Intrinsically linked graphs in projective space

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We examine graphs that contain a nontrivial link in every embedding into real projective space, using a weaker notion of unlink than was used in Flapan, et al [5]. We call such graphs intrinsically linked in $\mathbb{R}P^3$. We fully characterize such graphs with connectivity 0, 1 and 2. We also show that only one Petersen-family graph is intrinsically linked in $\mathbb{R}P^3$ and prove that $K_7$ minus any two edges is also minor-minimal intrinsically linked. In all, 597 graphs are shown to be minor-minimal intrinsically linked in $\mathbb{R}P^3$.

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1 Introduction

We can represent knots in $\mathbb{R}P^3$ as closed curves or unions of arcs in the closed 3–ball, $D^3$, such that the endpoints of the arcs lie on $\partial D^3$. Because $\mathbb{R}P^3$ can be obtained from $D^3$ by identifying antipodal points of $\partial D^3$, the set of endpoints of the arcs must be symmetric over the origin. Fix an arbitrary great circle as the equator. Using ambient isotopy, we can move the arcs so that all of the endpoints lie on the equator in general position. Then, the arcs can be projected onto the disc bounded by the equator with over- and under-crossings, as described by Drobotukhina [4] and Manturov [7].

Projective space has a nontrivial first homology group, $H_1(\mathbb{R}P^3) \cong \mathbb{Z}/2\mathbb{Z}$. The generator for the group, $g$, is the cycle originating from the line in $D^3$ that runs between the north and south poles. Mroczkowski [9] has shown that every knot in $\mathbb{R}P^3$ can be transformed into either the trivial cycle or $g$ by crossing changes and Reidemeister moves on an $\mathbb{R}P^2$ projection of the knot. This suggests that there exist two nonequivalent unknots in $\mathbb{R}P^3$. For the rest of the paper, we will refer to cycles...
that can be “unknotted” into a cycle homologous to $g$ as 1–homologous cycles and cycles that can be “unknotted” into a null-homologous cycle as 0–homologous cycles.

In $\mathbb{R}^3$, a two component link $L_1 \cup L_2$ is the unlink if and only if $L_1$ and $L_2$ are both the unknot and there exist $A, B \subset \mathbb{R}^3$, both homeomorphic to $B^3$, such that $A \cap B = \emptyset$, $L_1 \subset A$ and $L_2 \subset B$. Because $g$ cannot be contained within a sphere, using this definition in $\mathbb{R}P^3$ gives us a unique unlink consisting of two 0–homologous unknots. However, a 0–homologous unknot and a 1–homologous unknot in $\mathbb{R}P^3$ may be drawn in a projection onto $\mathbb{R}P^2$ with no crossings. On the other hand, two disjoint 1–homologous unknots will always cross. Consequently, two reasonable definitions for unlinks in $\mathbb{R}P^3$ exist.

Let $M$ be a 3–manifold.

**Definition 1** Let $L_1 \cup L_2$ be a two-component link in $M$. If $L_1$ and $L_2$ are both unknots and there exist $A, B \subset M$, both homeomorphic to $B^3$, such that $A \cap B = \emptyset$, $L_1 \subset A$ and $L_2 \subset B$, then $L_1$ and $L_2$ are strongly unlinked, and $L_1 \cup L_2$ is called the two-component unlink.

**Definition 2** Let $L_1 \cup L_2$ be a two-component link in $M$. If $L_1$ and $L_2$ are both unknots and there exists $A \subset M$ homeomorphic to $B^3$ such that $L_1 \subset A$ and $L_2 \subset A^C$, then $L_1$ and $L_2$ are unlinked, and $L_1 \cup L_2$ is a two-component unlink.

A two-component unlink will also be referred to as the trivial link, and a two-component link is nontrivial if it is not the two-component unlink.

Notice that Definition 1 and Definition 2 are equivalent when $M \cong \mathbb{R}^3$. Similarly, we can define strongly splittable and splittable by removing the condition that both components are unknots.

**Definition 3** Let $G$ be a graph. If every embedding of $G$ into $M$ contains a pair of cycles that form a nontrivial two-component link, then $G$ is intrinsically linked in $M$.

Graphs that are intrinsically linked in $\mathbb{R}^3$ have been completely classified through the work of Conway and Gordon [3], Sachs [15] and Roberston, Seymour and Thomas [14]. They have shown that a graph is intrinsically linked in $\mathbb{R}^3$ if and only if it contains one of the Petersen-family graphs (the 7 graphs obtained from $K_6$ by a sequence of $\Delta - Y$ and $Y - \Delta$ exchanges) as a minor.

Flapan, et al [5] classifies the set of all graphs that are intrinsically linked when using Definition 1. The complete minor-minimal set for intrinsic linking in any 3–manifold, $M$, is the same as in $\mathbb{R}^3$—namely, the Petersen-family graphs—when the
In intrinsically linked graphs in projective space, a two-component unlink is defined to be the union of cycles which bound discs that do not intersect. In \( \mathbb{R}P^3 \), their definition coincides with Definition 1.

However, \( K_6 \) embeds in the projective plane, as shown in Figure 1, so there exists an embedding of \( K_6 \) into projective space for which every two-component link is an unlink, as given by Definition 2. Thus, with this definition, \( K_6 \) is not intrinsically linked. For the remainder of this paper, unless otherwise noted, trivial and nontrivial links will be defined using Definition 2.

![Figure 1: An embedding of \( K_6 \) into \( \mathbb{R}P^2 \). The bounding circle is identified using the antipodal map to obtain \( \mathbb{R}P^3 \).](image)

In this paper, we will prove the following theorems.

**Theorem 4** Let \( \mathcal{P} \) be the set of all Petersen-family graphs excluding the graph obtained from \( K_{4,4} \) by removing an edge. Let \( A, B, G \) be graphs such that \( G \) has \( k \)-connectivity with vertex cut set \( \{v_1, \ldots, v_k\} \), \( G = A \cup B \) and \( V(A \cap B) = \{v_1, \ldots, v_k\} \).

