Quillen’s plus construction and the D(2) problem

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Given a finite connected 3–complex with cohomological dimension 2, we show it may be constructed up to homotopy by applying the Quillen plus construction to the Cayley complex of a finite group presentation. This reduces the D(2) problem to a question about perfect normal subgroups.

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1 Introduction

Given a finite cell complex one may ask what the minimal dimension of a finite cell complex in its homotopy type is. If $n \neq 2$ and the cell complex has cohomological dimension $n$ (with respect to all coefficient bundles), then the cell complex is in fact homotopy equivalent to a finite $n$–complex (a cell complex whose cells have dimension at most $n$). Although this has been known for around forty years (for $n > 2$ it is proved by Wall [13] and for $n = 1$ it follows from Swan [12] and Stallings [11]), it is an open question whether or not this holds when $n = 2$. This question is known as Wall’s D(2) problem:

Let $X$ be a finite 3–complex with $H^3(X; \beta) = 0$ for all coefficient bundles $\beta$. Must $X$ be homotopy equivalent to a finite 2–complex?

If $X$ (as above) is not homotopy equivalent to a finite 2–complex, we say it is a counterexample which solves the D(2) problem.

For connected $X$ with certain fundamental groups, it has shown been shown that $X$ must be homotopy equivalent to a finite 2–complex (see for example Johnson [7], Edwards [4] and Mannan [9]). However no general method has been forthcoming.

Also, whilst potential candidates for counterexamples have been constructed (see Beyl and Waller [1] and Bridson and Tweedale [2]), no successful method has yet emerged for verifying that they are not homotopy equivalent to finite 2–complexes.

In Section 2 we apply the Quillen plus construction to connected 2–complexes, resulting in cohomologically 2–dimensional 3–complexes. These are therefore candidates for
counterexamples which solve the D(2) problem. In Section 3 we show that in fact all finite connected cohomologically 2–dimensional 3–complexes arise this way, up to homotopy equivalence.

Finally, in Section 4 we use these results to reduce the D(2) problem to a question about perfect normal subgroups. This allows us to generalize existing approaches to the D(2) problem such as Johnson [8, Theorem I] and Harlander [6, Theorem 3.5].

Before moving on to the main argument we make a few notational points. All modules are right modules except where a left action is explicitly stated. The basepoint of a Cayley complex is always assumed to be its 0–cell.

If $X$ is a connected cell complex with basepoint, we denote its universal cover by $\tilde{X}$. Given two based loops $\gamma_1, \gamma_2 \in \pi_1(X)$ their product $\gamma_1 \gamma_2$ is the composition whose initial segment is $\gamma_2$ and final segment is $\gamma_1$. With this convention, we have a natural right action of $\pi_1(X)$ on the cells of $\tilde{X}$. Let $G = \pi_1(X)$. We can regard the associated chain complex of $\tilde{X}$ as an algebraic complex of right modules over $\mathbb{Z}[G]$. We follow [8] in denoting this algebraic complex $C_*(\tilde{X})$. Note that this differs from the convention in other texts. Thus in particular $C_*(X)$ and $C_*(\tilde{X})$ have the same underlying sequence of abelian groups, but the former is a sequence of modules over $\mathbb{Z}[G]$ whilst the latter is a sequence of modules over $\mathbb{Z}[\pi_1(\tilde{X})] = \mathbb{Z}$.

If $Y$ is a subcomplex of $X$ then $C_*(Y)$ is a sequence of right modules over $\pi_1(Y)$. Let $E = \pi_1(Y)$. The induced map $E \to G$ yields a left action of $E$ on $\mathbb{Z}[G]$. Thus we have an algebraic complex $C_*(Y) \otimes_E \mathbb{Z}[G]$ over $\mathbb{Z}[G]$. The inclusion $Y \subset X$ induces a chain map $C_*(Y) \otimes_E \mathbb{Z}[G] \to C_*(X)$. The complex $C_*(X, Y)$ is defined to be the relative chain complex associated to this chain map.

The basepoint allows us to interchange between coefficient bundles over $X$ and right modules over $\mathbb{Z}[G]$. Thus for a right module $N$ we have:

$$H^n(X; N) = H^n(C_*(X); N)$$

A left module over $\mathbb{Z}[G]$ may be regarded as a right module over $\mathbb{Z}[G]$, where right multiplication by a group element is defined to be left multiplication by its inverse. Hence a left module $M$ may also be regarded as a coefficient bundle and we have:

$$H_n(X; M) = H_n(C_*(X); M), \quad H_n(X, Y; M) = H_n(C_*(X, Y); M)$$

Given a finitely generated Abelian group $A$ we may regard it as a finitely generated module over $\mathbb{Z}$. Thus $A \otimes_{\mathbb{Z}} \mathbb{Q}$ is a finite dimensional vector space over $\mathbb{Q}$. The dimension of this vector space will be denoted $\text{rk}_{\mathbb{Z}}(A)$.

