The ℓ^2–homology of even Coxeter groups

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Given a Coxeter system (W, S), there is an associated CW–complex, denoted $\Sigma(W, S)$ (or simply Σ), on which W acts properly and cocompactly. This is the Davis complex. The nerve L of (W, S) is a finite simplicial complex. When L is a triangulation of $\mathbb{S}^3$, Σ is a contractible 4–manifold. We prove that when (W, S) is an *even* Coxeter system and L is a flag triangulation of $\mathbb{S}^3$, then the reduced ℓ^2–homology of Σ vanishes in all but the middle dimension.

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1 Introduction

The following conjecture is attributed to Singer.

Singer's Conjecture 1.1 *If M^n is a closed aspherical manifold, then the reduced ℓ^2–homology of $\widetilde{M}^n$, $\mathcal{H}_*(\widetilde{M}^n)$, vanishes for all $* \neq \frac{n}{2}$.*

Singer's conjecture holds for elementary reasons in dimensions ≤ 2. Indeed, top-dimensional cycles on manifolds are constant on each component, so a square-summable cycle on an infinite component is constant 0. As a result, Singer's Conjecture 1.1 in dimension ≤ 2 follows from Poincaré duality. In [9], Lott and Lück prove that it holds for those aspherical 3–manifolds for which Thurston's Geometrization Conjecture is true. (Hence, by Perelman, all aspherical 3–manifolds.) For details on ℓ^2–homology theory, see Davis and Moussoung [6], Davis and Okun [7] and Eckmann [8].

Let S be a finite set of generators. A *Coxeter matrix* on S is a symmetric $S \times S$ matrix $M = (m_{st})$ with entries in $\mathbb{N} \cup \{\infty\}$ such that each diagonal entry is 1 and each off diagonal entry is ≥ 2. The matrix M gives a presentation for an associated *Coxeter* group W:

$$W = \langle S \mid (st)^{m_{st}} = 1, \text{ for each pair } (s,t) \text{ with } m_{st} \neq \infty \rangle. \tag{1}$$

The pair (W, S) is called a *Coxeter system*. Denote by L the nerve of (W, S). In several papers (eg, [3; 4; 6]), M Davis describes a construction which associates to any Coxeter system (W, S), a simplicial complex $\Sigma(W, S)$, or simply Σ when the

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Coxeter system is clear, on which W acts properly and cocompactly. The two salient features of Σ are that (1) it is contractible and (2) it permits a cellulation under which the link of each vertex is L. It follows that if L is a triangulation of $\mathbb{S}^{n-1}$, Σ is an n–manifold. There is a special case of Singer's conjecture for such manifolds.

Singer's Conjecture for Coxeter groups 1.2 *Let (W,S) be a Coxeter system such that its nerve, L, is a triangulation of $\mathbb{S}^{n-1}$. Then*

$$\mathcal{H}_i(\Sigma(W,S)) = 0 \text{ for all } i \neq \frac{n}{2}.$$

In [7], Davis and Okun prove that if Conjecture 1.2 for *right-angled* Coxeter systems is true in some odd dimension n, then it is also true for right-angled systems in dimension $n+1$. (A Coxeter system is right-angled if generators either commute or have no relation.) They also show that Thurston's Geometrization Conjecture holds for these Davis 3–manifolds arising from right-angled Coxeter systems. Hence, the Lott and Lück result implies that Conjecture 1.2 for right-angled Coxeter systems is true for $n=3$ and, therefore, also for $n=4$. (Davis and Okun also show that Andreev's theorem [1, Theorem 2] implies Conjecture 1.2 in dimension 3 for right-angled systems.) In [11], the author geometrizes arbitrary 3–dimensional Davis manifolds and shows that Conjecture 1.2 in dimension 3 follows.

Right-angled Coxeter systems are specific examples of *even* Coxeter systems. We say a Coxeter system is even if for any two generators $s \neq t$, m_{st} is either even or infinite. The purpose of this paper is to prove the following:

Main Theorem 1.3 *Let (W,S) be an even Coxeter system whose nerve L is a flag triangulation of $\mathbb{S}^3$. Then $\mathcal{H}_i(\Sigma(W,S)) = 0$ for $i \neq 2$.*

In order to prove Main Theorem 1.3, we define a certain subspace Ω of Σ and its boundary $\partial\Omega$. In the right-angled case, Ω is a tubular neighborhood of a contractible 3–manifold. The vanishing of the ℓ^2–homology of the 3–manifold implies the vanishing of the ℓ^2–homology of the pair $(\Omega, \partial\Omega)$. This can be promoted, via an inductive argument, to show the vanishing of the ℓ^2–homology of the 4–manifold Σ except in dimension 2. If (W,S) is not right-angled, Ω is not a tubular neighborhood of a 3–manifold. We therefore subdivide Ω into subspaces we call "boundary collars," which are isomorphic to $B \times [0,1]$, where B is a component of $\partial\Omega$. We paint these boundary collars with finitely many *colors*, which can be categorized as even or odd. This painting is virtually invariant under the group action on Ω. Moreover, the intersection of two components with even colors is 2–acyclic and the intersection of an odd colored component with the union of all even colored components is acyclic. Then using Mayer–Vietoris, we are able to prove that $\mathcal{H}_*(\Omega, \partial\Omega) = 0$ for $* = 3, 4$.

Next, we prove that for any $V \subseteq S$, and any $t \in V$,

$$\mathcal{H}_i(\Sigma(W_V, V)) \cong \mathcal{H}_i(\Sigma(W_{V-t}, V - t)),$$

for $i = 3, 4$ and where W_V is the subgroup of W generated by the elements of V. It follows from induction and Poincaré duality that Main Theorem 1.3 is true.

2 Coxeter systems and the complex Σ

Coxeter systems Let (W, S) be a Coxeter system. Given a subset U of S, define W_U to be the subgroup of W generated by the elements of U. (W_U, U) is a Coxeter system. A subset T of S is *spherical* if W_T is a finite subgroup of W. In this case, we will also say that the subgroup W_T is spherical. We say the Coxeter system (W, S) is *even* if for any $s, t \in S$ with $s \neq t$, m_{st} is either even or infinite.