1. If \( k = 0 \) or \( 1 \), then \( G \) is minor-minimal intrinsically linked in \( \mathbb{R}P^3 \) if and only if \( A, B \in \mathcal{P} \).
2. If \( k = 2 \), then \( G \) is minor-minimal intrinsically linked in \( \mathbb{R}P^3 \) if and only if \( A', B' \in \mathcal{P} \), \( E(A) = E(A') \setminus \{(v_1, v_2)\} \) and \( E(B) = E(B') \setminus \{(v_1, v_2)\} \).

The theorem classifies intrinsically linked graphs with low connectivity. The first statement says that a graph that is disconnected (or with 1–connectivity) is intrinsically linked if and only if it is the disjoint union (or union along a vertex) of two Petersen-family graphs. The second statement is analogous for graphs with 2–connectivity, but the edge between the two vertices along which the Petersen-family graphs are joined is removed.
Theorem 5 The graph obtained by removing an edge from $K_{4,4}$ is minor-minimal intrinsically linked in $\mathbb{R} P^3$.

Theorem 6 The graphs obtained from $K_7$ by removing any two edges are minor-minimal intrinsically linked in $\mathbb{R} P^3$.

2 Definitions and notation

Before proceeding to our results, we begin with some elementary notation and definitions.

Definition 7 A graph $G = (V, E)$ is a set of vertices $V(G)$ and edges $E(G)$, where an edge is an unordered pair $(v_1, v_2)$ with $v_1, v_2 \in V$.

Definition 8 Let $G$ be a graph and $v_1, v_2, \ldots, v_n \in V(G)$ and

$$(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1) \in E(G)$$

such that $v_i \neq v_j$ for $i \neq j$. Then, the sequences of vertices $v_1, v_2, \ldots, v_n$ and edges $(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1)$ is an $n$–cycle in the graph $G$, denoted $v_1 v_2 \ldots v_n$.

In an abuse of notation, we will also refer to the image of a cycle $v_1 v_2 \ldots v_n$ in an embedding of the graph $G$ as the cycle $v_1 v_2 \ldots v_n$, when the distinction is clear.

The following notion of a graph minor allows us to specify when one graph contains another graph within it.

Definition 9 Let $G$ be a graph. Suppose $H$ is a graph such that $H$ can be obtained from $G$ by a sequence of the following three operations:

1. removal of an edge
2. removal of a vertex
3. contraction along an edge.

Then $H$ is called a minor of $G$, written $H \leq G$. If $H \leq G$ but $H \neq G$, then $H$ is called a proper minor of $G$, written $H < G$.

If $H \leq G$, we also call $G$ an expansion of $H$. 

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Nešetřil and Thomas [10] provide the following result for graph minors in \( \mathbb{R}^3 \), and the general result in arbitrary 3–manifolds can be proved by noticing that expansions preserve isotopy classes of cycles and links.

**Proposition 10** (J Nešetřil and R Thomas [10]) Let \( H \) be a graph that is intrinsically linked in a 3–manifold \( M \). If \( G \) is a graph such that \( H \subseteq G \), then \( G \) is also intrinsically linked in \( M \).

**Definition 11** A graph \( G \) is minor-minimal intrinsically linked in \( M \) if \( G \) is intrinsically linked in \( M \) and no proper minor of \( G \) is also intrinsically linked in \( M \).

In \( \mathbb{R}^3 \), the set of all minor-minimally intrinsically linked graphs is given by the seven Petersen-family graphs. These graphs are obtained from \( K_6 \) by \( \Delta - Y \) and \( Y - \Delta \) exchanges, where a \( \Delta - Y \) exchange is the removal of three edges \((v_1, v_2), (v_1, v_3),(v_2, v_3)\) and the addition of a vertex \( v \) along with the edges \((v, v_1), (v, v_2), (v, v_3)\). A \( Y - \Delta \) exchange is the reverse operation.

As a result of Robertson and Seymour’s proof of the Minor Theorem [13], the set of all minor-minimally intrinsically linked graphs in \( M \) is finite. This means that a full classification of minor-minimally intrinsically linked graphs in \( \mathbb{R}P^3 \) is possible. Because projective space has a simple first homology group, it may not be unrealistic to find a complete characterization for intrinsic linking.

### 3 Linked graphs with low connectivity

Exactly six of the seven Petersen-family graphs have embeddings into \( \mathbb{R}P^2 \), as shown by Glover, et al [6] and Archdeacon [1], and thus have linkless embeddings into \( \mathbb{R}P^3 \). We later show that the graph obtained by removing an edge from \( K_{4,4} \), which does not have a projective planar embedding, is in fact intrinsically linked in \( \mathbb{R}P^3 \).

Although not all Petersen-family graphs are intrinsically linked in \( \mathbb{R}P^3 \), we can use their intrinsic linking in \( \mathbb{R}^3 \) to deduce some facts about embeddings with no nontrivial two-component links.

**Lemma 12** Let \( P \) be a Petersen-family graph and \( v \) be a vertex of \( P \). If every cycle of \( P \setminus \{v\} \) is 0–homologous in an embedding \( f: P \to \mathbb{R}P^3 \), then \( f(P) \) contains a nontrivial link.

**Proof** Let \( L_1 \cup L_2 \) be a link with a projection onto a disc representing \( \mathbb{R}P^2 \) such that \( L_1 \) is affine and does not cross the boundary of the projection and \( L_2 \) is 1–homologous.
Take a point $p$ of $L_2$ that intersects the boundary of the projection (the line at infinity). Let $U$ be a sufficiently small neighborhood of $p$ in the projection such that $L_1$ does not intersect $U$ and $L_2$ intersects $\partial U$ in exactly two points, $p'$ and $q'$. Connect $p'$ and $q'$ with a line segment $s$ such that in the projection, $s$ crosses over every strand, and $s$ does not intersect the line at infinity. Define $L_2'$ as the cycle consisting of $s$ and the segment of $L_2$ that is not in $U$. Then, $L_2'$ is a 0–homologous cycle such that the linking number of $L_1 \cup L_2$ is the same as the linking number of $L_1 \cup L_2'$ (see Figure 2).