Finally given a group $G$ and elements $g, h \in G$, we follow the convention that $[g, h]$ denotes the element $ghg^{-1}h^{-1}$.
2 The plus construction applied to a Cayley complex

Let $\varepsilon = \langle g_1, \ldots, g_n | R_1, \ldots, R_m \rangle$ be a finite presentation for a group $E$. We say a normal subgroup of $E$ is finitely closed when it is the normal closure in $E$ of a finitely generated subgroup. Let $K \triangleleft E$ be finitely closed and perfect (so $K = [K, K]$). Let $K_\varepsilon$ denote the Cayley complex associated to $\varepsilon$.

Theorem 2.1 (Quillen; see Rosenberg [10, Theorem 5.2.2]) There is a 3–complex $K_\varepsilon^+$, containing $K_\varepsilon$ as a subcomplex, such that the inclusion $K_\varepsilon \hookrightarrow K_\varepsilon^+$ induces the quotient map $E \to E/K$ on fundamental groups and $H_n(K_\varepsilon^+, K_\varepsilon; M) = 0$ for all left modules $M$ over $\mathbb{Z}[E/K]$. Further, given another such 3–complex $X$, there is a homotopy equivalence $K_\varepsilon^+ \to X$ extending the identity map of the common subspace $K_\varepsilon$.

In fact we may construct $K_\varepsilon^+$ explicitly, using the fact that $K$ is finitely closed to ensure that we end up with a finite cell complex. Let $k_1, \ldots, k_r \in K$ generate a subgroup of $E$ whose normal closure (in $E$) is $K$. As $K = [K, K]$, each $k_i$ may be expressed as a product of commutators $k_i = \prod_{j=1}^{m_i} [a_{ij}, b_{ij}]$ with each $a_{ij}, b_{ij} \in K$. Then each $a_{ij}, b_{ij}$ may be represented by words $A_{ij}, B_{ij}$ in the $g_i$, $l = 1, \ldots, n$. For each $i = 1, \ldots, r$ attach a 2–cell $E_i$ to $K_\varepsilon$ whose boundary corresponds to the word $\prod_{j=1}^{m_i} [A_{ij}, B_{ij}]$. Denote the resulting chain complex $K'_{\varepsilon}$.

The chain complex $C_\ast(K_\varepsilon)$ may be written:

$C_\ast(K_\varepsilon): C_2(K_\varepsilon) \xrightarrow{\partial_2} C_1(K_\varepsilon) \xrightarrow{\partial_1} C_0(K_\varepsilon)$

The boundary map $\partial_2$ applied to a 2–cell is the Fox free differential $\partial: F(g_1, \ldots, g_n) \to C_1(K_\varepsilon)$, applied to the word which the 2–cell bounds (see Johnson [8, Section 48] and Fox [5]). Let $e_l$ denote the generator in $C_1(K_\varepsilon)$ representing the generator $g_l$. The free Fox differential is then characterized by:

(i) $\partial g_l = e_l$ for all $l = 1, \ldots, n$,

(ii) $\partial(AB) = \partial(A)B + \partial(B)$ for all words $A, B$.

Clearly the inclusion $K_\varepsilon \hookrightarrow K'_{\varepsilon}$ induces the quotient map $E \to E/K$ on fundamental groups. There is a right action of $\mathbb{Z}[E/K]$ on itself. Further there is a left action of $E$ on $\mathbb{Z}[E/K]$.

Lemma 2.2 As an algebraic complex of right $\mathbb{Z}[E/K]$ modules $C_\ast(K'_{\varepsilon})$ may be written:

$C_\ast(K'_{\varepsilon}): C_2(K_\varepsilon) \otimes E \mathbb{Z}[E/K] \oplus \mathbb{Z}[E/K]^r \xrightarrow{\partial_2 \oplus 0} C_1(K_\varepsilon) \otimes E \mathbb{Z}[E/K] \xrightarrow{\partial_1} C_0(K_\varepsilon) \otimes E \mathbb{Z}[E/K]$
Proof The boundary of $E_i$ is given by the free Fox differential $\partial$, applied to the word $\prod_{j=1}^{m_i}[A_{ij}, B_{ij}]$. However,
\[
\partial \prod_{j=1}^{m_i}[A_{ij}, B_{ij}] = \sum_{j=1}^{m_i}[\partial A_{ij} + \partial B_{ij} - \partial A_{ij} - \partial B_{ij}] = 0
\]
as each $A_{ij}, B_{ij}$ represents an element of $K$ and hence is trivial in $\pi_1(K^\epsilon_i) = E/K$. \qed

Each $E_i$ therefore generates an element of $H_2(K^\epsilon_i; \Z)$. By the Hurewicz isomorphism theorem we have isomorphisms $H_2(K^\epsilon_i; \Z) \cong \pi_2(K^\epsilon_i) \cong \pi_2(K^\epsilon_i)$ coming from the Hurewicz homomorphism and the covering map respectively. Let $\psi_i: S^2 \to K^\epsilon_i$ represent the element of $\pi_2(K^\epsilon_i)$ which corresponds to $E_i$ under these isomorphisms.

