition of $E(K)$

Let us denote by S_i the surface $\mathrm{cl}(\partial((R \times I) \cup N_1 \cup T_1 \cup \cdots \cup N_i) \setminus \partial E(K) \setminus R \times 0)$ and let F_{i+1} be the surface $\mathrm{cl}(\partial((R \times I) \cup N_1 \cup T_1 \cup \cdots \cup T_i) \setminus \partial E(K) \setminus R \times 0)$, where cl means the closure. When $i = k$, $F_{k+1} = F_1 = R$. Every S_i and F_i contains a Seifert surface for K; note that F_i or S_i may be disconnected.

The surfaces S_i and F_i, for $i = 1, 2, \ldots, k$ will be called *level surfaces*.

A level surface F_i is called a *thin surface* and a level surface S_i is called a *thick surface*.

Let $W_i =$ (collar of F_i) $\cup N_i \cup T_i$. W_i is divided by a copy of S_i into two compression bodies $A_i =$ (collar of F_i) $\cup N_i$ and $B_i =$ (collar of S_i) $\cup T_i$. Thus S_i describes a Heegaard splitting of W_i into compression bodies A_i and B_i, where $\partial_- A_1 = R$, $\partial_+ A_i = \partial_+ B_i$, $\partial_- B_i = \partial_- A_{i+1}$ ($i = 1, 2, \ldots, k-1$), $\partial_- B_k = R$. Thus we can write

$$E(K) = A_1 \cup_{S_1} B_1 \cup_{F_2} A_2 \cup_{S_2} B_2 \cup_{F_3} \cdots \cup_{F_k} A_k \cup_{S_k} B_k.$$

Figure 5 shows a schematic picture of a circular handle decomposition with level surfaces and compression bodies indicated.

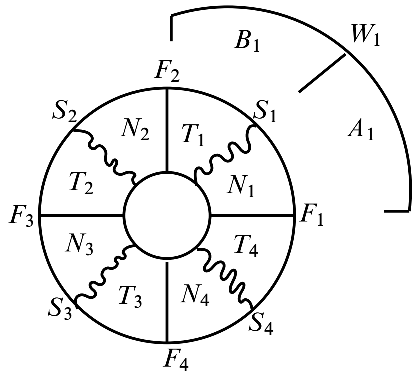

Figure 5: Splitting of $E(K)$ into compression bodies

We wish to find a decomposition in which the S_i are as simple as possible.

Definition 3.1 For a closed connected surface $\overline{S} \neq S^2$ define the complexity of $\overline{S}$, $c(\overline{S})$, to be $c(\overline{S}) = 1 - \chi(\overline{S})$. For a connected surface S with nonempty boundary we define the complexity, $c(S)$, to be $c(S) = 1 - \chi(\overline{S})$, where $\overline{S}$ denotes S with its boundary components capped off with disks. If $S = S^2$ or $S = D^2$, set $c(S) = 0$. If S is disconnected we define $c(S) = \Sigma(c(S_i))$ where S_i are the components of S.

Let K be a knot in S^3. Let D be a circular handle decomposition for $E(K)$. Define the *circular width of* $E(K)$ *with respect to the decomposition* D, $\mathrm{cw}(E(K), D)$, to be the set of integers $\{c(S_i), 1 \leq i \leq k\}$. Arrange each multiset of integers in monotonically nonincreasing order, and then compare the ordered multisets lexicographically. The *circular width of* $E(K)$, denoted $\mathrm{cw}(E(K))$, is the minimal circular width, $\mathrm{cw}(E(K), D)$ over all possible circular decompositions D for $E(K)$. $E(K)$ is in *circular thin position* if the circular width of the decomposition is the circular width of $E(K)$. If a knot K is fibered we define the circular width of K, $\mathrm{cw}(K)$, to be equal to zero.

Analogous to Scharlemann and Thompson [15], the following theorem holds:

Theorem 3.2 *If $E(K)$ is in circular thin position then:*

(1) *Each Heegaard splitting (A_i, B_i) is strongly irreducible.*
(2) *Each F_i is an incompressible surface in $E(K)$.*
(3) *Each S_i is a weakly incompressible surface in $E(K)$.*

The lemma below is needed in the proof of the theorem. A proof of the lemma can be found in [2].

Lemma 3.3 *Let* F *be a surface. If* F' *is obtained from* F *by a nontrivial compression then* $c(F') < c(F)$.

Proof of Theorem 3.2 (1) Suppose (A_i, B_i) is weakly reducible. Then there are nontrivial compressing disks $D_A \subset A_i$ and $D_B \subset B_i$ with $\partial D_A \cap \partial D_B = \varnothing$. Compress S_i towards A_i (resp. B_i) along D_A (resp. D_B) obtaining a surface S_i^A (resp. S_i^B) that divides A_i (B_i) into compression bodies H_1^A, H_2^A (resp. H_1^B, H_2^B) where $\partial_- H_1^A = F_i$, $\partial_+ H_1^A = S_i^A$, $\partial_+ H_2^A = S_i^A$ and $\partial_- H_2^A = S_i$ (resp. $\partial_- H_1^B = S_i$, $\partial_+ H_1^B = S_i^B$, $\partial_+ H_2^B = S_i^B$ and $\partial_- H_2^B = F_{i+1}$).

So we have obtained a new decomposition for $E(K)$ whose width is the original width except for the integer $c(S_i)$ that is replaced by $c(S_i^A)$ and $c(S_i^B)$. By Lemma 3.3 this new presentation has smaller circular width, which is a contradiction. Hence (A_i, B_i) is not weakly reducible.