Given $w \in W$, we call an expression $w = (s_1 s_2 \cdots s_n)$ *reduced* if there does not exist an integer $m < n$ with $w = (s'_1 s'_2 \cdots s'_m)$. Define the *length of* w, $l(w)$, to be the integer n such that $(s_1 s_2 \cdots s_n)$, is a reduced expression for w. Denote by $S(w)$ the set of elements of S which comprise a reduced expression for w. This set is well-defined [4, Proposition 4.1.1].

For $T \subseteq S$ and $w \in W$, the coset wW_T contains a unique element of minimal length. This element is said to be $(\varnothing, T)$–reduced. Moreover, it is shown in [2, Exercise 3, pages 31–32], that an element is $(\varnothing, T)$–reduced if and only if $l(wt) > l(w)$ for all $t \in T$. Likewise, we can define the $(T, \varnothing)$–reduced elements to be those w such that $l(tw) > l(w)$ for all $t \in T$. So given $X, Y \subseteq S$, we say an element $w \in W$ is (X, Y)–reduced if it is both $(X, \varnothing)$–reduced and $(\varnothing, Y)$–reduced.

Shortening elements of W We have the so-called "Exchange" condition (**E**) for Coxeter systems [2, Chapter 4, Section 1, Lemma 3; 4, Theorem 3.3.4]:

$$(\mathbf{E}) \qquad \begin{cases} \text{Given a reduced expression } w = (s_1 \cdots s_k) \text{ and an element } s \in S, \text{ either} \\ \ell(sw) = k + 1 \text{ or there is an index } i \text{ such that} \\ \qquad\qquad sw = (s_1 \cdots \widehat{s_i} \cdots s_k). \end{cases}$$

In the case of even Coxeter systems, the parity of a given generator in the set expressions for an element of W is well-defined. (We prove this herein in Lemma 3.4.) So, in (**E**), $s_i = s$; ie, if an element of $s \in S$ shortens a given element of W, it does so by deleting an instance of s in an expression for w.

It is also a fact about Coxeter groups [4, Theorem 3.4.2] that if two reduced expressions represent the same element, then one can be transformed into the other by replacing alternating subwords of the form $(sts\cdots)$ of length m_{st} by the alternating word $(tst\cdots)$ of length m_{st}. The proof of the first of the following two lemmas follows immediately from this.

Lemma 2.1 *Let $t \in S$, $w \in W_{S-t}$ and $v \in W$ with wtv reduced. If there exists an $r \in S(w)-S(v)$ with $(rt)^2 \neq 1$, then all r's appear to the left of all t's in any reduced expression for wtv.*

Lemma 2.2 *Let (W,S) be an even Coxeter system, let $t,s \in S$ be such that $2 < m_{st} < \infty$ and let $U_{st} = \{r \in S \mid m_{rt} = m_{rs} = 2\}$. Suppose that $tstw' = wtv$ (both reduced) where $w' \in W$, $w \in W_{S-t}$ and $S(v) \subset U_{st} \cup \{s,t\}$. Then $S(w) \subseteq U_{st} \cup \{s\}$.*

Proof Suppose that w is a counterexample of minimum length. w cannot start with an element of U_{st}, since if it did, multiplication on the left by this element would produce a shorter counterexample. Nor can w begin with s, since by the exchange condition, multiplication on the left by s would cancel an s in w', producing a shorter counterexample. Therefore, w must start with some r which either does not commute with t or does not commute with s. Lastly, suppose $w = (ru_1u_2\cdots u_nr')x$ where $u_i \in U_{st} \cup \{s\}$ $(i = 1,2,\dots,n)$, $r' \notin U_{st} \cup \{s\}$ and $x \in W_{S-t}$. Then, by the exchange condition, multiplication on the left by $(u_nu_{n-1}\cdots u_1r)$ produces a shorter counterexample. So by minimality we may also assume that every element appearing after r in w is from $U_{st} \cup \{s\}$.

If r does not commute with t, then by Lemma 2.1, r appears to the left of t in any reduced expression for wtv; a contradiction to $tstw' = wtv$. If r does commute with t but does not commute with s, then multiply both sides of $tstw' = wtv$ by t leaving $stw' = w''sv'$ (both reduced) where w'' begins with r, $S(v') \in U_{st} \cup \{s,t\}$ and $s \notin S(w'')$. Then, with t in Lemma 2.1 replaced by s, we have that r appears to the left of all s's in any reduced expression for wtv; a contradiction to $stw' = w''sv'$. □

The complex Σ Let (W,S) be an arbitrary Coxeter system. Denote by $\mathcal{S}$ the poset of spherical subsets of S, partially ordered by inclusion; and let $\mathcal{S}^{(k)} := \{T \in \mathcal{S} \mid \mathrm{Card}(T) = k\}$. Given a subset V of S, let $\mathcal{S}_{<V} := \{T \in \mathcal{S} \mid T \subset V\}$. Similar definitions exist for $>, \leq, \geq$. For any $w \in W$ and $T \in \mathcal{S}$, we call the coset wW_T a *spherical coset*. The poset of all spherical cosets we will denote by $W\mathcal{S}$.

The poset $\mathcal{S}_{>\varnothing}$ is an abstract simplicial complex, denote it by L, and call it the *nerve* of (W,S). The vertex set of L is S and a nonempty subset of vertices T spans a simplex of L if and only if T is spherical.

Let $K=|\mathcal{S}|$, the geometric realization of the poset $\mathcal{S}$. It is the cone on the barycentric subdivision of L, the cone point corresponding to the empty set, and thus a finite simplicial complex. Denote by $\Sigma(W,S)$, or simply Σ when the system is clear, the geometric realization of the poset $W\mathcal{S}$. This is the Davis complex. The natural action of W on $W\mathcal{S}$ induces a simplicial action of W on Σ which is proper and cocompact. K includes naturally into Σ via the map induced by $T\to W_T$, $T\in\mathcal{S}$. So we view K as a subcomplex of Σ and note that it is a strict fundamental domain for the action of W on Σ.