For each $i \in 1, \ldots, r$ we then attach a 3–cell $B_i$ to $K^\epsilon_i$ via the attaching map $\psi_i: \partial B_i \to K^\epsilon_i$. Let $K^\epsilon''_i$ denote the resulting 3–complex. Then we have that $C_*(K^\epsilon''_i)$ is
\[
C_*(K^\epsilon''_i) : \quad \Z[E/K]^\epsilon \xrightarrow{\partial_3} C_2(K^\epsilon_i) \otimes E \Z[E/K] \oplus \Z[E/K]^\epsilon \xrightarrow{\partial_2 \oplus 0} C_1(K^\epsilon_i) \otimes E \Z[E/K] \xrightarrow{\partial_1} C_0(K^\epsilon_i) \otimes E \Z[E/K]
\]
where $\partial_3$ is inclusion of the second summand.

Hence we have:

**Lemma 2.3** $H_*(K^\epsilon''_i, K^\epsilon_i; M) = 0$ for all left modules $M$ over $\Z[E/K]$.

**Proof** We have the following relative complex:
\[
C_*(K^\epsilon''_i, K^\epsilon_i) : \quad \Z[E/K]^\epsilon \xrightarrow{\sim} \Z[E/K]^\epsilon \to 0 \to 0 \quad \square
\]

Thus by Theorem 2.1 we may conclude that $K^\epsilon''_i$ has the homotopy type of $K^\epsilon_i$. 

**Lemma 2.4** The complex $K^\epsilon''_i$ is cohomologically 2–dimensional.

**Proof** The inclusion $\iota: K^\epsilon_i \to K^\epsilon''_i$ induces a chain homotopy equivalence:
\[
C_*(K^\epsilon_i) \otimes E \Z[E/K] \to C_*(K^\epsilon''_i) \quad \square
\]

**Corollary 2.5** We may choose $K^\epsilon_i$ to be the cohomologically 2–dimensional finite 3–complex $K^\epsilon''_i$. 

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3 Cohomologically 2–dimensional 3–complexes

Let $X$ be a finite connected 3–complex with $H^3(X; \beta) = 0$ for all coefficient bundles $\beta$. In this section we will show that up to homotopy, $X$ arises as the Quillen plus construction applied to a finite Cayley complex.

Let $T$ be a maximal tree in the 1–skeleton of $X$. The quotient map $X \to X/T$ is a homotopy equivalence. Hence we may assume without loss of generality that $X$ has one 0–cell. We take this to be the basepoint of $X$ and any complexes obtained from $X$ by adding or removing cells. Also we set $G = \pi_1(X)$ with respect to this basepoint.

Let $C_n(X)$ be denoted by

$$F_3 \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0$$

where the $F_i$, $i = 0, 1, 2, 3$, are free modules over $\mathbb{Z}[G]$ and the $\partial_i$ are linear maps over $\mathbb{Z}[G]$.

We have $H^3(X; F_3) = 0$ so in particular there exists $\phi$ such the following diagram commutes:

$$\begin{array}{ccc}
F_3 & \xrightarrow{\partial_3} & F_2 \\
\downarrow{1} & & \downarrow{\phi} \\
F_3 & \xrightarrow{\partial_3} & F_2
\end{array}$$

Hence $\partial_3$ is the inclusion of the first summand $\partial_3: F_3 \hookrightarrow \partial_3(F_3) \oplus S = F_2$, where $S$ is the kernel of $\phi$. Let $X'$ denote the wedge of $X$ with one disk for each 3–cell in $X$. Then the inclusion of cell complexes $X \hookrightarrow X'$ is a homotopy equivalence and:

$$C_*(X'): F_3 \xrightarrow{\partial'_3} F_2 \oplus F'_3 \xrightarrow{\partial'_2} F_1 \oplus F'_3 \xrightarrow{\partial'_1} F_0$$

Here $F'_3 \cong F_3$ and the maps are defined as follows:

$\partial'_1$ restricts to $\partial_1$ on $F_1$ and restricts to 0 on $F'_3$,

$$\partial'_2 = \begin{pmatrix} \partial_2 & 0 \\ 0 & 1 \end{pmatrix},$$

$\partial'_3$ is $\partial_3: F_3 \to F_2$ composed with the natural inclusion: $F_2 \hookrightarrow F_2 \oplus F'_3$.