(2) Suppose F_i is compressible. Let D be a compressing disk for a component of F_i. Let $F = \cup F_i$. By an innermost disk argument we can find a disk (which we will also call D) so that $D \cap F = \partial D \subset F_i$. D lies entirely inside either W_i or W_{i+1}, say the former. By Lemma 2.11 we have that (A_i, B_i) is weakly reducible, contradicting part (1).

(3) By part (2), F is incompressible. Hence we can assume that any compressing disk for S_i lies in W_i. Any pair of disjoint compressing disks for S_i in W_i would contradict (1). □

The converse of Theorem 3.2 is not always true. A knot exterior $E(K)$ could have a circular handle decomposition satisfying (1), (2) and (3) of Theorem 3.2, but such a decomposition need not be the thinnest.

Definition 3.4 A circular handle decomposition D for a knot exterior $E(K)$ is called a *circular locally thin* decomposition if the thin level surfaces F_i's are incompressible and the thick level surfaces S_i's are weakly incompressible.

If K has a circular thin decomposition in which all 1–handles are added before all 2–handles, we call K *almost fibered*.

Definition 3.5 K is *almost fibered* if there is a Seifert surface R so that $E(K)$ has a circular thin decomposition of the form

$$E(K) = \big((R \times I) \cup N_1 \cup T_1\big)/R \times 0 \sim R \times 1.$$

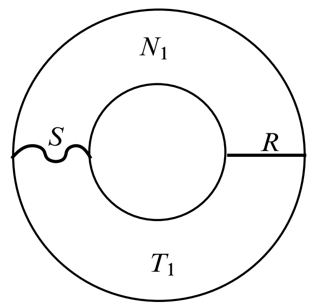

Figure 6: An almost fibered knot

Figure 6 shows a schematic picture of an almost fibered knot.

Examples of almost fibered knots are given by all nonfibered prime knots K up to 10 crossings. In [5] it is shown that these knots are handle number 1 and the decomposition arising from this handle number is realized by a minimal genus Seifert surface. Therefore all these knots are almost fibered.

Remark It is plausible to suspect that a minimal genus Seifert surface always arises as part of a thin circular handle decomposition of a knot exterior. The only evidence to support this suspicion are the cases of fibered knots and almost fibered knots up to ten crossing.

We can now state our main theorem:

Theorem 3.6 *Let $K \subset S^3$. At least one of the following holds:*

(1) *K is fibered.*

(2) *K is almost fibered.*

(3) *K contains a closed essential surface in its complement. Moreover this closed essential surface is in the complement of an incompressible Seifert surface for the knot.*

(4) *K has at least two nonisotopic incompressible Seifert surfaces.*

The proof of this theorem will follow from our definitions and a variation on a result of Waldhausen [17, Proposition 5.4], applied to the double of $E(K)$. The Waldhausen's result is the following:

Proposition 3.7 [17, Proposition 5.4] *Let M be an irreducible 3–manifold. In M let F and G be incompressible surfaces, such that $\partial F \subset \partial F \cap \partial G$, and $F \cap G$ consists of mutually disjoint simple closed curves, with transversal intersection at any curve which is not in ∂F. Suppose there is a surface H and a map $f\colon H \times I \to M$, such that $f|H \times 0$ is a covering map onto F, and*

$$f(\partial(H \times I) \setminus H \times 0) \subset G.$$

Then there is a surface $\widetilde{H}$ and an embedding $\widetilde{H} \times I \to M$, such that

$$\widetilde{H} \times 0 = \widetilde{F} \subset F, \mathrm{cl}(\partial(\widetilde{H} \times I) \setminus \widetilde{H} \times 0) = \widetilde{G} \subset G$$

(ie, a small piece of F is parallel to a small piece of G), and that moreover $\widetilde{F} \cap G = \partial \widetilde{F}$, and either $\widetilde{G} \cap F = \partial \widetilde{G}$, or $\widetilde{F}$ and $\widetilde{G}$ are disks.

We use this to prove:

Lemma 3.8 *Let M be an irreducible 3–manifold. Let F and G be isotopic, incompressible, closed, connected and disjoint surfaces in M. Then F and G are parallel, in other words they cobound a product region in M.*

Proof F and G have empty boundary and they are disjoint by hypothesis.

Since F and G are isotopic there exists $\mathcal{F}\colon M \times I \to M$ such that $\overline{f} := \mathcal{F}|M \times t$ is homeomorphism for every t, $\overline{f}_0 = \mathrm{Id}_M$ and $\overline{f}_1(F) = G$.

Define $f := \mathcal{F}|F \times I\colon F \times I \to M$. The restriction of f to $f \times 0$ is the identity on F, so is a covering map onto F. Moreover

$$f(\partial(F \times I) \setminus F \times 0) = f(F \times 1) = \overline{f}_1(F) = G.$$

So we can apply Proposition 3.7. There is a subsurface $\widetilde{F} \subset F$ which is parallel to a subsurface of $\widetilde{G} \subset G$. The intersection of $\widetilde{F}$ with G is precisely $\partial \widetilde{F}$. Since $F$and G are disjoint it follows that $\widetilde{F}$ is disjoint from G as well, therefore $\partial \widetilde{F} = \varnothing$.

Then $\widetilde{F}$ and $\widetilde{G}$ are not disks. So $\widetilde{G}$ intersects F in $\partial \widetilde{G}$, but F and G are disjoint, then it follows that $\partial \widetilde{G} = \varnothing$.

