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Intrinsically linked graphs in projective space

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We examine graphs that contain a nontrivial link in every embedding into real projective space, using a weaker notion of unlink than was used in Flapan, et al [5]. We call such graphs intrinsically linked in $\mathbb{R}P^3$. We fully characterize such graphs with connectivity 0, 1 and 2. We also show that only one Petersen-family graph is intrinsically linked in $\mathbb{R}P^3$ and prove that K_7 minus any two edges is also minor-minimal intrinsically linked. In all, 597 graphs are shown to be minor-minimal intrinsically linked in $\mathbb{R}P^3$.

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1 Introduction

We can represent knots in $\mathbb{R}P^3$ as closed curves or unions of arcs in the closed 3–ball, D^3, such that the endpoints of the arcs lie on ∂D^3. Because $\mathbb{R}P^3$ can be obtained from D^3 by identifying antipodal points of ∂D^3, the set of endpoints of the arcs must be symmetric over the origin. Fix an arbitrary great circle as the equator. Using ambient isotopy, we can move the arcs so that all of the endpoints lie on the equator in general position. Then, the arcs can be projected onto the disc bounded by the equator with over- and under-crossings, as described by Drobotukhina [4] and Manturov [7].

Projective space has a nontrivial first homology group, $H_1(\mathbb{R}P^3) \cong \mathbb{Z}/2\mathbb{Z}$. The generator for the group, g, is the cycle originating from the line in D^3 that runs between the north and south poles. Mroczkowski [9] has shown that every knot in $\mathbb{R}P^3$ can be transformed into either the trivial cycle or g by crossing changes and Reidemeister moves on an $\mathbb{R}P^2$ projection of the knot. This suggests that there exist two nonequivalent unknots in $\mathbb{R}P^3$. For the rest of the paper, we will refer to cycles

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that can be "unknotted" into a cycle homologous to g as 1*–homologous cycles* and cycles that can be "unknotted" into a null-homologous cycle as 0*–homologous cycles*.

In $\mathbb{R}^3$, a two component link $L_1 \cup L_2$ is the unlink if and only if L_1 and L_2 are both the unknot and there exist $A, B \subset \mathbb{R}^3$, both homeomorphic to B^3, such that $A \cap B = \varnothing$, $L_1 \subset A$ and $L_2 \subset B$. Because g cannot be contained within a sphere, using this definition in $\mathbb{R}P^3$ gives us a unique unlink consisting of two 0–homologous unknots. However, a 0–homologous unknot and a 1–homologous unknot in $\mathbb{R}P^3$ may be drawn in a projection onto $\mathbb{R}P^2$ with no crossings. On the other hand, two disjoint 1–homologous unknots will always cross. Consequently, two reasonable definitions for unlinks in $\mathbb{R}P^3$ exist.

Let M be a 3–manifold.

Definition 1 Let $L_1 \cup L_2$ be a two-component link in M. If L_1 and L_2 are both unknots and there exist $A, B \subset M$, both homeomorphic to B^3, such that $A \cap B = \varnothing$, $L_1 \subset A$ and $L_2 \subset B$, then L_1 and L_2 are *strongly unlinked*, and $L_1 \cup L_2$ is called the two-component unlink.

Definition 2 Let $L_1 \cup L_2$ be a two-component link in M. If L_1 and L_2 are both unknots and there exists $A \subset M$ homeomorphic to B^3 such that $L_1 \subset A$ and $L_2 \subset A^C$, then L_1 and L_2 are *unlinked*, and $L_1 \cup L_2$ is a *two-component unlink*.

A two-component unlink will also be referred to as the *trivial link*, and a two-component link is *nontrivial* if it is not the two-component unlink.

Notice that Definition 1 and Definition 2 are equivalent when $M \cong \mathbb{R}^3$. Similarly, we can define *strongly splittable* and *splittable* by removing the condition that both components are unknots.

Definition 3 Let G be a graph. If every embedding of G into M contains a pair of cycles that form a nontrivial two-component link, then G is *intrinsically linked in* M.

Graphs that are intrinsically linked in $\mathbb{R}^3$ have been completely classified through the work of Conway and Gordon [3], Sachs [15] and Roberston, Seymour and Thomas [14]. They have shown that a graph is intrinsically linked in $\mathbb{R}^3$ if and only if it contains one of the Petersen-family graphs (the 7 graphs obtained from K_6 by a sequence of $\triangle - Y$ and $Y - \triangle$ exchanges) as a minor.

Flapan, et al [5] classifies the set of all graphs that are intrinsically linked when using Definition 1. The complete minor-minimal set for intrinsic linking in any 3–manifold, M, is the same as in $\mathbb{R}^3$—namely, the Petersen-family graphs—when the two-component unlink is defined to be the union of cycles which bound discs that do not intersect. In $\mathbb{R}P^3$, their definition coincides with Definition 1.

However, K_6 embeds in the projective plane, as shown in Figure 1, so there exists an embedding of K_6 into projective space for which every two-component link is an unlink, as given by Definition 2. Thus, with this definition, K_6 is not intrinsically linked. For the remainder of this paper, unless otherwise noted, trivial and nontrivial links will be defined using Definition 2.

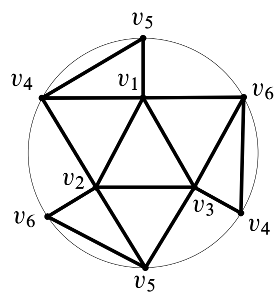


Figure 1: An embedding of K_6 into $\mathbb{R}P^2$. The bounding circle is identified using the antipodal map to obtain $\mathbb{R}P^3$.

In this paper, we will prove the following theorems.

