Algebraic & Geometric Topology 9 (2009) 723–729

Small curvature laminations in hyperbolic 3–manifolds

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We show that if $\mathcal{L}$ is a codimension-one lamination in a finite volume hyperbolic 3–manifold such that the principal curvatures of each leaf of $\mathcal{L}$ are all in the interval $(-\delta, \delta)$ for a fixed $\delta \in [0, 1)$ and no complementary region of $\mathcal{L}$ is an interval bundle over a surface, then each boundary leaf of $\mathcal{L}$ has a nontrivial fundamental group. We also prove existence of a fixed constant $\delta_0 > 0$ such that if $\mathcal{L}$ is a codimension-one lamination in a finite volume hyperbolic 3–manifold such that the principal curvatures of each leaf of $\mathcal{L}$ are all in the interval $(-\delta_0, \delta_0)$ and no complementary region of $\mathcal{L}$ is an interval bundle over a surface, then each boundary leaf of $\mathcal{L}$ has a noncyclic fundamental group.

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1 Introduction

In [9], Zeghib proved that any totally geodesic codimension-one lamination in a closed hyperbolic 3–manifold is a finite union of disjoint closed surfaces. In this paper we investigate whether a similar result holds for codimension-one laminations with small principal curvatures. We will prove the following theorems:

Theorem 1 *Let $\delta \in [0, 1)$. If $\mathcal{L}$ is a codimension-one lamination in a finite volume hyperbolic 3–manifold such that the principal curvatures of each leaf of $\mathcal{L}$ are everywhere in $(-\delta, \delta)$ for a fixed constant $\delta \in [0, 1)$ and no complementary region of $\mathcal{L}$ is an interval bundle over a surface, then each boundary leaf of $\mathcal{L}$ has a nontrivial fundamental group.*

Theorem 2 *There exists a fixed constant $\delta_0 > 0$ such that if $\mathcal{L}$ is a codimension-one lamination in a finite volume hyperbolic 3–manifold such that the principal curvatures of each leaf of $\mathcal{L}$ are everywhere in $(-\delta_0, \delta_0)$ and no complementary region is an interval bundle over a surface, then each boundary leaf of $\mathcal{L}$ has a noncyclic fundamental group.*

Published: 20 April 2009 DOI: 10.2140/agt.2009.9.723

2 Examples

Let $\mathcal{L}$ be a codimension-one lamination in a complete hyperbolic 3–manifold M. Let L be a leaf of $\mathcal{L}$ and endow it with the path metric induced from M. Let $\widetilde{L}$ be the universal cover of L and lift the inclusion $i_L\colon L \to M$ to a map $\widetilde{i_L}\colon \widetilde{L} \to \mathbb{H}^3$. A map $f\colon X \to Y$ from a metric space X to a metric space Y is a (k,c)–quasi-isometry if $\frac{1}{k} d_X(a,b) - c \le d_Y(f(a), f(b)) \le k d_X(a,b) + c$. The leaf L is *quasi-isometric* if the map $\widetilde{i}_L$ is a (k,c)–quasi-isometry for some k, c. The lamination $\mathcal{L}$ is *quasi-isometric* if each leaf of $\mathcal{L}$ is quasi-isometric for the same fixed constants k, c.

Let $\delta \in (0,1)$. If the principal curvatures of $\widetilde{i}_L(\widetilde{L})$ are everywhere in $(-\delta, \delta)$, then the map $\widetilde{i}_L$ is a (k,c)–quasi-isometry for constants k, c depending only on δ (see Thurston [8]. Also see Leininger [6] for an elementary proof).

The constant δ_0 in Theorem 2 is less than 1, so a lamination satisfying the hypotheses of Theorem 1 or Theorem 2 is necessarily quasi-isometric. Thus it makes sense to ask whether these results hold for general quasi-isometric laminations.

Quasi-isometric laminations with no compact leaves Cannon and Thurston [3] proved that the stable and unstable laminations of the suspension of a pseudo-Anosov homeomorphism of a closed surface are quasi-isometric, and each leaf is a plane or annulus in this case. In addition to these examples, Fenley [5] produced infinitely many examples of closed hyperbolic 3–manifolds with quasi-isometric laminations in which each leaf is an annulus, a mobius band, or a plane. Note that Theorem 2 implies that the examples of Cannon–Thurston and Fenley cannot have principal curvatures everywhere in the interval $(-\delta_0, \delta_0)$.

One can also ask if we need to require that no complementary region is an interval bundle over a surface.

