The equivariant J–homomorphism for finite groups at certain primes

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Suppose G is a finite group and p a prime, such that none of the prime divisors of G are congruent to 1 modulo p. We prove an equivariant analogue of Adams' result that $J' = J''$. We use this to show that the G–connected cover of $Q_G S^0$, when completed at p, splits up to homotopy as a product, where one of the factors of the splitting contains the image of the classical equivariant J–homomorphism on equivariant homotopy groups.

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1 Introduction

The study of the stable homotopy groups of spheres has dominated the attention of homotopy theorists since the inception of the subject. The work of Adams in his series of $J(X)$ papers [1; 2; 3; 4] was a major early breakthrough in this study. There, Adams developed a program for understanding the group $J(X)$, which is the quotient of the group of virtual bundles over a space X by the group of such bundles whose underlying spherical fibrations (obtained by compactifying fibers) are trivial. Thus, $J(X)$ can be thought of as the group of those stable spherical fibrations obtained from stable vector bundles over a base space. Specializing the base space to spheres amounts to studying the subgroups of the stable homotopy groups of spheres arising from the homotopy groups of the stable orthogonal groups, or the image of the classical J homomorphism.

Adams proposed to study $J(X)$ by defining two groups $J'(X)$ and $J''(X)$ equipped with canonical epimorphisms $J''(X) \to J(X) \to J'(X)$. By showing the composite of these epimorphisms to be an isomorphism, he could compute $J(X)$ by computing either $J''(X)$ or $J'(X)$, both of which are more easily studied. Adams completed his work modulo his eponymous conjecture, which was confirmed by Quillen [18].

The groups $J'(X)$ and $J''(X)$ can be described most easily after localizing at a given prime p. Then $J''(X)$ is obtained by finding a collection of virtual bundles with trivial underlying stable spherical fibrations. To be a little more precise, we let ψ^k denote the k–th Adams operation for an integer k. The Adams conjecture asserts then that for

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suitable choice of k, given a virtual bundle ξ, the underlying stable spherical fibrations of $\psi^k\xi$ and ξ are equivalent. Thus, $J''(X)$ is taken to be the quotient of the group $\widetilde{KO}(X)$ of zero-dimensional virtual bundles over X by the image of $\psi^k - 1$, and one immediately sees that there is an epimorphism from $J''(X)$ to $J(X)$.

The group $J'(X)$, in turn, is obtained by finding a condition satisfied by all virtual bundles with trivial underlying stable spherical fibrations. More precisely, Adams showed that if ξ is a virtual bundle of virtual dimension 0, with trivial underlying stable spherical fibration, then after applying a certain operation ρ^k to ξ, the resulting bundle could be written in the form $(\psi^k/1)(1+\zeta)$, for some virtual bundle ζ of virtual dimension 0. Thus, $J'(X)$ is taken to be the quotient of $\widetilde{KO}(X)$ by all bundles satisfying this very condition, and one sees that there is an epimorphism from $J(X)$ to $J'(X)$.

Adams' work was later rephrased in a geometric context by May [16], who considered the J–theory diagram (see Figure 1), in which each row is a fiber sequence and each space is implicitly localized away from an integer k.

$$\begin{tikzcd}
J^k \arrow[r, "\pi"] \arrow[d, "\alpha"] & BO \arrow[r, "\psi^k - 1"] \arrow[d, "\gamma"] & B\mathrm{Spin} \arrow[d, equal] & \\
\mathrm{SF} \arrow[r, "\tau"] \arrow[d, equal] & \mathrm{SF}/\mathrm{Spin} \arrow[r, "q"] \arrow[d, "f"] & B\mathrm{Spin} \arrow[r, "Bj"] \arrow[d, "g"] & B\mathrm{SF} \arrow[d, equal] \\
\mathrm{SF} \arrow[r] \arrow[d, "\varepsilon"] & BO_{\otimes} \arrow[r, "\tau"] \arrow[d, equal] & B(\mathrm{SF}; kO) \arrow[r, "q"] \arrow[d, "c(\psi^k)"] & B\mathrm{SF} \\
J^k_{\otimes} \arrow[r, "\pi"] & BO_{\otimes} \arrow[r, "\psi^k/1"] & B\mathrm{Spin}_{\otimes} &
\end{tikzcd}$$

Figure 1: May's J–theory diagram

In this diagram, the spaces BO, $B\mathrm{Spin}$, and $B\mathrm{SF}$ are classifying spaces for stable orthogonal bundles, stable bundles with spin structure, and stable spherical fibrations. The space $B(\mathrm{SF}; KO)$ classifies spherical fibrations ξ together with a KO–valued orientation, ie an orientation class $\mu \in \widetilde{KO}(T\xi)$, where $T\xi$ is the Thom space of ξ. These are all Hopf spaces, since such bundles and fibrations can be added by direct sum or fiberwise smash product. The spaces $BO_{\otimes}$ and $B\mathrm{Spin}_{\otimes}$ are equivalent as spaces to BO and $B\mathrm{Spin}$, but with Hopf space structure obtained from tensor products of bundles. The map Bj represents replacing a stable bundle with its underlying spherical fibration, obtained by compactifying fibers. The space SF of stable degree 1 self-maps of spheres is equivalent to $\Omega B\mathrm{SF}$. The map γ is obtained from the Adams conjecture, which gives an equivalence between $Bj(\psi^k(\xi))$ and $Bj(\xi)$ for a stable vector bundle ξ. The space $BO_\otimes$ can be realized as the fiber of the map from $B(\mathrm{SF}; KO)$ to $B\mathrm{SF}$; the map f is induced as a map of fiber sequences.

The map g comes from the Atiyah–Bott–Shapiro orientation of Spin–bundles [8]. To obtain the map $c(\psi^k)$, one considers the difference between a KO–oriented spherical fibration (ξ, μ) and the KO–oriented spherical fibration $(\xi, \psi^k(\mu))$. Since these have the same underlying spherical fibrations, the classifying map of this difference lifts to the fiber $BO_\otimes$, and, one can show, lifts further to $B\mathrm{Spin}_\otimes$. The composite $c(\psi^k)\circ g$ is denoted ρ^k and represents the Adams–Bott cannibalistic class, something particularly amenable to algebraic study.

Now, if we let $\sigma^k = f\circ\gamma$, then the central columns of the diagram above form a square, which we call the Adams–May square:

$$\begin{tikzcd} BO \arrow[r, "\psi^k-1"] \arrow[d, "\sigma^k"'] & B\mathrm{Spin} \arrow[d, "\rho^k"] \\ BO_\otimes \arrow[r, "\psi^k/1"] & B\mathrm{Spin}_\otimes \end{tikzcd}$$

May proved that after localizing at a prime, then for a suitable choice of k, this square is a pullback in the homotopy category. This is remarkable, in that pullbacks are rare in homotopy categories, where one typically instead encounters homotopy pullbacks. In particular, this implies that for a space X, one obtains, instead of just an exact sequence, an actual pullback diagram:

$$\begin{tikzcd} \widetilde{KO}(X)^\wedge_p \arrow[r, "\psi^k-1"] \arrow[d, "\sigma^k"'] & \widetilde{K\mathrm{Spin}}(X)^\wedge_p \arrow[d, "\rho^k"] \\ 1+\widetilde{KO}(X)^\wedge_p \arrow[r, "\psi^k/1"] & 1+\widetilde{K\mathrm{Spin}}(X)^\wedge_p \end{tikzcd}$$

Thus, if $\rho^k(\xi) = (\psi^k/1)(1+\zeta)$, then ξ is in the image of $\psi^k - 1$, implying Adams' result that $J'(X) = J''(X)$.

This geometric rephrasing of Adams' work yields extra information. Knowing that the Adams–May square is a pullback in the homotopy category implies that the composite $\varepsilon\circ\alpha$ in the left column of the J–theory diagram above induces a monomorphism on homotopy groups. Since the homotopy groups of J^k and $J^k_\otimes$ are abstractly isomorphic and finite, this implies $\varepsilon \circ \alpha$ is a weak equivalence, so that the p–localization of SF splits up to homotopy, with one factor being J^k. The image of this factor on homotopy groups is closely related to the image of the classical J–homomorphism in the stable homotopy groups of spheres. We should note that the existence of such a splitting of SF was first discovered by Sullivan, using different methods.

In [13], we constructed an equivariant version of the Adams–May square, and we considered the case when G is p–group, with $p \neq 2$. We proved that for suitable k, after restricting to G–connected covers and completing at the prime p, the equivariant Adams–May square becomes a pullback in the homotopy category. This is then an equivariant version of the statement $J' = J''$. Moreover, we used this result to show that p–completion of an equivariant analogue of SF splits as a product $J_p \times C_p$, where the fixed point subspaces of J_p capture the image of the equivariant J–homomorphism. One of the results of the current paper is to extend this result to the prime $p = 2$. We show that when G is a 2–group, then for $k = 3$, the equivariant Adams–May square becomes a pullback in the homotopy category after 2–completing the G–connected cover.

The case when the order of G is not a power of p is more subtle. In fact, if G is not a p–group, then the p–completion of the equivariant Adams–May square will not be a pullback in the homotopy category for any value of k. However, we show in this paper that when p is a fixed prime and none of the prime divisors of the order of G are congruent to 1 modulo p, there is a natural splitting of the p–completion of equivariant K–theory:

$$K_G(X)^{\wedge}_p \cong W_G(X) \times W_G^{\perp}(X)$$

This splitting, which can also be realized on the level of equivariant KO–theory, is compatible with Adams operations as well as the maps ρ^k and σ^k of the Adams–May square, so that the entire Adams–May square splits as a product of two squares. We show that every element in $W_G^{\perp}(X)$ can be written as a linear combination of elements of the form $(\psi^l - 1)(\xi)$ where l ranges over integers which are relatively prime to p and $|G|$. Thus, these elements have trivial underlying spherical fibrations by the Equivariant Adams Conjecture, and so do not contribute to the image of the J–homomorphism. Moreover, we show that for suitable k, the restriction of the Adams–May square to the W factor is a pullback. As a corollary, we obtain a splitting of the equivariant analogue of SF at the prime p.

A good understanding of the equivariant Adams–Bott class ρ^k lies at the heart of both this work and our previous work in [13]. There, we showed that when G is a p–group, then p–complete K_G–theory takes values in p–adic λ–rings, described by

Atiyah and Tall [9]. Thus, we were able to use their results to show that ρ^k induces an isomorphism on the Adams summand of p–complete K_G–theory. We would guess that Atiyah and Tall were primarily motivated in their study of λ–rings by an interest in giving a more conceptual and less computational proof of Adams' result in [2]. Their approach was certainly the right way to think about the equivariant case for p–groups. However, in the current paper, their work does not apply, and we are instead inspired by Adams' original work – we generalize the computational machinery he developed to reduce to the case of a p–group.

The condition that we impose on the order of G may seem unusual, but we make essential use of this condition in our arguments. As will be seen, we build from the simplest case when $G = \mathbb{Z}/q$, where q is a prime different from p. For this case, we need the number of units in G, which is $q-1$, to be invertible, and this requires that p does not divide $q-1$. While it is difficult to prove that one could not find a way around the obstacles we have encountered, we believe that a substantially new idea would be required. Indeed, we show that if some prime q divides $|G|$, and p divides $q-1$, then there cannot exist a splitting of p–completed equivariant KO–theory into two factors, in which one factor contains only linear combinations of elements of the form $(\psi^l - 1)(\xi)$, and such that for some k, the restriction of the Adams–May square to the other factor is a pullback. This does not preclude the existence of a splitting of equivariant SF, but it does suggest that the technique we use in this paper cannot be extended without a substantially different idea.