Theorem 4 *Let $\mathcal{P}$ be the set of all Petersen-family graphs excluding the graph obtained from $K_{4,4}$ by removing an edge. Let A, B, G be graphs such that G has k–connectivity with vertex cut set $\{v_1, \dots, v_k\}$, $G = A \cup B$ and $V(A \cap B) = \{v_1, \dots, v_k\}$.*

(1) *If $k = 0$ or 1, then G is minor-minimal intrinsically linked in $\mathbb{R}P^3$ if and only if $A, B \in \mathcal{P}$.*

(2) *If $k = 2$, then G is minor-minimal intrinsically linked in $\mathbb{R}P^3$ if and only if $A', B' \in \mathcal{P}$, $E(A) = E(A') \setminus \{(v_1, v_2)\}$ and $E(B) = E(B') \setminus \{(v_1, v_2)\}$.*

The theorem classifies intrinsically linked graphs with low connectivity. The first statement says that a graph that is disconnected (or with 1–connectivity) is intrinsically linked if and only if it is the disjoint union (or union along a vertex) of two Petersen-family graphs. The second statement is analogous for graphs with 2–connectivity, but the edge between the two vertices along which the Petersen-family graphs are joined is removed.

Theorem 5 *The graph obtained by removing an edge from $K_{4,4}$ is minor-minimal intrinsically linked in $\mathbb{R}P^3$.*

Theorem 6 *The graphs obtained from K_7 by removing any two edges are minor-minimal intrinsically linked in $\mathbb{R}P^3$.*

2 Definitions and notation

Before proceeding to our results, we begin with some elementary notation and definitions.

Definition 7 A graph $G = (V, E)$ is a set of vertices $V(G)$ and edges $E(G)$, where an edge is an unordered pair (v_1, v_2) with $v_1, v_2 \in V$.

Definition 8 Let G be a graph and $v_1, v_2, \dots, v_n \in V(G)$ and

$$(v_1, v_2), (v_2, v_3), \dots, (v_{n-1}, v_n), (v_n, v_1) \in E(G)$$

such that $v_i \neq v_j$ for $i \neq j$. Then, the sequences of vertices $v_1, v_2, \dots, v_n$ and edges $(v_1, v_2), (v_2, v_3), \dots, (v_{n-1}, v_n), (v_n, v_1)$ is an n–cycle in the graph G, denoted $v_1 v_2 \dots v_n$.

In an abuse of notation, we will also refer to the image of a cycle $v_1 v_2 \dots v_n$ in an embedding of the graph G as the cycle $v_1 v_2 \dots v_n$, when the distinction is clear.

The following notion of a graph minor allows us to specify when one graph contains another graph within it.

Definition 9 Let G be a graph. Suppose H is a graph such that H can be obtained from G by a sequence of the following three operations:

(1) removal of an edge

(2) removal of a vertex

(3) contraction along an edge.

Then H is called a *minor* of G, written $H \leq G$. If $H \leq G$ but $H \neq G$, then H is called a *proper minor* of G, written $H < G$.

If $H \leq G$, we also call G an *expansion* of H.

Nešetřil and Thomas [10] provide the following result for graph minors in $\mathbb{R}^3$, and the general result in arbitrary 3–manifolds can be proved by noticing that expansions preserve isotopy classes of cycles and links.

Proposition 10 (J Nešetřil and R Thomas [10]) *Let H be a graph that is intrinsically linked in a 3–manifold M. If G is a graph such that $H \leq G$, then G is also intrinsically linked in M.*

Definition 11 A graph G is *minor-minimal intrinsically linked in M* if G is intrinsically linked in M and no proper minor of G is also intrinsically linked in M.

In $\mathbb{R}^3$, the set of all minor-minimally intrinsically linked graphs is given by the seven Petersen-family graphs. These graphs are obtained from K_6 by $\triangle - Y$ and $Y - \triangle$ exchanges, where a $\triangle - Y$ exchange is the removal of three edges (v_1, v_2), (v_1, v_3), (v_2, v_3) and the addition of a vertex v along with the edges (v, v_1), (v, v_2), (v, v_3). A $Y - \triangle$ exchange is the reverse operation.

As a result of Robertson and Seymour's proof of the Minor Theorem [13], the set of all minor-minimally intrinsically linked graphs in M is finite. This means that a full classification of minor-minimally intrinsically linked graphs in $\mathbb{R}P^3$ is possible. Because projective space has a simple first homology group, it may not be unrealistic to find a complete characterization for intrinsic linking.

3 Linked graphs with low connectivity

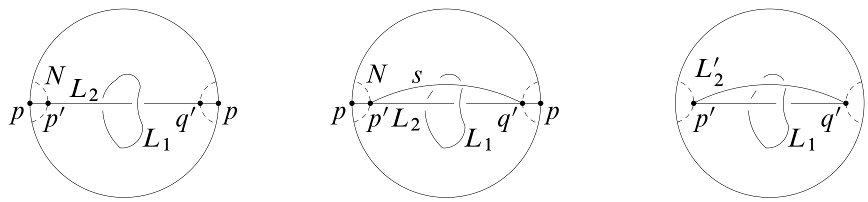
Exactly six of the seven Petersen-family graphs have embeddings into $\mathbb{R}P^2$, as shown by Glover, et al [6] and Archdeacon [1], and thus have linkless embeddings into $\mathbb{R}P^3$. We later show that the graph obtained by removing an edge from $K_{4,4}$, which does not have a projective planar embedding, is in fact intrinsically linked in $\mathbb{R}P^3$.

Although not all Petersen-family graphs are intrinsically linked in $\mathbb{R}P^3$, we can use their intrinsic linking in $\mathbb{R}^3$ to deduce some facts about embeddings with no nontrivial two-component links.

Lemma 12 *Let P be a Petersen-family graph and v be a vertex of P. If every cycle of $P \backslash \{v\}$ is 0–homologous in an embedding $f\colon P \to \mathbb{R}P^3$, then $f(P)$ contains a nontrivial link.*

Proof Let $L_1 \cup L_2$ be a link with a projection onto a disc representing $\mathbb{R}P^2$ such that L_1 is affine and does not cross the boundary of the projection and L_2 is 1–homologous.

