

Small curvature laminations in hyperbolic 3–manifolds

WILLIAM BRESLIN

We show that if \mathcal{L} is a codimension-one lamination in a finite volume hyperbolic 3–manifold such that the principal curvatures of each leaf of \mathcal{L} are all in the interval $(-\delta, \delta)$ for a fixed $\delta \in [0, 1)$ and no complementary region of \mathcal{L} is an interval bundle over a surface, then each boundary leaf of \mathcal{L} has a nontrivial fundamental group. We also prove existence of a fixed constant $\delta_0 > 0$ such that if \mathcal{L} is a codimension-one lamination in a finite volume hyperbolic 3–manifold such that the principal curvatures of each leaf of \mathcal{L} are all in the interval $(-\delta_0, \delta_0)$ and no complementary region of \mathcal{L} is an interval bundle over a surface, then each boundary leaf of \mathcal{L} has a noncyclic fundamental group.

57M50

1 Introduction

In [9], Zeghib proved that any totally geodesic codimension-one lamination in a closed hyperbolic 3–manifold is a finite union of disjoint closed surfaces. In this paper we investigate whether a similar result holds for codimension-one laminations with small principal curvatures. We will prove the following theorems:

Theorem 1 *Let $\delta \in [0, 1)$. If \mathcal{L} is a codimension-one lamination in a finite volume hyperbolic 3–manifold such that the principal curvatures of each leaf of \mathcal{L} are everywhere in $(-\delta, \delta)$ for a fixed constant $\delta \in [0, 1)$ and no complementary region of \mathcal{L} is an interval bundle over a surface, then each boundary leaf of \mathcal{L} has a nontrivial fundamental group.*

Theorem 2 *There exists a fixed constant $\delta_0 > 0$ such that if \mathcal{L} is a codimension-one lamination in a finite volume hyperbolic 3–manifold such that the principal curvatures of each leaf of \mathcal{L} are everywhere in $(-\delta_0, \delta_0)$ and no complementary region is an interval bundle over a surface, then each boundary leaf of \mathcal{L} has a noncyclic fundamental group.*

2 Examples

Let \mathcal{L} be a codimension-one lamination in a complete hyperbolic 3-manifold M . Let L be a leaf of \mathcal{L} and endow it with the path metric induced from M . Let \tilde{L} be the universal cover of L and lift the inclusion $i_L: L \rightarrow M$ to a map $\tilde{i}_L: \tilde{L} \rightarrow \mathbb{H}^3$. A map $f: X \rightarrow Y$ from a metric space X to a metric space Y is a (k, c) -quasi-isometry if $\frac{1}{k}d_X(a, b) - c \leq d_Y(f(a), f(b)) \leq kd_X(a, b) + c$. The leaf L is *quasi-isometric* if the map \tilde{i}_L is a (k, c) -quasi-isometry for some k, c . The lamination \mathcal{L} is *quasi-isometric* if each leaf of \mathcal{L} is quasi-isometric for the same fixed constants k, c .

Let $\delta \in (0, 1)$. If the principal curvatures of $\tilde{i}_L(\tilde{L})$ are everywhere in $(-\delta, \delta)$, then the map \tilde{i}_L is a (k, c) -quasi-isometry for constants k, c depending only on δ (see Thurston [8]). Also see Leininger [6] for an elementary proof).

The constant δ_0 in Theorem 2 is less than 1, so a lamination satisfying the hypotheses of Theorem 1 or Theorem 2 is necessarily quasi-isometric. Thus it makes sense to ask whether these results hold for general quasi-isometric laminations.

Quasi-isometric laminations with no compact leaves Cannon and Thurston [3] proved that the stable and unstable laminations of the suspension of a pseudo-Anosov homeomorphism of a closed surface are quasi-isometric, and each leaf is a plane or annulus in this case. In addition to these examples, Fenley [5] produced infinitely many examples of closed hyperbolic 3-manifolds with quasi-isometric laminations in which each leaf is an annulus, a mobius band, or a plane. Note that Theorem 2 implies that the examples of Cannon–Thurston and Fenley cannot have principal curvatures everywhere in the interval $(-\delta_0, \delta_0)$.