Our paper is organized as follows. In Section 2, we give just enough background to understand the equivariant J–theory diagram, and we formally state the main result of the paper; we also give a proof of this theorem, using results from later in the paper. In Section 3, we define our generalized Adams summand in equivariant K–theory, and in Section 4, we study the equivariant Adams–Bott map ρ^k. In Section 5, we study the map of equivariant classifying spaces $B_G\mathrm{Spin} \to B_G O$, in order to prove that $\psi^k - 1$ lifts to $B_G\mathrm{Spin}$. In Section 6, we construct the map ρ^k, and prove that ρ^k and σ^k become homotopic maps after restricting to G–connected covers. Finally, in Section 7, which is an appendix, we prove a somewhat technical result (Corollary 2.3) about the zeroth space of the equivariant p–complete sphere spectrum, and we show why our technique for proving Theorem 2.10 cannot be extended to the case where one of the prime divisors of G is congruent to 1 modulo p.

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2 Background and main result

Some of the material in this section comes straight out of our work in [13], and more details can be found there. Our first task is to understand the equivariant analogues of the classifying spaces involved in the J–theory diagram. It is useful to begin as generally as possible, so we suppose A and G to be compact Lie groups. A principal (G,A)–bundle is a principal A–bundle $p\colon E \to B$, where p is also a map of G–spaces, and the actions of G and A on E commute. Waner [20] constructed a classifying space $B_G A$ for principal (G,A)–bundles. A G–map $\rho\colon G \to A$ endows $B_G A$ with a canonical G–fixed basepoint. One tool for studying a based G–space such as $B_G A$ is to pass to G–connected covers.

Notation 2.1 Given a based G–space X, let $\mathcal{S}_G X$ denote the G–connected cover of X. That is, $\mathcal{S}_G X$ is a based G–space equipped with a based G–map $\mathcal{S}_G X \to X$ such that for each $H \le G$, the restriction $(\mathcal{S}_G X)^H \to X^H$ induces a homotopy equivalence between $(\mathcal{S}_G X)^H$ and the basepoint component of X^H.

One uses Elmendorf's construction [12] to prove the existence and uniqueness up to homotopy of the G–connected cover.

In [13], we found the following simple model for $\mathcal{S}_G B_G A$. Suppose $\rho\colon G \to A$ is a distinguished homomorphism, determining a basepoint for $B_G A$. Then G acts on A by conjugation: $g \cdot a = \rho(g) a \rho(g)^{-1}$. This in turn induces an action of G on BA, with $(BA)^H = B(A^H)$. Thus, we may view BA as a G–connected G–space. The following proposition is obtained by combining Lemmas 2.19 and 2.20 in [13].

Proposition 2.2 *The G–connected cover of $B_G A$ at the basepoint obtained from a homomorphism $\rho\colon G \to A$ is equivariantly homotopy equivalent to BA, where the action on BA is induced by ρ.*

In [13], we constructed stabilization maps to build classifying spaces for various types of stable bundles. These are denoted $B_G O$, $B_G \mathrm{SO}$, $B_G \mathrm{Spin}$, $B_G U$, and $B_G \mathrm{SU}$. We proved that each of these classifying spaces has a weak G–Hopf space structure. Weak, here, means that the associativity and unit diagrams commute up to homotopy when restricted to finite G–CW complexes. One has obvious maps such as $B_G \mathrm{Spin} \to B_G \mathrm{SO}$ and $B_G \mathrm{SO} \to B_G O$ classifying forgetful functors.

Building on work of Waner [20], we also developed equivariant analogues for the classifying spaces of $\mathcal{F}$–fibrations, which May described in [15]. A *set of admissible fibers* $\mathbb{F}$ is a set of based-spaces $\{F_\lambda\}$, where each space is endowed with a left base-point preserving action of a closed subgroup H_λ of G, such that whenever K is subconjugate to H_λ, the K–space obtained from F_λ is also in $\mathbb{F}$. There is then a notion of a $G\mathcal{F}(\mathbb{F})$–fibration, generalizing an $\mathcal{F}$–fibration. In such fibrations, the fiber over a point with isotropy group K is a K–space F which is equivalent to one of the K–spaces F_λ in the set $\mathbb{F}$. There is a classifying space $B_G(\mathbb{F})$ classifying such $G\mathcal{F}(\mathbb{F})$–fibrations.

For example, one could take V to be a finite-dimensional complex inner product space. For an action λ of a subgroup $H \le G$ on V, we denote the one-point compactification of the representation V_λ by S^{V_λ}. The collection of such spaces forms an admissible set of fibers $\mathbb{S}^V$, and hence determines a classifying space $B_G(\mathbb{S}^V)$. Restricting to those actions H on V such that V^H is nonzero, we could consider the set of all p–completions $(S^{V_\lambda})^\wedge_p$, along with the restrictions of such spaces to subconjugates of H. We denote the associated set of fibers $(\mathbb{S}^V)^\wedge_p$, and the associated classifying space $B_G((\mathbb{S}^V)^\wedge_p)$. We constructed stable classifying spaces $B_G(\mathbb{S})$ and $B_G(\mathbb{S}^\wedge_p)$ obtained by taking colimits of the above classifying spaces over stabilization maps. We also constructed a fiberwise completion map $k^\wedge_p$ from $B_G(\mathbb{S})$ to $B_G(\mathbb{S}^\wedge_p)$.

Finally, we constructed a space $B_G(\mathbb{S}^\wedge_p; KO^\wedge_p)$ which classifies $G\mathcal{F}(\mathbb{S}^\wedge_p)$–fibrations equipped with a $KO^\wedge_p$–orientation. As in the nonequivariant case, the $G\mathcal{F}(\mathbb{S}^\wedge_p)$–fibration obtained from a (G, Spin)–bundle has an Atiyah–Bott–Shapiro $KO^\wedge_p$–orientation, and we thus obtain a map $g^\wedge_p\colon B_G\mathrm{Spin} \to B_G(\mathbb{S}^\wedge_p; KO^\wedge_p)$, which is of fundamental importance in the J–theory diagram. We let $Bj^\wedge_p$ denote the composite of $g^\wedge_p$ with the map $q\colon B_G(\mathbb{S}^\wedge_p; KO^\wedge_p) \to B_G(\mathbb{S}^\wedge_p)$ representing the forgetful functor. Then $Bj^\wedge_p$ represents the forgetful functor from equivariant Spin–bundles to $G\mathcal{F}(\mathbb{S}^\wedge_p)$–fibrations.

Now, in the middle row in the J–theory diagram Figure 1, we used the identification between $\Omega B\mathrm{SF}$ and SF to identify the left-hand term. We thus need to consider what SF should mean in an equivariant context. The equivariant analogue of QS^0 is given by $Q_G S^0 = \mathrm{colim}_{V_i}\, \Omega^{V_i} S^{V_i}$ where V_i runs over a suitable sequence of complex G–representations. We denote this space F_G. The group G acts on F_G by conjugation, with the basepoint determined by the identity map. There are two natural choices for an equivariant analogue of SF: the basepoint component of F_G, or the G–connected cover: $\mathcal{S}_G F_G$. In this paper, we will work with $\mathcal{S}_G F_G$. In Section 7.1, we will prove the following corollary, which is the best available generalization of the nonequivariant result $\Omega B\mathrm{SF} \simeq \mathrm{SF}$.

Corollary 2.3 *Let p be a prime number. The space $\mathcal{S}_G \Omega B_G(\mathbb{S}^\wedge_p)$ is G–equivalent to $(\mathcal{S}_G F_G)^\wedge_p$. Also, $\pi_n(B_G(\mathbb{S}^\wedge_p)^H)$ is p–complete for all $n \ge 1$. Thus, $[X, B_G(\mathbb{S}^\wedge_p)^H]_G$ is p–complete whenever X is a finite G–connected G–CW complex.*

Both equivariantly and nonequivariantly, Spin bundles have KO–orientations, so that the source of the map g in the J–theory diagram Figure 1 is $B\mathrm{Spin}$. Conveniently, it is not hard to show that the map $\psi^k - 1\colon BO \to BO$ lifts uniquely to $B\mathrm{Spin}$. We show in Section 5.2 that the map $\psi^k - 1\colon B_G O \to B_G O$ also lifts to $B_G\mathrm{Spin}$, and that the restriction of this lift to $\mathcal{S}_G B_G O$ is unique.

Arguably the most important tool in the construction of the J–theory diagram in Figure 1 is the validity of the Adams Conjecture. Recall that this implies that after suitable localization, a certain collection of bundles are not detected by the J–homomorphism. On the classifying-space level, this allows us to define the map γ in the J–theory diagram. We now turn to considering these ideas in the equivariant context.

The following is tom Dieck's version of the Equivariant Adams Conjecture, quoted verbatim. Here, G is a finite group, E an orthogonal G–bundle, $S(E)$ its associated (stable) spherical fibration, and k is odd.

Theorem 2.4 [11, Theorem 11.3.8] *There exist stable G–maps $f\colon S(E) \to S(\psi^k E)$ such that f^H has for all $H < G$ a degree which divides a power of k (k prime to $|G|$).*

Definition 2.5 Let $T_G(X)$ denote the subgroup of $\widetilde{KO}_G(X)$ generated by the images of the maps $\psi^k - 1$, where k ranges over all odd integers relatively prime to p and $|G|$. Let $T_G(X)^\wedge_p$ denote the p–completion of $T_G(X)$.

Thus, the Equivariant Adams Conjecture implies that elements in $T_G(X)$ are not detected by the equivariant J–homomorphism.

Remark 2.6 Note that in the nonequivariant case, it is possible to choose a single integer k which topologically generates the units in $\mathbb{Z}^\wedge_p$, at least for p odd. With such a choice, the image of $\psi^k - 1$ contains the images of $\psi^l - 1$ for all other integers l relatively prime to p (after p–completion). Therefore, the nonequivariant analogue of $T_G(X)^\wedge_p$ can be described quite simply as the image of a single map $\psi^k - 1$. Similarly, if G is a p–group and k is a topological generator of the units in $\mathbb{Z}^\wedge_p$, then for a G–space X, $T_G(X)^\wedge_p$ can be again realized as the image of a single map $\psi^k - 1$. On the other hand, if G is not a p–group, this is no longer in general possible. One needs the images of $\psi^k - 1$ for several values of k to generate all of $T_G(X)^\wedge_p$.

For now, we simply take k to be an odd number prime to p and $|G|$. We wish to use the Equivariant Adams Conjecture to define the map γ^k in the equivariant J–theory diagram. By Corollary 2.3, when X is a finite G–connected G–CW complex, the group

$[X, B_G(\mathbb{S}^\wedge_p)]$ is p–complete. Therefore, if k is prime to p and $|G|$, then it follows from Theorem 2.4 and a standard $\lim^1$–argument that the composite $Bj^\wedge_p \circ (\psi^k - 1)$ becomes null-homotopic after restricting to $\mathcal{S}_G B_G O$. By choosing a null-homotopy and restricting to $\mathcal{S}_G B_G O$, we obtain a map γ^k from $\mathcal{S}_G B_G O$ to the G–connected cover of the fiber of $Bj^\wedge_p$.

Remark 2.7 The map γ^k is determined by a choice of null-homotopy, so we have some flexibility in defining it. We will only need to use this flexibility for the J–theory diagram at $p = 2$. See Remark 6.10 for details.