Small curvature laminations with simply connected boundary leaves Let S be a closed totally geodesic embedded surface in a closed hyperbolic 3–manifold M. Let $N(S) = S \times [0,1]$ be a closed embedded neighborhood of S in M. If the neighborhood $N(S)$ is small then the surfaces $S \times t$ will have small principal curvatures. Since $\pi_1(S)$ is left-orderable, there exist faithful representations $\rho\colon \pi_1(S) \to \mathrm{Homeo}([0,1])$ such that some points have trivial stabilizers (see Calegari [2]) The foliated bundle whose holonomy is ρ has a leaf which is simply connected. Replace $N(S)$ with this foliated bundle. We can blow up the simply connected leaf and remove the interior to get a lamination which is C^∞ close to the original (so that the leaves have small principal curvatures) and such that some boundary leaf is simply connected. See Calegari [1] to see why the foliated bundle can be embedded in M so that the leaves are smooth. Note that this lamination has a complementary region which is an interval bundle over a surface.

Small curvature laminations with no compact leaves One may also construct small curvature laminations in closed hyperbolic 3–manifolds with no compact leaves. The author would like to thank Chris Leininger for describing the following construction. The idea is to construct a small curvature branched surface in a closed hyperbolic 3–manifold which has an irrational point in the space of projective classes of measured laminations carried by the branched surface. A lamination corresponding to this irrational point will contain no compact leaves. There are totally geodesic immersed closed surfaces in the figure-eight knot complement M_8 arbitrarily close to any plane in the tangent bundle (see Reid [7]). Using this and the fact that $\pi_1(M_8)$ is LERF, one can find two such surfaces which lift to embedded surfaces S_1 and S_2 in a finite cover M of M_8 which intersect in a nonseparating (in both surfaces) simple closed geodesic l at an arbitrarily small angle. Flatten out the intersection to get a branched surface with small principal curvatures in which S_1 connects one side of S_2 to the other side. The branched surface has three branch sectors (an annulus, $S_1 \setminus l$, and $S_2 \setminus l$) and one branch equation ($x_1 = x_2 + x_3$). A solution to the branch equation in which two coordinates are not rationally related (eg, $x_1 = 1/2$, $x_2 = 1/\pi$, $x_3 = 1/2 - 1/\pi$) will correspond to a lamination with no compact leaves which can be isotoped to have small principal curvatures. Since the leaves do not have any cusps, we can fill the cusps of M to get a small curvature lamination in a closed hyperbolic 3–manifold with no compact leaves.

3 Proof of Theorem 1

Let $\epsilon > 0$ be so small that if P_1, P_2, P_3 are three disjoint smoothly embedded planes in hyperbolic 3–space with principal curvatures in $(-1, 1)$ which intersect the same ϵ–ball, then one of the P_i separates the other two.

Let $\mathcal{L}$ be a codimension-one lamination in a finite volume hyperbolic 3–manifold M such that the principal curvatures of each leaf are everywhere in the interval $(-\delta, \delta)$ for some $\delta \in (0, 1)$. Assume that no complementary region of $\mathcal{L}$ is an interval bundle over a surface. Let $\widetilde{\mathcal{L}}$ be the lift of $\mathcal{L}$ to $\mathbb{H}^3$. Since every leaf of $\mathcal{L}$ has principal curvatures everywhere in $(-\delta, \delta)$, the lamination $\mathcal{L}$ is a quasi-isometric lamination, and cannot be a foliation of M by Fenley [4].

Let L_0 be a boundary leaf of $\mathcal{L}$. Suppose, for contradiction, that $\pi_1(L_0)$ is trivial, which implies that L_0 has infinite area. Since M is closed, L_0 must intersect some fixed compact ball in M infinitely many times. Thus given any integer k, we can find a point y_k in L_0 such that the next leaf over on the boundary side of L_0 is within $1/k$ of y_k.

Let $\widetilde{L}_0$ be a lift of L_0 to $\mathbb{H}^3$. Lift the points y_k to a fixed fundamental domain of $\widetilde{L}_0$ and call them y_k. Let $\widetilde{L}_k$ be the next leaf over from $\widetilde{L}_0$ which is within $1/k$ of y_k. We now have a sequence of leaves $\widetilde{L}_k$ in $\widetilde{\mathcal{L}}$ on the boundary side of $\widetilde{L}_0$ such that for each k the distance from $\widetilde{L}_k$ to y_k is less than $1/k$, and there is no leaf of $\mathcal{L}$ between $\widetilde{L}_0$ and $\widetilde{L}_k$. We also have that $\partial\widetilde{L}_0 \neq \partial\widetilde{L}_k$ for all k, because otherwise the region between L_0 and L_k would be an interval bundle in the complement of $\mathcal{L}$.