![Figure 2: Conversion of a link consisting of an affine knot and 1–homologous knot into one consisting of two 0–homologous knots](image)

Consider $f(P)$. Using crossing changes and ambient isotopy, we may assume that the embedding for the subgraph $P\backslash \{v\}$ is affine so that $f(P\backslash \{v\})$ does not intersect the boundary of the projection (in other words, it does not pass through the line at infinity), $v$ lies on the boundary of the projection, and no point besides $v$ lies on the line at infinity.

Define

$$\lambda \equiv \sum_{L_1 \cup L_2 \text{ is a two-component link in } f(P)} \text{lk}(L_1, L_2) \pmod{2},$$

where $\text{lk}(L_1, L_2)$ is the linking number of $L_1 \cup L_2$. The previous observation shows that there exists an affine embedding of $P$ for which $\lambda$ is unchanged. Because crossing changes do not affect $\lambda$, the results of Conway and Gordon [3] and Sachs [15] for $K_6$ and Petersen graphs in $\mathbb{R}^3$, respectively, imply that $\lambda \equiv 1 \pmod{2}$ for the embedding $f$ into $\mathbb{R}P^3$. Hence, the embedding must contain a two-component link with nonzero linking number, proving the lemma.

Lemma 12 allows us to completely classify intrinsically linked graphs in $\mathbb{R}P^3$ with connectivity 0, 1 and 2, assuming that $K_{4,4} \backslash \{e\}$ is intrinsically linked in $\mathbb{R}P^3$. 

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Proposition 13 Let $G = A \cup B$ be a 2–connected graph with vertex cut set $V(A \cap B) = \{v_1, v_2\}$. Let $\overline{A} = A \cup \{(v_1, v_2)\}$ and $\overline{B} = B \cup \{(v_1, v_2)\}$. If $G$ is minor-minimal intrinsically linked in $\mathbb{R}P^3$, then $\overline{A}$ and $\overline{B}$ are intrinsically linked in $\mathbb{R}^3$.

Proof Suppose $\overline{A}$ is not intrinsically linked in $\mathbb{R}^3$. Since $G$ is minor-minimal, $\overline{B} < G$ has a linkless embedding, $f$, in $\mathbb{R}P^3$. Let $g$ be an embedding of a closed 3–ball with interior $D$ into $\mathbb{R}P^3$ such that $f((v_1, v_2)) \subset g(D)$, only the vertices $v_1$ and $v_2$ intersect $\partial D$, and $f(\overline{B} \setminus \{(v_1, v_2)\})$ is in the complement of $g(D)$. Take a linkless embedding, $h$, of $\overline{A}$ in $\mathbb{R}^3 \cong D$. Then, $g \circ h$ is a linkless embedding of $\overline{A}$. Using ambient isotopy on $g \circ h$, we may assume that the arcs $f((v_1, v_2))$ and $g \circ h((v_1, v_2))$ coincide. The union of these two embeddings produces a linkless embedding of $G \cup (v_1, v_2)$ into $\mathbb{R}P^3$. \qed

Proposition 14 Let $G = (P_1 \cup P_2) \setminus \{(v_1, v_2)\}$ be a graph, where $P_1, P_2 \in \mathcal{P}$ and $V(P_1 \cap P_2) = \{v_1, v_2\}$. Then $G$ is intrinsically linked in $\mathbb{R}P^3$.

Proof Notice that both $P_1$ and $P_2$ are minors of $G$. Embed $G$ in $\mathbb{R}P^3$. By Lemma 12, if $P_i$ does not contain any nontrivial links, then $P_i \setminus \{v_i\}$ must contain a 1–homologous cycle, for $i = 1, 2$. This results in two disjoint 1–homologous cycles. Hence, $G$ is linked. \qed

The previous two propositions prove Theorem 4 for $k = 2$, assuming Theorem 5. The results for $k = 0$ and $k = 1$ are proved similarly, and Theorem 5 is proved in the following section.

For the case $k = 0$, it is easy to see that there are $\binom{5}{3} = 21$ minor-minimal intrinsically linked graphs in $\mathbb{R}P^3$. When $k = 1$, it is necessary to count the different number of vertex classes in each graph to determine the number of ways a pair of Petersen-family graphs may be glued along a vertex. From Table 1, the number of minor-minimal intrinsically linked graphs with 1–connectivity in $\mathbb{R}P^3$ is determined to be 91.

Define the vertex flipping number (VFN) for some vertex pair $\{x_1, x_2\}$ as

$$
\text{VFN}(x_1, x_2) = \begin{cases} 
0 & \text{if } x_1 \neq x_2, \\
1 & \text{otherwise,}
\end{cases}
$$

where $x_1 \neq x_2$ is equivalence under a graph isomorphism. Counting the number of minor-minimal intrinsically linked graphs in Theorem 4 when $k = 2$ requires attention to the VFN of vertex pair classes, where two pairs of vertices are equivalent if there is a graph isomorphism taking one pair to the other. For each pair $\{x_1, x_2\} \subseteq E(G_1), \{y_1, y_2\} \subseteq E(G_2)$ of vertex pair classes for two graphs $G_1, G_2$, the number of ways to glue $G_1$.
Table 1: Petersen-family graphs and the number of vertices, up to equivalence under graph isomorphism