One can also ask if we need to require that no complementary region is an interval bundle over a surface.

Small curvature laminations with simply connected boundary leaves Let S be a closed totally geodesic embedded surface in a closed hyperbolic 3-manifold M . Let $N(S) = S \times [0, 1]$ be a closed embedded neighborhood of S in M . If the neighborhood $N(S)$ is small then the surfaces $S \times t$ will have small principal curvatures. Since $\pi_1(S)$ is left-orderable, there exist faithful representations $\rho: \pi_1(S) \rightarrow \text{Homeo}([0, 1])$ such that some points have trivial stabilizers (see Calegari [2]). The foliated bundle whose holonomy is ρ has a leaf which is simply connected. Replace $N(S)$ with this foliated bundle. We can blow up the simply connected leaf and remove the interior to get a lamination which is C^∞ close to the original (so that the leaves have small principal curvatures) and such that some boundary leaf is simply connected. See Calegari [1] to see why the foliated bundle can be embedded in M so that the leaves are smooth. Note that this lamination has a complementary region which is an interval bundle over a surface.

Small curvature laminations with no compact leaves One may also construct small curvature laminations in closed hyperbolic 3-manifolds with no compact leaves. The author would like to thank Chris Leininger for describing the following construction. The idea is to construct a small curvature branched surface in a closed hyperbolic 3-manifold which has an irrational point in the space of projective classes of measured laminations carried by the branched surface. A lamination corresponding to this irrational point will contain no compact leaves. There are totally geodesic immersed closed surfaces in the figure-eight knot complement M_8 arbitrarily close to any plane in the tangent bundle (see Reid [7]). Using this and the fact that $\pi_1(M_8)$ is LERF, one can find two such surfaces which lift to embedded surfaces S_1 and S_2 in a finite cover M of M_8 which intersect in a nonseparating (in both surfaces) simple closed geodesic l at an arbitrarily small angle. Flatten out the intersection to get a branched surface with small principal curvatures in which S_1 connects one side of S_2 to the other side. The branched surface has three branch sectors (an annulus, $S_1 \setminus l$, and $S_2 \setminus l$) and one branch equation ($x_1 = x_2 + x_3$). A solution to the branch equation in which two coordinates are not rationally related (eg, $x_1 = 1/2$, $x_2 = 1/\pi$, $x_3 = 1/2 - 1/\pi$) will correspond to a lamination with no compact leaves which can be isotoped to have small principal curvatures. Since the leaves do not have any cusps, we can fill the cusps of M to get a small curvature lamination in a closed hyperbolic 3-manifold with no compact leaves.

3 Proof of Theorem 1

Let $\epsilon > 0$ be so small that if P_1, P_2, P_3 are three disjoint smoothly embedded planes in hyperbolic 3-space with principal curvatures in $(-1, 1)$ which intersect the same ϵ -ball, then one of the P_i separates the other two.

Let \mathcal{L} be a codimension-one lamination in a finite volume hyperbolic 3-manifold M such that the principal curvatures of each leaf are everywhere in the interval $(-\delta, \delta)$ for some $\delta \in (0, 1)$. Assume that no complementary region of \mathcal{L} is an interval bundle over a surface. Let $\tilde{\mathcal{L}}$ be the lift of \mathcal{L} to \mathbb{H}^3 . Since every leaf of \mathcal{L} has principal curvatures everywhere in $(-\delta, \delta)$, the lamination \mathcal{L} is a quasi-isometric lamination, and cannot be a foliation of M by Fenley [4].