Take a point p of L_2 that intersects the boundary of the projection (the line at infinity). Let U be a sufficiently small neighborhood of p in the projection such that L_1 does not intersect U and L_2 intersects ∂U in exactly two points, p' and q'. Connect p' and q' with a line segment s such that in the projection, s crosses over every strand, and s does not intersect the line at infinity. Define L_2' as the cycle consisting of s and the segment of L_2 that is not in U. Then, L_2' is a 0–homologous cycle such that the linking number of $L_1 \cup L_2$ is the same as the linking number of $L_1 \cup L_2'$ (see Figure 2).



(a) p' and q' in a small neighborhood of p (b) s crosses over all other arcs (c) L_2' constructed from L_2

Figure 2: Conversion of a link consisting of an affine knot and 1–homologous knot into one consisting of two 0–homologous knots

Consider $f(P)$. Using crossing changes and ambient isotopy, we may assume that the embedding for the subgraph $P\backslash\{v\}$ is affine so that $f(P\backslash\{v\})$ does not intersect the boundary of the projection (in other words, it does not pass through the line at infinity), v lies on the boundary of the projection, and no point besides v lies on the line at infinity.

Define
$$\lambda \equiv \sum_{\substack{L_1 \cup L_2 \text{ is a} \\ \text{two-component} \\ \text{link in } f(P)}} \mathrm{lk}(L_1, L_2) \pmod 2,$$

where $\mathrm{lk}(L_1, L_2)$ is the linking number of $L_1 \cup L_2$. The previous observation shows that there exists an affine embedding of P for which λ is unchanged. Because crossing changes do not affect λ, the results of Conway and Gordon [3] and Sachs [15] for K_6 and Petersen graphs in $\mathbb{R}^3$, respectively, imply that $\lambda \equiv 1 \pmod 2$ for the embedding f into $\mathbb{R}P^3$. Hence, the embedding must contain a two-component link with nonzero linking number, proving the lemma. □

Lemma 12 allows us to completely classify intrinsically linked graphs in $\mathbb{R}P^3$ with connectivity 0, 1 and 2, assuming that $K_{4,4} \setminus \{e\}$ is intrinsically linked in $\mathbb{R}P^3$.

Proposition 13 *Let $G = A \cup B$ be a 2–connected graph with vertex cut set $V(A \cap B) = \{v_1, v_2\}$. Let $\bar{A} = A \cup \{(v_1, v_2)\}$ and $\bar{B} = B \cup \{(v_1, v_2)\}$. If G is minor-minimal intrinsically linked in $\mathbb{R}P^3$, then $\bar{A}$ and $\bar{B}$ are intrinsically linked in $\mathbb{R}^3$.*

Proof Suppose $\bar{A}$ is not intrinsically linked in $\mathbb{R}^3$. Since G is minor-minimal, $\bar{B} < G$ has a linkless embedding, f, in $\mathbb{R}P^3$. Let g be an embedding of a closed 3–ball with interior D into $\mathbb{R}P^3$ such that $f((v_1, v_2)) \subset g(\bar{D})$, only the vertices v_1 and v_2 intersect ∂D. and $f(\bar{B} \setminus \{(v_1, v_2)\})$ is in the complement of $g(D)$. Take a linkless embedding, h, of $\bar{A}$ in $\mathbb{R}^3 \cong D$. Then, $g \circ h$ is a linkless embedding of $\bar{A}$. Using ambient isotopy on $g \circ h$, we may assume that the arcs $f((v_1, v_2))$ and $g \circ h((v_1, v_2))$ coincide. The union of these two embeddings produces a linkless embedding of $G \cup (v_1, v_2)$ into $\mathbb{R}P^3$. □

Proposition 14 *Let $G = (P_1 \cup P_2) \setminus \{(v_1, v_2)\}$ be a graph, where $P_1, P_2 \in \mathcal{P}$ and $V(P_1 \cap P_2) = \{v_1, v_2\}$. Then G is intrinsically linked in $\mathbb{R}P^3$.*

Proof Notice that both P_1 and P_2 are minors of G. Embed G in $\mathbb{R}P^3$. By Lemma 12, if P_i does not contain any nontrivial links, then $P_i \setminus \{v_i\}$ must contain a 1–homologous cycle, for $i = 1, 2$. This results in two disjoint 1–homologous cycles. Hence, G is linked. □

The previous two propositions prove Theorem 4 for $k = 2$, assuming Theorem 5. The results for $k = 0$ and $k = 1$ are proved similarly, and Theorem 5 is proved in the following section.

For the case $k = 0$, it is easy to see that there are $\binom{6}{2} = 21$ minor-minimal intrinsically linked graphs in $\mathbb{R}P^3$. When $k = 1$, it is necessary to count the different number of vertex classes in each graph to determine the number of ways a pair of Petersen-family graphs may be glued along a vertex. From Table 1, the number of minor-minimal intrinsically linked graphs with 1–connectivity in $\mathbb{R}P^3$ is determined to be 91.

Define the *vertex flipping number* (VFN) for some vertex pair $\{x_1, x_2\}$ as

$$\mathrm{VFN}(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 \ x_2, \\ 1 & \text{otherwise}, \end{cases}$$

where $x_1 \ x_2$ is equivalence under a graph isomorphism. Counting the number of minor-minimal intrinsically linked graphs in Theorem 4 when $k = 2$ requires attention to the VFN of vertex pair classes, where two pairs of vertices are equivalent if there is a graph isomorphism taking one pair to the other. For each pair $\{x_1, x_2\} \subseteq E(G_1), \{y_1, y_2\} \subseteq E(G_2)$ of vertex pair classes for two graphs G_1, G_2, the number of ways to glue G_1

Graph	Vertex Classes
K_6	1
$K_{3,3,1}$	2
P_7	3
P_8	4
P_9	2
Petersen	1

Table 1: Petersen-family graphs and the number of vertices, up to equivalence under graph isomorphism

and G_2 along the specified vertex pairs is $\mathrm{VFN}(x_1, x_2)\,\mathrm{VFN}(y_1, y_2) + 1$. Table 2 lists the number of vertex pair classes of each type, and the number of minor-minimal intrinsically linked graphs in $\mathbb{R}P^3$ is 469.