For any element $w\in W$, write wK for the w–translate of K in Σ. Let $w,w'\in W$ and consider $wK\cap w'K$. This intersection is nonempty if and only if $V=S(w^{-1}w')$ is a spherical subset. In fact, $wK\cap w'K$ is simplicially isomorphic to $|\mathcal{S}_{\geq V}|$, the geometric realization of $\mathcal{S}_{\geq V}:=\{V'\in\mathcal{S}\mid V\subseteq V'\}$.

A cubical structure on Σ For each $w\in W$, $T\in\mathcal{S}$, denote by $w\mathcal{S}_{\leq T}$ the subposet $\{wW_V\mid V\subseteq T\}$ of $W\mathcal{S}$. Put $n=\mathrm{Card}(T)$. $|w\mathcal{S}_{\leq T}|$ has the combinatorial structure of a subdivision of an n–cube. We identify the subsimplicial complex $|w\mathcal{S}_{\leq T}|$ of Σ with this coarser cubical structure and call it a *cube of type* T. Note that the vertices of these cubes correspond to spherical subsets $V\in\mathcal{S}_{\leq T}$. (For details on this cubical structure, see Moussoung [10].)

A cellulation of Σ by Coxeter cells Σ has a coarser cell structure: its cellulation by "Coxeter cells." (For reference, see Davis [4], Davis and Okun [7] and Davis, Dymara, Januszkiewicz and Okun [5].) Suppose that $T\in\mathcal{S}$; then by definition W_T is finite. Take the canonical representation of W_T on $\mathbb{R}^{\mathrm{Card}(T)}$ and choose a point x in the interior of a fundamental chamber. The *Coxeter cell of type* T is defined as the convex hull C, in $\mathbb{R}^{\mathrm{Card}(T)}$, of W_Tx (a generic W_T–orbit). The vertices of C are in 1–1 correspondence with the elements of W_T. Furthermore, a subset of these vertices is the vertex set of a face of C if and only if it corresponds to the set of elements in a coset of the form wW_V, where $w\in W_T$ and $V\subset T$. Hence, the poset of nonempty faces of C is naturally identified with the poset $W_T\mathcal{S}_{\leq T}:=\{wW_V|w\in W_T, V\subseteq T\}$. Therefore, we can identify the simplicial complex $\Sigma(W_T,T)$ with the barycentric subdivision of the Coxeter cell of type T.

Now, for each $T\in\mathcal{S}^{(k)}$ and $w\in W$, the poset $(W\mathcal{S})_{\leq wW_T}$ is isomorphic to the poset $W_T\mathcal{S}_{\leq T}$ via the map $vW_V\to w^{-1}vW_V$. Thus, the subcomplex of $\Sigma(W,S)$ which is obtained from the poset $(W\mathcal{S})_{\leq wW_T}$ may be identified with the barycentric subdivision of the Coxeter k–cell of type T. In this way, we put a cell structure on Σ which is coarser than the simplicial structure by identifying each simplicial subcomplex $|(W\mathcal{S})_{\leq wW_T}|$ with a Coxeter cell of type T.

We will write Σ_{cc}, when necessary, to denote the Davis complex equipped with this cellulation by Coxeter cells. Under this cellulation, the vertices of Σ_{cc} correspond to cosets of $W_\varnothing$, ie to elements from W; and 1–cells correspond to cosets of W_s, $s \in S$. The features of this cellulation are summarized by the following, from [4].

Proposition 2.3 *There is a natural cell structure on Σ so that*

- *its vertex set is W, its 1–skeleton is the Cayley graph of (W, S) and its 2–skeleton is a Cayley 2–complex.*
- *each cell is a Coxeter cell.*
- *the link of each vertex is isomorphic to L (the nerve of (W, S)) and so if L is a triangulation of $\mathbb{S}^{n-1}$, Σ is a topological n–manifold.*
- *a subset of W is the vertex set of a cell if and only if it is a spherical coset.*
- *the poset of cells is $W\mathcal{S}$.*

It will be our convention to use the term "vertices" for vertices in the cellulation of Σ by Coxeter cells or for vertices in L and to use "0–simplices" for 0–simplices in K or translates of K.

Ruins The following subspaces are defined in [5]. Let (W, S) be a Coxeter system. For any $U \subseteq S$, let $\mathcal{S}(U) = \{T \in \mathcal{S} | T \subseteq U\}$ and let $\Sigma(U)$ be the subcomplex of Σ_{cc} consisting of all cells of type T, with $T \in \mathcal{S}(U)$.

Given $T \in \mathcal{S}(U)$, define three subcomplexes of $\Sigma(U)$:

$\Omega(U, T)$: the union of closed cells of type T', with $T' \in \mathcal{S}(U)_{\geq T}$,

$\widehat{\Omega}(U, T)$: the union of closed cells of type T'', $T'' \in \mathcal{S}(U)$, $T'' \notin \mathcal{S}(U)_{\geq T}$,

$\partial\Omega(U, T)$: the cells of $\Omega(U, T)$ of type T'', with $T'' \notin \mathcal{S}(U)_{\geq T}$.

The pair $(\Omega(U, T), \partial\Omega(U, T))$ is called the (U, T)*–ruin*. For $T = \varnothing$, we have $\Omega(U, \varnothing) = \Sigma(U)$ and $\partial\Omega(U, \varnothing) = \varnothing$.

The subspace Ω Let $t \in S$. We call the (S, t)–ruin a *one-letter ruin*. Denote $U := \{s \in S \mid m_{st} < \infty\}$, ie U is the vertex set of the star of t in L. From this point on, U will denote this subset of S. 1–cells in $\Omega(S, t)$ are of type u where $u \in U$. So two vertices w, v in a component of $\Omega(S, t)$, thought of as group elements of W, have the property that $v = wp$, where $p \in W_U$. Thus, the path components of $\Omega(S, t)$ are indexed by the cosets W/W_U. Denote by Ω the path-component of $\Omega(S, t)$ with vertex set corresponding W_U. The action of W_U on Σ restricts to an action on Ω. Put $K(U) := K \cap \Omega$ and note that the W_U–translates of $K(U)$ cover Ω, ie $\Omega = \bigcup_{w \in W_U} wK(U)$. Let $\partial\Omega := \Omega \cap \partial\Omega(S,t)$. Coxeter 1–cells of $\partial\Omega(S,t)$ are of type u where $u \in U - t$; so the path components of $\partial\Omega$ are indexed by the cosets W_U / W_{U-t}.