Let k be so large that $1/k < \epsilon/8$. Since $\widetilde{L}_k$ eventually diverges from $\widetilde{L}_0$ we can find a point $x_k \in \widetilde{L}_0$ such that the distance from x_k to $\widetilde{L}_k$ is exactly $\epsilon/8$. Let b_k be the $(\epsilon/32)$–ball tangent to $\widetilde{L}_0$ at x_k on the boundary side of $\widetilde{L}_0$.

We will show that infinitely many of the balls b_k are disjointly embedded in M, contradicting the fact that M has finite volume. Suppose that $\gamma(b_l) \cap b_k \neq \varnothing$ for some integers l, k and some γ in $\pi_1(M)$. Note that $\gamma(\widetilde{L}_0) \neq \widetilde{L}_0$, since L_0 has trivial fundamental group. Now $\widetilde{L}_0$, $\widetilde{L}_k$, and $\gamma(\widetilde{L}_0)$ all intersect some ϵ–ball, so we must have that one of them separates the other two. Since there are no leaves of $\widetilde{\mathcal{L}}$ between $\widetilde{L}_0$ and $\widetilde{L}_k$, and $\gamma(\widetilde{L}_0)$ is closer to x_k than $\widetilde{L}_k$, we must have that $\widetilde{L}_0$ separates $\widetilde{L}_k$ and $\gamma(\widetilde{L}_0)$ (see Figure 1(a)). Also note that $\widetilde{L}_0$, $\widetilde{L}_k$, and $\gamma(\widetilde{L}_l)$ are all on the boundary side of $\gamma(\widetilde{L}_0)$ (ie, the side which contains the ball $\gamma(b_l)$).

Now we will show no matter where γ sends $\widetilde{L}_l$, we get a contradiction. We cannot have $\gamma(\widetilde{L}_l) = \widetilde{L}_k$, because this would imply that $\gamma^{-1}(\widetilde{L}_0)$ separates $\widetilde{L}_l$ and $\widetilde{L}_0$. Thus we have $\gamma(\widetilde{L}_l) \neq \widetilde{L}_k$.

Since $\widetilde{L}_0$, $\widetilde{L}_k$, and $\gamma(\widetilde{L}_l)$ all intersect some fixed ϵ–ball, we must have that one of them separates the other two. We cannot have that $\gamma(\widetilde{L}_l)$ separates $\widetilde{L}_0$ and $\widetilde{L}_k$, because there are no leaves of $\widetilde{\mathcal{L}}$ between $\widetilde{L}_0$ and $\widetilde{L}_k$ (See Figure 1(b)). If $\widetilde{L}_0$ separates $\widetilde{L}_k$ and $\gamma(\widetilde{L}_l)$, then $\gamma(\widetilde{L}_l)$ is between $\widetilde{L}_0$ and $\gamma(\widetilde{L}_0)$, so that $d(x_l, \widetilde{L}_l) = d(\gamma(x_l), \gamma(\widetilde{L}_l)) \leq \epsilon/16$ which is a contradiction (see Figure 1(c)). Thus $\widetilde{L}_0$ cannot separate $\widetilde{L}_k$ and $\gamma(\widetilde{L}_l)$. If $\widetilde{L}_k$ separates $\widetilde{L}_0$ and $\gamma(\widetilde{L}_l)$, then $\gamma^{-1}(\widetilde{L}_k)$ separates $\widetilde{L}_0$ and $\widetilde{L}_l$ which is a contradiction (see Figure 1(d)). Thus $\widetilde{L}_k$ cannot separate $\widetilde{L}_0$ and $\gamma(\widetilde{L}_l)$. We have shown that $\widetilde{L}_l$ has nowhere to go under the map γ, so that $\gamma(b_l) \cap \gamma(b_k) = \varnothing$ for any integers l, k and any $\gamma \in \pi_1(M)$. This implies that M contains infinitely many disjoint $(\epsilon/32)$–balls, contradicting the fact that M has finite volume. □

4 Proof of Theorem 2

Let $\epsilon > 0$ be so small that if P_1, P_2, P_3 are three disjoint smoothly embedded planes in hyperbolic 3–space with principal curvatures in $(-1, 1)$ which intersect the same