	Vertex Pair Classes		
Graph	Total	VFN $= 0$	VFN $= 1$
K_6	1	1	0
$K_{3,3,1}$	3	2	1
P_7	5	2	3
P_8	10	3	7
P_9	6	4	2
Petersen	2	2	0

Table 2: Petersen-family graphs and the number of vertex pairs, up to equivalence under graph isomorphism

4 $K_{4,4}$ with an edge removed

In this section, we prove that the graph obtained by removing an edge from $K_{4,4}$ is intrinsically linked in $\mathbb{R}P^3$.

We will need the following observation.

Proposition 15 *For every embedding into $\mathbb{R}P^3$, $K_{3,2}$ has an even number of 1–homologous 4–cycles.*

Proof Whenever two cycles C_1 and C_2 intersect along an arc, D, we can define the sum of C_1 and C_2 to be $C_1 \cup C_2 \setminus D$. Then, the result can be obtained by noting that the sum of two 0–homologous cycles and the sum of two 1–homologous cycles are 0–homologous cycles, and the sum of a 0–homologous cycle with a 1–homologous cycle is 1–homologous. □

The combinatorial observation in Proposition 15 is used to prove the following characterization of homology classes of cycles in $K_{3,3}$, similar to one given by O'Donnol [11] for embeddings of $K_{3,3}$ into $\mathbb{R}^3$ and a simple closed curve in its complement.

Lemma 16 *If a graph G isomorphic to $K_{3,3}$ is embedded in $\mathbb{R}P^3$ such that at least one of its cycles is 1–homologous, then the homology classes of all of the 4–cycles in the embedding of G have one of two possibilities:*

(1) *A cycle is 1–homologous if and only if it passes through a specified edge, (u, v), of the graph. We call (u, v) the* including edge *and the homology pattern of the embedding a 4–*pattern*.*

(2) *A cycle is 1–homologous if and only if it does not pass through two of the edges in $F \subset E(G)$, where F is a specified set of three mutually disjoint edges of G. We call F the set of* excluding edges *and the homology pattern of the embedding a 6–*pattern*.*

Proof Let $\{a_1, a_2, a_3\} \subset V(G)$ and $\{b_1, b_2, b_3\} \subset V(G)$ be the partition sets of G. Suppose G contains a 1–homologous cycle. Then, it must contain a 1–homologous 4–cycle C_1. Let H be a subgraph of G isomorphic to $K_{3,2}$ that contains C_1. By Proposition 15, H must contain two 1–homologous 4–cycles. Without loss of generality, they are the cycles $a_1b_1a_2b_2$ and $a_1b_1a_2b_3$. It also must be the case that the cycle $a_1b_2a_2b_3$ is 0–homologous.

Now, consider the subgraph induced by $\{a_1, a_2, a_3, b_1, b_2\}$. By Proposition 15, one of the two cycles $a_1b_1a_3b_2$ and $a_2b_1a_3b_2$ is 1–homologous, and the other is 0–homologous. Since interchanging a_1 and a_2 does not affect the choices made up to this point, without loss of generality, the cycle $a_1b_1a_3b_2$ is 1–homologous and the cycle $a_2b_1a_3b_2$ is 0–homologous.

Next, consider the subgraph induced by $\{a_1, a_3, b_1, b_2, b_3\}$. Since the cycle $a_1b_1a_3b_2$ is 1–homologous, then either the cycle $a_1b_1a_3b_3$ is also 1–homologous and the cycle $a_1b_2a_3b_3$ is 0–homologous, or the cycle $a_1b_1a_3b_3$ is 0–homologous and the cycle $a_1b_2a_3b_3$ is 1–homologous.

Case 1 Cycle $a_1b_1a_3b_3$ is 1–homologous and cycle $a_1b_2a_3b_3$ is 0–homologous.

Applying Proposition 15 to all of the other subgraphs of G isomorphic to $K_{3,2}$ forces the last two cycles, $a_2b_1a_3b_3$ and $a_2b_2a_3b_3$, to be 0–homologous. Observe that a cycle in G is 1–homologous if and only if it includes the edge (a_1, b_1). Hence, this embedding of G has a 4–pattern, with (a_1, b_1) as its including edge.

Case 2 The cycle $a_1b_1a_3b_3$ is 0–homologous and the cycle $a_1b_2a_3b_3$ is 1–homologous.

Again, by using Proposition 15 on the remaining $K_{3,2}$ subgraphs of G, the cycles $a_2b_1a_3b_3$ and $a_2b_2a_3b_3$ must be 1–homologous. A 4–cycle of G is 0–homologous if and only if it contains two edges from the set $F = \{(a_1, b_3), (a_2, b_2), (a_3, b_1)\}$. The set F is the set of excluding edges, and the embedding is a 6–pattern. □

Theorem 5 *The graph G obtained by removing an edge from $K_{4,4}$ is minor-minimal intrinsically linked in $\mathbb{R}P^3$.*

Proof Consider an embedding of $G = K_{4,4} \setminus \{(a_1, b_1)\}$, where

$$\{a_1, a_2, a_3, a_4\},\ \{b_1, b_2, b_3, b_4\} \subset V(G)$$

are the partition sets.

Let A be the subgraph induced by $\{a_2, a_3, a_4, b_2, b_3, b_4\}$, B be the subgraph induced by $\{a_1, a_2, a_3, b_2, b_3, b_4\}$ and C be the subgraph induced by $\{a_2, a_3, a_4, b_1, b_2, b_3\}$. By Lemma 16, A contains no 1–homologous cycles, is a 4–pattern or is a 6–pattern.

Case 1 The subgraph A contains no 1–homologous cycles.