If we restrict our attention to cubes of type T, where $T \subseteq T'$ for some $T' \in \mathcal{S}_{\geq t}$, Ω is a cubical complex and $\partial\Omega$ is a subcomplex. Moreover, if B is a component of $\partial\Omega$, the space $D := B \times [0,1]$ is isomorphic to the union of the w–translates of $K(U)$ where w is a vertex of B. We call such subspaces *boundary collars*. It is clear that the collection of boundary collars covers Ω. We denote by $\partial_{\text{in}}(D)$ the end of this product which does not lie in $\partial\Omega$; the 0–simplices of $\partial_{\text{in}}(D)$ correspond to elements of $\mathcal{S}_{\geq t}$. The boundary collars intersect along subsets of these "inner" boundaries.

3 The ℓ^2–homology of $\Omega(S,t)$

Here and for the remainder of the paper, we require that (W,S) be an even Coxeter system with nerve L. Fix $t \in S$ and let U, Ω and $\partial\Omega$ be defined as in Section 2.

Any $s \in U$ has the property that $m_{st} < \infty$. Let $S' := \{s \in U \mid m_{st} > 2\}$, and assume that S' is not empty. The group W_U has the following properties.

Lemma 3.1 *Suppose that L is flag. Then for $s, s' \in S'$, either $s = s'$, or $m_{ss'} = \infty$.*

Proof Suppose that $s \neq s'$ and that $m_{ss'} < \infty$. Then $\{s,s'\} \in \mathcal{S}$, and since s, s' are both in U, the vertices corresponding to s, s' and t are pairwise connected in L. L is a flag complex, so this implies that $\{s,s',t\} \in \mathcal{S}$. But

$$\frac{1}{m_{ss'}} + \frac{1}{m_{st}} + \frac{1}{m_{ts'}} \leq \frac{1}{m_{ss'}} + \frac{1}{4} + \frac{1}{4} \leq 1.$$

This contradicts $\{s,s',t\}$ being a spherical subset. So we must have that $m_{ss'} = \infty$. □

Corollary 3.2 *Let $s \in S'$ and let $T \in \mathcal{S}_{\geq\{s,t\}}$. Then $m_{ut} = m_{us} = 2$ for $u \in T - \{s,t\}$.*

Let L_{st} denote the link in L of the edge connecting the vertices s and t. The above Corollary states that the generators in the vertex set of L_{st} commute with both s and t. As in Lemma 2.2, denote this set of generators by U_{st}.

Of particular interest to us will be elements of W_U with a reduced expression of the form $tst\cdots st$ for some $s \in S'$. Since W is even, this expression is unique, and we have the following lemma.

Lemma 3.3 *Let $s \in S'$ and let $u \in W_{\{s,t\}}$ be such that $u = tst\cdots st$, is a reduced expression beginning and ending with t. Then u is $(U-t, U-t)$–reduced.*

Lemma 3.4 *Let $V, T \subseteq S$ and consider the function $g_{VT}\colon W_V \to W_T$ induced by the following rule: $g_{VT}(s) = s$ if $s \in V \cap T$ and $g_{VT}(s) = e$ (the identity element of W) for $s \in V - T$. Then g_{VT} is a homomorphism.*

Proof We show that g_{VT} respects the relations in W_V. Let $s, u \in V$ be such that $(su)^m = 1$. Then

$$g_{VT}((su)^m) = \begin{cases} (su)^m & \text{if } s \in T, u \in T, \\ s^m & \text{if } s \in T, u \notin T, \\ u^m & \text{if } u \in T, s \notin T, \\ e & \text{if } s \notin T, u \notin T. \end{cases}$$

In all cases, since (W_V, V) is even, $g_T((su)^m) = e$. □

Then with $T \in \mathcal{S}_{\geq t}$ and U as above, we define an action of W_U on the set of cosets W_T / W_{T-t}: For $w \in W_U$ and $v \in W_T$, define

$$w \cdot v W_{T-t} = g_{UT}(w) v W_{T-t}. \tag{2}$$

Coloring boundary collars Set

$$A = \prod_{T \in \mathcal{S}_{\geq t}} W_T / W_{T-t}.$$

We call A the set of colors and note that it is a finite set. The action defined in Equation (2) extends to a diagonal W_U–action on A; for $w \in W_U$ and $a \in A$, write $w \cdot a$ to denote w acting on a. Let $\overline{e}$ be the element of A defined by taking the trivial coset W_{T-t} for each $T \in \mathcal{S}_{\geq t}$. Vertices of Ω correspond to group elements of W_U, so we paint the vertices of Ω by defining a map $c\colon W_U \to A$ with the rule $c(w) := w \cdot \overline{e}$.

Remark 3.5 If an element $w \in W_U$ does not contain t in any reduced expression, then w acts trivially on the element $\overline{e}$, ie $w \cdot \overline{e} = \overline{e}$.

We paint the space $wK(U)$ with $c(w)$. In this way, all of Ω is colored with some element of A. For vertices w and w' of the same component B of $\partial\Omega$, $h = w^{-1}w' \in W_{U-t}$. So $c(w') = c(wh) = wh \cdot \overline{e} = w \cdot \overline{e} = c(w)$, where the third equality follows from Remark 3.5. Therefore all of $D = B \times [0,1]$ is painted with $c(w)$. Note that each component of $\partial\Omega$ is monochromatic while each "inner" boundary is not.

Lemma 3.6 *Let $D = B \times [0,1]$ and $D' = B' \times [0,1]$ be boundary collars where B and B' are different components of $\partial\Omega$. Suppose that the vertices of B and B' have the same color. Then $D \cap D' = \varnothing$.*

Proof Suppose, by way of contradiction, that $D \cap D' \neq \varnothing$, ie there exist vertices $w \in B$, $w' \in B'$ such that $c(w) = c(w')$ and $wK(U) \cap w'K(U) \neq \varnothing$. Let $V = S(v)$, where $v = w^{-1}w'$, and since w and w' are from different components of $\partial\Omega$, $t \in V$. Now $c(w) = c(w') \Rightarrow w \cdot \overline{e} = wv \cdot \overline{e} \Rightarrow \overline{e} = v \cdot \overline{e}$. Thus, for any $T \in \mathcal{S}_{\geq t}$, we have that

$$v \cdot W_{T-t} = W_{T-t}. \tag{3}$$

But since $v \in W_V$, the action of v on W_V / W_{V-t} defined in (2) is left multiplication by v. But by Equation (3), we have that $v \in W_{V-t}$ which is a contradiction. □

Now for $c \in A$, define the *c–collar*, F_c, to be the disjoint union of the boundary collars $D = B \times [0, 1]$ where each component B of $\partial\Omega$ has the color c. The collection of c–collars is a finite cover of Ω.