The only possible subsurfaces of F and G with empty boundary are themselves. Therefore F and G are parallel. □

We apply Lemma 3.8 to the double of a knot exterior to obtain:

Lemma 3.9 *Let K be a knot in S^3. If F and G are disjoint incompressible isotopic Seifert surfaces in $E(K)$ then F and G are parallel, that is they cobound a product region.*

Proof Let M be the double of $E(K)$, i.e, M is constructed by taking two disjoint copies of $E(K)$ and gluing them together along their boundary. M is an irreducible manifold.

Let F' (and G') be the closed surface in M obtained by gluing along the boundary two disjoint copies F_1 and F_2 of F (two disjoint copies G_1 and G_2 of G). Notice that F' and G' are disjoint, incompressible (see Lemma 2.4) and isotopic. By Lemma 3.8, F' and G' are parallel.

By construction the intersection of the images of $\partial E(K)$ in M with F' and G' cobound a product annulus A in the image of $\partial E(K)$, which is contained in the product region bounded by F' and G'. Hence we can split M along $\partial E(K)$ to recover the manifold $E(K)$. The surfaces F and G inherit the parallelism of G' and F'. Therefore the surfaces F and G cobound a product region in $E(K)$. □

We can now prove Theorem 3.6.

Proof of Theorem 3.6 Let D be a circular thin decomposition of $E(K)$.

$$D = \big((R \times I) \cup N_1 \cup T_1 \cup \cdots \cup N_j \cup T_j\big)/R \times 0 \sim R \times 1.$$

Suppose K is not fibered and not almost fibered. Then $j > 1$; so there is at least one thin level surface, F_2, different from R.

Consider $F_2 = \mathrm{cl}(\partial(R \times I \cup N_1 \cup T_1) \setminus \partial E(K) \setminus R \times 0)$. F_2 is an incompressible surface in $E(K)$, by part (2) of Theorem 3.2.

Suppose F_2 is not connected. Then F_2 contains a closed component. Since each of its components is incompressible, (3) holds.

Otherwise F_2 is an incompressible Seifert surface. If F_2 is isotopic to $F_1 = R$, by Lemma 3.9 they are parallel, so they bound a product on one side. This implies that the decomposition is not thin, since it can be replaced on one side by a product (see Figure 7). Therefore F_2 is not isotopic to R, and (4) holds. □

4 Behavior of circular width

We describe two ways to construct new manifolds from $E(K_1)$ and $E(K_2)$. We will analyze the effect of the constructions on the circular width.

We note that the width, for compact orientable 3–manifolds, is "additive" under connected sum of 3–manifolds: $w(M_1 \sharp M_2) = w(M_1) \cup w(M_2)$ (see Scharlemann and Thompson [15]). In [14], Scharlemann and Schultens analyze the behavior of generalized Heegaard splittings by cutting along a family of essential annuli.

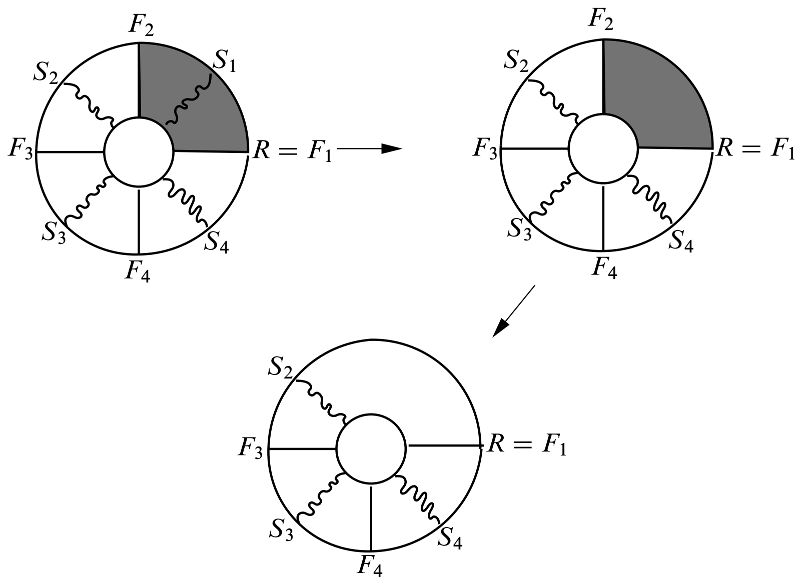

Figure 7: Circular width decreases when there are two isotopic Seifert surfaces.

4.1 Connected sum of knots

For definitions of connected sum of knots and boundary connected sum of Seifert surfaces, see Section 2.

Let us consider the knot exteriors $E(K_1)$ and $E(K_2)$. Assume they have the following circular handle decompositions starting with Seifert surfaces R_1 and R_2, respectively:

$$E(K_1) = \big((R_1 \times I) \cup N_1 \cup T_1 \cup N_2 \cup T_2 \cup \cdots \cup N_k \cup T_k\big)/R_1 \times 0 \sim R_1 \times 1$$

with level surfaces S_1, $F_2, \ldots, F_k$, S_k, F_{k+1} and

$$E(K_2) = \big((R_2 \times I) \cup O_1 \cup W_1 \cup O_2 \cup W_2 \cup \cdots \cup O_l \cup W_l\big)/R_2 \times 0 \sim R_2 \times 1$$

with level surfaces $P_1, G_2, \ldots, G_l$, P_l, G_{l+1}.