By Lemma 12, if the embedding is linkless, the subgraph induced by $\{a_1, a_2, a_3, a_4, b_2, b_3, b_4\}$ must contain a 1–homologous cycle. Because A does not contain any 1–homologous cycles, all such cycles must pass through a_1. Consider the subgraph induced by $\{a_1, a_2, a_3, b_2, b_3, b_4\}$. This $K_{3,3}$ subgraph must then have a 4–pattern. Without loss of generality, the including edge is (a_1, b_2).

Similarly, the subgraph induced by $\{a_2, a_3, a_4, b_1, b_2, b_3\}$ contains a 4–pattern with including edge (a_2, b_1). Then, $a_1b_2a_2b_3$ and $b_1a_2b_2a_3$ are disjoint 1–homologous cycles.

Case 2 The subgraph A contains a 4–pattern.

Without loss of generality, A has (a_4, b_4) as its including edge.

Subcase 2.1 Either B or C has a 6–pattern.

The subgraph B cannot have a 6–pattern as then subgraph induced by $\{a_2, a_3, b_2, b_3, b_4\}$ would contain a 1–homologous cycle, contradicting that all 1–homologous cycles in A pass through its including edge.

Subcase 2.2 Both B and C contain no 1–homologous cycles.

It is easy to see that all 1–homologous cycles of G must pass through the edge (a_4, b_4) by looking at the other four $K_{3,3}$ subgraphs of G and noticing that each subgraph must have a 1–homologous cycle by the including edge in A. If any subgraph of G (not including B and C) has a 6–pattern or a 4–pattern with an including edge that is not (a_4, b_4), then this would force a 1–homologous cycle in B or C. By Lemma 12, since all 1–homologous cycle pass through a_4, G is linked.

Subcase 2.3 Both B and C have 4–patterns.

If B contains a 4–pattern, then its including edge must pass through a_1. Otherwise, A contains a 1–homologous cycle disjoint from its including edge. Similarly, if C contains a 4–pattern, then its including edge must pass through b_1. The subgraph B has its including edge passing through a_1 and C has its including edge passing through b_1. So we can find disjoint 1–homologous cycles in G.

Subcase 2.4 One of B or C has a 4–pattern and the other contains no 1–homologous cycles.

Without loss of generality, assume that B has a 4–pattern and C contains no 1–homologous cycles. By the previous subcase, the including edge in B has a_1 as an endpoint. We claim that the subgraphs induced by $\{a_2, a_3, a_4, b_1, b_2, b_4\}$ and $\{a_2, a_3, a_4, b_1, b_3, b_4\}$ must have 4–patterns: both contain 1–homologous cycles due to A having a 4–pattern, and if either contained a 6–pattern, there would be a 1–homologous cycle in C. Any edge with b_1 as an endpoint cannot be an including edge for these two graphs, since then C would contain a 1–homologous cycle. Consequently, both subgraphs must have (a_4, b_4) as its including edge. Otherwise, there would be a 1–homologous 4–cycle in A that does not have (a_4, b_4) as one of its edges.

If the including edge in B does not have b_4 as its other endpoint, because the subgraph induced by $\{a_2, a_3, a_4, b_1, b_2, b_4\}$ has (a_4, b_4) as its including edge, G contains disjoint 1–homologous links. Otherwise, since the cycles $a_i b_2 a_4 b_4$ and $a_i b_3 a_4 b_4$ are 1–homologous from A and cycles $a_1 b_2 a_i b_4$ and $a_1 b_3 a_i b_4$ are 1–homologous from B, then the subgraph induced by $\{a_1, a_i, a_4, b_2, b_3, b_4\}$ has a 4–pattern with (a_i, b_4) as its including edge, for $i = 2, 3$. In this case, we have shown that all 1–homologous cycles pass through b_4, so by Lemma 12, G is linked.

Case 3 The subgraph A has a 6–pattern.

Without loss of generality, the excluding edges in A are (a_i, b_i) for $i = 2, 3, 4$. Then, every $K_{3,3}$ subgraph of G shares a $K_{3,2}$ with A, so it must contain a 1–homologous cycle.

Subcase 3.1 Both B and C contain 4–patterns.

If B contains a 4–pattern, its including edge must pass through b_4. Otherwise, B contains a 1–homologous cycle from the 6–pattern in A that does not pass through its including edge. Since the subgraph induced by $\{a_2, a_3, b_2, b_3, b_4\}$ contains a 1–homologous cycle by A, then B has it including edge passing through a_2 or a_3. Let (a_i, b_4) be the including edge in B.

Likewise, if C has a 4–pattern, its including edge must be (a_4, b_j), where $j = 2$ or 3. Then, it is easy to see that G contains disjoint 1–homologous cycles. If C has a 6–pattern, then let $k = 2, 3$, $k \neq i$. Then, the subgraph induced by $\{a_k, a_4, b_1, b_2, b_3\}$ contains a 1–homologous 4–cycle, one of which must pass through b_1. The 4–cycle that is disjoint from this cycle is also 1–homologous by the including edge in B, so G is linked.

Subcase 3.2 Either B or C contain a 6–pattern.

Without loss of generality, assume that B has a 6–pattern. One of its excluding edges must be (a_1, b_4) since cycle $a_2b_2a_3b_3$ is 0–homologous by A, and (a_1, b_4) is the only edge in B that is disjoint from this cycle. Note that if (a_2, b_2) and (a_3, b_3) are also excluding edges, then all cycles in the subgraph induced by $\{a_1, a_2, a_4, b_2, b_3, b_4\}$ through (a_2, b_3) are 1–homologous. We saw in the Subcase 2.1 that when there is a 4–pattern in a $K_{3,3}$ that is one adjacent (differs by one vertex) to a $K_{3,3}$ with a 6–pattern, then the graph is linked. Otherwise, (a_2, b_3) and (a_3, b_2) are the other excluding edges in B.