Let L_0 be a boundary leaf of \mathcal{L} . Suppose, for contradiction, that $\pi_1(L_0)$ is trivial, which implies that L_0 has infinite area. Since M is closed, L_0 must intersect some fixed compact ball in M infinitely many times. Thus given any integer k , we can find a point y_k in L_0 such that the next leaf over on the boundary side of L_0 is within $1/k$ of y_k .

Let \tilde{L}_0 be a lift of L_0 to \mathbb{H}^3 . Lift the points y_k to a fixed fundamental domain of \tilde{L}_0 and call them y_k . Let \tilde{L}_k be the next leaf over from \tilde{L}_0 which is within $1/k$ of y_k . We now have a sequence of leaves \tilde{L}_k in $\tilde{\mathcal{L}}$ on the boundary side of \tilde{L}_0 such that for each k the distance from \tilde{L}_k to y_k is less than $1/k$, and there is no leaf of \mathcal{L} between \tilde{L}_0 and \tilde{L}_k . We also have that $\partial\tilde{L}_0 \neq \partial\tilde{L}_k$ for all k , because otherwise the region between L_0 and L_k would be an interval bundle in the complement of \mathcal{L} .

Let k be so large that $1/k < \epsilon/8$. Since \tilde{L}_k eventually diverges from \tilde{L}_0 we can find a point $x_k \in \tilde{L}_0$ such that the distance from x_k to \tilde{L}_k is exactly $\epsilon/8$. Let b_k be the $(\epsilon/32)$ -ball tangent to \tilde{L}_0 at x_k on the boundary side of \tilde{L}_0 .

We will show that infinitely many of the balls b_k are disjointly embedded in M , contradicting the fact that M has finite volume. Suppose that $\gamma(b_l) \cap b_k \neq \emptyset$ for some integers l, k and some γ in $\pi_1(M)$. Note that $\gamma(\tilde{L}_0) \neq \tilde{L}_0$, since L_0 has trivial fundamental group. Now \tilde{L}_0, \tilde{L}_k , and $\gamma(\tilde{L}_0)$ all intersect some ϵ -ball, so we must have that one of them separates the other two. Since there are no leaves of $\tilde{\mathcal{L}}$ between \tilde{L}_0 and \tilde{L}_k , and $\gamma(\tilde{L}_0)$ is closer to x_k than \tilde{L}_k , we must have that \tilde{L}_0 separates \tilde{L}_k and $\gamma(\tilde{L}_0)$ (see Figure 1(a)). Also note that \tilde{L}_0, \tilde{L}_k , and $\gamma(\tilde{L}_l)$ are all on the boundary side of $\gamma(\tilde{L}_0)$ (ie, the side which contains the ball $\gamma(b_l)$).

Now we will show no matter where γ sends \tilde{L}_l , we get a contradiction. We cannot have $\gamma(\tilde{L}_l) = \tilde{L}_k$, because this would imply that $\gamma^{-1}(\tilde{L}_0)$ separates \tilde{L}_l and \tilde{L}_0 . Thus we have $\gamma(\tilde{L}_l) \neq \tilde{L}_k$.

Since \tilde{L}_0, \tilde{L}_k , and $\gamma(\tilde{L}_l)$ all intersect some fixed ϵ -ball, we must have that one of them separates the other two. We cannot have that $\gamma(\tilde{L}_l)$ separates \tilde{L}_0 and \tilde{L}_k , because there are no leaves of $\tilde{\mathcal{L}}$ between \tilde{L}_0 and \tilde{L}_k (See Figure 1(b)). If \tilde{L}_0 separates \tilde{L}_k and $\gamma(\tilde{L}_l)$, then $\gamma(\tilde{L}_l)$ is between \tilde{L}_0 and $\gamma(\tilde{L}_0)$, so that $d(x_l, \tilde{L}_l) = d(\gamma(x_l), \gamma(\tilde{L}_l)) \leq \epsilon/16$ which is a contradiction (see Figure 1(c)). Thus \tilde{L}_0 cannot separate \tilde{L}_k and $\gamma(\tilde{L}_l)$. If \tilde{L}_k separates \tilde{L}_0 and $\gamma(\tilde{L}_l)$, then $\gamma^{-1}(\tilde{L}_k)$ separates \tilde{L}_0 and \tilde{L}_l which is a contradiction (see Figure 1(d)). Thus \tilde{L}_k cannot separate \tilde{L}_0 and $\gamma(\tilde{L}_l)$. We have shown that \tilde{L}_l has nowhere to go under the map γ , so that $\gamma(b_l) \cap \gamma(b_k) = \emptyset$ for any integers l, k and any $\gamma \in \pi_1(M)$. This implies that M contains infinitely many disjoint $(\epsilon/32)$ -balls, contradicting the fact that M has finite volume. \square