Now, for a given odd number k which is relatively prime to p and $|G|$, the equivariant J–theory diagram is displayed in Figure 2, and this should be compared to Figure 1. Given a map $f\colon X \to Y$, $\mathrm{Fib}(f)$ denotes the homotopy fiber of f.

$$\begin{tikzcd}
\mathcal{S}_G\mathrm{Fib}(\psi^k - 1) \arrow[r] \arrow[d, "\alpha^k"] & \mathcal{S}_G B_G O \arrow[r, "\psi^k - 1"] \arrow[d, "\gamma^k"] & \mathcal{S}_G B_G \mathrm{Spin} \arrow[d, equal] & \\
(\mathcal{S}_G F_G)^\wedge_p \arrow[r] \arrow[d, equal] & \mathcal{S}_G\mathrm{Fib}(Bj^\wedge_p) \arrow[r] \arrow[d, "f"] & \mathcal{S}_G B_G \mathrm{Spin} \arrow[r, "Bj^\wedge_p"] \arrow[d, "g^\wedge_p"] & \mathcal{S}_G B_G(\mathbb{S}^\wedge_p) \arrow[d, equal] \\
(\mathcal{S}_G F_G)^\wedge_p \arrow[r] & \mathcal{S}_G\mathrm{Fib}(q) \arrow[r, "\tau"] \arrow[d, hook, "\iota"] & \mathcal{S}_G B_G(\mathbb{S}^\wedge_p; KO^\wedge_p) \arrow[r, "q"] \arrow[d, "c(\Psi^k)"] & \mathcal{S}_G B_G(\mathbb{S}^\wedge_p) \\
& \mathcal{S}_G B_G O_{\otimes}{}^\wedge_p \arrow[r, "\psi^k/1"] & \mathcal{S}_G B_G O_{\otimes}{}^\wedge_p. &
\end{tikzcd}$$

Figure 2: The equivariant J–theory diagram

In this diagram, the first three rows are G–connected covers of fiber sequences; this follows for the second and third row by Corollary 2.3. The map α^k is the map of homotopy fibers determined by γ^k. The map f is the map of fibers determined by $g^\wedge_p$. The space $B_G O_\otimes$ is the same space as $B_G O$, but with Hopf-space structure determined by tensor products of bundles of virtual dimension 1. Thus, $\psi^k/1$ represents the operation of taking such a bundle μ and replacing it with $\psi^k(\mu)/\mu$. The map $c(\Psi^k)$ (see Section 6) represents replacing a $KO^\wedge_p$–oriented $G\mathcal{F}(\mathbb{S}^\wedge_p)$–fibration (E, μ) with the virtual bundle $\psi^k(\mu)/\mu$. A $KO^\wedge_p$–orientation of a trivialized $G\mathcal{F}(\mathbb{S}^\wedge_p)$–fibration determines a virtual bundle over the base space; this gives a geometric interpretation of the map ι. The map ρ^k is defined to be $c(\Psi^k) \circ g^\wedge_p$; this represents the Adams–Bott cannibalistic class. As in the nonequivariant setting, we let $\sigma^k = \iota \circ f \circ \gamma^k$.

As we show in Section 5.2 and Section 6.1, there are homotopy lifts of $\psi^k/1$ and ρ^k to $\mathcal{S}_G B_G \mathrm{Spin}_{\otimes}{}^\wedge_p$. We show in Corollary 5.4 that for a G–connected space X, if a map $X \to B_G O$ lifts to $B_G \mathrm{Spin}$, then that lift is unique up to homotopy. The same argument applies to $\mathcal{S}_G B_G \mathrm{Spin}_{\otimes}{}^\wedge_p$, so that the square on the right below commutes up to homotopy. This induces the map ε^k on the left.

$$\begin{tikzcd}
(\mathcal{S}_G F_G)^\wedge_p \arrow[r] \arrow[d, dotted, "\varepsilon^k"] & \mathcal{S}_G \mathrm{Fib}(Bj^\wedge_p) \arrow[r] \arrow[d, "i\circ f"] & \mathcal{S}_G B_G \mathrm{Spin} \arrow[d, "\rho^k"] \\
\mathcal{S}_G \mathrm{Fib}(\psi^k/1)^\wedge_p \arrow[r] & \mathcal{S}_G B_G O_{\otimes}{}^\wedge_p \arrow[r, "\psi^k/1"] & \mathcal{S}_G B_G \mathrm{Spin}_{\otimes}{}^\wedge_p
\end{tikzcd}$$

One might hope that for suitably chosen k, the composite $\varepsilon^k \circ \alpha^k$ would induce an equivalence, thus giving an equivariant splitting of $(\mathcal{S}_G F_G)^\wedge_p$. We proved this in [13] assuming that p is odd and the order of G is a power of p. But, in the general case, there is no good choice for k, as we remarked in Remark 2.6. The fact that, in the nonequivariant setting, the image of the classical J–homomorphism at the prime p could be realized as the homotopy groups of $\mathrm{Fib}(\psi^k - 1)$ for a particular value of k turns out to be something of a happy accident that carries over only partially into the equivariant setting. More generally, $\mathrm{Fib}(\psi^k - 1)$ is simply the wrong space to be called J. We need to define a different space.

We will look at a small summand of p–complete KO_G–theory, which determines a factor of the space $B_G O^\wedge_p$. This summand, as we will see, carries all the homotopical information we need; using the Equivariant Adams Conjecture, one can show that the restriction of the J–homomorphism to the complementary summand is trivial.

To prepare for our definition, we first recall the Adams splitting of p–complete KO–theory. In the p–adic metric, the Adams operations act continuously on p–complete KO–theory, so one can define Adams operations ψ^α, where α is any p–adic integer. In particular, one can choose α to be a $(p-1)$–st root of unity, so that $KO(X)^\wedge_p$ naturally splits as a product of eigenspaces. By Brown representability, $BO^\wedge_p$ then splits as a product $W \times W^\perp$, where W represents the summand of $KO(X)^\wedge_p$ corresponding to the elements fixed by ψ^α. The space W is often called the Adams summand of $BO^\wedge_p$.

Equivariantly, the Adams operations again act continuously on p–complete KO_G–theory. Thus, one can define ψ^α where α is a $(p-1)$–st root of unity, so one could again define a splitting just as before. However, when G is not a p–group, one can in fact whittle down p–complete KO_G–theory quite a bit further. In Section 3 (see in particular Remark 3.11), we construct a natural splitting of p–complete KO_G–theory:

$$KO_G(X)^\wedge_p \cong W_G(X) \oplus WO_G^\perp(X)$$

When G is not a p–group, our group $W_G(X)$ is significantly smaller than the fixed points of ψ^α. In particular, if $G = Q \times P$, where P is a Sylow p–subgroup, then $W_G(S^{2m})$ is isomorphic to a tensor product $W(Q) \otimes W_P(S^{2m})$, where $W(Q) \subseteq R(Q)$ consists of those representations which are fixed by ψ^k for all k relatively prime to the order of Q. (See Remark 3.8). As we explain in Remark 3.11, the splitting of $KO_G(X)^\wedge_p$ leads to a splitting of $\mathcal{S}_G B_G \mathrm{SO}^\wedge_p$ as a product $W_G \times WO_G^\perp$. We let $(W_G)_\otimes$ and $(WO_G^\perp)_\otimes$ denote the (homotopically equivalent) corresponding factors of $\mathcal{S}_G B_G \mathrm{SO}_\otimes{}^\wedge_p$.

We are now ready to define our space J_G. Let p be a prime and suppose G is a finite group such that none of the prime divisors of $|G|$ are congruent to 1 modulo p. If, for example $p = 2$, then G must be a 2–group, and we simply define J_G to be $\mathcal{S}_G\mathrm{Fib}(\psi^3 - 1)^\wedge_2$ and $J_{G\otimes}$ to be $\mathcal{S}_G\mathrm{Fib}(\psi^3/1)^\wedge_2$. If p is odd, let k be an odd integer which is relatively prime to $|G|$ and which is also a topological generator in $\mathbb{Z}^\wedge_p$. Because Adams operations commute, the operation $\psi^k - 1$ can be restricted to a natural transformation on $W_G(X)$, and so induces a self-map of the representing space W_G. We define J_G to be the fiber of this map. Similarly, the operation $\psi^k/1$ induces a self-map of $W_{G\otimes}$, and we define $J_{G\otimes}$ to be the fiber of this map.

Remark 2.8 It should be remembered that the spaces J_G and W_G are defined in terms of the prime p and the integer k. To simplify notation, we have not incorporated these data into the notation for these spaces.

Since p is odd, the maps $B_G\mathrm{Spin} \to B_G\mathrm{SO}$ and $B_G\mathrm{SO} \to B_G O$ are p–equivalences (by Lemma 5.2 and Lemma 5.3), so we will refer just to $B_G\mathrm{SO}$. Now we have the (homotopy) commutative diagrams in Figure 3.

We could define the classical J–homomorphism as the map $\pi_n(\mathrm{Spin}) \to \pi_n(\mathrm{SF})$, induced by taking homotopy groups of the map $\mathrm{Spin} \to \mathrm{SF}$. Nonequivariantly, the homotopy groups of J are equal to the image of the classical J–homomorphism at an odd prime, and contain this image when $p = 2$. If we replace Spin and SF by their equivariant counterparts Spin and $\mathcal{S}_G F_G$, then for each $H \leq G$, we have a J–homomorphism $\pi_n(\mathrm{Spin}^H) \to \pi_n(\mathcal{S}_G F_G^H)$. We now show that, at a prime p, the homotopy groups of the fixed-point subspaces J_G^H are equal to the image of this J–homomorphism when p is odd, and contain this image when $p = 2$. We have the following maps of fiber sequences.

$$\begin{tikzcd}
\mathrm{Spin} \arrow[r] \arrow[d, equal] & \mathrm{Fib}(\psi^k - 1) \arrow[r] \arrow[d] & B_G O \arrow[r, "\psi^k - 1"] \arrow[d] & B_G\mathrm{Spin} \arrow[d, equal] & \\
\mathrm{Spin} \arrow[r, "\Omega Bj"] & \Omega B_G(\mathbb{S}^\wedge_p) \arrow[r] & \mathrm{Fib}(Bj) \arrow[r] & B_G\mathrm{Spin} \arrow[r, "Bj"] & B_G(\mathbb{S}^\wedge_p)
\end{tikzcd}$$

$$\begin{tikzcd}
J_G \arrow[r] \arrow[d] & W_G \arrow[r, "\psi^k-1"] \arrow[d] & W_G \arrow[d] \\
\mathcal{S}_G\mathrm{Fib}(\psi^k-1)^\wedge_p \arrow[r] & \mathcal{S}_G B_G\mathrm{SO}^\wedge_p \arrow[r, "\psi^k-1"] & \mathcal{S}_G B_G\mathrm{SO}^\wedge_p,
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{S}_G\mathrm{Fib}(\psi^k/1)^\wedge_p \arrow[r] \arrow[d] & \mathcal{S}_G B_G\mathrm{SO}_\otimes{}^\wedge_p \arrow[r, "\psi^k/1"] \arrow[d] & \mathcal{S}_G B_G\mathrm{SO}_\otimes{}^\wedge_p \arrow[d] \\
(J_{G\otimes}) \arrow[r] & W_{G\otimes} \arrow[r, "\psi^k/1"] & W_{G\otimes}.
\end{tikzcd}$$

Figure 3

Here, we have used Proposition 2.2 to identify $\Omega B_G\mathrm{Spin}$ with Spin and $\Omega B_G\mathrm{SO}$ with SO. Also, note that by Corollary 2.3, $\pi_n(\Omega B_G(\mathbb{S}^\wedge_p)^H) \cong \pi_n((\mathcal{S}_G F_G)^{\wedge H}_p)$ for all $n \geq 1$.

Proposition 2.9 *For any $H \leq G$ and any $n \geq 1$, the image of*

$$\pi_n(\Omega Bj)\colon \pi_n(\mathrm{Spin}^H) \to \pi_n(\Omega B_G(\mathbb{S}^\wedge_p)^H) \cong \pi_n((\mathcal{S}_G F_G)^{\wedge H}_p)$$

is contained in the image of

$$\pi_n(J^H_G) \to \pi_n(\mathrm{Fib}(\psi^k-1)^{\wedge H}_p) \to \pi_n(\Omega B_G(\mathbb{S}^\wedge_p)^H) \cong \pi_n((\mathcal{S}_G F_G)^{\wedge H}_p).$$

When p is odd, this containment is actually an equality.