<table>
<thead>
<tr>
<th>Graph</th>
<th>Vertex Classes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_6$</td>
<td>1</td>
</tr>
<tr>
<td>$K_{3,3,1}$</td>
<td>2</td>
</tr>
<tr>
<td>$P_7$</td>
<td>3</td>
</tr>
<tr>
<td>$P_8$</td>
<td>4</td>
</tr>
<tr>
<td>$P_9$</td>
<td>2</td>
</tr>
<tr>
<td>Petersen</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 2: Petersen-family graphs and the number of vertex pairs, up to equivalence under graph isomorphism

<table>
<thead>
<tr>
<th>Graph</th>
<th>Vertex Pair Classes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Total</td>
</tr>
<tr>
<td>$K_6$</td>
<td>1</td>
</tr>
<tr>
<td>$K_{3,3,1}$</td>
<td>3</td>
</tr>
<tr>
<td>$P_7$</td>
<td>5</td>
</tr>
<tr>
<td>$P_8$</td>
<td>10</td>
</tr>
<tr>
<td>$P_9$</td>
<td>6</td>
</tr>
<tr>
<td>Petersen</td>
<td>2</td>
</tr>
</tbody>
</table>

4 $K_{4,4}$ with an edge removed

In this section, we prove that the graph obtained by removing an edge from $K_{4,4}$ is intrinsically linked in $\mathbb{R}P^3$.

We will need the following observation.

Proposition 15 For every embedding into $\mathbb{R}P^3$, $K_{3,2}$ has an even number of 1–homologous 4–cycles.

Proof Whenever two cycles $C_1$ and $C_2$ intersect along an arc, $D$, we can define the sum of $C_1$ and $C_2$ to be $C_1 \cup C_2 \setminus D$. Then, the result can be obtained by noting that
the sum of two 0–homologous cycles and the sum of two 1–homologous cycles are 0–homologous cycles, and the sum of a 0–homologous cycle with a 1–homologous cycle is 1–homologous.

The combinatorial observation in Proposition 15 is used to prove the following characterization of homology classes of cycles in $K_{3,3}$, similar to one given by O’Donnell [11] for embeddings of $K_{3,3}$ into $\mathbb{R}^3$ and a simple closed curve in its complement.

**Lemma 16** If a graph $G$ isomorphic to $K_{3,3}$ is embedded in $\mathbb{R}P^3$ such that at least one of its cycles is 1–homologous, then the homology classes of all of the 4–cycles in the embedding of $G$ have one of two possibilities:

1. A cycle is 1–homologous if and only if it passes through a specified edge, $(u, v)$, of the graph. We call $(u, v)$ the including edge and the homology pattern of the embedding a 4–pattern.

2. A cycle is 1–homologous if and only if it does not pass through two of the edges in $F \subseteq E(G)$, where $F$ is a specified set of three mutually disjoint edges of $G$. We call $F$ the set of excluding edges and the homology pattern of the embedding a 6–pattern.

**Proof** Let $\{a_1, a_2, a_3\} \subseteq V(G)$ and $\{b_1, b_2, b_3\} \subseteq V(G)$ be the partition sets of $G$. Suppose $G$ contains a 1–homologous cycle. Then, it must contain a 1–homologous 4–cycle $C_1$. Let $H$ be a subgraph of $G$ isomorphic to $K_{3,2}$ that contains $C_1$. By Proposition 15, $H$ must contain two 1–homologous 4–cycles. Without loss of generality, they are the cycles $a_1b_1a_2b_2$ and $a_1b_1a_2b_3$. It also must be the case that the cycle $a_1b_2a_2b_3$ is 0–homologous.

Now, consider the subgraph induced by $\{a_1, a_2, a_3, b_1, b_2\}$. By Proposition 15, one of the two cycles $a_1b_1a_3b_2$ and $a_2b_1a_3b_2$ is 1–homologous, and the other is 0–homologous. Since interchanging $a_1$ and $a_2$ does not affect the choices made up to this point, without loss of generality, the cycle $a_1b_2a_3b_2$ is 1–homologous and the cycle $a_2b_1a_3b_2$ is 0–homologous.

Next, consider the subgraph induced by $\{a_1, a_3, b_1, b_2, b_3\}$. Since the cycle $a_1b_1a_3b_2$ is 1–homologous, then either the cycle $a_1b_1a_3b_3$ is also 1–homologous and the cycle $a_1b_2a_3b_3$ is 0–homologous, or the cycle $a_1b_1a_3b_3$ is 0–homologous and the cycle $a_1b_2a_3b_3$ is 1–homologous.

**Case 1** Cycle $a_1b_1a_3b_3$ is 1–homologous and cycle $a_1b_2a_3b_3$ is 0–homologous.
Applying Proposition 15 to all of the other subgraphs of $G$ isomorphic to $K_{3,2}$ forces the last two cycles, $a_2b_1a_3b_3$ and $a_2b_2a_3b_3$, to be 0–homologous. Observe that a cycle in $G$ is 1–homologous if and only if it includes the edge $(a_1, b_1)$. Hence, this embedding of $G$ has a 4–pattern, with $(a_1, b_1)$ as its including edge.

**Case 2** The cycle $a_1b_1a_3b_3$ is 0–homologous and the cycle $a_1b_2a_3b_3$ is 1–homologous.

Again, by using Proposition 15 on the remaining $K_{3,2}$ subgraphs of $G$, the cycles $a_2b_1a_3b_3$ and $a_2b_2a_3b_3$ must be 1–homologous. A 4–cycle of $G$ is 0–homologous if and only if it contains two edges from the set $F = \{(a_1, b_3), (a_2, b_2), (a_3, b_1)\}$. The set $F$ is the set of excluding edges, and the embedding is a 6–pattern. 

**Theorem 5** The graph $G$ obtained by removing an edge from $K_{4,4}$ is minor-minimal intrinsically linked in $\mathbb{R}P^3$.

**Proof** Consider an embedding of $G = K_{4,4}\\setminus\{(a_1, b_1)\}$, where

$$\{a_1, a_2, a_3, a_4\}, \{b_1, b_2, b_3, b_4\} \subset V(G)$$

are the partition sets.