Even and odd collars Let $T = \{t\}$ and consider the homomorphism $g_{UT} \colon W_U \to W_t$ defined in Lemma 3.4. Under g_{UT}, an element $w \in W_U$ is sent to the identity in W_t if w has an even number of t's present in some factorization (and therefore, all factorizations) as a product of generators from U and an element $w \in W_U$ is sent to $t \in W_t$ if w has an odd number of t's present in factorizations. Thus, we call a vertex w *even* if $g_{UT}(w) = e$; *odd* if $g_{UT}(w) = t$. If two vertices w and w' are such that $c(w) = c(w')$, then clearly $g_{UT}(w) = g_{UT}(w')$, so we may also classify the colors as even or odd. A c–collar is even or odd as c is even or odd and we refer to it as an "even or odd collar."

Of fundamental importance will be how these collars intersect. By Remark 3.5, we know that in order for the vertices of a Coxeter cell to support two different colors, this cell must be of type $T \in \mathcal{S}_{\geq t}$. But, for a cell to support two different *even* vertices, v and v', this cell must be of type $T \in \mathcal{S}_{\geq \{s,t\}}$ for exactly one $s \in S'$ (uniqueness is given by Corollary 3.2). Moreover, $w = v^{-1}v'$ has the properties that (1) $\{s, t\} \subseteq S(w)$ and that (2) it contains at least two, and an even number of t's in any factorization as a product of generators. We call such w *t–even*.

Example 3.7 The following is representative of our situation. Suppose $L = \mathbb{S}^1$, and $U = \{t, r, s \mid (rt)^2 = 1, (st)^4 = 1\}$. Ω is represented in Figure 1. The black dots represent the vertices of the Coxeter cellulation, with the vertices e and tst labeled. The even collars are shaded. Even boundary collars intersect in a 0–simplex corresponding to the spherical subset $\{s, t\}$. The intersection of one odd collar and all evens is the inner boundary of the odd collar.

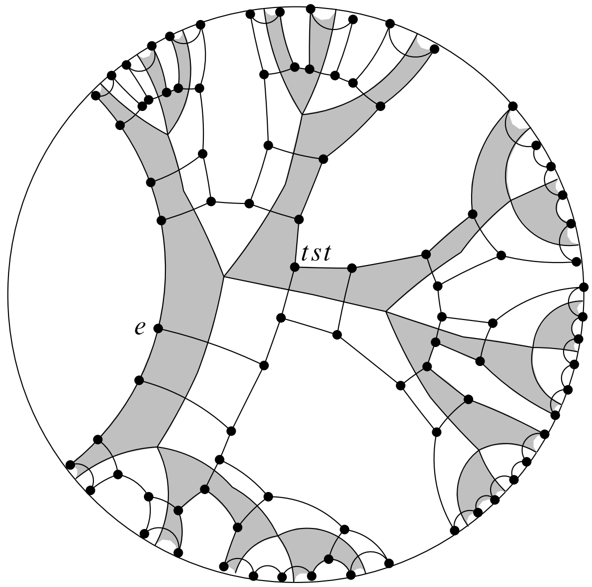

Figure 1: Even and odd colors of Ω

The intersection of even collars We now focus solely on our case: L a flag triangulation of $\mathbb{S}^3$. Let D_0 denote the boundary collar containing the vertex e. Fix $s \in S'$ and let D_2 denote the boundary collar containing the vertex u, where $u \in W_{\{s,t\}}$ is t–even and has a reduced expression ending in t. We study $D_0 \cap D_2$.

Lemma 3.8 *Let $W' := W_{U_{st}}$, where $U_{st} = \{r \in S \mid m_{rt} = m_{rs} = 2\}$, and let $K' = K(U) \cap uK(U)$. Denote by $W'K'$ the orbit of K' under W'. Then $D_0 \cap D_2 = W'K'$.*

Proof For any $w \in W'$, the vertex w is in the same component of $\partial\Omega$ as e (by Remark 3.5), and therefore $wK(U) \subset D_0$. $wu = uw$, so wu is in the same component of $\partial\Omega$ as u and $wuK(U) \subset D_2$. Thus $wK' = wK(U) \cap wuK(U) \subset D_0 \cap D_2$.

Now let σ be a 0–simplex in $D_0 \cap D_2$. Then there exist $w, w' \in W_{U-t}$ such that $\sigma \in wK(U) \cap uw'K(U)$, ie σ is simultaneously the w– and uw'–translate of a 0–simplex σ' in $K(U)$. Let V be the spherical subset to which σ' corresponds and let $v \in W_V$ be such that $uw' = wv$. $c(e) = c(w)$ and $c(u) = c(uw')$, so w and uw' are differently colored even vertices of a cell of type V. By the paragraph preceding Example 3.7, $\{s', t\} \subseteq S(v) \subseteq V$ for exactly one $s' \in S'$ and v is t–even.

Claim 1 $s' = s$.

Proof Since $w' \in W_{U-t}$, $c(u) = c(uw') = c(wv)$, ie u and wv act the same on every coordinate of $\overline{e}$. Consider the $\{s,t\}$–coordinate. $u \in W_{\{s,t\}}$ is t–even, so $u \cdot W_s = uW_s$ and $uW_s \neq W_s$. But if $s \notin S(v)$, then v being t–even and $w \in W_{U-t}$ imply that $wv \cdot W_s = W_s$; which contradicts u and wv having the same color. So Claim 1 is true, and as a result $V \in \mathcal{S}_{\geq\{s,t\}}$ and $\sigma' \in K'$. Moreover, by Corollary 3.2, $V \subseteq U_{st} \cup \{s,t\}$. It remains to show that σ is in the W'–orbit of K'. □

Claim 2 $S(w) \subseteq (U_{st} \cup \{s\})$.