Proof When $p=2$, J_G is defined to be equal to $\mathrm{Fib}(\psi^k-1)^\wedge_2$ (where $k=3$), so the statement is obvious. Suppose p is odd. The given map $\pi_n(\Omega Bj)$ factors through $\pi_n(\mathrm{Spin}^H)^\wedge_p$. Since p is odd,

$$\pi_n(\mathrm{Spin}^H)^\wedge_p \cong \pi_{n+1}(B_G\mathrm{SO}^H)^\wedge_p \cong \widetilde{W}_H(S^{n+1}) \oplus \widetilde{WO}^\perp_H(S^{n+1}).$$

By Lemma 3.12, the image of $\widetilde{WO}^\perp_H(S^{n+1})$ is trivial. But the map from $\pi_n(\Omega W^H_G)$ to $\pi_n(\mathrm{Fib}(\psi^k-1)^{\wedge H}_p)$ factors through $\pi_n(J^H_G)$.

The second statement follows since $\psi^k - 1\colon B_G\mathrm{SO}^\wedge_p \to B_G\mathrm{SO}^\wedge_p$ induces an injection on π_n of H–fixed points for all $n \geq 1$. Therefore, the same can be said of $\psi^k-1\colon W_G \to W_G$. This implies that every element in $\pi_n(J_G)$ lifts to $\pi_n(\Omega W_G)$. □

We now state and prove our main theorem, citing the results that we will need in later sections.

Theorem 2.10 *Suppose G is a finite group, p a prime, and none of the prime divisors of the order of G are congruent to 1 modulo p. If p is odd, suppose k is an odd integer which is relatively prime to the order of G, and which topologically generates the units in $\mathbb{Z}_p^\wedge$. If $p = 2$, let $k = 3$.*

Then the space J_G is, up to homotopy, a direct factor of $(\mathcal{S}_G F_G)_p^\wedge$; more precisely, the composite

$$J_G \longrightarrow \mathcal{S}_G Fib(\psi^k - 1)_p^\wedge \xrightarrow{\alpha^k} (\mathcal{S}_G F_G)_p^\wedge \xrightarrow{\varepsilon^k} \mathcal{S}_G Fib(\psi^k/1)_p^\wedge \longrightarrow (J_{G\otimes})$$

is an equivariant weak homotopy equivalence. (The first and last maps are identities if $p = 2$.)

If $p = 2$, so that G is a 2–group, then the displayed square is a pullback diagram in the homotopy category:

$$\begin{tikzcd}
\mathcal{S}_G B_G O_p^\wedge \arrow[r, "\psi^k - 1"] \arrow[d, "\sigma^k"'] & \mathcal{S}_G B_G \mathrm{Spin}_p^\wedge \arrow[d, "\rho^k"] \\
\mathcal{S}_G B_G O_{\otimes}{}_p^\wedge \arrow[r, "\psi^k/1"] & \mathcal{S}_G B_G \mathrm{Spin}_p^\wedge
\end{tikzcd}$$

If p is odd, then the displayed square, which represents the components of the four maps $\psi^k - 1, \psi^k/1, \sigma^k, \rho^k$ on the Adams summands, is a pullback diagram in the homotopy category:

$$\begin{tikzcd}
W_G \arrow[r, "\psi^k - 1"] \arrow[d, "\sigma^k"'] & W_G \arrow[d, "\rho^k"] \\
W_{G\otimes} \arrow[r, "\psi^k/1"] & W_{G\otimes}
\end{tikzcd}$$

Proof First, suppose p is odd. Then the diagram in Figure 4 commutes up to homotopy.

By Corollary 6.9, the maps σ^k and ρ^k induce the same map on the n–th homotopy groups of H–fixed points for $H \leq G$ and for $n \geq 1$. By Theorem 4.1, the right composite of vertical maps (and therefore the middle vertical composite) induces an equivalence on homotopy groups. Therefore, the left composite of vertical maps induces an equivalence on homotopy groups.

$$\begin{tikzcd}
J_G \arrow[r] \arrow[d] & W_G \arrow[r, "\psi^k-1"] \arrow[d] & W_G \arrow[d] \\
\mathcal{S}_G\mathrm{Fib}(\psi^k-1)^\wedge_p \arrow[r] \arrow[d, "\varepsilon^k\circ\alpha^k"] & \mathcal{S}_G B_G\mathrm{SO}^\wedge_p \arrow[r, "\psi^k-1"] \arrow[d, "\sigma^k"] & \mathcal{S}_G B_G\mathrm{SO}^\wedge_p \arrow[d, "\rho^k"] \\
\mathcal{S}_G\mathrm{Fib}(\psi^k/1)^\wedge_p \arrow[r] \arrow[d] & \mathcal{S}_G B_G\mathrm{SO}_\otimes{}^\wedge_p \arrow[r, "\psi^k/1"] \arrow[d] & \mathcal{S}_G B_G\mathrm{SO}_\otimes{}^\wedge_p \arrow[d] \\
J_{G\otimes} \arrow[r] & W_{G\otimes} \arrow[r, "\psi^k/1"] & W_{G\otimes}.
\end{tikzcd}$$

Figure 4

Now suppose $p=2$, so $k=3$. In this case, the following diagram commutes up to homotopy:

$$\begin{tikzcd}
\mathcal{S}_G\mathrm{Fib}(\psi^3-1)^\wedge_2 \arrow[r] \arrow[d, "\varepsilon^3\circ\alpha^3"] & \mathcal{S}_G B_G O^\wedge_2 \arrow[r, "\psi^3-1"] \arrow[d, "\sigma^3"] & \mathcal{S}_G B_G\mathrm{Spin}^\wedge_2 \arrow[d, "\rho^3"] \\
\mathcal{S}_G\mathrm{Fib}(\psi^3/1)^\wedge_2 \arrow[r] & \mathcal{S}_G B_G O_\otimes{}^\wedge_2 \arrow[r, "\psi^3/1"] & \mathcal{S}_G B_G\mathrm{Spin}_\otimes{}^\wedge_2
\end{tikzcd}$$

By the last part of Theorem 4.2, the map ρ^3 on the right induces an equivalence on π_n of H–fixed points for $n\geq 2$. By Lemma 5.2,

$$\mathcal{S}_G B_G O \simeq (\mathcal{S}_G B_G\mathrm{SO})\times(\mathcal{S}_G B_G O(1)).$$

By Corollary 6.9 and Theorem 4.2, the map σ^3 is a weak equivalence on the factor $(\mathcal{S}_G B_G\mathrm{SO})^\wedge_2$, while by Lemma 6.11, σ^3 is a weak equivalence on the factor $\mathcal{S}_G B_G O(1)^\wedge_2$. Thus, σ^3 is a weak equivalence, so $\varepsilon^3\circ\alpha^3$ is a weak equivalence. □

In Section 7.2 of our appendix, we will prove the following, in order to explain why we have been unable to remove the condition on the prime divisors of $|G|$.

Theorem 2.11 *Suppose p is a prime, G is a finite group such that one of the prime divisors of the order of G is congruent to 1 modulo p, and k is an odd integer prime to p and $|G|$. Suppose given a natural splitting*

$$KO_G(X)^\wedge_p \cong A(X)\times B(X)$$

which is compatible with Adams operations and ρ^k. *Finally, suppose that for each space* X, $B(X) \subseteq T_G(X)^\wedge_p$ *(see Definition 2.5). Then there is a space* X *and a pair of elements* $\xi, \zeta \in A(X)$ *of virtual dimension 0 such that* $\rho^k(\xi) = (\psi^k/1)(1+\zeta)$, *but* ξ *is not in the image of* $\psi^k - 1$. *Thus, the restriction of the Adams–May square to* $A(X)$ *cannot be a pullback diagram.*

3 A splitting of equivariant K–theory

Throughout this section, let p be an odd prime, and let G be a finite group of order n, where none of the prime divisors of n are congruent to 1 modulo p. Let $G_c\mathcal{T}$ be the category whose objects are finite G–connected G–CW complexes.

In this section, as advertised, we construct a natural splitting of $K_G(X)^\wedge_p$ for $X \in G_c\mathcal{T}$, generalizing the Adams splitting of $K(X)^\wedge_p$. Since p is odd, it will be easy to compare $K_G(X)^\wedge_p$ with $KO_G(X)^\wedge_p$.

Construction 3.1 We sketch a construction of the nonequivariant Adams splitting. First, one shows that the action of the Adams operations ψ^k are p–adically continuous in k, so one can define a new set of Adams operations ψ^α for p–adic integers α. If α is a $(p-1)$–st root of unity, then one can define a projection onto the fixed-points of ψ^α by the formula $\frac{1}{p-1}\sum_{j=1}^{p-1}\psi^{\alpha^j}$. Thus, the fixed points of ψ^α form a direct summand of $K(X)^\wedge_p$, called the Adams summand.

In the general equivariant setting, it is no longer the case that the action of the Adams operations ψ^k are p–adically continuous in k. However, as we show in Lemma 3.2, if we restrict k to lie in a certain multiplicative submonoid of $\mathbb{Z}$, then the Adams operations are again p–adically continuous in k. Furthermore, this submonoid is dense in $\mathbb{Z}^\wedge_p$, so one can again define Adams operations for any p–adic integer. However, unless k happens to lie in our multiplicative submonoid, one is *not* guaranteed that the new Adams operation for an ordinary integer k will coincide with the traditional operation ψ^k (see Remark 3.3 for a specific example.) To distinguish the new operations from the old, we will denote the new operations by $\widetilde{\psi}^k$.

One might expect that we would then define $W_G(X)$ as the fixed points of $\widetilde{\psi}^\alpha$, where α is a $(p-1)$–st root of unity. Our goal, however, is to find a summand $W_G(X)$ of $K_G(X)^\wedge_p$ such that the complementary summand lies in $T_G(X)$, and so that ρ^k restricts to an isomorphism; for this, we need $W_G(X)$ to be significantly smaller than just the fixed points of $\widetilde{\psi}^\alpha$. In fact, we will want $W_G(X)$ to consist of those elements which are fixed by $\widetilde{\psi}^\alpha$, and for which the action of ψ^k and $\widetilde{\psi}^k$ coincide for a certain collection of integers k (see Lemma 3.9). To get this smaller summand, we define a finite collection of commuting projections and let π denote the product; this leads us to our Definition 3.5 of $W_G(X)$. In Lemma 3.12, we show that the elements in the complementary summand which have an underlying real structure are themselves contained in $T_G(X)$, as was our goal.

Now, we may write $n = n'_p p^{r(n)}$, where $p \nmid n'_p$ and $r(n)$ is the number of factors of p in n. Let $M(n'_p) = 1 + n'_p\mathbb{Z}$ denote the multiplicative monoid of all integers congruent to 1 modulo n'_p; note that $M(n'_p)$ is dense in $\mathbb{Z}^\wedge_p$. The monoid $M(n'_p)$ acts on $K_G(X)^\wedge_p$ by $k \cdot \zeta = \psi^k(\zeta)$ for $k \in M(n'_p)$ and $\zeta \in K_G(X)^\wedge_p$.

Lemma 3.2 *The action of $M(n'_p)$ on $K_G(X)^\wedge_p$ ($X \in G_c\mathcal{T}$) is p–adically continuous.*

Proof We need to show that if $k_1 - k_2$ is divisible by a sufficiently large power of p, then $\psi^{k_1}(\xi) - \psi^{k_2}(\xi)$ is divisible by a specified large power of p. First, suppose L is a line bundle over X. For each $x \in X^H$, the fiber $L_x^{|G|}$ of $L^{|G|}$ over x has trivial H–action. It follows from [13, Corollary 2.10] that $L^{|G|} = \pi^*(L')$, where $\pi\colon X \to X/G$ is the projection to the orbit space and L' is a nonequivariant line bundle over X/G. In turn, $L' \cong f_{L'}^*([H])$ for some classifying map $f_{L'}\colon X/G \to \mathbb{C}P^m$, where $[H]$ is the canonical line bundle over $\mathbb{C}P^m$. Letting $f_L = f_{L'} \circ \pi$, we have $L^{|G|} = f_L^*([H])$.