Let $A$ be the subgraph induced by $\{a_2, a_3, a_4, b_2, b_3, b_4\}$, $B$ be the subgraph induced by $\{a_1, a_2, a_3, b_2, b_3, b_4\}$ and $C$ be the subgraph induced by $\{a_2, a_3, a_4, b_1, b_2, b_3\}$.

By Lemma 16, $A$ contains no 1–homologous cycles, is a 4–pattern or is a 6–pattern.

**Case 1** The subgraph $A$ contains no 1–homologous cycles.

By Lemma 12, if the embedding is linkless, the subgraph induced by $\{a_1, a_2, a_3, a_4, b_2, b_3, b_4\}$ must contain a 1–homologous cycle. Because $A$ does not contain any 1–homologous cycles, all such cycles must pass through $a_1$. Consider the subgraph induced by $\{a_1, a_2, a_3, b_2, b_3, b_4\}$. This $K_{3,3}$ subgraph must then have a 4–pattern. Without loss of generality, the including edge is $(a_1, b_2)$.

Similarly, the subgraph induced by $\{a_2, a_3, a_4, b_1, b_2, b_3\}$ contains a 4–pattern with including edge $(a_2, b_1)$. Then, $a_1b_2a_2b_3$ and $b_1a_2b_2a_3$ are disjoint 1–homologous cycles.

**Case 2** The subgraph $A$ contains a 4–pattern.

Without loss of generality, $A$ has $(a_4, b_4)$ as its including edge.
Subcase 2.1  Either $B$ or $C$ has a 6–pattern.

The subgraph $B$ cannot have a 6–pattern as then subgraph induced by $\{a_2, a_3, b_2, b_3, b_4\}$ would contain a 1–homologous cycle, contradicting that all 1–homologous cycles in $A$ pass through its including edge.

Subcase 2.2  Both $B$ and $C$ contain no 1–homologous cycles.

It is easy to see that all 1–homologous cycles of $G$ must pass through the edge $(a_4, b_4)$ by looking at the other four $K_{3,3}$ subgraphs of $G$ and noticing that each subgraph must have a 1–homologous cycle by the including edge in $A$. If any subgraph of $G$ (not including $B$ and $C$) has a 6–pattern or a 4–pattern with an including edge that is not $(a_4, b_4)$, then this would force a 1–homologous cycle in $B$ or $C$. By Lemma 12, since all 1–homologous cycle pass through $a_4$, $G$ is linked.

Subcase 2.3  Both $B$ and $C$ have 4–patterns.

If $B$ contains a 4–pattern, then its including edge must pass through $a_1$. Otherwise, $A$ contains a 1–homologous cycle disjoint from its including edge. Similarly, if $C$ contains a 4–pattern, then its including edge must pass through $b_1$. The subgraph $B$ has its including edge passing through $a_1$ and $C$ has its including edge passing through $b_1$. So we can find disjoint 1–homologous cycles in $G$.

Subcase 2.4  One of $B$ or $C$ has a 4–pattern and the other contains no 1–homologous cycles.

Without loss of generality, assume that $B$ has a 4–pattern and $C$ contains no 1–homologous cycles. By the previous subcase, the including edge in $B$ has $a_1$ as an endpoint. We claim that the subgraphs induced by $\{a_2, a_3, a_4, b_1, b_2, b_4\}$ and $\{a_2, a_3, a_4, b_1, b_3, b_4\}$ must have 4–patterns: both contain 1–homologous cycles due to $A$ having a 4–pattern, and if either contained a 6–pattern, there would be a 1–homologous cycle in $C$. Any edge with $b_1$ as an endpoint cannot be an including edge for these two graphs, since then $C$ would contain a 1–homologous cycle. Consequently, both subgraphs must have $(a_4, b_4)$ as its including edge. Otherwise, there would be a 1–homologous 4–cycle in $A$ that does not have $(a_4, b_4)$ as one of its edges.

If the including edge in $B$ does not have $b_4$ as its other endpoint, because the subgraph induced by $\{a_2, a_3, a_4, b_1, b_2, b_4\}$ has $(a_4, b_4)$ as its including edge, $G$ contains disjoint 1–homologous links. Otherwise, since the cycles $a_1b_2a_4b_4$ and $a_1b_3a_4b_4$ are 1–homologous from $A$ and cycles $a_1b_2a_ib_4$ and $a_1b_3a_ib_4$ are 1–homologous from $B$, then the subgraph induced by $\{a_1, a_i, a_4, b_2, b_3, b_4\}$ has a 4–pattern with $(a_i, b_4)$ as its including edge, for $i = 2, 3$. In this case, we have shown that all 1–homologous cycles pass through $b_4$, so by Lemma 12, $G$ is linked.
Case 3  The subgraph $A$ has a 6–pattern.

Without loss of generality, the excluding edges in $A$ are $(a_i, b_i)$ for $i = 2, 3, 4$. Then, every $K_{3,3}$ subgraph of $G$ shares a $K_{3,2}$ with $A$, so it must contain a 1–homologous cycle.

Subcase 3.1  Both $B$ and $C$ contain 4–patterns.

If $B$ contains a 4–pattern, its including edge must pass through $b_4$. Otherwise, $B$ contains a 1–homologous cycle from the 6–pattern in $A$ that does not pass through its including edge. Since the subgraph induced by $\{a_2, a_3, b_2, b_3, b_4\}$ contains a 1–homologous cycle by $A$, then $B$ has it including edge passing through $a_2$ or $a_3$. Let $(a_i, b_4)$ be the including edge in $B$.