Proof Take a reduced expression for u which ends in t. If this expression begins with s, multiply u on the left by s, so that we have $suw' = swv$. The only change this can effect on $S(w)$ is either adding or subtracting an s, which is inconsequential to our claim. So, we may assume that u has a reduced expression of the form $tst\cdots st$ as described in Lemma 3.3. Hence, u is $(U-t, U-t)$–reduced and uw' has a reduced expression beginning with the subword tst. wv has a reduced expression of the form $w''tv'$ where $w'' \in W_{U-t}$, $S(v') \subset U_{st} \cup \{s,t\}$ and where the difference between $S(w)$ and $S(w'')$ is contained in $U_{st} \cup \{s\}$. Claim 2 then follows from Lemma 2.2 applied to w''. □

We now finish the proof of Lemma 3.8. If $s \notin S(w)$, then $w \in W'$ and we are done since σ is the w–translate of σ'. If $s \in S(w)$, then w may be written as qs, with $q \in W'$ and since $s \in V$, $qsW_V = qW_V$. So σ is also the q–translate of σ'. □

Proposition 3.9 $(D_0 \cap D_2) \cong \Sigma(W', U_{st})$, *an infinite connected* 2*–manifold.*

Proof Since $S(u) = \{s,t\}$, K' is the geometric realization of the poset $\mathcal{S}_{\geq\{s,t\}} = \{V \in \mathcal{S} | \{s,t\} \subseteq V\}$. By Lemma 3.8, $(D_0 \cap D_2) \cong |W'\mathcal{S}_{\geq\{s,t\}}|$, and by Corollary 3.2, $\mathcal{S}_{\geq\{s,t\}}$ is isomorphic to $\mathcal{S}(U_{st})$ via the map $T \to T - \{s,t\}$. So $(D_0 \cap D_2) \cong |W'\mathcal{S}(U_{st})| = \Sigma(W', U_{st})$.

Simplices in L_{st} correspond to spherical subsets $T \in \mathcal{S}$ such that neither s nor t is contained in T but $T \cup \{s,t\} \in \mathcal{S}$. So by Corollary 3.2, the vertex set of a simplex of L_{st} corresponds to a spherical subset of $\mathcal{S}(U_{st})$. Conversely, given a spherical subset $T \in \mathcal{S}(U_{st})$, $W_{T\cup\{s,t\}} = W_T \times W_{\{s,t\}}$, which is finite. So T corresponds to a simplex of L_{st}. Thus, L_{st} is the nerve of the Coxeter system (W', U_{st}). Since L triangulates $\mathbb{S}^3$, L_{st} triangulates $\mathbb{S}^1$. It follows from Proposition 2.3 that $\Sigma(W', U_{st})$ is a contractible 2–manifold. □

Corollary 3.10 *Let $c, c' \in A$ be even. Then $\mathcal{H}_2(F_c \cap F_{c'}) = 0$.*

Proof Suppose that $F_c \neq F_{c'}$ are both even collars and $F_c \cap F_{c'} \neq \varnothing$. Then there exist even vertices v and v' with $vK(U) \cap v'K(U) \neq \varnothing$. Let $w = v^{-1}v'$ and put $T = S(v^{-1}v')$. T is a spherical subset, and v and v' are both vertices of a cell of type T. So we have exactly one $s \in S'$ with $\{s,t\} \subseteq T$. Factor w as $w = xq$ where $x \in W_{\{s,t\}}$ is t–even and $q \in W_{T-\{s,t\}}$. Now, x may not have a reduced expression ending in t. If it does not, then xs does and it is in the same boundary collar as x and w. So let

$$u = \begin{cases} x & \text{if } x \text{ has a reduced expression ending in } t, \\ xs & \text{otherwise.} \end{cases}$$

Then $vK(U) \cap v'K(U) \subseteq vK(U) \cap vuK(U)$. Act on the left by v^{-1} and we are in the situation studied in Lemma 3.8 and Proposition 3.9. So $F_c \cap F_{c'}$ is the disjoint union of infinite connected 2–manifolds. As a result, any 2–cycle must be constant 0. □

Remark 3.11 If W is right-angled, or if $S' = \varnothing$, then $W_U = W_{U-t} \times W_t$ and there is one even and one odd collar.

Multiple even collars Suppose that $D_1, D_2, \dots, D_n, D_e$ are even boundary collars. Then

$$D_e \cap \left(\bigcup_{j=1}^{n} D_j \right) = (D_e \cap D_1) \cup \cdots \cup (D_e \cap D_n),$$

and suppose that for some $1 \leq i < k \leq n$ we have that $(D_e \cap D_i)$ and $(D_e \cap D_k)$ are not disjoint. Let σ be a 0–simplex contained in $D_e \cap D_i \cap D_k$ corresponding to a coset of the form vW_T. Then there exists $w, w' \in W_T$ such that $v \in D_e$, $vw \in D_i$, $vw' \in D_k$ and $\sigma \in vK(U) \cap vwK(U) \cap vw'K(U)$. These three vertices are differently colored even vertices of a cell of type T, so $\{s,t\} \subseteq T$ for exactly one $s \in S'$ and both w and w' are t–even. Then, as in the proof of Corollary 3.10, it follows that $D_e \cap D_i = D_e \cap D_k \cong |W'\mathcal{S}_{\geq \{s,t\}}|$. So Corollary 3.10 generalizes to the following:

Corollary 3.12 *Let $F_{c_1}, F_{c_2}, \dots, F_{c_n}, F_{c_e}$ be even collars. Then*

$$\mathcal{H}_2\left(F_{c_e} \cap \left(\bigcup_{j=1}^{n} F_{c_j} \right) \right) = 0.$$

Lemma 3.13 *Define*

$$\partial_{\text{in}}(F_c) := \coprod_{D \subset F_c} \partial_{\text{in}}(D).$$

Let $\mathcal{F}_E$ denote the union of all even collars and let F_o be odd, then $F_o \cap \mathcal{F}_E = \partial_{\text{in}}(F_o)$.