Similarly, if G does not contain any nontrivial links, then C must have (a_4, b_1), (a_2, b_3) and (a_3, b_2) as excluding edges. Hence, $a_1b_4a_2b_2$ and $a_4b_1a_3b_3$ are disjoint 1–homologous cycles. So G is linked.

The graph G is minor-minimal since any proper minor of G embeds in the projective plane, as shown by Glover, et al [6] and Archdeacon [1]. □

5 K_7 minus two edges

We now prove that any graph obtained by removing two edges from K_7 is minor-minimal intrinsically linked in $\mathbb{R}P^3$. There are two cases of Theorem 6: when the two edges are adjacent and when the two edges are nonadjacent. We will use the following lemma.

Lemma 17 *Given a linkless embedding of K_6, no K_4 subgraph can have all 0–homologous cycles.*

Proof Consider an embedding of K_6 for which there is a K_4 subgraph with all cycles 0–homologous. By using crossing changes and ambient isotopy, this K_4 subgraph can be deformed so that it does not touch the line at infinity, and so that there are no crossings on it in a projection. Denote the vertices of this K_4 by $\{v_1, v_2, v_3, v_4\}$ and denote the vertices not in the K_4 by v_5 and v_6. One can deform the edge (v_5, v_6) so that it is contained in the line at infinity, so that v_6 is placed at 12 o'clock and 6 o'clock, and so that v_5 is placed at 3 o'clock and 9 o'clock. We may assume the edge (v_5, v_6) goes from 12 o'clock to 3 o'clock (see Figure 3).

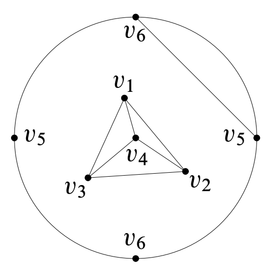


Figure 3: We may deform the embedded graph to be in this position (not all edges are shown).

Now, we claim that the edges connecting v_6 to the K_4 can be deformed (using crossing changes and ambient isotopy) so that they are straight lines in the projection that connect to the K_4 either from 12 o'clock or from 6 o'clock. We may assume that the edge connecting to v_4 is under all of the other edges of the K_4 in the projection. We will justify the claim for the edge (v_6, v_1). Consider the embedded cycle formed the two (additional) edges e_1 and e_2, where e_1 connects v_1 to the 12 o'clock v_6, and e_2 connects v_1 to the 6 o'clock v_6, where both e_1 and e_2 are straight edges in the projection. This cycle is 1–homologous. The edge (v_1, v_6) from the K_6 embedding breaks up this cycle into two cycles, one formed by e_1 and (v_1, v_6) and the other formed by e_2 and (v_1, v_6). One of these two cycles must be 1–homologous, and the other must be 0–homologous. If the cycle formed by e_1 and (v_1, v_6) is 0–homologous, then (v_1, v_6) can be deformed, using crossing change and ambient isotopy, to e_1. Similarly, if the cycle formed by e_2 and (v_1, v_6) is 0–homologous, then (v_1, v_6) can be deformed to e_2. This established our claim. It similarly follows that the edges connecting v_5 to the K_4 can be deformed (using crossing changes and ambient isotopy) so that they are straight lines in the projection that connect to the K_4 from either 3 o'clock or 9 o'clock.

Now, it cannot be the case that all of the edges connecting v_6 to the K_4 are incident to 12 o'clock, for then v_1, v_2, v_3, v_4 and v_6 would induce a K_5 with all cycles 0–homologous, which cannot occur in a linkless embedding of K_6 by Lemma 12. Similarly, all of the edges cannot be incident 6 o'clock, nor can all of the edges emanating from K_5 be incident to 3 o'clock, nor can they all be incident to 9 o'clock. Thus, there must be exactly 1, 2 or 3 edges from the K_4 incident to 12 o'clock, and exactly 1, 2 or 3 edges from the K_4 incident to 3 o'clock. In all cases but one, there are a pair of disjoint 1–homologous cycles. These disjoint 1–homologous cycles would have been present in the original embedding of K_6. For example, if only (v_1, v_6) is incident to 12 o'clock, and only (v_2, v_5) is incident to 3 o'clock, then (v_1, v_6, v_3) and (v_2, v_5, v_4) form disjoint 1–homologous cycles.

The only case that does not lead to disjoint 1–homologous cycles is the case when exactly 1 edge from K_4 is incident to 12 o'clock (6 o'clock) and exactly one edge from K_4 is incident to 3 o'clock (9 o'clock), and these two edges are incident to the same vertex in the K_4. By symmetry, we may assume the edge (v_1, v_6) is incident to 12 o'clock, and the edge (v_1, v_5) is incident to 3 o'clock. Then, all 1–homologous cycles pass through v_1, so by Lemma 12, this embedding is linked. □

An easy consequence of Lemma 17 is as follows:

Corollary 18 *The graph on 9 vertices obtained by pasting together two copies of $K_{3,1,1,1}$ along the three vertices that are mutually nonadjacent is intrinsically linked in $\mathbb{R}P^3$.*

Theorem 19 *The graph obtained from K_7 by removing two edges incident to a common vertex is minor-minimal intrinsically linked in $\mathbb{R}P^3$.*

Proof Let G be the graph obtained from K_7 by removing two edges incident to a common vertex. Let $v_1, v_2, .., v_7$ denote the vertices of G, with v_7 connected only to v_1, v_2, v_3 and v_4. Embed G. By the previous result, if the embedding is linkless, the K_4 induced on $\{v_1, v_2, v_3, v_4\}$ must contain a 1–homologous 3–cycle. By a homology argument, then there must be a 1–homologous 3–cycle through the vertex v_7. Without loss of generality, we may assume the 3–cycle is (v_1, v_2, v_7). If the embedding is linkless, then by the previous result, the K_4 induced by $\{v_3, v_4, v_5, v_6\}$ must contain a 1–homologous cycle, but this forces two disjoint 1–homologous cycles. Thus, the embedding cannot be linkless.