Likewise, if $C$ has a 4–pattern, its including edge must be $(a_4, b_j)$, where $j = 2$ or 3. Then, it is easy to see that $G$ contains disjoint 1–homologous cycles. If $C$ has a 6–pattern, then let $k = 2, 3, k \neq i$. Then, the subgraph induced by $\{a_k, a_4, b_1, b_2, b_3\}$ contains a 1–homologous 4–cycle, one of which must pass through $b_1$. The 4–cycle that is disjoint from this cycle is also 1–homologous by the including edge in $B$, so $G$ is linked.

Subcase 3.2  Either $B$ or $C$ contain a 6–pattern.

Without loss of generality, assume that $B$ has a 6–pattern. One of its excluding edges must be $(a_1, b_4)$ since cycle $a_2b_2a_3b_3$ is 0–homologous by $A$, and $(a_1, b_4)$ is the only edge in $B$ that is disjoint from this cycle. Note that if $(a_2, b_2)$ and $(a_3, b_3)$ are also excluding edges, then all cycles in the subgraph induced by $\{a_1, a_2, a_4, b_2, b_3, b_4\}$ through $(a_2, b_3)$ are 1–homologous. We saw in the Subcase 2.1 that when there is a 4–pattern in a $K_{3,3}$ that is one adjacent (differs by one vertex) to a $K_{3,3}$ with a 6–pattern, then the graph is linked. Otherwise, $(a_2, b_3)$ and $(a_3, b_2)$ are the other excluding edges in $B$.

Similarly, if $G$ does not contain any nontrivial links, then $C$ must have $(a_4, b_1)$, $(a_2, b_3)$ and $(a_3, b_2)$ as excluding edges. Hence, $a_1b_4a_2b_2$ and $a_4b_1a_3b_3$ are disjoint 1–homologous cycles. So $G$ is linked.

The graph $G$ is minor-minimal since any proper minor of $G$ embeds in the projective plane, as shown by Glover, et al [6] and Archdeacon [1].

5  $K_7$ minus two edges

We now prove that any graph obtained by removing two edges from $K_7$ is minor-minimal intrinsically linked in $\mathbb{R}P^3$. There are two cases of Theorem 6: when the two
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edges are adjacent and when the two edges are nonadjacent. We will use the following lemma.

**Lemma 17** Given a linkless embedding of $K_6$, no $K_4$ subgraph can have all 0–homologous cycles.

**Proof** Consider an embedding of $K_6$ for which there is a $K_4$ subgraph with all cycles 0–homologous. By using crossing changes and ambient isotopy, this $K_4$ subgraph can be deformed so that it does not touch the line at infinity, and so that there are no crossings on it in a projection. Denote the vertices of this $K_4$ by $\{v_1, v_2, v_3, v_4\}$ and denote the vertices not in the $K_4$ by $v_5$ and $v_6$. One can deform the edge $(v_5, v_6)$ so that it is contained in the line at infinity, so that $v_6$ is placed at 12 o’clock and 6 o’clock, and so that $v_5$ is placed at 3 o’clock and 9 o’clock. We may assume the edge $(v_5, v_6)$ goes from 12 o’clock to 3 o’clock (see Figure 3).

![Figure 3: We may deform the embedded graph to be in this position (not all edges are shown).](image_url)

Now, we claim that the edges connecting $v_6$ to the $K_4$ can be deformed (using crossing changes and ambient isotopy) so that they are straight lines in the projection that connect to the $K_4$ either from 12 o’clock or from 6 o’clock. We may assume that the edge connecting to $v_4$ is under all of the other edges of the $K_4$ in the projection. We will justify the claim for the edge $(v_6, v_1)$. Consider the embedded cycle formed the two (additional) edges $e_1$ and $e_2$, where $e_1$ connects $v_1$ to the 12 o’clock $v_6$, and $e_2$ connects $v_1$ to the 6 o’clock $v_6$, where both $e_1$ and $e_2$ are straight edges in the projection. This cycle is 1–homologous. The edge $(v_1, v_6)$ from the $K_6$ embedding breaks up this cycle into two cycles, one formed by $e_1$ and $(v_1, v_6)$ and the other formed by $e_2$ and $(v_1, v_6)$. One of these two cycles must be 1–homologous, and the other must be 0–homologous. If the cycle formed by $e_1$ and $(v_1, v_6)$ is 0–homologous, then $(v_1, v_6)$ can be deformed, using crossing change and ambient isotopy, to $e_1$. Similarly,
if the cycle formed by $e_2$ and $(v_1, v_6)$ is 0–homologous, then $(v_1, v_6)$ can be deformed to $e_2$. This established our claim. It similarly follows that the edges connecting $v_5$ to the $K_4$ can be deformed (using crossing changes and ambient isotopy) so that they are straight lines in the projection that connect to the $K_4$ from either 3 o’clock or 9 o’clock.

Now, it cannot be the case that all of the edges connecting $v_6$ to the $K_4$ are incident to 12 o’clock, for then $v_1, v_2, v_3, v_4$ and $v_6$ would induce a $K_5$ with all cycles 0–homologous, which cannot occur in a linkless embedding of $K_6$ by Lemma 12. Similarly, all of the edges cannot be incident to 12 o’clock, nor can all of the edges emanating from $K_5$ be incident to 3 o’clock, nor can they all be incident to 9 o’clock. Thus, there must be exactly 1, 2 or 3 edges from the $K_4$ incident to 12 o’clock, and exactly 1, 2 or 3 edges from the $K_4$ incident to 3 o’clock. In all cases but one, there are a pair of disjoint 1–homologous cycles. These disjoint 1–homologous cycles would have been present in the original embedding of $K_6$. For example, if only $(v_1, v_6)$ is incident to 12 o’clock, and only $(v_2, v_5)$ is incident to 3 o’clock, then $(v_1, v_6, v_3)$ and $(v_2, v_5, v_4)$ form disjoint 1–homologous cycles.