Let $K = K_1 \sharp K_2$. There is a natural way to obtain a circular handle decomposition for $E(K)$ as follows. Starting with the Seifert surface $R = R_1 \sharp_\partial R_2$ for K, we attach the sequence of handles corresponding to $E(K_1)$, ie, we attach N_i and T_i, along the R_1 summand of R. Then we attach the sequence of handles corresponding to $E(K_2)$, ie, we attach O_j and W_j, along the R_2 component of R. So we have:

$$E(K) = \big((R \times I) \cup N_1 \cup T_1 \cup N_2 \cup T_2 \cup \cdots \cup N_k \cup T_k \cup O_1 \\ \cup W_1 \cup O_2 \cup W_2 \cup \cdots \cup O_l \cup W_l\big)/R \times 0 \sim R \times 1$$

with the following level surfaces:

- Q_i a boundary connected sum $S_i \sharp_\partial R_2$ of S_i and R_2 for $i = 1, 2, \ldots, k$,
- Σ_i a boundary connected sum $F_i \sharp_\partial R_2$ of F_i and R_2 for $i = 2, 3, \ldots, k+1$,
- Γ_j a boundary connected sum $R_1 \sharp_\partial P_j$ of R_1 and P_j for $j = 1, 2, .., l$,
- Ω_j a boundary connected sum $R_1 \sharp_\partial G_j$ of R_1 and G_j for $j = 2, 3, \ldots, l+1$.

Figure 8 is a schematic picture of the induced circular handle decomposition in a complement of a connected sum of two knots.

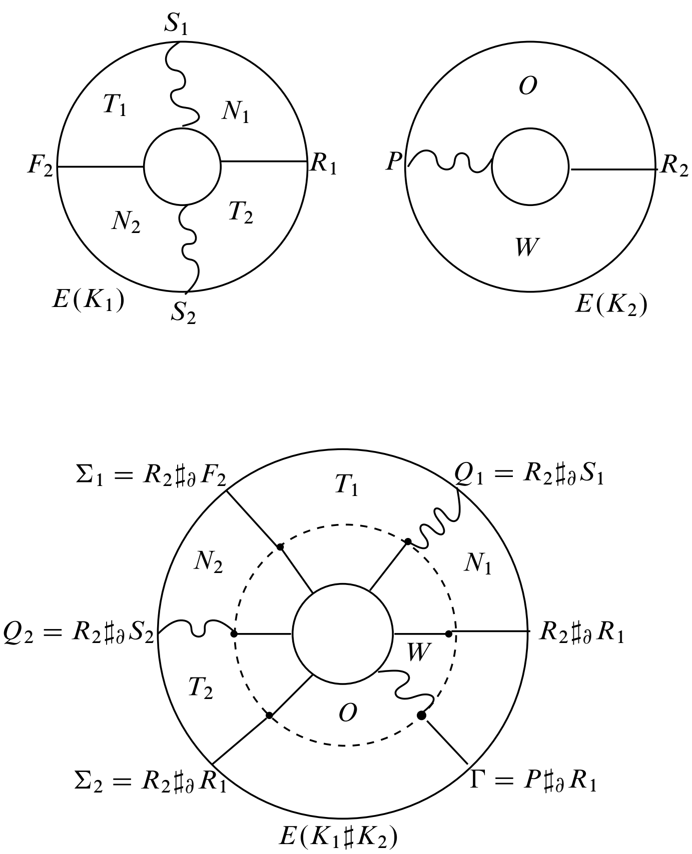

Figure 8: (a) Circular handle decomposition for $E(K_1)$ (b) Circular handle decomposition for $E(K_2)$ (c) Induced circular handle decomposition for $E(K_1 \sharp K_2)$

The complexity $c(S) = 1 - \chi(S)$ applied to a boundary connected sum $S_1 \sharp_\partial S_2$ becomes equal to

$$\begin{aligned} c(S_1 \sharp_\partial S_2) = 1 - \chi(S_1 \sharp_\partial S_2) &= 2\,\mathrm{genus}(S_1 \sharp_\partial S_2) - 1 \\ &= 2\,\mathrm{genus}(S_1) + 2\,\mathrm{genus}(S_2) - 1 \\ &= 1 - \chi(S_1) + 2\,\mathrm{genus}(S_2) = c(S_1) + 2\,\mathrm{genus}(S_2). \end{aligned}$$

Hence the complexity assigned to the thick level surfaces $S_i \sharp_\partial R_2$ and $R_1 \sharp_\partial P_j$ is given by

$$\begin{aligned} c(S_i \sharp_\partial R_2) &= c(S_i) + 2\,\mathrm{genus}(R_2), && \text{for } i = 1, 2, 3, \dots, k, \\ c(R_1 \sharp_\partial P_j) &= c(P_i) + 2\,\mathrm{genus}(R_1), && \text{for } j = 1, 2, 3, \dots, l. \end{aligned}$$

Therefore the circular width of $E(K_1 \sharp K_2)$ with respect to the circular handle decomposition D (the one induced by attaching first handles of $E(K_1)$), $\mathrm{cw}(E(K_1 \sharp K_2), D)$, is the set

$$\big\{c(S_1) + 2\,\mathrm{genus}(R_2), \dots, c(S_k) + 2\,\mathrm{genus}(R_2), \\ c(P_1) + 2\,\mathrm{genus}(R_1), \dots, c(P_l) + 2\,\mathrm{genus}(R_1)\big\}$$

arranged in monotonically nonincreasing order.