Now, if k_1 and k_2 are integers in $M(n'_p)$, and $k_1 - k_2 = kn'_p$, then so long as $p^{r(n)}$ divides k, we have:

$$\begin{aligned}\psi^{k_1}(L) - \psi^{k_2}(L) &= L^{k_2}(L^{kn'_p} - 1)\\ &= L^{k_2}((L^{|G|})^{kp^{-r(n)}} - 1) = L^{k_2}(f_L^*([H]^{kp^{-r(n)}} - 1))\end{aligned}$$

Letting $x = [H] - 1$, we have $[H]^{kp^{-r(n)}} - 1 = (x+1)^{kp^{-r(n)}} - 1$. Since $x^{m+1} \in \widetilde{K}(\mathbb{C}P^m)$ is trivial, we only need to consider the binomial coefficients

$$\binom{kp^{-r(n)}}{t}$$

for $1 \le t \le m$. The p–adic valuation of the denominator $t!$ of this coefficient is bounded, while the p–adic valuation of the term $kp^{-r(n)}$ in the numerator can be made as large as we like by making the p–adic valuation of k large.

The case of sums of line bundles is immediate. Now suppose E is a bundle over X. Then by the equivariant splitting principle, there is a G–map $f\colon X' \to X$ so that $f^*\colon K_G(X)^\wedge_p \to K_G(X')^\wedge_p$ is injective and takes E to a sum of line bundles. Then $f^*((\psi^{k_1} - \psi^{k_2})(E)) = (\psi^{k_1} - \psi^{k_2})(f^*E)$ will be divisible by a specified power of p provided that $k_1 - k_2$ is divisible by a sufficiently large power of p. The set of elements in $K_G(X)^\wedge_p$ which are not divisible by p^a for some a is compact in the p–adic topology, so its image under f^* is compact and does not contain 0 (since f^* is injective). The complement of this image is an open neighborhood U of 0 in $K_G(X')^\wedge_p$, which must contain all elements divisible by p^b for some b. By definition of U, $(f^*)^{-1}(U)$ consists entirely of elements divisible by p^a. Thus, by ensuring that $f^*(\psi^{k_1} - \psi^{k_2})(E)$ is divisible by p^b, we have $(\psi^{k_1} - \psi^{k_2})(E)$ divisible by p^a. □

It follows that we have a natural action of $\mathbb{Z}^{\wedge\times}_p$ on $K_G(X)^\wedge_p$. We denote the operation of $k \in \mathbb{Z}^{\wedge\times}_p$ on $K_G(X)^\wedge_p$ as $\widetilde{\psi}^k$.

Remark 3.3 For $k \in M(n'_p)$, we of course have $\widetilde{\psi}^k = \psi^k$, but $\widetilde{\psi}^k$ is not in general the same as ψ^k. For example, suppose that $X = S^2$, $p = 3$, $G = \mathbb{Z}/5$, and $k = 2$. Then $\widetilde{K}_G(S^2)^\wedge_p$ is a free $R(G)^\wedge_p$ module, generated by the Bott class y, and $R(G) = \mathbb{Z}[z]/(z^5 - 1)$. Thus,

$$\psi^2(zy) = z^2\psi^2(y) = 2z^2 y.$$

To compute $\widetilde{\psi}^2$, we must approximate 2 in $\mathbb{Z}^\wedge_p$ by a sequence of elements $a_0, a_1, a_2, \ldots$ which are all congruent to 1 modulo 5 (eg $1, 11, 11, 56, 326, 731, \ldots$). So $\psi^{a_i}(zy) = z^{a_i} a_i y = z a_i y$, since $z^{a_i} = z$. Since $\lim a_i = 2$ by construction,

$$\widetilde{\psi}^2(zy) = \lim \psi^{a_i}(zy) = \lim z a_i y = 2zy.$$

Remark 3.4 Suppose G contains a normal Sylow p–subgroup P, with $Q = G/P$, so $|Q| = n'_p$. Then $K_G(X)^\wedge_p$ is an $R(G)$–module, and hence by restriction an $R(Q)$–module. For each $k \in M(n'_p)$, $k \equiv 1 \bmod |Q|$; thus ψ^k acts as the identity on Q–representations, so ψ^k is an $R(Q)$–module homomorphism. Thus, for each $k \in \mathbb{Z}^\wedge_p$, $\widetilde{\psi}^k$ is an $R(Q)$–module homomorphism. In particular, $\widetilde{\psi}^\alpha$ acts as an $R(Q)$–module homomorphism.

Also, it follows from Lemma 3.2 that the action of the entire monoid $\mathbb{Z}$ on $K_P(X)^\wedge_p$ (by Adams operations) is p–adically continuous in k (since $n'_p = 1$.) Thus, for an ordinary integer k, the restriction of $\widetilde{\psi}^k$ to $K_P(X)^\wedge_p$ coincides with ψ^k. Moreover, because ψ^k acts on $K(S^{2m})$ as multiplication by k^m, $\widetilde{\psi}^\alpha$ acts on $K(S^{2m})^\wedge_p$ as multiplication by α^m.

We now turn to defining the commuting projections on $K_G(X)^\wedge_p$ whose product will have image equal to $W_G(X)$. The elements of $W_G(X)$ will consist of those elements which are fixed by $\widetilde{\psi}^\alpha$ (where $\alpha \in \mathbb{Z}^\wedge_p$ is a primitive $(p-1)^{\text{st}}$ root of unity), and for which the action of ψ^k and $\widetilde{\psi}^k$ coincide for each k relatively prime to n'_p. We first define π_0 to be

$$\frac{1}{p-1}\sum_{j=0}^{p-2}\widetilde{\psi}^{\alpha^j}.$$

Since $\widetilde{\psi}^\alpha \circ \pi_0 = \pi_0$, π_0 is a projection, and its image consists of those elements $x \in K_G(X)^\wedge_p$ fixed by $\widetilde{\psi}^\alpha$.

Now, suppose that k is an integer satisfying the following conditions:

(1) $k \equiv 1 \bmod p^{r(n)}$.

(2) k determines a unit in $\mathbb{Z}/n$.

Let b_k be the order of k in $(\mathbb{Z}/n)^\times$. Then $b_k = \prod q^{r_q-1}(q-1)$ where q runs over the prime divisors of n'_p, and r_q is the exponent of q in the prime factorization. Since none of these prime divisors are congruent to 1 modulo p, it follows that b_k is not divisible by p, and is therefore invertible in $\mathbb{Z}^\wedge_p$. So, we have a natural operation π_k on $K_G(X)^\wedge_p$ given by

$$\frac{1}{b_k}\sum_{j=0}^{b_k-1}\psi^{k^j}\widetilde{\psi}^{k^{-j}}.$$

Also, $k^{b_k} \in M(n'_p)$, so $\psi^{k^{b_k}}\widetilde{\psi}^{k^{-b_k}} = \widetilde{\psi}^{k^{b_k}}\widetilde{\psi}^{k^{-b_k}} = 1$. This implies $\psi^k\widetilde{\psi}^{k^{-1}}\pi_k = \pi_k$, so π_k is also a projection.

Since $\psi^k\pi_k = \widetilde{\psi}^k\pi_k$, the image of π_k consists solely of elements $x \in K_G(X)^\wedge_p$ for which $\psi^k(x) = \widetilde{\psi}^k(x)$, and all such elements are fixed by π_k, hence in the image of π_k. Thus, each of the projections π_k does part of the job of whittling down $K_G(X)^\wedge_p$. We now simply choose a finite set S_n of odd integers k satisfying conditions (1) and (2) above, and generating the group of units in $\mathbb{Z}/n'_p$. Since the projections π_k and π_0 all commute, we have a natural projection π given by $\pi_0\prod_{k\in S_n}\pi_k$ on $K_G(X)^\wedge_p$.

Definition 3.5 Let $W_G(X)$ and $W_G^\perp(X)$ denote the image and kernel of π. Thus,

$$K_G(X)^\wedge_p \cong W_G(X) \oplus W_G^\perp(X).$$

As usual, we let $\widetilde{W}_G(X)$ and $\widetilde{W}_G^\perp(X)$ denote the kernels of the maps $W_G(X) \to W_G(*)$ and $W_G^\perp(X) \to W_G^\perp(*)$.

Definition 3.6 If Q is a group whose order is relatively prime to p, let $W(Q) \subseteq R(Q)$ consist of those representations fixed by ψ^k for all k relatively prime to the order of Q.

We include the following examples to help the reader get accustomed to our notation.

Examples 3.7 (1) (G is a p–group.) If G is a p–group, then $n'_p = 1$, so we can take S_n to be empty. Thus, in this case, $W_G(X)$ simply coincides with the $\widetilde{\psi}^\alpha$ fixed-points. If $G = \mathbb{Z}/p$, m is some positive integer, and k is an integer such that $\alpha \equiv k \bmod p$, then $\widetilde{W}_G(S^{2m})$ is generated over $\mathbb{Z}_p^\wedge$ by

$$\big(z + \alpha^m z^k + \alpha^{2m} z^{k^2} + \cdots + \alpha^{m(p-2)} z^{k^{p-2}}\big) x_m,$$

where x_m is a generator of $\widetilde{K}(S^{2m})$.

(2) (G is cyclic of prime order $q \neq p$.) If G is cyclic of order $n = q$, where $q \neq p$ is a prime which is not congruent to 1 modulo p, and $m \geq 1$ is some integer. Then, $K_G(S^{2m})_p^\wedge \cong R(G) \otimes K(S^{2m})_p^\wedge$. As we observed in Remark 3.4, $\widetilde{\psi}^\alpha$ acts by $R(G)$–module homomorphisms, and acts on $K(S^{2m})_p^\wedge$ as multiplication by α^m. Thus, the $\widetilde{\psi}^\alpha$–fixed points of $K_G(S^{2m})_p^\wedge$ are trivial unless $p-1|m$, in which case $\widetilde{\psi}^\alpha$ acts as the identity. Now, we may choose an odd integer k which generates $(\mathbb{Z}/q)^\times$, and take S_n to be $\{k\}$. The actions of ψ^k and $\widetilde{\psi}^k$ on $K_G(S^{2m})_p^\wedge$ correspond under the isomorphism $K_G(S^{2m})_p^\wedge \cong R(G) \otimes K(S^{2m})_p^\wedge$ to $\psi^k \otimes \psi^k$ and $1 \otimes \psi^k$ respectively. It follows that the group of elements in $K_G(S^{2m})_p^\wedge$ for which ψ^k and $\widetilde{\psi}^k$ coincide is isomorphic to $W(G) \otimes K(S^{2m})_p^\wedge$. Now, $W(G)$ is a two-dimensional free Abelian group generated by the trivial representation and the regular representation. Thus, $\widetilde{W}_G(S^{2m})$ is trivial unless $p-1|m$, in which case it is isomorphic to $W(G)_p^\wedge = \mathbb{Z}_p^\wedge \oplus \mathbb{Z}_p^\wedge$.