Thus $\partial'_3$ is the inclusion into the first summand $\partial'_3: F_3 \hookrightarrow \partial'_3F_3 \oplus S \oplus F'_3$.

Let $m$ denote the number of 2–cells in $X$. The submodule $S \oplus F'_3 \subset (\partial'_3F_3 \oplus S) \oplus F'_3$ is isomorphic to $S \oplus F_3 \cong F_2$ and hence has a basis $x_1, \ldots, x_m \in F_2 \oplus F'_3$. 

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The cell complex $X'$ has one 0–cell, so $F_0 \cong \mathbb{Z}[G]$. Let $n$ denote the number of 1–cells in $X'$. Then each 1–cell corresponds to a generator $g_i$, $i \in [1, \ldots, n]$ of $G$. Let $\{e_1, \ldots, e_n\}$ form the corresponding basis for $F_1 \oplus F'_1$.

Let $r$ denote the number of 2–cells in $X'$. The attaching map for each 2–cell maps the boundary of a disk round a word in the $g_i$. For each 2–cell let $R_j$, $j \in [1, \ldots, r]$ denote this word. Let $\{E_1, \ldots, E_r\}$ form the corresponding basis for $F_2 \oplus F'_2$. Thus we have a presentation $G = \langle g_1, \ldots, g_n \mid R_1, \ldots, R_r \rangle$.

We may therefore express each $x_i$ as a linear combination of the $E_j$. Thus for some integers $v_i$ and sequences $j_1, \ldots, j_{i v_i} \in \{1, \ldots, r\}$ we have

$$x_i = \sum_{l=1}^{v_i} E_{j_{i l}} \lambda_{i l} \sigma_{i l}$$

with each $\lambda_{i l} \in G$ and $\sigma_{i l} \in \{1, -1\}$. For each $i \in [1, \ldots, m]$, $l \in [1, \ldots, v_i]$ let $w_{il}$ be a word in the $g_k$, $k = 1, \ldots, n$, representing $\lambda_{i l}$. Now for each $i = 1, \ldots, m$, let:

$$S_i = \prod_{l=1}^{v_i} w_{il}^{-1} R_{j_{i l}} \sigma_{i l} \ w_{il}$$

For each $i \in [1, \ldots, m]$, attach a 2–cell $a_i$ to $X'$ by mapping the boundary of a disk round a word in the 1–skeleton of $X'$ corresponding to the word $S_i$. Let $Z$ denote the resulting finite cell complex. Note that each word $S_i$ corresponds to a trivial element of $G$, so the inclusion $X' \subset Z$ induces an isomorphism $\pi_1(X') \cong \pi_1(Z)$. Hence we may write $C_*(Z)$:

$$C_*(Z) : \quad F_3 \xrightarrow{\partial_*^2} (F_2 \oplus F'_2) \oplus F'_2 \xrightarrow{(\partial'_2 \ \partial'_3)} (F_1 \oplus F'_3) \xrightarrow{\partial'_1} F_0$$

where $\partial'_3$ is understood to be $\partial'_3 : F_3 \rightarrow (F_2 \oplus F'_3)$ composed with the natural inclusion $(F_2 \oplus F'_3) \hookrightarrow (F_2 \oplus F'_3) \oplus F'_2$.

For $i = 1, \ldots, m$ let $A_i$ be the basis element of $F'_2$ corresponding to the 2–cell $a_i$. Recall the Fox free differential, $\partial'$. We have:

$$\partial'^2 A_i = \partial S_i = \sum_{l=1}^{v_i} \partial(w_{il}^{-1} R_{j_{i l}} \sigma_{i l} w_{il}) = \sum_{l=1}^{v_i} \partial'_2 E_{j_{i l}} \lambda_{i l} \sigma_{i l} = \partial'_2 x_i$$

Thus $A_i - x_i$ represents a class in $H_2(\widetilde{Z(2)}; \mathbb{Z})$ which is isomorphic to $\pi_2(Z(2))$ via the Hurewicz isomorphism composed with the map $\pi_2(Z(2)) \rightarrow \pi_2(Z(3))$ induced by the covering map. Let $\psi_i : S^2 \rightarrow Z(2)$ represent the corresponding element of $\pi_2(Z(2))$.

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Then for each \( i = 1, \ldots, m \) we may attach a 3–cell \( b_i \) to \( Z \) via the map \( \psi_i \). We denote the resulting complex \( X'' \).