The only case that does not lead to disjoint 1–homologous cycles is the case when exactly 1 edge from $K_4$ is incident to 12 o’clock (6 o’clock) and exactly one edge from $K_4$ is incident to 3 o’clock (9 o’clock), and these two edges are incident to the same vertex in the $K_4$. By symmetry, we may assume the edge $(v_1, v_6)$ is incident to 12 o’clock, and the edge $(v_1, v_5)$ is incident to 3 o’clock. Then, all 1–homologous cycles pass through $v_1$, so by Lemma 12, this embedding is linked.

An easy consequence of Lemma 17 is as follows:

**Corollary 18** The graph on 9 vertices obtained by pasting together two copies of $K_{3,1,1,1}$ along the three vertices that are mutually nonadjacent is intrinsically linked in $\mathbb{R}P^3$.

**Theorem 19** The graph obtained from $K_7$ by removing two edges incident to a common vertex is minor-minimal intrinsically linked in $\mathbb{R}P^3$.

**Proof** Let $G$ be the graph obtained from $K_7$ by removing two edges incident to a common vertex. Let $v_1, v_2, \ldots, v_7$ denote the vertices of $G$, with $v_7$ connected only to $v_1, v_2, v_3$ and $v_4$. Embed $G$. By the previous result, if the embedding is linkless, the $K_4$ induced on $\{v_1, v_2, v_3, v_4\}$ must contain a 1–homologous 3–cycle. By a homology argument, then there must be a 1–homologous 3–cycle through the vertex $v_7$. Without loss of generality, we may assume the 3–cycle is $(v_1, v_2, v_7)$. If the embedding is
linkless, then by the previous result, the $K_4$ induced by \{v_3, v_4, v_5, v_6\} must contain a 1–homologous cycle, but this forces two disjoint 1–homologous cycles. Thus, the embedding cannot be linkless.

The graph $G$ is minor-minimal since any proper minor of $G$ embeds in the projective plane by Glover, et al [6] and Archdeacon [1].

**Theorem 20** The graph obtained from $K_7$ by removing two nonadjacent edges is minor-minimal intrinsically linked in $\mathbb{R}P^3$.

**Proof** Let $K$ be the graph obtained from $K_7$ by removing two nonadjacent edges.

Label the vertices of $K_7$ as $v_1, v_2, ..., v_7$, and suppose edges $(v_4, v_5)$ and $(v_6, v_7)$ are removed to result in the graph $K$. Embed $K$, and suppose the embedding is linkless. We claim that the 4–cycle $(v_4, v_5, v_6, v_7)$ cannot be 1–homologous. If it were, then a cycle of the form $(v_i, v_j)$ is 1–homologous, for $i, j \in \{4, 5, 6, 7\}$, with $i \neq j$.

Without loss of generality, suppose $(v_1, v_5, v_7)$ is 1–homologous; then the subgraph induced by $v_2, v_3, v_4, v_6$ forms $K_4$, and since $K$ contracts onto $K_6$, by Lemma 17, there must be a disjoint 1–homologous cycle, which is a contradiction. Similarly, the following 3–cycles must also be 0–homologous: $(v_i, v_4, v_7)$, $(v_j, v_5, v_7)$, $(v_k, v_5, v_6)$ and $(v_m, v_4, v_6)$, where $i, j, k, m \in \{1, 2, 3\}$. It follows that for the subgraph induced by the vertices $v_1, v_4, v_5, v_7, v_3, v_6$, every cycle is 0–homologous. Since this subgraph contracts onto $K_4$, and since $K$ contracts onto $K_6$ (by contracting the same edge), it follows from Lemma 17 that there must be nonsplittable links in the embedding. It follows that $K$ is intrinsically linked in $\mathbb{R}P^3$.

The graph $K$ is minor-minimal for intrinsic linking since any proper minor of $K$ is projective planar, as shown by Glover, et al [6] and Archdeacon [1] or does not contain any intrinsically $\mathbb{R}^3$–linked graphs as a minor, so there exists a linkless embedding of every proper minor into a 3–ball.

The graph $K$ is minor-minimal for intrinsic linking in $\mathbb{R}P^3$ since any proper minor of $K$ is either projective planar [1; 6] or becomes ($\mathbb{R}^2$) planar after the removal of a vertex (and hence is not intrinsically linked in space). In either case no minor is intrinsically linked in $\mathbb{R}P^3$. 

\section{6 Other intrinsically linked graphs}

It is not too hard to see that $\Delta - Y$ exchanges preserve intrinsic linking as in $\mathbb{R}^3$ (see Motwani, Raghunathan and Saran [8]), so any graph generated from a known
intrinsically $\mathbb{R}P^3$–linked graph by a sequence of $\triangle - Y$ exchanges is also intrinsically linked. Corollary 18 provides a graph with several $\triangle$ subgraphs.

Notice that two copies of $K_{3,1,1,1}$ glued along the three mutually nonadjacent vertices is the same as gluing two copies of $K_6$ along three vertices $v_1, v_2, v_3$ and then removing the triangle composed of the three edges between $v_1, v_2,$ and $v_3$. For notational convenience, each copy of $K_6$ with the triangle removed will be referred to as $K_6 \cdot \cdot$, and $K_6 \cdot \cdot K_6$ denotes the gluing of two copies of $K_6 \cdot \cdot$ along the three vertices that are mutually nonadjacent. By Corollary 18, $K_6 \cdot \cdot K_6$ is intrinsically linked in $\mathbb{R}P^3$.

In general, we define $G \cdot \cdot$ to be a graph with three marked vertices which are mutually nonadjacent. If $G_1 \cdot \cdot$ and $G_2 \cdot \cdot$ are two such graphs, then $G_1 \cdot \cdot G_2$ is a graph obtained by gluing the two graphs along the three marked vertices of $G_1 \cdot \cdot$ and $G_2 \cdot \cdot$. The resulting graph may not be unique if permutation of the marked vertices does not yield a graph isomorphism of each $G_i \cdot \cdot, i = 1, 2$. In such cases, we will differentiate between the (up to) three distinct graphs by a subscript, as in $G_1 \cdot \cdot_1 G_2, G_1 \cdot \cdot_2 G_2$ and $G_1 \cdot \cdot_3 G_2$.