(3) (G is cyclic of order $q_1 q_2$, for primes q_1, q_2 not equal to p.) Suppose G is cyclic of order $n = q_1 q_2$, where q_1 and q_2 are two primes, neither of which is equal to p or congruent to 1 modulo p, and suppose $m \geq 1$. Using the same arguments as in the last example, $\widetilde{W}_G(S^{2m})$ is trivial unless $p-1|m$, and in this case is isomorphic to $W(G)_p^\wedge$. This time, $W(G) \subseteq R(G)$ is a four-dimensional free Abelian group, generated by the trivial representation 1, the regular representation $\sum_{i=1}^{q_1 q_2} z^s$, and the representations $\sum_{i=1}^{q_1} z^{q_2 s}$ and $\sum_{i=1}^{q_2} z^{q_1 s}$. Thus, $\widetilde{W}_G(S^{2m})$ is trivial when $p-1 \nmid m$, and is isomorphic to $\mathbb{Z}_p^{\wedge \oplus 4}$ when $p-1|m$.

(4) (G is cyclic of order relatively prime to p.) If G is cyclic of order n'_p, where $p \nmid n'_p$ and no prime divisor of n'_p is congruent to 1 modulo p, then one may be able to guess from the above that $\widetilde{W}_G(S^{2m})$ is trivial when $p-1 \nmid m$ and is isomorphic to $\mathbb{Z}_p^{\wedge \oplus \sigma(n'_p)}$ when $p-1|m$. Here, $\sigma(n'_p)$ is the number of divisors of n'_p.

(5) (G is a permutation group.) Let us now consider a noncyclic group. Suppose G is the permutation group Σ_t, so $n = t!$, and p is any prime such that none of the prime divisors of $m!$ are congruent to 1 modulo p (eg $p \geq t/2$.) Suppose k satisfies conditions (1) and (2), ie k is relatively prime to $n!$, and $k \equiv 1 \bmod p^{r(n)}$. Then k

is relatively prime to all the numbers from 1 to t. We claim ψ^k acts as the identity on $R(G)$. Indeed, if χ_V is the character of a complex G–representation V, then $\chi_{\psi^k V}(\sigma) = \chi_V(\sigma^k)$. By writing σ as a product of disjoint cycles, it is easy to see that σ^k is conjugate to σ in S_t (since k is relatively prime to every possible cycle length.) Therefore, $\chi_{\psi^k V} = \chi_V$, so that $\psi^k V = V$. Thus, ψ^k acts as an $R(G)$–module homomorphism.

We calculate $\widetilde{\psi}^k$ by p–adically approximating k with numbers in $M(n'_p)$. Suppose $l \equiv 1 \bmod n'_p$, and also $l \equiv k \bmod p^{r(n)}$. Then since $k \equiv 1 \bmod p^{r(n)}$, it follows that $l \equiv 1 \bmod n$. Therefore, ψ^l acts the identity on $R(G)$ for any $l \in M(n'_p)$ which is sufficiently close p–adically to k. Thus, $\widetilde{\psi}^k$ acts as an $R(G)$–module homomorphism. It follows easily that ψ^k and $\widetilde{\psi}^k$ coincide on $K_G(S^{2m})^\wedge_p$, so that $\widetilde{W}_G(S^{2m})$ is just the $\widetilde{\psi}^\alpha$ invariants.

Remark 3.8 Suppose G splits as a product $Q \times P$, where P is a Sylow p–subgroup. Then for $m \geq 1$,

$$\widetilde{K}_G(S^{2m})^\wedge_p \cong R(G)^\wedge_p \cong R(Q) \otimes \widetilde{K}_P(S^{2m})^\wedge_p.$$

By Remark 3.4, $\widetilde{\psi}^\alpha$ acts as an $R(Q)$–algebra homomorphism, so by Examples 3.7 (1), the fixed points of $\widetilde{\psi}^\alpha$ are $R(Q) \otimes \widetilde{W}_P(S^{2m})$. Also by Remark 3.4, each $\widetilde{\psi}^k$ acts as an $R(Q)$–algebra homomorphism, and the actions of $\widetilde{\psi}^k$ and ψ^k on $\widetilde{K}_P(S^{2m})^\wedge_p$ coincide. Thus,

$$\widetilde{W}_G(S^{2m}) \cong W(Q) \otimes \widetilde{W}_P(S^{2m}).$$

In general, we have the following description of $W_G(X)$.

Lemma 3.9 *For $X \in G_c\mathcal{T}$, the subgroup $W_G(X)$ of $K_G(X)^\wedge_p$ consists precisely of those elements which are fixed by $\widetilde{\psi}^\alpha$, and for which the action of ψ^k and $\widetilde{\psi}^k$ coincide for each k relatively prime to n'_p.*

Proof Clearly such elements are fixed by π and therefore in the image of π. For the converse, note that $\widetilde{\psi}^\alpha \pi = \pi$, so elements in the image of π are fixed by $\widetilde{\psi}^\alpha$. Now suppose given k relatively prime to n'_p. Though k may not be in S_n, we can find an l which is a product of elements in S_n such that $kl \equiv 1 \bmod n'_p$. Therefore, $\psi^{kl} = \widetilde{\psi}^{kl}$ by Remark 3.3, and the actions of ψ^l and $\widetilde{\psi}^l$ coincide on the image of π. Thus on the image of π, we have

$$\psi^k \widetilde{\psi}^l = \psi^k \psi^l = \psi^{kl} = \widetilde{\psi}^{kl} = \widetilde{\psi}^k \widetilde{\psi}^l.$$

Since l is a unit in $\mathbb{Z}^\wedge_p$, $\widetilde{\psi}^l$ is invertible. Thus, $\psi^k = \widetilde{\psi}^k$ on the image of π. □

Corollary 3.10 *For $X \in G_c\mathcal{T}$, the subgroup $W_G(X)$ is contained in the image of the complexification map*

$$c\colon KO_G(X)^\wedge_p \to K_G(X)^\wedge_p.$$

Proof Suppose x is in the image of π. Since $\alpha^{(p-1)/2} = -1$ in $\mathbb{Z}^\wedge_p$, and $x = \widetilde{\psi}^\alpha x$ by Lemma 3.9, we have $x = \widetilde{\psi}^{-1}(x)$. Applying Lemma 3.9 again yields $\psi^{-1}(x) = \widetilde{\psi}^{-1}(x)$, since -1 is relatively prime to n'_p. Thus

$$cr(x) = (1 + \psi^{-1})(x) = 2x.$$

But 2 is a unit since p is odd. □

Remark 3.11 Since $rc = 2$ is a unit, the complexification map c is one-to-one. We let $WO^\perp_G(X) \subseteq KO_G(X)^\wedge_p$ denote the preimage under c of $W^\perp_G(X)$. By Corollary 3.10, we have

$$KO_G(X)^\wedge_p \cong W_G(X) \oplus WO^\perp_G(X).$$

By equivariant Brown representability [17, Chapter XIII, Theorem 3.1], the restrictions of the functors $\widetilde{W}_G(-)$ and $\widetilde{WO}^\perp_G(-)$ to finite G–connected based G–CW complexes are representable by G–connected G–spaces W_G and $WO^\perp_G$. Thus, we have

$$\mathcal{S}_G B_G O^\wedge_p \simeq W_G \times WO^\perp_G.$$

The following lemma, together with the Equivariant Adams Conjecture, implies that $WO^\perp_G(X)$ does not contribute to the image of J.

Lemma 3.12 $WO^\perp_G(X) \subseteq T_G(X)^\wedge_p$.

Proof We will show that each element $\zeta \in W^\perp_G(X)$ can be p–adically approximated by linear combinations of the form $\sum(\psi^{k_i} - 1)(\zeta_i)$, where k_i is an odd integer relatively prime to p and $|G|$ and $\zeta_i \in \widetilde{K}_G(X)^\wedge_p$. Assuming this, then if ξ is in $WO^\perp_G(X)$, we have $c\xi \in W^\perp_G(X)$, so $c\xi$ can be approximated as a linear combination $\sum(\psi^{k_i} - 1)(\zeta_i)$. Then $2\xi = rc\xi$ is approximated by $\sum r(\psi^{k_i} - 1)(\zeta_i)$. Since c is one-to-one and both c and cr commute with ψ^{k_i}, it follows that r commutes with ψ^{k_i}, so 2ξ is approximated by $\sum(\psi^{k_i} - 1)r(\zeta_i)$. Thus, since 2 is invertible, $\xi \in T_G(X)^\wedge_p$.

Now, recall that $W^\perp_G(X)$ is generated by the kernels of π_0 and π_k, $k \in S_n$ (S_n was defined in the paragraph prior to Definition 3.5). The kernel of π_0 is the image of

$$\pi_0 - 1 = \frac{1}{p-1} \sum_{j=1}^{p-1} (\widetilde{\psi}^{\alpha^j} - 1),$$

so it suffices to p–adically approximate elements in the image of $\widetilde{\psi}^{\alpha^j}-1$. Suppose $\zeta\in\widetilde{K}_G(X)^\wedge_p$. For each j, α^j can be p–adically approximated by some odd integer $k_j\in M(n'_p)$, and since p does not divide α^j, p does not divide k_j. Then $\psi^{k_j}(\zeta)$ is a p–adic approximation to $\widetilde{\psi}^{\alpha^j}(\zeta)$, so $(\psi^{k_j}-1)(\zeta)$ is a p–adic approximation to $(\widetilde{\psi}^{\alpha^j}-1)(\zeta)$.

Now, for each $k\in S_n$, the kernel of π_k is the image of

$$\pi_k-1=\frac{1}{b_k}\sum_{j=1}^{b_k}\big((\psi^{k^j}-1)\widetilde{\psi}^{k^{-j}}+(\widetilde{\psi}^{k^j}-1)\big).$$

Again, since p does not divide k, k^j can be p–adically approximated by odd integers in $M(n'_p)$ which are not divisible by p. So any element of the form $(\widetilde{\psi}^{k^j}-1)(\zeta)$ can be p–adically approximated as above. Elements in the image of $\psi^{k^j}-1$ are already in the needed form. □

We end this section with a technical lemma that we will need later.

Lemma 3.13 *Suppose that P is a p–group, and q is a prime not congruent to 0 or 1 modulo p. Then ψ^q-q induces an isomorphism on $W_P(S^{2m})$.*

Proof It suffices to show that $\psi^{q^j}-q^j$ induces an isomorphism for some j, since

$$\begin{aligned}\psi^{q^j}-q^j&=(\psi^q-q)(\psi^{q^{j-1}}+q\psi^{q^{j-2}}+\cdots+q^{j-1})\\&=(\psi^{q^{j-1}}+q\psi^{q^{j-2}}+\cdots+q^{j-1})(\psi^q-q).\end{aligned}$$

Because q is not congruent to 0 or 1 modulo p, we have $q\equiv\alpha^l \bmod p$ for some l with $1\le l\le p-2$. Let $\beta=\alpha^l$. Then since $p|(q-\beta)$, it follows by an inductive argument that $p^{j+1}|(q^{p^j}-\beta^{p^j})=(q^{p^j}-\beta)$. Therefore, the sequence of integers q^{p^j} p–adically converges to β. Thus, the sequence of maps $\psi^{q^{p^j}}-q^{p^j}$ p–adically converges to $\widetilde{\psi}^\beta-\beta$ (in this case, $M(n'_p)=M(1)=\mathbb{Z}$). Since $\widetilde{\psi}^\beta$ acts as the identity on $W_P(S^{2m})$ and $p\nmid(1-\beta)$, $\widetilde{\psi}^\beta-\beta$ induces an isomorphism on $W_P(S^{2m})$. Since $W_P(S^{2m})$ has finite rank over $\mathbb{Z}^\wedge_p$ and the units in $\mathbb{Z}^\wedge_p$ form an open subset, it follows that $\psi^{q^{p^j}}-q^{p^j}$ must be an isomorphism for j sufficiently large. □

4 The Adams–Bott map ρ^k

As in Section 3, let $G_c\mathcal{T}$ denote the category whose objects are finite G–connected G–CW complexes. For a finite group G and a space $X\in G_c\mathcal{T}$, let ρ^k_c denote the complex Adams–Bott cannibalistic class:

$$\rho_c^k\colon K_G(X) \to K_G(X)_k (= K_G(X)[1/k])$$

Then ρ_c^k is natural and is exponential in the sense that $\rho_c^k(\xi_1 \oplus \xi_2) = \rho_c^k(\xi_1) \cdot \rho_c^k(\xi_2)$. When L is a line bundle,

$$\rho_c^k(L) = 1 + L + L^2 + \cdots + L^{k-1} = \frac{L^k - 1}{L - 1}.$$

While our primary interest lies in the corresponding map ρ^k for KO_G–theory, the map ρ_c^k is computationally easier to deal with, and the results we will need about ρ^k will generally be derived from counterparts about ρ_c^k.