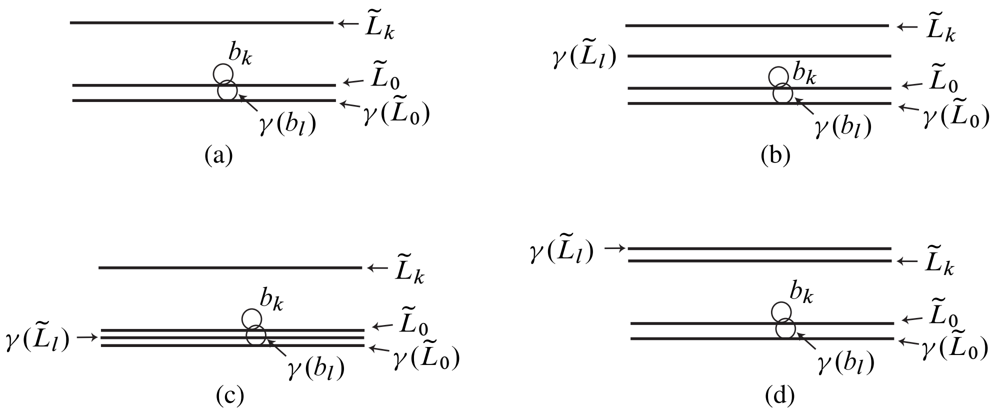

Figure 1: (a) $\widetilde{L}_0$ separates $\widetilde{L}_k$ and $\gamma(\widetilde{L}_0)$. (b) $\gamma(\widetilde{L}_l)$ cannot separate $\widetilde{L}_0$ and $\widetilde{L}_k$. (c) $\widetilde{L}_0$ cannot separate $\widetilde{L}_k$ and $\gamma(\widetilde{L}_l)$. (d) $\widetilde{L}_k$ cannot separate $\widetilde{L}_0$ and $\gamma(\widetilde{L}_l)$.

ϵ–ball, then one of the P_i separates the other two. Let $\delta_0 > 0$ be so small that if a smooth curve $\gamma\colon (-\infty, \infty) \to \mathbb{H}^3$ in $\mathbb{H}^3$ with endpoints in $\partial\mathbb{H}^3$ has curvature at most δ_0 at each point, then $\gamma(t)$ is in the $(\epsilon/2)$–neighborhood of the geodesic of $\mathbb{H}^3$ with the same endpoints.

Let $\mathcal{L}$ be a codimension-one lamination in a finite volume hyperbolic 3–manifold M such that the principal curvatures of each leaf are everywhere in the interval $(-\delta_0, \delta_0)$. Assume that no complementary region of $\mathcal{L}$ is an interval bundle over a surface. Let $\widetilde{\mathcal{L}}$ be the lift of $\mathcal{L}$ to $\mathbb{H}^3$. As in the proof of Theorem 1, $\mathcal{L}$ cannot be a foliation. Let L_0 be a boundary leaf of $\mathcal{L}$. Suppose, for contradiction, that $\pi_1(L_0)$ is cyclic, which implies that L_0 has infinite area. Since M is closed, L_0 must intersect some fixed compact ball in M infinitely many times. Also, by Theorem 1, we know that $\pi_1(L_0)$ is nontrivial, so that $\pi_1(L_0) \approx \mathbb{Z}$.

Let $\widetilde{L}_0$ be a lift of L_0 to $\mathbb{H}^3$. Since L_0 intersects a fixed compact ball in M infinitely many times, we can find a sequence of points y_k in $\widetilde{L}_0$ such that the closest leaf of $\widetilde{\mathcal{L}}$ to y_k on the boundary side of $\widetilde{L}_0$ is within $1/k$ of y_k. Let $\widetilde{L}_k$ be the leaf which is closest to y_k on the boundary side of $\widetilde{L}_0$. Note that there is no leaf of $\widetilde{\mathcal{L}}$ between $\widetilde{L}_0$ and $\widetilde{L}_k$. We have $\partial\widetilde{L}_0 \neq \partial\widetilde{L}_k$ for all k, because the complement of $\mathcal{L}$ contains no interval bundle components. We may assume that all y_k are contained in a fixed fundamental domain $\mathcal{D}$ of $\widetilde{L}_0$, and that y_k converge to a point $y_\infty \in \partial\widetilde{L}_0$.

For k large enough we have $\partial\widetilde{L}_0 \neq \partial\widetilde{L}_k$ and $d(y_k, \widetilde{L}_k) \leq \epsilon/8$, so that we can find a point x_k such that $d(x_k, \widetilde{L}_k) = \epsilon/8$.

Case 1 We can choose the sequence of points $x_k \in \widetilde{L}_0$ to be contained in a fixed fundamental domain D of $\widetilde{L}_0$ such that x_k exit an end of D whose projection to M has infinite area.