**Lemma 3.1**  The inclusion \( i: X' \subset X'' \) is a homotopy equivalence.

**Proof**  Starting with \( X' \), for each \( i \) we attached a 2–cell \( a_i \) with contractible boundary in \( X' \), and then attached a 3–cell \( b_i \) with \( a_i \) as a free face. Thus \( X'' \) is obtained from \( X' \) through a series of cell expansions and the inclusion \( X' \subset X'' \) is a simple homotopy equivalence. \( \square \)

Let \( Y \) denote the subcomplex of \( X'' \) consisting of the 1–skeleton, \( X''(1) \), together with the \( a_i, i = 1, \ldots, m \). Let \( \varepsilon \) denote the group presentation \( \langle g_1, \ldots, g_n \mid S_1, \ldots, S_m \rangle \) and let \( E \) denote the underlying group. By construction we have \( Y = \mathcal{K}_\varepsilon \).

Let \( k_1, \ldots, k_r \in E \) denote the elements represented by the words \( R_1, \ldots, R_r \). Let \( K \) denote the normal closure in \( E \) of \( k_1, \ldots, k_r \). By construction then, \( K \) is finitely closed and we have a short exact sequence of groups:

\[
1 \to K \to E \to G \to 1
\]

**Lemma 3.2**  \( K \) is a perfect group.

**Proof**  Clearly \( \mathbb{Z}[G] \) is a right module over itself and there is a left action of \( E \) on \( \mathbb{Z}[G] \). The algebraic complex \( C_*(\mathcal{K}_\varepsilon) \otimes_E \mathbb{Z}[G] \) is given by:

\[
F_2' \xrightarrow{\partial'_2} (F_1 \oplus F'_3) \xrightarrow{\partial'_1} F_0
\]

Now consider \( C_*(X') \):

\[
F_3 \xrightarrow{\partial'_3} F_2 \oplus F'_3 \xrightarrow{\partial'_2} F_1 \oplus F'_3 \xrightarrow{\partial'_1} F_0
\]

As \( \tilde{X}' \) is simply connected, we have \( \ker(\partial'_1) = \operatorname{Im}(\partial'_2) \).

Recall that \( F_2 \oplus F'_3 = \partial'_2(F_2) \oplus S \oplus F'_3 \) and that \( S \oplus F'_3 \) has basis \( x_1, \ldots, x_m \). Clearly \( \partial'_2 \) restricts to 0 on \( \partial'_2(F_2) \), so \( \ker(\partial'_1) = \operatorname{Im}(\partial'_2) \) which is generated by the \( \partial'_2(x_i) \).

Also recall that \( \partial'_2x_i = \partial''_2A_i \). Hence \( \ker(\partial'_1) = \operatorname{Im}(\partial''_2) \) and \( H_1(C_*(\mathcal{K}_\varepsilon) \otimes_E \mathbb{Z}[G]) = 0 \).

However by restricting coefficients \( C_*(\mathcal{K}_\varepsilon) \) may be regarded as an algebraic complex of free modules over \( \mathbb{Z}[K] \). Hence we have

\[
K/[K, K] = H_1(K, \mathbb{Z}) = H_1(C_*(\mathcal{K}_\varepsilon) \otimes_K \mathbb{Z}) = H_1(C_*(\mathcal{K}_\varepsilon) \otimes_E \mathbb{Z}[G]) = 0
\]

where \( \mathbb{Z} \) is regarded as having a trivial left \( K \)–action. \( \square \)
Lemma 3.3 \(X'' = \mathcal{K}_{e}^{+}\) where \(+\) is taken with respect to \(K\).

**Proof** We may identify \(\mathcal{K}_{e}\) with the subcomplex \(Y \subset X''\). The inclusion \(\ell: \mathcal{K}_{e} \hookrightarrow X''\) then induces the quotient map \(E \to E / K\) on fundamental groups. By Theorem 2.1 it is sufficient to show that \(H_{*}(X'', Y; M) = 0\) for all left coefficient modules \(M\).

Let \(\ell_{*}: C_{*}(\mathcal{K}_{e}) \otimes_{E} \mathbb{Z}[G] \to C_{*}(X'')\) be the chain map induced by the inclusion \(\ell: \mathcal{K}_{e} \hookrightarrow X''\). We have the following commutative diagram:

\[
\begin{array}{ccc}
F_2' & \xrightarrow{\partial''_2} & (F_1' + F_3') \\
\downarrow{\ell_2} & & \downarrow{\ell_1} \\
(F_2 + F_3') & \xrightarrow{(\partial''_2, \partial''_3)} & (F_1 + F_3')
\end{array}
\]

where \(F_2''\) has a basis \(D_1, \ldots, D_m\) corresponding to the \(3\)-cells \(b_1, \ldots, b_m\), so for \(i = 1, \ldots, m\) we have \(\partial''_3(D_i) = A_i - x_i\). Here \(\ell_0\) and \(\ell_1\) are the identity maps and \(\ell_2\) is the inclusion of the second summand.