The graph G is minor-minimal since any proper minor of G embeds in the projective plane by Glover, et al [6] and Archdeacon [1]. □

Theorem 20 *The graph obtained from K_7 by removing two nonadjacent edges is minor-minimal intrinsically linked in $\mathbb{R}P^3$.*

Proof Let K be the graph obtained from K_7 by removing two nonadjacent edges.

Label the vertices of K_7 as $\{v_1, v_2, ..., v_7\}$, and suppose edges (v_4, v_5) and (v_6, v_7) are removed to result in the graph K. Embed K, and suppose the embedding is linkless. We claim that the 4–cycle (v_4, v_7, v_5, v_6) cannot be 1–homologous. If it were, then a cycle of the form (v_1, v_i, v_j) is 1–homologous, for $i, j \in \{4, 5, 6, 7\}$, with $i \neq j$. Without loss of generality, suppose (v_1, v_5, v_7) is 1–homologous; then the subgraph induced by $\{v_2, v_3, v_4, v_6\}$ forms K_4, and since K contracts onto K_6, by Lemma 17, there must be a disjoint 1–homologous cycle, which is a contradiction. Similarly, the following 3–cycles must also be 0–homologous: (v_i, v_4, v_7), (v_j, v_5, v_7), (v_k, v_5, v_6) and (v_m, v_4, v_6), where $i, j, k, m \in \{1, 2, 3\}$. It follows that for the subgraph induced by the vertices $\{v_1, v_4, v_7, v_5, v_6\}$, every cycle is 0–homologous. Since this subgraph contracts onto K_4, and since K contracts onto K_6 (by contracting the same edge), it follows from Lemma 17 that there must be nonsplittable links in the embedding. It follows that K is intrinsically linked in $\mathbb{R}P^3$.

The graph K is minor-minimal for intrinsic linking since any proper minor of K is projective planar, as shown by Glover, et al [6] and Archdeacon [1] or does not contain any intrinsically $\mathbb{R}^3$–linked graphs as a minor, so there exists a linkless embedding of every proper minor into a 3–ball.

The graph K is minor-minimal for intrinsic linking in $\mathbb{R}P^3$ since any proper minor of K is either projective planar [1; 6] or becomes ($\mathbb{R}^2$) planar after the removal of a vertex (and hence is not intrinsically linked in space). In either case no minor is intrinsically linked in $\mathbb{R}P^3$. □

6 Other intrinsically linked graphs

It is not too hard to see that $\triangle - Y$ exchanges preserve intrinsic linking as in $\mathbb{R}^3$ (see Motwani, Raghunathan and Saran [8]), so any graph generated from a known intrinsically $\mathbb{R}P^3$–linked graph by a sequence of $\triangle - Y$ exchanges is also intrinsically linked. Corollary 18 provides a graph with several $\triangle$ subgraphs.

Notice that two copies of $K_{3,1,1,1}$ glued along the three mutually nonadjacent vertices is the same as gluing two copies of K_6 along three vertices v_1, v_2, v_3 and then removing the triangle composed of the three edges between v_1, v_2, and v_3. For notational convenience, each copy of K_6 with the triangle removed will be referred to as $K_6 \therefore$, and $K_6 \therefore K_6$ denotes the gluing of two copies of $K_6 \therefore$ along the three vertices that are mutually nonadjacent. By Corollary 18, $K_6 \therefore K_6$ is intrinsically linked in $\mathbb{R}P^3$.

In general, we define $G \therefore$ to be a graph with three marked vertices which are mutually nonadjacent. If $G_1 \therefore$ and $G_2 \therefore$ are two such graphs, then $G_1 \therefore G_2$ is a graph obtained by gluing the two graphs along the three marked vertices of $G_1 \therefore$ and $G_2 \therefore$. The resulting graph may not be unique if permutation of the marked vertices does not yield a graph isomorphism of each $G_i \therefore$, $i = 1, 2$. In such cases, we will differentiate between the (up to) three distinct graphs by a subscript, as in $G_1 \therefore_1 G_2$, $G_1 \therefore_2 G_2$ and $G_1 \therefore_3 G_2$.

Proposition 21 *There are 18 intrinsically linked graphs in $\mathbb{R}P^3$ with 3–connectivity that can be obtained from $K_6 \therefore K_6$ by $\triangle - Y$ exchanges.*

Proof Figure 4 shows $K_6 \therefore$ with the marked vertices (d, e, and f) shown as open circles, and the edges which were removed from K_6 shown as dotted lines. All other edges are not shown. In the subsequent figures, only edges added by $\triangle - Y$ exchange are shown.

The graphs obtained by repeated $\triangle - Y$ exchanges on $K_6 \therefore$ are shown in Figure 4. Consequently, there are 18 intrinsically linked graphs in $\mathbb{R}P^3$ obtained from $K_6 \therefore K_6$. The graphs $P_{7B} \therefore P_{7B}$, $P_{7B} \therefore P_{8B}$ and $P_{8B} \therefore P_{8B}$ each have two different configurations. The configuration where the Y subgraphs of the two copies of $P_{7B} \therefore$ are glued together along a shared vertex is $P_{7B} \therefore_1 P_{7B}$, and the configuration where they are not is $P_{7B} \therefore_2 P_{7B}$. Similarly, if the Y in $P_{7B} \therefore$ shares a vertex with a Y in $P_{8B} \therefore$, we have $P_{7B} \therefore_1 P_{8B}$, and otherwise, we have $P_{7B} \therefore_2 P_{8B}$. If the Y subgraphs are paired up in $P_{8B} \therefore P_{8B}$, we have $P_{8B} \therefore_1 P_{8B}$, and otherwise, we have $P_{8B} \therefore_2 P_{8B}$. □

Remark 22 The graphs $P_{7A} \therefore K_6$, $P_{7A} \therefore P_{7A}$, $P_{7A} \therefore P_{7B}$, $P_{7A} \therefore P_{8B}$ and $P_{7A} \therefore P_{9B}$ are intrinsically linked in $\mathbb{R}P^3$, but all contain $K_{4,4}$ with an edge removed as a minor. Hence, they are not minor-minimal intrinsically linked in $\mathbb{R}P^3$.