4 Proof of Theorem 2

Let $\epsilon > 0$ be so small that if P_1, P_2, P_3 are three disjoint smoothly embedded planes in hyperbolic 3-space with principal curvatures in $(-1, 1)$ which intersect the same

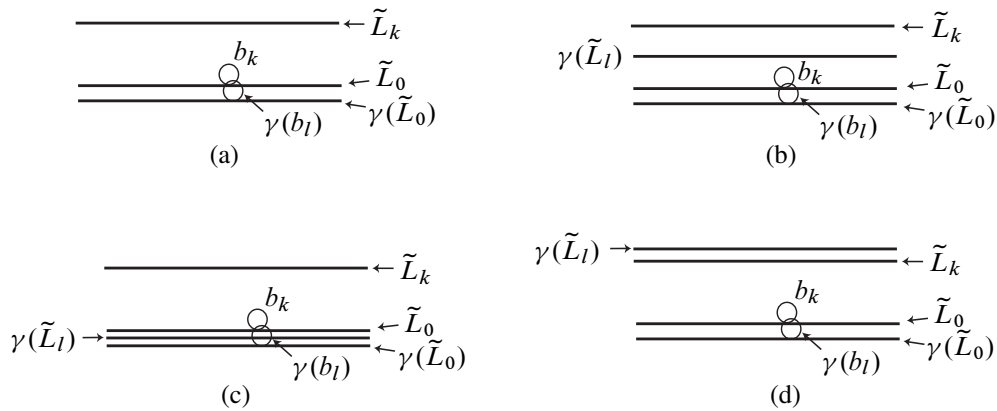


Figure 1: (a) \tilde{L}_0 separates \tilde{L}_k and $\gamma(\tilde{L}_0)$. (b) $\gamma(\tilde{L}_l)$ cannot separate \tilde{L}_0 and \tilde{L}_k . (c) \tilde{L}_0 cannot separate \tilde{L}_k and $\gamma(\tilde{L}_l)$. (d) \tilde{L}_k cannot separate \tilde{L}_0 and $\gamma(\tilde{L}_l)$.

ϵ -ball, then one of the P_i separates the other two. Let $\delta_0 > 0$ be so small that if a smooth curve $\gamma: (-\infty, \infty) \rightarrow \mathbb{H}^3$ in \mathbb{H}^3 with endpoints in $\partial\mathbb{H}^3$ has curvature at most δ_0 at each point, then $\gamma(t)$ is in the $(\epsilon/2)$ -neighborhood of the geodesic of \mathbb{H}^3 with the same endpoints.

Let \mathcal{L} be a codimension-one lamination in a finite volume hyperbolic 3-manifold M such that the principal curvatures of each leaf are everywhere in the interval $(-\delta_0, \delta_0)$. Assume that no complementary region of \mathcal{L} is an interval bundle over a surface. Let $\tilde{\mathcal{L}}$ be the lift of \mathcal{L} to \mathbb{H}^3 . As in the proof of Theorem 1, \mathcal{L} cannot be a foliation. Let L_0 be a boundary leaf of \mathcal{L} . Suppose, for contradiction, that $\pi_1(L_0)$ is cyclic, which implies that L_0 has infinite area. Since M is closed, L_0 must intersect some fixed compact ball in M infinitely many times. Also, by Theorem 1, we know that $\pi_1(L_0)$ is nontrivial, so that $\pi_1(L_0) \approx \mathbb{Z}$.