We have found an upper bound for the circular width of $E(K_1 \sharp K_2)$. The following results hold by our construction. The optimal bound occurs if R_1 and R_2 are minimal genus Seifert surfaces.

Proposition 4.1

$$\mathrm{cw}(E(K_1 \sharp K_2)) \leq \big\{c(S_1) + 2\,\mathrm{genus}(R_2), \dots, c(S_k) + 2\,\mathrm{genus}(R_2), \\ c(P_1) + 2\,\mathrm{genus}(R_1), \dots, c(P_l) + 2\,\mathrm{genus}(R_1)\big\}.$$

Special cases occur when one of the knots is fibered.

Corollary 4.2 *Suppose K_1 is fibered and K_2 is nonfibered. Then*

$$\mathrm{cw}(E(K_1 \sharp K_2)) \leq \{c(P_1) + 2\,\mathrm{genus}(R_1), \dots, c(P_l) + 2\,\mathrm{genus}(R_1)\}.$$

Corollary 4.3 *If K_1 and K_2 are both fibered then* $\mathrm{cw}(E(K_1 \sharp K_2)) = 0$.

Proof The connected sum of two fibered knots is fibered and $\mathrm{cw}(K) = 0$ when K is fibered. □

Here is one case when the equality Proposition 4.1 holds:

Corollary 4.4 *Let K_1 be a fibered knot with fiber R_1. Let K_2 be an almost fibered knot whose thin circular handle decomposition consists of one 1–handle and one 2–handle. Let R_2 and S_2 be the thin and thick level surfaces, respectively, of $E(K_2)$. Suppose that R_2 is a minimal genus Seifert surface. Then the circular width of $E(K_1 \sharp K_2)$ is given by*

$$\mathrm{cw}(E(K_1 \sharp K_2)) = \{c(S_2) + 2\,\mathrm{genus}(R_1)\}.$$

Moreover the knot $K_1 \sharp K_2$ is almost fibered.

Proof $E(K_1 \sharp K_2)$ inherits a circular handle decomposition D from $E(K_1)$ and $E(K_2)$, which consists of one 1–handle and one 2–handle. The circular handle decomposition D has the thin level surface $R_1 \sharp_\partial R_2$, which is a minimal genus Seifert surface for $K_1 \sharp K_2$; and the thick level surface $R_1 \sharp_\partial S_2$. By Corollary 4.2, $\{c(S_2) + 2\,\mathrm{genus}(R_1)\}$ is an upper bound for the circular width of $E(K_1 \sharp K_2)$. If there were a thinner circular handle decomposition for $E(K_1 \sharp K_2)$, it would have a minimal genus Seifert surface as a thin level surface. The number of 1–handles would have to be fewer than those in D, hence zero. Thus $K_1 \sharp K_2$ would be fibered. But the connected sum of two knots is fibered if and only if both knots are fibered. Thus, we have that $\mathrm{cw}(E(K_1 \sharp K_2)) = \{c(S_2) + 2\,\mathrm{genus}(R_1)\}$. □

Suppose that the given circular handle decompositions for $E(K_1)$ and $E(K_2)$ are thin. A natural question arises: Is the circular handle decomposition induced on $E(K_1 \sharp K_2)$ a thin circular decomposition? In order to address this question we need to prove:

Lemma 4.5

(1) *The boundary connected sum of two incompressible Seifert surfaces is incompressible*

(2) *The boundary connected sum of an incompressible Seifert surface and a weakly incompressible Seifert surface is a weakly incompressible Seifert surface.*

Proof Let K_i be a nontrivial knot in S^3 and F_i its Seifert surface, for $i = 1, 2$. Consider $K = K_1 \sharp K_2$ the connected sum of the knots and $F = F_1 \sharp_\partial F_2$ the boundary connected sum of the surfaces. Let Σ be the decomposition sphere. Notice that $\Sigma \cap F$ is a properly embedded arc α in $E(K)$.

(1) See Gabai [3].

(2) Suppose F_1 is incompressible and F_2 is weakly incompressible. Assume that F is not weakly incompressible. Then there exist compressing disks D_1 and D_2 lying on opposite sides of F with $\partial D_1 \cap \partial D_2 = \varnothing$. Moreover we can choose them so that $D_1 \cup D_2$ meets Σ in a minimal number of arcs. Notice that $(D_1 \cup D_2) \cap \Sigma$ is nonempty since both disks cannot be contained in F_1 or F_2 at the same time. Consider β_1 an arc in $D_1 \cap \Sigma$ which is outermost in D_1, so β_1 cuts off a disk in D_1 with boundary β_1 and $\beta_1' \subset \partial D_1$. If β_1' is trivial in F_1 or F_2 it can be pushed across Σ (taking any other arcs with it) and reducing $|(D_1 \cup D_2) \cap \Sigma|$. Thus β_1' is essential in F_1 or F_2. β_1' is not essential in F_1 since F_1 is incompressible. Hence β_1' is essential in F_2. Capping off β_1 in a neighborhood of Σ gives rise to a compressing disk D_1' for F_2.

If $D_2 \cap \Sigma = \varnothing$, then D_2 and D_1' are compressing disks for F_2 lying on opposite sites with $\partial D_2 \cap \partial D_1' = \varnothing$, which contradicts the weak incompressibility of F_2.

If $D_2 \cap \Sigma \neq \varnothing$, we can proceed as we did with D_1 to conclude that an outermost arc β_2 cuts off a disk in D_2 with boundary consisting of the arcs β_2 and $\beta_2' \in \partial D_2$, where β_2' is an essential arc in F_2. Cap off β_2 to obtain a compression disk D_2' for F_2. Figure 9 illustrates how β_1 and β_2 must lie in Σ.