To show that the remaining 13 intrinsically linked graphs in $\mathbb{R}P^3$ are minor-minimal, we will use the following result from Brouwer, et al [2] and Ozawa and Tsutsumi [12]:

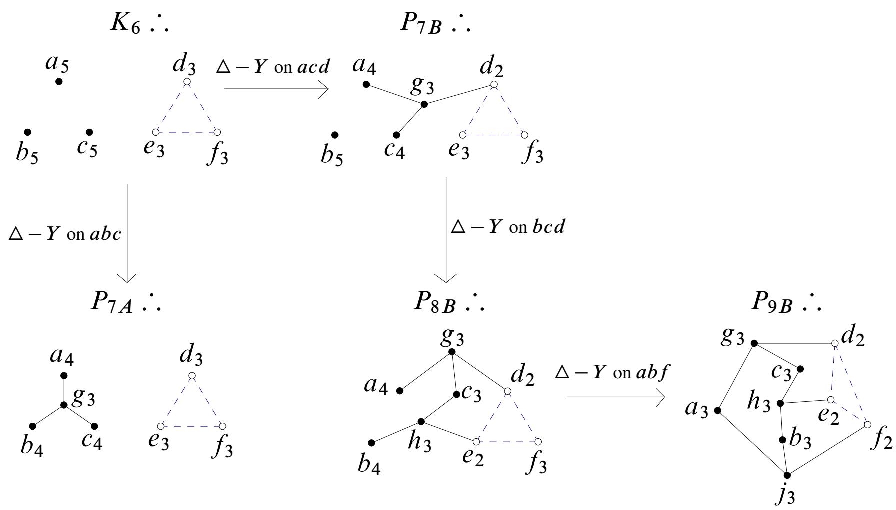


Figure 4: Graphs obtained from performing $\triangle - Y$ exchanges on $K_6 \therefore$ and each subsequent graph, with subscripts denoting degree of the vertex. The open circles represent the three marked vertices and dashed edges represent edges removed from the original K_6.

Theorem 23 *Let P be a property preserved under $\triangle - Y$ exchange. Let G be a graph that contains at least one degree three vertex and is minor-minimal with respect to P. Let G' be a graph obtained from G by a $Y - \triangle$ exchange. If G' has property P, then G' is also minor-minimal with respect to P.*

Thus, we need only show that no proper minor of $P_{9B} \therefore P_{9B}$ is intrinsically linked in $\mathbb{R}P^3$.

Theorem 24 *The 13 graphs obtained from $\triangle - Y$ exchange on $K_6 \therefore K_6$ which do not contain $P_{7A} \therefore$ as a subgraph are minor-minimal intrinsically linked in $\mathbb{R}P^3$.*

Proof We will show that no proper minor of $P_{9B} \therefore P_{9B}$ is intrinsically linked in $\mathbb{R}P^3$.

Consider $P_{9B} \therefore P_{9B}$ as drawn in Figure 5. The only pair of linked cycles in this embedding is $c_1 h_1 e a_1 g_1$ and $f i_1 b_1 d b_2 i_2$. There are two vertex equivalence classes in $P_{9B} \therefore P_{9B}$: $\{d, e, f\}$ and $V(P_{9B} \therefore P_{9B}) \setminus \{d, e, f\}$.

To check for minor-minimality, it suffices to show that removing or contracting any edge in $P_{9B} \therefore P_{9B}$ results in graph that is not intrinsically linked in $\mathbb{R}P^3$.

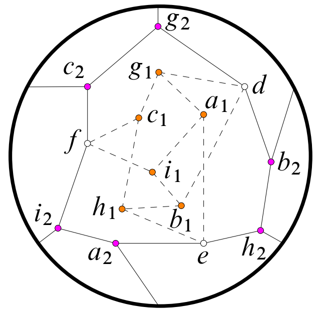


Figure 5: $P_{9B} \therefore P_{9B}$ connected along the marked vertices d, e, f. One copy of $P_{9B} \therefore$ is denoted by dashed edges.

There are two edge classes (up to graph isomorphism) that need to be considered. Removing edge (a_1, e) or edge (c_1, h_1) from the embedding in Figure 5 results in a linkless embedding. Contracting edge (f, c_1) or edge (a_1, i_1) in Figure 5 results in a linkless embedding since the edge contractions send vertices on the (only) two linked cycles to the same point, thus eliminating the nontrivial link. □

7 Remarks

Using the weaker definition of unlinked components in Definition 2 allows the use of 1–homologous cycles to reduce the number of crossings in a graph embedding in projective space. Thus, intrinsically linked graphs in $\mathbb{R}P^3$ are more complex. Unlike in $\mathbb{R}^3$, where there are simple arguments showing that there are no minor-minimal intrinsically linked graphs with connectivity 0, 1 or 2, such graphs exist in projective space. Using careful combinatorics, one can show that there are 21 disconnected graphs, 91 graphs with 1–connectivity and 469 graphs with 2–connectivity which are minor-minimal intrinsically linked in $\mathbb{R}P^3$. It is not too hard to see that $\triangle - Y$ exchanges preserve intrinsic linking as in $\mathbb{R}^3$, so we predict that there are many more minor-minimal intrinsically linked graphs than the ones we have observed in this paper. In particular, the graphs obtained by removing two edges from K_7 have a myriad of triangles in which we can perform a $\triangle - Y$ exchange, leading to more intrinsically linked graphs. Some of these graphs have $K_{4,4}$ with an edge removed as a minor, but others have yet to be explored fully. It would be of interest to see which of these are in fact minor-minimal intrinsically linked in $\mathbb{R}P^3$.

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