**Proposition 21** There are 18 intrinsically linked graphs in $\mathbb{R}P^3$ with 3–connectivity that can be obtained from $K_6 \cdot \cdot K_6$ by $\triangle - Y$ exchanges.

**Proof** Figure 4 shows $K_6 \cdot \cdot$, with the marked vertices ($d, e,$ and $f$) shown as open circles, and the edges which were removed from $K_6$ shown as dotted lines. All other edges are not shown. In the subsequent figures, only edges added by $\triangle - Y$ exchange are shown.

The graphs obtained by repeated $\triangle - Y$ exchanges on $K_6 \cdot \cdot$, are shown in Figure 4. Consequently, there are 18 intrinsically linked graphs in $\mathbb{R}P^3$ obtained from $K_6 \cdot \cdot K_6$. The graphs $P_7B \cdot \cdot P_7B, P_7B \cdot \cdot P_8B$ and $P_8B \cdot \cdot P_8B$ each have two different configurations. The configuration where the $Y$ subgraphs of the two copies of $P_7B \cdot \cdot$ are glued together along a shared vertex is $P_7B \cdot \cdot_1 P_7B$, and the configuration where they are not is $P_7B \cdot \cdot_2 P_7B$. Similarly, if the $Y$ in $P_7B \cdot \cdot$ shares a vertex with a $Y$ in $P_8B \cdot \cdot$, we have $P_7B \cdot \cdot_1 P_8B$, and otherwise, we have $P_7B \cdot \cdot_2 P_8B$. If the $Y$ subgraphs are paired up in $P_8B \cdot \cdot P_8B$, we have $P_8B \cdot \cdot_1 P_8B$, and otherwise, we have $P_8B \cdot \cdot_2 P_8B$.

**Remark 22** The graphs $P_7A \cdot : K_6, P_7A \cdot : P_7A, P_7A \cdot : P_7B, P_7A \cdot : P_8B$ and $P_7A \cdot : P_9B$ are intrinsically linked in $\mathbb{R}P^3$, but all contain $K_{4,4}$ with an edge removed as a minor. Hence, they are not minor-minimal intrinsically linked in $\mathbb{R}P^3$.

To show that the remaining 13 intrinsically linked graphs in $\mathbb{R}P^3$ are minor-minimal, we will use the following result from Brouwer, et al [2] and Ozawa and Tsutsumi [12]:

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Figure 4: Graphs obtained from performing $\Delta - Y$ exchanges on $K_6$ and each subsequent graph, with subscripts denoting degree of the vertex. The open circles represent the three marked vertices and dashed edges represent edges removed from the original $K_6$.

Theorem 23  Let $P$ be a property preserved under $\Delta - Y$ exchange. Let $G$ be a graph that contains at least one degree three vertex and is minor-minimal with respect to $P$. Let $G'$ be a graph obtained from $G$ by a $Y - \Delta$ exchange. If $G'$ has property $P$, then $G'$ is also minor-minimal with respect to $P$.

Thus, we need only show that no proper minor of $P_{9B}$. $P_{9B}$ is intrinsically linked in $\mathbb{R}P^3$.

Theorem 24  The 13 graphs obtained from $\Delta - Y$ exchange on $K_6$ as a subgraph are minor-minimal intrinsically linked in $\mathbb{R}P^3$.

Proof  We will show that no proper minor of $P_{9B}$. $P_{9B}$ is intrinsically linked in $\mathbb{R}P^3$.

Consider $P_{9B}$. $P_{9B}$ as drawn in Figure 5. The only pair of linked cycles in this embedding is $c_1h_1e_1g_1$ and $f_1i_1d_1b_2i_2$. There are two vertex equivalence classes in $P_{9B}$. $P_{9B}$: $\{d, e, f\}$ and $V(P_{9B}$. $P_{9B}) \setminus \{d, e, f\}$.

To check for minor-minimality, it suffices to show that removing or contracting any edge in $P_{9B}$. $P_{9B}$ results in graph that is not intrinsically linked in $\mathbb{R}P^3$.

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There are two edge classes (up to graph isomorphism) that need to be considered. Removing edge \((a_1, e)\) or edge \((c_1, h_1)\) from the embedding in Figure 5 results in a linkless embedding. Contracting edge \((f, c_1)\) or edge \((a_1, i_1)\) in Figure 5 results in a linkless embedding since the edge contractions send vertices on the (only) two linked cycles to the same point, thus eliminating the nontrivial link.

\[ \square \]

7 Remarks

Using the weaker definition of unlinked components in Definition 2 allows the use of \(1\)–homologous cycles to reduce the number of crossings in a graph embedding in projective space. Thus, intrinsically linked graphs in \(\mathbb{R}P^3\) are more complex. Unlike in \(\mathbb{R}^3\), where there are simple arguments showing that there are no minor-minimal intrinsically linked graphs with connectivity 0, 1 or 2, such graphs exist in projective space. Using careful combinatorics, one can show that there are 21 disconnected graphs, 91 graphs with 1–connectivity and 469 graphs with 2–connectivity which are minor-minimal intrinsically linked in \(\mathbb{R}P^3\). It is not too hard to see that \(\triangle - Y\) exchanges preserve intrinsic linking as in \(\mathbb{R}^3\), so we predict that there are many more minor-minimal intrinsically linked graphs than the ones we have observed in this paper. In particular, the graphs obtained by removing two edges from \(K_7\) have a myriad of triangles in which we can perform a \(\triangle - Y\) exchange, leading to more intrinsically linked graphs. Some of these graphs have \(K_{4,4}\) with an edge removed as a minor, but others have yet to be explored fully. It would be of interest to see which of these are in fact minor-minimal intrinsically linked in \(\mathbb{R}P^3\).
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