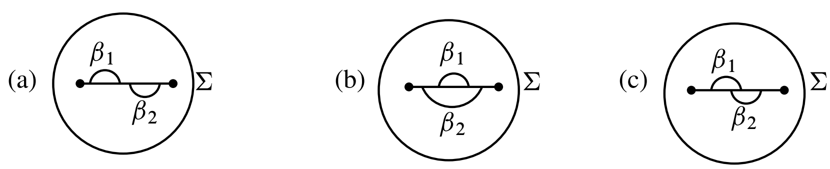

Figure 9: All possibilities for β_1 and β_2

In cases (a)–(b) we see that $\partial D_1'$ and $\partial D_2'$ can be made disjoint, contradicting weakly incompressibility of F_2.

In case (c), $\partial D_1'$ and $\partial D_2'$ intersect in one point. Let B be the union of the bicollar of D_1' and the bicollar of D_2' along the square where they intersect. Let $P = \partial B$. We can slightly move P so that F_2 cuts it into two hemispheres, each one on opposite sides, and $P \cap F_2$ is a single curve c. The curve c cuts off from F_2 a punctured torus. If c is inessential then either c bounds a disk in F_2 or c is boundary parallel. c can not bound a disk in F_2, otherwise F_2 would be a torus. c is not boundary parallel, otherwise K_2 bounds a disk implying that K_2 is the unknot. Thus, c is essential in F_2. By cutting P into the two hemispheres and pushing the two boundaries apart, we produce disjoint compressing disks on opposite sides of F_2, contradicting weakly incompressibility. □

Hence we have the following corollary:

Corollary 4.6 *If $E(K_1)$ and $E(K_2)$ are provided with a circular thin decomposition, then $E(K_1 \sharp K_2)$ inherits a circular handle decomposition in which Q_i and Q_j are incompressible and Σ_i and Σ_j are weakly incompressible. In other words the circular handle decomposition inherited by $E(K_1 \sharp K_2)$ is circular locally thin.*

4.2 Boundary sum of knot exteriors

Now let us consider another operation on the exterior $E(K_1)$ and $E(K_2)$ of two knots K_1 and K_2 in S^3. Let R_1 and R_2 be Seifert surfaces for K_1 and K_2, respectively.

Let M be the orientable closed 3–manifold obtained from $E(K_1)$ and $E(K_2)$ by identifying their boundaries via a homeomorphism $h\colon \partial E(K_1) \to \partial E(K_2)$ such that $h(l_1) = l_2$, where $l_i = \partial R_i$ for $i = 1, 2$. Under this identification R_1 and R_2 are glued together along their boundary via h obtaining a closed embedded surface in M, the *boundary sum* of $E(K_1)$ and $E(K_2)$. Let us denote this surface by $F = R_1 \cup_\partial R_2$. Recall F is called a *boundary sum* of the Seifert surfaces R_1 and R_2. (See Definition 2.3.)

Suppose $E(K_i)$ is provided with a circular handle decomposition, for $i = 1, 2$:

$$E(K_1) = \big((R_1 \times I) \cup N_1 \cup T_1 \cup N_2 \cup T_2 \cup \cdots \cup N_k \cup T_k\big)/R_1 \times 0 \sim R_1 \times 1$$

with level surfaces S_1, $F_2, \dots, F_k$, S_k, F_{k+1}, and

$$E(K_2) = \big((R_2 \times I) \cup O_1 \cup W_1 \cup O_2 \cup W_2 \cup \cdots \cup O_l \cup W_l\big)/R_2 \times 0 \sim R_2 \times 1$$

with level surfaces P_1, $G_2, \dots, G_k$, P_k, G_{l+1}.

There is a natural way to obtain a circular handle decomposition for M, starting with $(R_1 \cup_\partial R_2) \times I$. We attach the sequence of handles corresponding to $E(K_1)$, ie, we attach N_i and T_i, along the R_1 component of $R_1 \cup_\partial R_2$. Then we attach the sequence of handles of $E(K_2)$, ie, O_j and W_j, along the R_2 component of $R_1 \cup_\partial R_2$. So we have:

$$\begin{aligned} M = ((R_1 \cup_\partial R_2) \times I) \cup N_1 \cup T_1 \cup N_2 \cup T_2 \cup \cdots \cup N_k \cup T_k \cup O_1 \\ \cup W_1 \cup O_2 \cup W_2 \cup \cdots \cup O_l \cup W_l \cup B_3 \end{aligned}$$

with level surfaces:

- Q_i which is a boundary sum $R_2 \cup_\partial S_i$ for $i = 1, 2, \dots, k$,
- Σ_i which is a boundary sum $R_2 \cup_\partial F_i$ for $i = 2, 3, \dots, k+1$,

- Γ_j which is a boundary sum $R_1 \cup_\partial P_j$ for $j = 1, 2, \dots, l$,
- Ω_j which is a boundary sum $R_1 \cup_\partial G_j$ for $j = 2, 3, \dots, l+1$,
- B_3 a collection of 3–handles.