Let b_k be the $(\epsilon/32)$–ball tangent to $\widetilde{L}_0$ at x_k on the boundary side of $\widetilde{L}_0$. For k large enough, say all k, the generator of $\mathrm{stab}_{\pi_1(M)}(\widetilde{L}_0)$ moves the center of b_k a distance of at least ϵ. Thus we can assume that $\gamma(b_l) \cap b_k = \varnothing$ for any integers l, k and any $\gamma \in \mathrm{stab}_{\pi_1(M)}(\widetilde{L}_0)$.

We may now proceed as in the proof of Theorem 1 to show that $\gamma(b_l) \cap b_k = \varnothing$ for any integers l, k and any $\gamma \in \pi_1(M)$. This again contradicts the fact that M has finite volume.

Case 2 We cannot choose the sequence of points x_k as in Case 1.

If infinitely many of the leaves $\widetilde{L}_k$ were distinct, then we would be able to find a sequence of points as described in Case 1. Thus $\widetilde{L}_k = \widetilde{L}_+$ for some fixed leaf $\widetilde{L}_+ \in \widetilde{\mathcal{L}}$.

Let U be the component of the complement in $\partial\widetilde{L}_0$ of the fixed point(s) of the generator of $\mathrm{stab}_{\pi_1(M)}(\widetilde{L}_0)$ which contains the point y_∞. We will now show that $\partial\widetilde{L}_+ \cap \partial\widetilde{L}_0$ must contain U.

Suppose that $\partial\widetilde{L}_+ \cap \partial\widetilde{L}_0$ does not contain U. Since $d(y_k, \widetilde{L}_+) < 1/k$ and $\partial\widetilde{L}_+ \cap \partial\widetilde{L}_0$ does not contain U, we can find a sequence of points x_k in $\widetilde{L}_0$ which converge to a point $x_\infty \in U$ with $d(x_k, \widetilde{L}_+) = \epsilon/8$. Since the point x_∞ cannot be a fixed point of the generator of $\mathrm{stab}_{\pi_1(M)}(\widetilde{L}_0)$, a tail of the sequence x_k must be contained in a fixed fundamental domain of $\widetilde{L}_0$. This contradicts the fact that we are in Case 2. Thus $\partial\widetilde{L}_+ \cap \partial\widetilde{L}_0$ must contain U, hence must contain the fixed point(s) of the generator of $\mathrm{stab}_{\pi_1(M)}(\widetilde{L}_0)$.

If the generator of $\mathrm{stab}_{\pi_1(M)}(\widetilde{L}_0)$ is parabolic, then it has only one fixed point. This implies that $\partial\widetilde{L}_+ = \partial\widetilde{L}_0$, giving us a contradiction.

If the generator of $\mathrm{stab}_{\pi_1(M)}(\widetilde{L}_0)$ is loxodromic, then we can argue as above to find a leaf $\partial\widetilde{L}_-$ of $\widetilde{\mathcal{L}}$ which contains the other component of complement in $\partial\widetilde{L}_0$ of the fixed points of the generator of $\mathrm{stab}_{\pi_1(M)}(\widetilde{L}_0)$. So $\partial\widetilde{L}_+$ and $\partial\widetilde{L}_-$ both contain the endpoints of the axis of the generator of $\mathrm{stab}_{\pi_1(M)}(\widetilde{L}_0)$. Since the principal curvatures of $\widetilde{L}_0$, $\widetilde{L}_+$, and $\widetilde{L}_-$ are all in the interval $(-\delta_0, \delta_0)$, and $\partial\widetilde{L}_0$, $\partial\widetilde{L}_+$, $\partial\widetilde{L}_-$ all contain the endpoints of the axis of the generator of $\mathrm{stab}_{\pi_1(M)}$, we must have that $\widetilde{L}_0$, $\widetilde{L}_+$, and $\widetilde{L}_-$ all intersect some fixed ϵ–ball. Thus one of the three separates the other two. This gives us a contradiction since $\widetilde{L}_+$ and $\widetilde{L}_-$ are on the same side of $\widetilde{L}_0$ (ie, the boundary side) and there are no leaves of $\mathcal{L}$ between $\widetilde{L}_0$ and $\widetilde{L}_+$ or between $\widetilde{L}_0$ and $\widetilde{L}_-$. $\square$

Acknowledgements This work was partially supported by the NSF grants DMS-0135345 and DMS-0602191.

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Received: 9 February 2009 Revised: 6 March 2009