We have that \((F_2 + F_3') = \partial''_3 F_3 + (S + F_3')\). Hence we have \((F_2 + F_3') \oplus F_2'' = \partial''_2 F_3 + (S + F_3') \oplus F_2''\).

The submodule \((S + F_3')\) has basis \(x_1, \ldots, x_m\). The submodule \(F_2''\) has basis \(A_1, \ldots, A_m\). Also \(\partial''_2 F_2''\) has basis \(A_1 - x_1, \ldots, A_m - x_m\). Hence we have the following equality of submodules: \((S + F_3') \oplus F_2'' = \partial''_2 F_2'' \oplus F_2''\).

Thus:

\[(F_2 + F_3') \oplus F_2'' = \partial''_2 F_3 \oplus \partial''_2 F_2'' \oplus F_2''\]

The relative chain complex \(C_{*}(X'', Y)\) is therefore given by

\[F_3 \oplus F_2'' \xrightarrow{\partial''_2} F_3 \oplus \partial''_2 F_2'' \to 0 \to 0\]

and \(H_{*}(X'', Y; M) = 0\) for all left coefficient modules \(M\) as required. \(\square\)

As \(X \sim X''\), we have proved the following theorem:

**Theorem 3.4** Let \(X\) be a finite connected \(3\)-complex with \(H^3(X; \beta) = 0\) for all coefficient bundles \(\beta\). Then \(X\) has the homotopy type of \(\mathcal{K}_{e}^{+}\) for some finite presentation \(e\) of a group \(E\), where \(+\) is taken with respect to some perfect finitely closed normal subgroup \(K \triangleleft E\).

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4 Implications for the D(2) problem

The D(2) problem asks if every finite cohomologically 2–dimensional 3–complex must be homotopy equivalent to a finite 2–complex. Clearly a counterexample must have a connected component which is also cohomologically 2–dimensional but not homotopy equivalent to a finite 2–complex. By Theorem 3.4 this component must have the homotopy type of $\mathcal{K}_e^+$ for some finite presentation $e$ of a group $E$, where $+$ is taken with respect to some perfect finitely closed normal subgroup $K \triangleleft E$.

Conversely, by Corollary 2.5, given any finite presentation $e$ of a group $E$ together with some perfect finitely closed normal subgroup $K \triangleleft E$ we have a cohomologically 2–dimensional finite 3–complex, $\mathcal{K}_e^+$. It follows that the D(2) problem is equivalent to:

Given a finite presentation $e$ for a group $E$, and a finitely closed perfect normal subgroup $K \triangleleft E$, must $\mathcal{K}_e^+$ be homotopy equivalent to a finite 2–complex?

Suppose that we have a homotopy equivalence $\mathcal{K}_e^+ \sim Y$ for some finite 2–complex $Y$. Let $T$ be a maximal tree in the 1–skeleton of $Y$. The quotient map $Y \to Y/T$ is a homotopy equivalence so $Y \sim \mathcal{K}_G$ for some finite presentation $G$ of $\pi_1(Y) = \pi_1(\mathcal{K}_e^+) = E/K$.

Hence the affirmative answer to the D(2) problem would be equivalent to:

For all finitely presented groups $E$ and all perfect finitely closed normal subgroups $K \triangleleft E$ and all finite presentations $e$ of $E$, there exists a finite presentation $G$ of $E/K$ and a homotopy equivalence $\mathcal{K}_e^+ \sim \mathcal{K}_G$ inducing the identity $1: E/K \to E/K$ on fundamental groups.

Lemma 4.1 The following are equivalent:

(i) There exists a homotopy equivalence $\mathcal{K}_e^+ \sim \mathcal{K}_G$ inducing the identity $1: E/K \to E/K$ on fundamental groups.

(ii) There exists a chain homotopy equivalence $C_*(\mathcal{K}_e^+) \sim C_*(\mathcal{K}_G)$ over $\mathbb{Z}[E/K]$.

Proof (i) $\Rightarrow$ (ii) is immediate. Conversely, from (ii) we have a chain homotopy equivalence between the algebraic complexes associated to a finite cohomologically 2–dimensional 3–complex and a finite 2–complex (with respect to an isomorphism of fundamental groups). To show that (ii) $\Rightarrow$ (i) we must construct a homotopy equivalence between the spaces, inducing the same isomorphism on fundamental groups. For finite fundamental groups this is done in [8, Proof of Theorem 59.4]. The same argument holds for all finitely presented fundamental groups [8, Appendix B, Proof of Weak Realization Theorem].
From the proof of Lemma 2.4, $C_*(K_e) \otimes E \mathbb{Z}[E / K] \sim C_*(K_e^+)$.