As usual, let $\widetilde{K}_G(X)$ denote the kernel of the augmentation homomorphism $K_G(X) \to K_G(*) \cong R(G)$, and let $1 + \widetilde{K}_G(X)$ denote the set of elements in $K_G(X)$ which map to 1 under the augmentation. By naturality, ρ_c^k restricts to a map $\rho_c^k\colon \widetilde{K}_G(X) \to 1 + \widetilde{K}_G(X)_k$. By the splitting principle, ρ_c^k commutes with Adams operations. Therefore, so long as $p \nmid k$, so that $k \in \mathbb{Z}_p^{\wedge\times}$, ρ_c^k restricts to a map $\rho_c^k\colon \widetilde{W}_G(X) \to 1 + \widetilde{W}_G(X)$ (see Definition 3.5). In Section 3, we showed that $\widetilde{W}_G(X)$ is also a summand of $\widetilde{KO}_G(X)_p^\wedge$, and as above, ρ^k restricts to a map $\rho^k\colon \widetilde{W}_G(X) \to 1 + \widetilde{W}_G(X)$.

Adams' computations involving the maps ρ^k lie at the heart of his study of the groups $J(X)$. He found a connection between these maps and the sequence α_t satisfying:

$$\log \frac{e^x - 1}{x} = \sum_{t=0}^{\infty} \alpha_t \frac{x^t}{t!}$$

In particular, he proved [2, Theorem 5.18] that if $x \in \widetilde{KO}(S^{4m})$, then

$$\rho^k(x) = 1 + \frac{1}{2}(k^{2m} - 1)\alpha_{2m} x. \tag{4-1}$$

Using his techniques, one can similarly show that for $x \in \widetilde{K}(S^{2m})$,

$$\rho_c^k(x) = 1 + (k^m - 1)\alpha_m x. \tag{4-2}$$

Using these calculations, one can show that for the right choice of k, ρ^k induces an isomorphism on the Adams summand of KO–theory (discussed in Construction 3.1). Our goal in this section will be to prove the following equivariant version of this fact, which we used for the proof of our main result, Theorem 2.10.

Theorem 4.1 *Suppose G is a finite group, p is an odd prime, and none of the prime divisors of the order of G are congruent to 1 modulo p. Also, suppose that k is an integer which is relatively prime to the order of G, and which topologically generates* *the units in $\mathbb{Z}_p^\wedge$. Then for each integer $m \geq 1$, the map $\rho^k\colon \widetilde{W}_G(S^{2m}) \to 1+\widetilde{W}_G(S^{2m})$ is an isomorphism. Thus, the composite map*

$$W_G \longrightarrow \mathcal{S}_G B_G \mathrm{SO}_p^\wedge \xrightarrow{\rho^k} \mathcal{S}_G B_G \mathrm{SO}_{\otimes}{}_p^\wedge \longrightarrow W_{G\otimes}$$

is an equivariant weak homotopy equivalence.

In the case when $p=2$, our condition on the prime divisors of $|G|$ is equivalent to saying that G must be a 2–group. In this section, we will also prove the following theorem, which we used for the proof of Theorem 2.10. Here, $IK\mathrm{SO}_G(X)$ denotes the kernel of the natural augmentation map $K\mathrm{SO}_G(X) \to K\mathrm{SO}(*) \cong \mathbb{Z}$.

Theorem 4.2 *Suppose G is a 2–group. Then for $m \geq 1$, the map*

$$\rho^3\colon IK\mathrm{SO}_G(S^m)_2^\wedge \to 1 + IK\mathrm{SO}_G(S^m)_2^\wedge$$

is an isomorphism. Thus, the map $\rho^3\colon (\mathcal{S}_G B_G \mathrm{SO})_2^\wedge \to (\mathcal{S}_G B_G \mathrm{SO}_\otimes)_2^\wedge$ is a weak equivalence. Moreover, the map $\rho^3\colon (\mathcal{S}_G B_G \mathrm{Spin})_2^\wedge \to (\mathcal{S}_G B_G \mathrm{Spin}_\otimes)_2^\wedge$ induces an isomorphism on π_n of H–fixed points for $n \geq 2$, and any $H \leq G$.

As a first step toward proving Theorem 4.1, we generalize Adams' arguments in [2] to find an equivariant version of Equation (4-2) above (see Equation (4-3) in Section 4.1). We show in Section 4.2 how this calculation alone can be used to prove that $\rho_c^k\colon \widetilde{W}_G(S^{2m}) \to 1+\widetilde{W}_G(S^{2m})$ is an isomorphism when G is cyclic of prime order q, where $q \neq p$ and $q \not\equiv 1 \bmod p$. In Section 4.3, we show how to simplify the expressions obtained from Equation (4-3) when we apply ρ_c^k to generators of $\widetilde{W}_G(S^{2m})$. In Section 4.4, we define and analyze a commutative diagram which reduces the study of ρ_c^k for cyclic groups G to the case of cyclic p–groups, which we considered in [13]. Finally, in Section 4.5 and Section 4.6, we put all our previous results together to prove Theorem 4.1 and Theorem 4.2.

4.1 An equivariant version of Equation (4-2)

Much of Adams' study in [2] can be described in purely algebraic terms, and in this way can be generalized to our context. For this subsection, we simply consider a cyclic group G of order n. Let $R(n)$ denote the representation ring of G; algebraically, $R(n)$ is just $\mathbb{Z}[z]/(z^n-1)$. In Definition 4.4, we describe a sequence of linear self-maps α_r on $R(n)\otimes\mathbb{Q}$, which we will use for the equivariant study of ρ^k in much the same way Adams used the coefficients α_r in the nonequivariant context. Indeed, if G is trivial, so that $n=1$ and $R(n)=\mathbb{Z}$, then our map α_r is just multiplication by the coefficient α_r that Adams studied. More generally, $\widetilde{K}_G(S^{2m})$ is a free module over $R(n) = \mathbb{Z}[z]/(z^n - 1)$ on one generator x_m, and as we show in Proposition 4.12, we have the following analogue of Equation (4-2) above:

$$\rho_c^k(z^s x_m) = 1 + (k^m \alpha_m(z^{ks}) - \alpha_m(z^s))x_m. \tag{4-3}$$

In order to prove this result, we work in an algebraic setting, identifying $K_G(\mathbb{C}P^m)$ with $R(n)[y]/(y^{m+1})$, and defining algebraic maps ε, ψ^k, ch and bh corresponding to maps Adams considered in [2]. The main relation between these maps is proved in Lemma 4.10, which is strictly analogous to [2, Proposition 5.6].

For an Abelian group M and any integer k, let $M_k = M \otimes \mathbb{Z}[1/k]$, and let $M_{\mathbb{Q}} = M \otimes \mathbb{Q}$. Let ψ^k act on $R(n)$ by taking z to z^k. As in Section 3, we may write n in the form $n'_p p^{r(n)}$, where p does not divide n'_p. Let $\Phi_d(z)$ be the d–th cyclotomic polynomial, so that $z^n - 1 = \prod_{d|n} \Phi_d(z)$. Let $R^d = \mathbb{Z}[z]/\Phi_d(z)$. Thus, $R^d_{\mathbb{Q}}$ is just the field $\mathbb{Q}(\zeta)$, where ζ is a primitive d–th root of unity, and $z \in R(n)_{\mathbb{Q}}$ corresponds to $\zeta \in R^d_{\mathbb{Q}}$. The following lemma is well-known and not hard to prove.

Lemma 4.3 *After inverting n, $R(n)$ splits as a product of the rings R^d, so we have $R(n)_n \cong \prod_{d|n} R^d_n$, and $R(n)_{\mathbb{Q}} \cong \prod_{d|n} R^d_{\mathbb{Q}}$.*

Definition 4.4

- For each triple (r, s, d), where r is a positive integer, s is any integer, and d is a positive divisor of n, we define $\alpha_{r,s,d} \in R^d_{\mathbb{Q}}$ by the equation below.

$$\sum_{r=1}^{\infty} \alpha_{r,s,d} \frac{x^r}{r!} = \begin{cases} \log \frac{e^x - 1}{x} & d \mid s \\ \log \frac{z^s e^x - 1}{z^s - 1} & d \nmid s \end{cases} \tag{4-4}$$

- Let $\alpha_{r,s} \in R(n)_{\mathbb{Q}} \cong \prod_{d|n} R^d_{\mathbb{Q}}$ be the element determined by $\alpha_{r,s,d}$ for each divisor d of n.
- Let $\alpha_r \colon R(n)_{\mathbb{Q}} \to R(n)_{\mathbb{Q}}$ be the linear map taking z^s to $\alpha_{r,s}$.
- Let $\alpha \colon R(n)_{\mathbb{Q}}[\![x]\!] \to R(n)_{\mathbb{Q}}[\![x]\!]$ be the group homomorphism defined by

$$\alpha\left(\sum_{r=0}^{\infty} c_r x^r\right) = \sum_{r=1}^{\infty} \alpha_r(c_r) x^r.$$

Notation 4.5 We will write

$$\log \frac{e^x - 1}{x} = \sum_{t=0}^{\infty} \alpha'_t \frac{x^t}{t!},$$

so that $\alpha'_t \in \mathbb{Q}$ denotes the coefficient Adams referred to as α_t. Note that $\alpha_{t,s,1} = \alpha'_t$.

We may identify $K_G(\mathbb{C}P^m)$ with $R(n)[y]/(y^{m+1})$, where $y+1$ represents the canonical line bundle. Thus, the elements representing line bundles are of the form $z^s(y+1)^r$. We will use Lemma 4.8 below to show that the following definition makes sense.

Definition 4.6 For k relatively prime to n, let $\rho_c^k\colon R(n)[y] \to (R(n)_k[\![y]\!])^\times$ be the group homomorphism determined by

$$\rho_c^k(z^s(y+1)^r) = 1 + z^s(y+1)^r + z^{2s}(y+1)^{2r} + \cdots + z^{(k-1)s}(y+1)^{(k-1)r}.$$

Remark 4.7 By our construction, the diagram below commutes:

$$\begin{tikzcd}
R(n)[y] \arrow[r] \arrow[d, "\rho_c^k"] & R(n)[y]/y^{m+1} \arrow[r, "\cong"] & K_G(\mathbb{C}P^m) \arrow[d, "\rho_c^k"] \\
(R(n)_k[\![y]\!])^\times \arrow[r] & R(n)_k[y]/y^{m+1} \arrow[r, "\cong"] & K_G(\mathbb{C}P^m)_k
\end{tikzcd}$$

Lemma 4.8 *If k is relatively prime to n, then*

$$1 + z^s(y+1)^r + z^{2s}(y+1)^{2r} + \cdots + z^{(k-1)s}(y+1)^{(k-1)r}$$

is a unit in $R(n)_k[\![y]\!]$.

Proof For a ring S, an element in $S[\![y]\!]$ is a unit if and only if its constant coefficient is a unit, so it suffices to show that $1 + z^s + \cdots + z^{(k-1)s}$ is a unit in $R(n)_k$. We will show this by showing that the map $R(n)_k \to R(n)_k$ given by multiplication by $1 + z^s + \cdots + z^{(k-1)s}$ has determinant k, and is therefore a $\mathbb{Z}_k$–module isomorphism. (After proving this, we realized that Hirata and Kono proved the same result about this determinant in [14].)