Let \tilde{L}_0 be a lift of L_0 to \mathbb{H}^3 . Since L_0 intersects a fixed compact ball in M infinitely many times, we can find a sequence of points y_k in \tilde{L}_0 such that the closest leaf of $\tilde{\mathcal{L}}$ to y_k on the boundary side of \tilde{L}_0 is within $1/k$ of y_k . Let \tilde{L}_k be the leaf which is closest to y_k on the boundary side of \tilde{L}_0 . Note that there is no leaf of $\tilde{\mathcal{L}}$ between \tilde{L}_0 and \tilde{L}_k . We have $\partial\tilde{L}_0 \neq \partial\tilde{L}_k$ for all k , because the complement of \mathcal{L} contains no interval bundle components. We may assume that all y_k are contained in a fixed fundamental domain \mathcal{D} of \tilde{L}_0 , and that y_k converge to a point $y_\infty \in \partial\tilde{L}_0$.

For k large enough we have $\partial\tilde{L}_0 \neq \partial\tilde{L}_k$ and $d(y_k, \tilde{L}_k) \leq \epsilon/8$, so that we can find a point x_k such that $d(x_k, \tilde{L}_k) = \epsilon/8$.

Case 1 We can choose the sequence of points $x_k \in \tilde{L}_0$ to be contained in a fixed fundamental domain D of \tilde{L}_0 such that x_k exit an end of D whose projection to M has infinite area.

Let b_k be the $(\epsilon/32)$ -ball tangent to \tilde{L}_0 at x_k on the boundary side of \tilde{L}_0 . For k large enough, say all k , the generator of $\text{stab}_{\pi_1(M)}(\tilde{L}_0)$ moves the center of b_k a distance of at least ϵ . Thus we can assume that $\gamma(b_l) \cap b_k = \emptyset$ for any integers l, k and any $\gamma \in \text{stab}_{\pi_1(M)}(\tilde{L}_0)$.

We may now proceed as in the proof of Theorem 1 to show that $\gamma(b_l) \cap b_k = \emptyset$ for any integers l, k and any $\gamma \in \pi_1(M)$. This again contradicts the fact that M has finite volume.

Case 2 We cannot choose the sequence of points x_k as in Case 1.

If infinitely many of the leaves \tilde{L}_k were distinct, then we would be able to find a sequence of points as described in Case 1. Thus $\tilde{L}_k = \tilde{L}_+$ for some fixed leaf $\tilde{L}_+ \in \tilde{\mathcal{L}}$.

Let U be the component of the complement in $\partial\tilde{L}_0$ of the fixed point(s) of the generator of $\text{stab}_{\pi_1(M)}(\tilde{L}_0)$ which contains the point y_∞ . We will now show that $\partial\tilde{L}_+ \cap \partial\tilde{L}_0$ must contain U .

Suppose that $\partial\tilde{L}_+ \cap \partial\tilde{L}_0$ does not contain U . Since $d(y_k, \tilde{L}_+) < 1/k$ and $\partial\tilde{L}_+ \cap \partial\tilde{L}_0$ does not contain U , we can find a sequence of points x_k in \tilde{L}_0 which converge to a point $x_\infty \in U$ with $d(x_k, \tilde{L}_+) = \epsilon/8$. Since the point x_∞ cannot be a fixed point of the generator of $\text{stab}_{\pi_1(M)}(\tilde{L}_0)$, a tail of the sequence x_k must be contained in a fixed fundamental domain of \tilde{L}_0 . This contradicts the fact that we are in Case 2. Thus $\partial\tilde{L}_+ \cap \partial\tilde{L}_0$ must contain U , hence must contain the fixed point(s) of the generator of $\text{stab}_{\pi_1(M)}(\tilde{L}_0)$.