The complexity $c(S) = 1 - \chi(S)$ applied to a sum $S_1 \cup_\partial S_2$ is equal to

$$\begin{aligned} c(S_1 \cup_\partial S_2) = 1 - \chi(S_1 \cup_\partial S_2) &= 2\,\mathrm{genus}(S_1 \cup_\partial S_2) - 1 \\ &= 2\,\mathrm{genus}(S_1) + 2\,\mathrm{genus}(S_2) - 1 \\ &= 1 - \chi(S_1) + 2\,\mathrm{genus}(S_2) = c(S_1) + 2\,\mathrm{genus}(S_2). \end{aligned}$$

Hence the complexity assigned to the thick level surfaces $S_i \cup_\partial R_2$ and $R_1 \cup_\partial P_j$ for is given by

$$\begin{aligned} c(S_i \cup_\partial R_2) &= c(S_i) + 2\,\mathrm{genus}(R_2), &&\text{for } i = 1, 2, 3, \dots, k, \\ c(R_1 \cup_\partial P_j) &= c(P_i) + 2\,\mathrm{genus}(R_1), &&\text{for } j = 1, 2, 3, ..l. \end{aligned}$$

Defining circular width, as well as circular width with respect to a circular handle decomposition, in the obvious way for the closed manifold M we have that the circular width of M with respect to the circular handle decomposition D (the one induced by attaching first handles of $E(K_1)$), $\mathrm{cw}(E(K_1) \cup_\partial E(K_2), D)$, is the set

$$\big\{c(S_1) + 2\,\mathrm{genus}(R_2), \dots, c(S_k) + 2\,\mathrm{genus}(R_2), \\ c(P_1) + 2\,\mathrm{genus}(R_1), \dots, c(P_l) + 2\,\mathrm{genus}(R_1)\big\}$$

arranged in monotonically nonincreasing order.

Therefore we have an upper bound for the circular width of the manifold $M = E(K_1) \cup_\partial E(K_2)$.

Proposition 4.7

$$\mathrm{cw}(M) \leq \big\{c(S_1) + 2\,\mathrm{genus}(R_2), \dots, c(S_k) + 2\,\mathrm{genus}(R_2), \\ c(P_1) + 2\,\mathrm{genus}(R_1), \dots, c(P_l) + 2\,\mathrm{genus}(R_1)\big\}.$$

In the case when both knots K_1 and K_2 are fibered then the manifold M is fibered as well; indeed:

Remark If K_1 and K_2 are fibered knots in S^3, then the manifold $M = E(K_1) \cup_h E(K_2)$ is fibered and $\mathrm{cw}(M) = 0$.

If either K_1 or K_2 is fibered, say K_1, the circular width of M is bounded in the obvious way:

Corollary 4.8 $\mathrm{cw}(M) \leq \{c(S_1) + 2\,\mathrm{genus}(R_2), \dots, c(S_k) + 2\,\mathrm{genus}(R_2)\}$.

Here is one case when the equality in Proposition 4.7 holds:

Corollary 4.9 *Let K_1 be a fibered knot with fiber R_1. Let K_2 be an almost fibered knot whose thin circular handle decomposition consists of one 1–handle and one 2–handle. Let R_2 and S_2 be the thin and thick level surfaces, respectively of $E(K_2)$. Suppose that R_2 is a minimal genus Seifert surface. Then the circular width of $M = E(K_1) \cup_\partial E(K_2)$ is given by*

$$\mathrm{cw}(M) = \{c(S_2) + 2\,\mathrm{genus}(R_1)\}.$$

To prove this corollary we need to guarantee that the boundary sum of two minimal genus Seifert surfaces is minimal genus in its homology class in the manifold M. To accomplish this we invoke the following result proved by Gabai [4].

Proposition 4.10 [4, Corollary 8.3] *If M is obtained by performing zero frame surgery on a knot K in S^3, then M is prime and the genus of K is the minimum genus of any embedded, oriented nonseparating surface.*

Recall that the *genus* of a knot K in S^3 is the minimal genus over all Seifert surfaces for K.

If F is a surface properly embedded in $E(K)$, the genus of F is the genus of the closed surface obtained by capping off the boundary components of F.

Proposition 4.10 implies that a genus g knot K cannot have an orientable surface in the homology class of the Seifert surface with genus less than g, even if the surface is allowed to have more than one boundary component.

Here is the result we need:

Lemma 4.11 *Let K_1 and K_2 be two knots in S^3 and let R_1 and R_2 be minimal genus Seifert surfaces for K_1 and K_2, respectively. Then $R_1 \cup_\partial R_2$ is minimal genus in its homology class $[G]$ in $M = E(K_1) \cup_\partial E(K_2)$.*

Proof If $R_1 \cup_\partial R_2$ is not minimal genus in its homology class $[G]$, let R be a representative in $[G]$ with smaller genus. Let T be the separating incompressible torus given by the image of $\partial E(K_i)$ in M. R intersects T in a finite collection of essential closed curves. These curves are parallel to the preferred longitude which is $(R_1 \cup_\partial R_2) \cap T$. Since T separates M, after splitting M along T the surface R is split into the pieces $R \cap E(K_1)$ and $R \cap E(K_2)$. Since R is of smaller genus than $R_1 \cup_\partial R_2$, then either the genus of $R \cap E(K_1)$ or $R \cap E(K_2)$ is smaller than R_1 or R_2. Contradicting Proposition 4.10. Therefore $R_1 \cup_\partial R_2$ is minimal genus in its homology class $[G]$ in M. □

Now we can provide a proof for Corollary 4.9.