Proof Since F_o is a disjoint union of boundary collars, it suffices to show that $D \cap \mathcal{F}_E = \partial_{\mathrm{in}}(D)$ for some boundary collar $D \subset F_o$.

$\supseteq$: Let σ be a 0–simplex in $\partial_{\mathrm{in}}(D)$. Then σ corresponds to a coset of the form wW_V where $V \in \mathcal{S}_{\geq t}$ and $w \in W_U$ is an odd vertex of D. Consider the even vertex wt. Then since $t \in V$, $wW_V = wtW_V$, and $\sigma \in wtK(U) \subset \mathcal{F}_E$.

$\subseteq$: Now suppose that σ is a 0–simplex contained in $D \cap \mathcal{F}_E$. Then there exists a spherical subset V and cosets $wW_V = w'W_V$ where w is odd and w' is even. Let $v = w^{-1}w'$. Since w is odd and w' is even, v must contain an odd number of t's in any of its reduced expressions. Therefore $t \in V$ and $\sigma \in \partial_{\mathrm{in}}(D)$. □

As before, let $\mathcal{F}_E$ denote the union of all even collars, and now let $\mathcal{F}_O$ denote the union of a subcollection of odd collars. Let $\mathcal{F}_{E'} = \mathcal{F}_E \cup \mathcal{F}_O$ and let F_o be an odd collar not included in $\mathcal{F}_O$. Then by Lemma 3.13,

$$F_o \cap \mathcal{F}_{E'} = (F_o \cap \mathcal{F}_E) \cup (F_o \cap \mathcal{F}_O) = \partial_{\mathrm{in}}(F_o) \cup (F_o \cap \mathcal{F}_O).$$

Any 0–simplex in F_o which is also in a different collar must be of the form wW_V, where w is a vertex of F_o and $V \in \mathcal{S}_{\geq t}$. Therefore $(F_o \cap \mathcal{F}_O) \subset \partial_{\mathrm{in}}(F_o)$ and $F_o \cap \mathcal{F}_{E'} = \partial_{\mathrm{in}}(F_o)$.

It is clear from the product structure on boundary collars that $\partial_{\mathrm{in}}(F_o) \cong F_o \cap \partial\Omega$, the latter a disjoint collection of components of $\partial\Omega$. Since L is flag, we have a 1–1 correspondence between Coxeter cells of any component of $\partial\Omega$ and cells of $\Sigma(W_{U-t}, U-t)_{\mathrm{cc}}$. Denote by L_t the link in L of the vertex corresponding to t, it is a triangulation of $\mathbb{S}^2$ and it is isomorphic to the nerve of $(W_{U-t}, U-t)$. Then since Singer's Conjecture for Coxeter groups 1.2 is true in dimension 3 [11, Corollary 4.4],

$$\mathcal{H}_i(F_o \cap \mathcal{F}_{E'}) = 0 \tag{4}$$

for all i.

Proposition 3.14 *Let (W,S) be an even Coxeter system whose nerve, L is a flag triangulation of $\mathbb{S}^3$. Let $t \in S$. Then $\mathcal{H}_i(\Omega(S,t), \partial\Omega(S,t)) = 0$ for $i = 3, 4$.*

Proof It suffices to calculate $\mathcal{H}_*(\Omega, \partial\Omega)$. We first show that $\mathcal{H}_4(\Omega, \partial\Omega) = 0$. Consider the long exact sequence of the pair $(\Omega, \partial\Omega)$:

$$\to \mathcal{H}_4(\Omega) \to \mathcal{H}_4(\Omega, \partial\Omega) \to \mathcal{H}_3(\partial\Omega) \to$$

Ω is a 4–dimensional manifold with infinite boundary, so $\mathcal{H}_4(\Omega) = 0$ and $\mathcal{H}_3(\partial\Omega) = 0$. Then by exactness, $\mathcal{H}_4(\Omega, \partial\Omega) = 0$.

Let $\mathcal{F}_{E'}$ denote the union of a collection of even collars or the union of all evens and a collection of odd collars. Let F_c be a collar not contained in $\mathcal{F}_{E'}$ (if $\mathcal{F}_{E'}$ is not all the even colors, require that F_c be an even color). Let $\partial_{E'} = \mathcal{F}_{E'} \cap \partial\Omega$ and let $\partial_{F_c} = F_c \cap \partial\Omega$. Note that $\partial_{E'} \cap \partial_{F_c} = \varnothing$ and consider the relative Mayer–Vietoris sequence of the pair $(\mathcal{F}_{E'} \cup F_c, \partial_{E'} \cup \partial_{F_c})$:

$$\cdots \to \mathcal{H}_3(\mathcal{F}_{E'}, \partial_{E'}) \oplus \mathcal{H}_3(F_c, \partial_{F_c}) \to \mathcal{H}_3(\mathcal{F}_{E'} \cup F_c, \partial_{E'} \cup \partial_{F_c}) \to \mathcal{H}_2(\mathcal{F}_{E'} \cap F_c) \to \cdots$$

Assume $\mathcal{H}_3(\mathcal{F}_{E'}, \partial_{E'}) = 0$. Each color retracts onto its boundary, so $\mathcal{H}_3(F_c, \partial_{F_c}) = 0$. If F_c is even, the last term vanishes by Corollary 3.12, if F_c is odd, the last term vanishes by Equation (4). In either case, exactness implies that $\mathcal{H}_3(\mathcal{F}_{E'} \cup F_c, \partial_{E'} \cup \partial_{F_c}) = 0$. It follows from induction that $\mathcal{H}_3(\Omega, \partial\Omega) = 0$. □

4 The ℓ^2–homology of Σ

Lemma 4.1 *Let $V \subseteq S$ and let $T \subseteq V$ be a spherical subset with* $\mathrm{Card}(T) = 2$. *Then* $\mathcal{H}_4(\Omega(V,T), \partial\Omega(V,T)) = 0$.

Proof If $\mathcal{S}(V)^{(4)}_{>T} = \varnothing$, then $\Omega(V,T)$ does not contain 4–dimensional cells, and we are done. So assume that $\mathcal{S}(V)^{(4)}_{>T} \neq \varnothing$. The codimension 1 faces of 4–cells of $\Omega(V,T)$ are either faces of one other 4–cell in $\Omega(V,T)$ (Σ is a 4–manifold), or they are free faces, ie they are not faces of any other 4–cell in $\Omega(V,T)$.