**Theorem 4.2** The following two statements are equivalent:

(i) Let $X$ be a a finite 3–complex with $H^3(X; \beta) = 0$ for all coefficient bundles $\beta$. Then $X$ is homotopy equivalent to a finite 2–complex.

(ii) Let $K$ be a perfect finitely closed normal subgroup of a finitely presented group $E$. For each finite presentation $\epsilon$ of $E$, there exists a finite presentation $\mathcal{G}$ of $E / K$, such that we have a chain homotopy equivalence over $\mathbb{Z}[E / K]$:

$$C_*(K_e) \otimes E \mathbb{Z}[E / K] \rightarrow C_*(K_\mathcal{G})$$

Suppose we have a short exact sequence

$$1 \rightarrow L \rightarrow F \rightarrow G \rightarrow 1$$

where $G$ is a finitely presented group and $F$ is a free group generated by elements $g_1, \ldots, g_n$. Let $R_1, \ldots, R_m$ be elements of $L$.

**Definition 4.3** $\langle g_1, \ldots, g_n \mid R_1, \ldots, R_m \rangle$ is called a finite partial presentation for $G$ when the normal closure $N_F(R_1, \ldots, R_m)$ surjects onto $L/[L, L]$ under the quotient map $L \rightarrow L/[L, L]$.

Note that a finite partial presentation $\epsilon = \langle g_1, \ldots, g_n \mid R_1, \ldots, R_m \rangle$ as above is an actual finite presentation of some group $E$, so it has a well defined Cayley complex $K_\epsilon$.

Let $K$ denote the kernel of the homomorphism $E \rightarrow G$ sending each $g_i$ to the corresponding element in $G$. If $G$ is finitely presented then it is finitely presented on the generators in $\epsilon$ [3, Chapter 1, Proposition 17]. As $K$ is the normal closure in $E$ of the images of this finite set of relators we have that $K$ is finitely closed.

Further $K$ is perfect as every $k \in K$ may be lifted to an element of $L$ which may be written in the form $ab$ where $a \in [L, L]$ and $b \in N_F(R_1, \ldots, R_m)$. Thus $k$ is equal to the image of $a$ in $E$, so $k \in [K, K]$. Thus a finite partial presentation $\epsilon$ of a finitely presented group $G$ may be viewed as a presentation satisfying the hypothesis’ of statement (ii) in Theorem 4.2.

Conversely, given $\epsilon$ as in statement (ii) of Theorem 4.2, we have that $\epsilon$ is a finite partial presentation of $E / K$ (as $K = [K, K]$), and $E / K$ is finitely presented (as $K$ is finitely closed).
Thus statement (ii) is equivalent to:

(ii)’ Given a finite partial presentation $\varepsilon$ of a finitely presented group $G$, there exists a finite presentation $\mathcal{G}$ of $G$, such that we have a chain homotopy equivalence

$$C_*(\mathcal{K}_\varepsilon) \otimes_E \mathbb{Z}[G] \to C_*(\mathcal{K}_\mathcal{G})$$

where $E$ is the group presented by $\varepsilon$ and each $x \in E$ acts on $\mathbb{Z}[G]$ by left multiplication by its image in $G$.

One approach to the D(2) problem is to use Euler characteristic as an obstruction. That is, given a finite cohomologically 2–dimensional 3–complex $X$, if we can show that every finite 2–complex $Y$ with $\pi_1(Y) = \pi_1(X)$ satisfies $\chi(X) < \chi(Y)$ then clearly $X$ cannot be homotopy equivalent to any such $Y$. It has been shown that certain constructions involving presentations of a group would allow one to construct such a space [6, Theorem 3.5]. A candidate for such a space is given in [2]. In light of Corollary 2.5 and Theorem 3.4 we are able to generalize this approach.

The deficiency $\text{Def}(\mathcal{G})$ of a finite presentation $\mathcal{G}$ is the number of generators minus the number of relators. We say a presentation of a group is minimal if it has the maximal possible deficiency. A finitely presented group $G$ always has a minimal presentation, because an upper bound for the deficiency of a presentation is given by $\text{rk}_\mathbb{Z}(G/[G, G])$. The deficiency $\text{Def}(G)$ of a finitely presented group $G$ is defined to be the deficiency of a minimal presentation.

Again let $K \triangleleft E$ be a perfect finitely closed normal subgroup. Then if $\varepsilon$ is a finite presentation of $E$ and $\mathcal{G}$ is a finite presentation for $E/K$ we have:

$$\chi(K^+_\varepsilon) = \chi(K_\varepsilon) = 1 - \text{Def}(\varepsilon), \quad \chi(K_\mathcal{G}) = 1 - \text{Def}(\mathcal{G})$$

**Lemma 4.4** If $\text{Def}(E) > \text{Def}(E/K)$ then given a minimal presentation $\varepsilon$ of $E$ we have that $\chi(K^+_\varepsilon) < \chi(K_\mathcal{G})$ for any finite presentation $\mathcal{G}$ of $E/K$.