Proof of Corollary 4.9 M inherits a circular handle decomposition D from $E(K_1)$ and $E(K_2)$, which consists of one 1–handle and one 2–handle. The circular handle decomposition D has the thin level surface $R_1 \cup_\partial R_2$, which is minimal genus in its homology class $[G]$ in M (by Lemma 4.11), and the thick level surface $R_1 \cup_\partial S_2$. By Corollary 4.8, $\{c(S_2) + 2\,\text{genus}(R_1)\}$ is an upper bound for the circular width of M. If there were a thinner circular handle decomposition for M, it would have a minimal genus surface as a thin level surface. The number of 1–handles would have to be fewer than those in D, hence zero. Thus M would be fibered. But this is not possible unless both knots K_1 and K_2 are fibered. Thus we have that $\text{cw}(M) = \{c(S_2) + 2\,\text{genus}(R_1)\}$. □

If we assume that $E(K_1)$ and $E(K_2)$ are in circular thin position, then we can ask if the inherited presentation for M is circular thin as well or even locally circular thin. Under some conditions we are able to prove local circular thinness. We need the analog to Lemma 4.5. First we introduce some definitions which are similar to those in Johnson and Thompson [7].

Definition 4.12 Let S be a Seifert surface for a knot K in S^3. S is *boundary compressible* if there is a disk $D \subset E(K)$ such that ∂D consists of an essential α arc in S and an arc β in $\partial E(K)$. If S is not boundary compressible, it is said to be *boundary incompressible*. D is a *boundary compressing disk*.

S is *strongly boundary compressible* if there are boundary compressing disks D_1 and D_2 on opposite sides of S with disjoint boundaries, or a boundary compressing disk and a compressing disk on opposite sides of S with disjoint boundaries.

S is *weakly boundary incompressible* if S is not strongly boundary compressible and S is not strongly compressible.

Remark An incompressible Seifert surface is boundary incompressible. If not, using the boundary compressing disk and the irreducibility of $E(K)$, we can find a compressing disk of the Seifert surface since $\partial E(K)$ is a torus.

The following Lemma is a variation of Lemma 3 in [7]. The main change is made on the separability of S. It is replaced by the hypothesis of being 2–sided. The proof is similar, but for the sake of completeness we include it here.

Lemma 4.13

(1) *If $F_1 \subset E(K_1)$ is an incompressible Seifert surface and $F_2 \subset E(K_2)$ is a weakly boundary incompressible Seifert surface, then $F = F_1 \cup_\partial F_2$ is a weakly incompressible surface in $M = E(K_1) \cup_\partial E(K_2)$.*

(2) *If F_i is an incompressible Seifert surface in $E(K_i)$ for $i = 1, 2$ then $F = F_1 \cup_\partial F_2$ is incompressible in $M = E(K_1) \cup_\partial E(K_2)$.*

Proof Let T be the image of $\partial E(K_i)$ in M. T is a separating incompressible torus embedded in M.

(1) Suppose F is strongly compressible. Then there exist nontrivial compressing disks D_1 and D_2 lying on opposite sides of F with $\partial D_1 \cap \partial D_2 = \varnothing$ and $(D_1 \cup D_2) \cap T$ a minimal collection of arcs. Notice that $(D_1 \cup D_2) \cap T \neq \varnothing$. Otherwise D_1 and D_2 are both contained in $E(K_2)$ because F_1 is incompressible. This contradicts the assumption that F_2 is weakly incompressible.

Then $D_1 \cap T = \varnothing$ and $D_2 \cap T = \varnothing$ cannot happen at the same time, hence

- $D_2 \subset E(K_2)$ and $D_1 \cap T \neq \varnothing$ or
- $D_1 \cap T \neq \varnothing$ and $D_2 \cap T \neq \varnothing$.

Let $\alpha_1 \in D_1 \cap T$ be an outermost arc in D_1, and let α_1' be the arc in ∂D_1 so that $\alpha_1 \cup \alpha_1'$ cuts off a disk. If α_1' is trivial in F_1 or F_2, we can push it across T (taking any other arc with it), reducing $(D_1 \cup D_2) \cap T$. So α_1' is essential in F_1 or F_2.

If α_1' is in F_1 then the disk bounded by α_1 and α_1' is a boundary compressing disk for F_1. Because F_1 is boundary incompressible this is not possible, so α_1' must be in F_2. Hence D_1 contains a boundary compressing disk D_1' for F_2. D_1' lies on the same side as D_1.

If $D_2 \subset E(K_2)$ then it is a compressing disk for F_2. D_2 is on the opposite side of D_1' and $\partial D_1'$ is disjoint from ∂D_2. This contradicts the weak incompressibility of F_2.

If $D_2 \cap T$ is nonempty then, as with D_1, an outermost arc argument implies that there is a boundary compressing disk D_2' for F_2. D_2' lies on the same side as D_2. The disks D_1' and D_2' are disjoint and on opposite sides of F_2. This contradicts that F_2 is weakly boundary incompressible.

Therefore $R = R_1 \cup_\partial R_2$ is weakly incompressible.

(2) The proof of this case is similar but easier. □

Hence, we have the following:

Corollary 4.14 *If $E(K_1)$ and $E(K_2)$ are provided with circular thin decompositions respectively, and if we further assume that the thin level surfaces for both decompositions are boundary incompressible and the thick levels for both decompositions are weakly boundary incompressible, then $M = E(K_1) \cup_h E(K_2)$ inherits a circular locally thin decomposition.*

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Department of Mathematics, University of California
Davis, CA, 95616, USA

fabiola@math.ucdavis.edu

http://www.math.ucdavis.edu/~fabiola

Received: 20 October 2008 Revised: 1 December 2008