Suppose that cells of type $T' \in \mathcal{S}(V)^{(4)}_{>T}$ have a codimension one face of type R which is a face of another 4–cell in $\Omega(V,T)$ of type T''. Then any relative 4–cycle must be constant on adjacent cells of type T' and T'', where $T' = R \cup \{r\}$, and $T'' = R \cup \{s\}$, $R \in \mathcal{S}(V)^{(3)}_{>T}$ and $r, s \in V$. Since L is flag and 3–dimensional, $m_{rs} = \infty$. So in this case, there is a sequence of adjacent 4–cells with vertex sets $W_{T'}, W_{T''}, sW_{T'}, srW_{T''}, srsW_{T'}, srsrW_{T''}, \ldots$. Hence, this constant must be 0.

Now suppose that for a given 4–cell of $\Omega(V,T)$, every codimension one face is free. This cell has faces not contained in $\partial\Omega(V,T)$, so relative 4–cycles cannot be supported on this cell. □

Let $V \subseteq S$, be arbitrary; $T \subseteq V$ spherical, $\Omega := \Omega(V,T)$, $\partial\Omega := \partial\Omega(V,T)$. Recall that $\Sigma(V)$ is the subcomplex of Σ_{cc} consisting of cells of type T', with $T' \subseteq V$. We have excision isomorphisms (as in [5])

$$C_*(\Omega(V,T), \partial\Omega) \cong C_*(\Sigma(V), \widehat{\Omega}(V,T)), \tag{5}$$

and for any $s \in T$ and $T' := T - s$,

$$C_*(\Sigma(V-s), \widehat{\Omega}(V-s, T')) \cong C_*(\widehat{\Omega}(V,T), \widehat{\Omega}(V,T')). \tag{6}$$

Set $\widehat{\Omega} := \widehat{\Omega}(V,T)$, and $\widehat{\Omega}' := \widehat{\Omega}(V,T')$. Consider the long, weakly exact sequence of the triple $(\Sigma(V), \widehat{\Omega}, \widehat{\Omega}')$:

$$\cdots \to \mathcal{H}_*(\widehat{\Omega}, \widehat{\Omega}') \to \mathcal{H}_*(\Sigma(V), \widehat{\Omega}') \to \mathcal{H}_*(\Sigma(V), \widehat{\Omega}) \to \cdots$$

By Equations (5) and (6), the left hand term excises to the homology of the $(V-s, T')$–ruin, the right hand term to that of the (V,T)–ruin and the middle term to that of the (V,T')–ruin; leaving the sequence:

$$\cdots \to \mathcal{H}_*(\Omega(V-s,T'),\partial) \to \mathcal{H}_*(\Omega(V,T'),\partial) \to \mathcal{H}_*(\Omega(V,T),\partial) \to \cdots \tag{7}$$

Proposition 4.2 *Let (W,S) be an even Coxeter system, whose nerve L is a flag triangulation of $\mathbb{S}^3$. Let $V \subseteq S$ and $t \in V$. Then*

$$\mathcal{H}_i(\Omega(V,t), \partial\Omega(V,t)) = 0, \tag{8}$$

for $i = 3,4$.

Proof It is clear that $\mathcal{H}_i(\Omega(V,t)) = 0$ for $i = 3,4$ whenever $\mathrm{Card}(V) \leq 2$, so we may assume that $\mathrm{Card}(V) > 2$. We show Equation (8) by induction on $\mathrm{Card}(S-V)$, Proposition 3.14 giving us the base case. Let $V = V' \cup s$ and $t \in V'$. Assume (8) holds for V. If $m_{st} = \infty$ then $(\Omega(V',t),\partial) = (\Omega(V,t),\partial)$ and we are done. Otherwise, consider the sequence in Equation (7), taking $T = \{s,t\}$, $T' = \{t\}$:

$$\begin{array}{ccccccc} 0 & \to & \mathcal{H}_4(\Omega(V',t),\partial) & \to & \mathcal{H}_4(\Omega(V,t),\partial) & \to & \mathcal{H}_4(\Omega(V,\{s,t\}),\partial) \to \\ & \to & \mathcal{H}_3(\Omega(V',t),\partial) & \to & \mathcal{H}_3(\Omega(V,t),\partial) & \to & \cdots \end{array}$$

$\mathcal{H}_i(\Omega(V,t),\partial) = 0$ for $i = 3,4$ by assumption and $\mathcal{H}_4(\Omega(V,\{s,t\}),\partial) = 0$ by Lemma 4.1. So by exactness, $\mathcal{H}_4(\Omega(V',t),\partial) = 0$. □

Main Theorem 4.3 *Let (W,S) be an even Coxeter system whose nerve L is a flag triangulation of $\mathbb{S}^3$. Then*

$$\mathcal{H}_i(\Sigma) = 0 \text{ for } i \neq 2.$$

Proof Let $V \subseteq S$ and $t \in V$. Consider the following form of (7), where $T = \{t\}$:

$$\begin{array}{ccccccc} 0 & \to & \mathcal{H}_4(\Sigma(V-t)) & \to & \mathcal{H}_4(\Sigma(V)) & \to & \mathcal{H}_4(\Omega(V,t),\partial) \to \\ & \to & \mathcal{H}_3(\Sigma(V-t)) & \to & \mathcal{H}_3(\Sigma(V)) & \to & \mathcal{H}_3(\Omega(V,t),\partial) \to \cdots \end{array}$$

By Proposition 4.2, $\mathcal{H}_i(\Omega(V,t),\partial) = 0$ for $i = 3,4$. So by exactness,

$$\mathcal{H}_i(\Sigma(V-t)) \cong \mathcal{H}_i(\Sigma(V)),$$

for $i = 3,4$. It follows that $\mathcal{H}_i(\Sigma) \cong \mathcal{H}_i(\Sigma(\varnothing)) = 0$ for $i = 3,4$ and hence, by Poincaré duality, $\mathcal{H}_i(\Sigma) = 0$ for $i \neq 2$. □

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