**Proof** $\chi(K_\mathcal{G}) = 1 - \text{Def}(\mathcal{G}) \geq 1 - \text{Def}(E/K) > 1 - \text{Def}(E) = 1 - \text{Def}(\varepsilon) = \chi(K^+_\varepsilon)$. □

Suppose we have a short exact sequence of groups

$$1 \to K \to E \to G \to 1$$

with $E, G$ finitely presented. Then given a finite presentation for $E$, the images in $G$ of the generators will generate $G$. We may present $G$ on these generators with a finite set of relators [3, Chapter 1, Proposition 17]. Let $k_1, \ldots, k_r$ denote the elements of $K$ represented by these relators. Then $K$ is the normal closure in $E$ of $k_1, \ldots, k_r$ and so
$K$ is finitely closed in $E$. In particular $K/[K, K]$ is generated by the $k_1, \ldots, k_r$ as a right module over $\mathbb{Z}[G]$ (where $G$ acts on $K/[K, K]$ by conjugation). Let $\text{rk}_G(K)$ denote the minimal number of elements required to generate $K/[K, K]$ over $\mathbb{Z}[G]$.

**Theorem 4.5** The following statements are equivalent:

(i) There exists a connected finite cohomologically $2$–dimensional $3$–complex $X$, such that for all finite connected $2$–complexes $Y$ with $\pi_1(Y) = \pi_1(X)$ we have $\chi(X) < \chi(Y)$.

(ii) There exists a short exact sequence of groups $1 \rightarrow K \rightarrow E \rightarrow G \rightarrow 1$ with $E$, $G$ finitely presented and:

$$\text{rk}_G(K) + \text{Def}(G) < \text{Def}(E)$$

**Proof** (i) $\Rightarrow$ (ii) By Theorem 3.4, $X$ is homotopy equivalent to $K^+_\varepsilon$ for some finite presentation $\varepsilon$ of some group $E$ and some perfect finitely closed normal subgroup $K$. Let $G = E/K$. We have a short exact sequence:

$$1 \rightarrow K \rightarrow E \rightarrow G \rightarrow 1$$

As $K$ is finitely closed, $G$ is finitely presented. As $K$ is perfect we have $\text{rk}_G(K) = 0$. Let $\mathcal{G}$ be some finite presentation of $G$. We have:

$$1 - \text{Def}(\varepsilon) = \chi(K^+_\varepsilon) < \chi(K_\mathcal{G}) = 1 - \text{Def}(\mathcal{G})$$

Thus $\text{Def}(\mathcal{G}) < \text{Def}(\varepsilon)$. As $\mathcal{G}$ was chosen arbitrarily, we have $\text{Def}(G) < \text{Def}(\varepsilon) \leq \text{Def}(E)$. Hence $0 + \text{Def}(G) < \text{Def}(E)$ as required.

(ii) $\Rightarrow$ (i) We start with the short exact sequence $1 \rightarrow K \rightarrow E \rightarrow G \rightarrow 1$. Let $k_1, \ldots, k_r \in K$ generate $K/[K, K]$ over $\mathbb{Z}[G]$, where $r = \text{rk}_G(K)$. Let $K'$ denote the normal closure in $E$ of $k_1, \ldots, k_r$. Then we have a short exact sequence:

$$1 \rightarrow K/K' \rightarrow E/K' \rightarrow G \rightarrow 1$$

Then $K = K'[K, K]$ so $K/K'$ is perfect. From the discussion preceding this theorem we know that $K$ is finitely closed in $E$, so $K/K'$ must be finitely closed in $E/K'$. Also $E/K'$ may be presented by taking a minimal presentation of $E$ and adding $r$ relators (representing to $k_1, \ldots, k_r$). Hence:

$$\text{Def}(E/K') \geq \text{Def}(E) - \text{rk}_G(K) > \text{Def}(G)$$

Take a minimal presentation $\varepsilon$ of $E/K'$ and let $X = K^+_\varepsilon$, where $+$ is taken with respect to $K/K'$. Any finite connected $2$–complex $Y$ with $\pi_1(Y) = \pi_1(X)$ is homotopy equivalent to $K_\mathcal{G}$ for some finite presentation $\mathcal{G}$ of $G$. Therefore by Lemma 4.4 we have $\chi(X) < \chi(Y)$ as required.

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We note that Michael Dyer proved (ii) \(\Rightarrow\) (i) in the case where \(H^3(G; \mathbb{Z}[G]) = 0\) and \(E\) is a free group whose generators are the generating set for some minimal presentation of \(G\) [6, Theorem 3.5].

**References**


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