If the generator of $\text{stab}_{\pi_1(M)}(\tilde{L}_0)$ is parabolic, then it has only one fixed point. This implies that $\partial\tilde{L}_+ = \partial\tilde{L}_0$, giving us a contradiction.

If the generator of $\text{stab}_{\pi_1(M)}(\tilde{L}_0)$ is loxodromic, then we can argue as above to find a leaf $\partial\tilde{L}_-$ of $\tilde{\mathcal{L}}$ which contains the other component of complement in $\partial\tilde{L}_0$ of the fixed points of the generator of $\text{stab}_{\pi_1(M)}(\tilde{L}_0)$. So $\partial\tilde{L}_+$ and $\partial\tilde{L}_-$ both contain the endpoints of the axis of the generator of $\text{stab}_{\pi_1(M)}(\tilde{L}_0)$. Since the principal curvatures of \tilde{L}_0 , \tilde{L}_+ , and \tilde{L}_- are all in the interval $(-\delta_0, \delta_0)$, and $\partial\tilde{L}_0$, $\partial\tilde{L}_+$, $\partial\tilde{L}_-$ all contain the endpoints of the axis of the generator of $\text{stab}_{\pi_1(M)}$, we must have that \tilde{L}_0 , \tilde{L}_+ , and \tilde{L}_- all intersect some fixed ϵ -ball. Thus one of the three separates the other two. This gives us a contradiction since \tilde{L}_+ and \tilde{L}_- are on the same side of \tilde{L}_0 (ie, the boundary side) and there are no leaves of \mathcal{L} between \tilde{L}_0 and \tilde{L}_+ or between \tilde{L}_0 and \tilde{L}_- . \square

Acknowledgements This work was partially supported by the NSF grants DMS-0135345 and DMS-0602191.

References

- [1] **D Calegari**, *Leafwise smoothing laminations*, *Algebr. Geom. Topol.* 1 (2001) 579–585 MR1875608
- [2] **D Calegari**, *Circular groups, planar groups, and the Euler class*, from: “Proceedings of the Casson Fest”, (C Gordon, Y Rieck, editors), *Geom. Topol. Monogr.* 7 (2004) 431–491 MR2172491
- [3] **J W Cannon**, **W P Thurston**, *Group invariant Peano curves*, *Geom. Topol.* 11 (2007) 1315–1355 MR2326947
- [4] **S R Fenley**, *Quasi-isometric foliations*, *Topology* 31 (1992) 667–676 MR1174265
- [5] **S R Fenley**, *Foliations with good geometry*, *J. Amer. Math. Soc.* 12 (1999) 619–676 MR1674739
- [6] **C J Leininger**, *Small curvature surfaces in hyperbolic 3-manifolds*, *J. Knot Theory Ramifications* 15 (2006) 379–411 MR2217503
- [7] **A W Reid**, *Totally geodesic surfaces in hyperbolic 3-manifolds*, *Proc. Edinburgh Math. Soc.* (2) 34 (1991) 77–88 MR1093177
- [8] **W P Thurston**, *The geometry and topology of three-manifolds*, Princeton Univ. Math. Dept. Lecture Notes (1979) Available at <http://msri.org/publications/books/gt3m/>
- [9] **A Zeghib**, *Laminations et hypersurfaces géodésiques des variétés hyperboliques*, *Ann. Sci. École Norm. Sup.* (4) 24 (1991) 171–188 MR1097690

Department of Mathematics, University of Michigan
530 Church Street, Ann Arbor 48109-1043, United States

breslin@umich.edu

<http://www.math.lsa.umich.edu/people/facultyDetail.php?unique=breslin>

Received: 9 February 2009 Revised: 6 March 2009