We may compute the above determinant by computing the determinant of the vector-space map

$$\mathbb{C}[z]/(z^n - 1) \to \mathbb{C}[z]/(z^n - 1)$$

given by multiplication by $1 + z^s + \cdots + z^{(k-1)s}$. Now, $\mathbb{C}[z]/(z^n - 1) \cong \prod_{i=1}^n \mathbb{C}$, and z corresponds to the sequence $(\zeta_0, \zeta_1, \ldots, \zeta_{n-1})$, where ζ_i runs through the n–th roots of unity (with $\zeta_0 = 1$). In $\prod_{i=1}^n \mathbb{C}$, multiplication by $1 + z^s + \cdots + z^{(k-1)s}$ corresponds to the map which, on the i–th factor, is multiplication by $1 + \zeta_i^s + \cdots + \zeta_i^{(k-1)s}$. This map then has determinant

$$\prod_{i=0}^{n-1} (1 + \zeta_i^s + \cdots + \zeta_i^{(k-1)s}) = k \cdot \prod_{i=1}^{n-1} \frac{\zeta_i^{ks} - 1}{\zeta_i^s - 1}.$$

Since k is relatively prime to n, the denominators and numerators in the above product cancel, and we are left with just k. □

Notations 4.9 Let $\varepsilon\colon R(n)_k[y] \to R(n)_k$ be the $R(n)_k$–module homomorphism sending y to 0, let $\psi^k\colon R(n)[y] \to R(n)[y]$ be the ring homomorphism sending $z^s(y+1)^r$ to $z^{ks}(y+1)^{kr}$, and let $\mathrm{ch}\colon R(n)_k[y] \to R(n)_{\mathbb{Q}}[\![x]\!]$ be the $R(n)_k$–algebra homomorphism determined by $\mathrm{ch}(y) = e^x - 1$ (so $\mathrm{ch}(y+1) = e^x$). Since $e^x - 1$ is divisible by x in $R(n)_{\mathbb{Q}}[\![x]\!]$, ch extends to an $R(n)_k$–algebra homomorphism from $R(n)_k[\![y]\!]$ to $R(n)_{\mathbb{Q}}[\![x]\!]$. Note also that ch restricts to a homomorphism of multiplicative groups:

$$\mathrm{ch}\colon 1 + yR(n)_k[\![y]\!] \to 1 + xR(n)_{\mathbb{Q}}[\![x]\!].$$

For a divisor d of n, let $\mathrm{bh}^d\colon R(n)[y] \to 1 + xR^d_{\mathbb{Q}}[\![x]\!]$ be the group homomorphism determined by

$$\mathrm{bh}^d(z^s(y+1)^r) = \begin{cases} \frac{e^{rx}-1}{rx} & d \mid s, \\ \frac{z^s e^{rx}-1}{z^s-1} & d \nmid s. \end{cases}$$

Note that when $d \nmid s$, $z^s - 1$ is invertible since $R^d_{\mathbb{Q}}$ is a field. By Lemma 4.3, $R(n)_{\mathbb{Q}} \cong \prod_{d|n} R^d_{\mathbb{Q}}$, so we may let

$$\mathrm{bh}\colon R(n)[y] \to 1 + xR(n)_{\mathbb{Q}}[\![x]\!]$$

be the group homomorphism determined by the maps bh^d.

The following lemma is our algebraic analogue of Adams' result [2, Proposition 5.6].

Lemma 4.10 *For each $\zeta \in R(n)[y]$,*

$$\mathrm{bh}(\zeta) \cdot \mathrm{ch}(\rho^k_c(\zeta)) = \rho^k_c(\varepsilon(\zeta)) \cdot \mathrm{bh}(\psi^k(\zeta)).$$

Thus, if $\zeta = z^s y^m$ for some s and some $m \geq 1$, then

$$\log \mathrm{bh}(\zeta) + \log \mathrm{ch}(\rho^k_c(\zeta)) = \log \mathrm{bh}(\psi^k(\zeta)).$$

Proof It suffices to prove the equality in $(R^d_{\mathbb{Q}}[\![x]\!])^\times$ for each d. Since both sides are exponential, it suffices to consider the generators $z^s(y+1)^r$. When $d|s$, we have $z^s = 1$ in $R^d_{\mathbb{Q}}$, so we have

$$\begin{aligned} &\mathrm{bh}^d(z^s(y+1)^r) \cdot \mathrm{ch}(\rho^k_c(z^s(y+1)^r)) \\ &= \frac{e^{rx}-1}{rx} \cdot \frac{e^{krx}-1}{e^{rx}-1} = k \cdot \frac{e^{krx}-1}{krx} = \rho^k_c(\varepsilon(z^s(y+1)^r)) \cdot \mathrm{bh}^d(\psi^k(z^s(y+1)^r)). \end{aligned}$$

When $d \nmid s$, we have

$$\begin{aligned}
&\mathrm{bh}^d(z^s(y+1)^r)\cdot \mathrm{ch}(\rho_c^k(z^s(y+1)^r))\\
&\qquad = \frac{z^s e^{rx}-1}{z^s-1}\cdot\frac{z^{ks}e^{krx}-1}{z^s e^{rx}-1}\\
&\qquad = \frac{z^{ks}-1}{z^s-1}\cdot\frac{z^{ks}e^{krx}-1}{z^{ks}-1} = \rho_c^k(\varepsilon(z^s(y+1)^r))\cdot \mathrm{bh}^d(\psi^k(z^s(y+1)^r)).
\end{aligned}$$

The second statement follows since $\varepsilon(z^s y^m) = 0$ when $m \geq 1$. □

Lemma 4.11 *For each $\zeta \in R(n)[y]$, we have $\alpha(\mathrm{ch}(\zeta)) = \log(\mathrm{bh}(\zeta))$, where α is the homomorphism of Definition 4.4.*

Proof In $R^d_{\mathbb{Q}}[\![x]\!]$, we have

$$\alpha(\mathrm{ch}(z^s(y+1)^r)) = \alpha(z^s e^{rx}) = \begin{cases}\log\frac{e^{rx}-1}{rx} & d\mid s,\\ \log\frac{z^s e^{rx}-1}{z^s-1} & d\nmid s\end{cases} = \log(\mathrm{bh}^d(z^s(y+1)^r)). \quad □$$

Proposition 4.12 *Suppose $m \geq 1$. Let $\hat{x}_m$ denote a generator of $\widetilde{K}(S^{2m})$, and let $x_m \in \widetilde{K}_G(S^{2m})$ be the element corresponding to $1 \otimes \hat{x}_m \in R(n) \otimes \widetilde{K}(S^{2m})$. If $\gcd(k,n) = 1$, then the map $\rho_c^k\colon \widetilde{K}_G(S^{2m}) \to 1 + \widetilde{K}_G(S^{2m})_k$ takes $z^s x_m$ to $1 + \big(k^m\alpha_{m,ks} - \alpha_{m,s}\big)\, x_m$.*

Proof First, recall that $\widetilde{K}_G(\mathbb{C}P^m) \cong R(n)[y]/(y^{m+1})$, and the submodule $\widetilde{K}_G(S^{2m})$ of $\widetilde{K}_G(\mathbb{C}P^m)$ corresponds under this isomorphism to the submodule generated by y^m. We may then identify x_m with y^m. We will prove the claim by showing that the map $\rho_c^k\colon \widetilde{K}_G(\mathbb{C}P^m) \to 1 + \widetilde{K}_G(\mathbb{C}P^m)_k$ takes $z^s y^m$ to $1 + (k^m\alpha_m(z^{ks}) - \alpha_m(z^s))y^m$. By Remark 4.7, we may use our algebraic description of ρ_c^k given in Definition 4.6. By Lemma 4.10 and Lemma 4.11, we have:

$$\begin{aligned}
&\log \mathrm{ch}(\rho_c^k(z^s y^m))\\
&\qquad = \log \mathrm{bh}(\psi^k(z^s y^m)) - \log \mathrm{bh}(z^s y^m)\\
&\qquad = \alpha\big(\mathrm{ch}(\psi^k(z^s y^m) - z^s y^m)\big) = \alpha\big(\mathrm{ch}\,\big(z^{ks}((y+1)^k-1)^m - z^s y^m\big)\big)\\
&\qquad = \alpha\big(z^{ks}(e^{kx}-1)^m - z^s(e^x-1)^m\big) = \alpha\big((k^m z^{ks} - z^s)x^m + \cdots\big)\\
&\qquad = \big(k^m\alpha_{m,ks} - \alpha_{m,s}\big)x^m + \cdots
\end{aligned}$$

It follows that

$$\mathrm{ch}(\rho_c^k(z^s y^m)) = 1 + \big(k^m\alpha_{m,ks} - \alpha_{m,s}\big)x^m + \cdots,$$

and therefore in $1 + \widetilde{K}_G(\mathbb{C}P^m)_{\mathbb{Q}}$, we must have

$$\rho_c^k(z^s y^m) = 1 + \left(k^m \alpha_{m,ks} - \alpha_{m,s}\right) y^m. \qquad \square$$

In the corollary below, the map labelled $+1$ is the map that takes x to $1+x$.

Corollary 4.13 *If* $\gcd(k,n) = 1$ *and* $m \geq 1$, *then the diagram below commutes:*

$$\begin{tikzcd}
R(n) \otimes \widetilde{K}(S^{2m}) \arrow[r, "\cong"] \arrow[d, "\alpha_m \otimes 1"] & \widetilde{K}_G(S^{2m}) \arrow[rr, "\rho_c^k"] & & 1 + \widetilde{K}_G(S^{2m})_k \arrow[d, hook] \\
R(n)_{\mathbb{Q}} \otimes \widetilde{K}(S^{2m}) \arrow[r, "\cong"] & \widetilde{K}_G(S^{2m})_{\mathbb{Q}} \arrow[r, "+1"] & 1 + \widetilde{K}_G(S^{2m})_{\mathbb{Q}} \arrow[r, "\psi^k/1"] & 1 + \widetilde{K}_G(S^{2m})_{\mathbb{Q}}
\end{tikzcd}$$

Proof First, $\psi^k(x_m) = k^m x_m$. Thus, chasing $z^s \otimes \hat{x}_m$ counterclockwise, we get

$$(\psi^k/1)(1 + \alpha_m(z^s)x_m) = \frac{1 + \psi^k(\alpha_{m,s}) \cdot k^m x_m}{1 + \alpha_{m,s} \cdot x_m} = 1 + \left(k^m \alpha_{m,ks} - \alpha_{m,s}\right) \cdot x_m.$$

Going clockwise, we get the same by Proposition 4.12. $\square$

Remark 4.14 For an integer $m \geq 1$, the multiplication on $\widetilde{K}_G(S^{2m})$ is trivial. It follows that the map $1 + \widetilde{K}_G(S^{2m}) \to \widetilde{K}_G(S^{2m})$ taking $1+x$ to x is an isomorphism. We will sometimes use the notation ρ_c^k to denote the composite

$$\widetilde{K}_G(S^{2m}) \xrightarrow{\rho_c^k} 1 + \widetilde{K}_G(S^{2m})_k \longrightarrow \widetilde{K}_G(S^{2m})_k .$$

In this usage, ρ_c^k is an additive homomorphism.

Also, note that for an element $x \in \widetilde{K}_G(S^{2m})$,

$$(\psi^k/1)(1+x) = \frac{1 + \psi^k x}{1+x} = 1 + (\psi^k x - x) = 1 + (\psi^k - 1)x.$$

Thus, Corollary 4.13 implies that the diagram below commutes:

$$\begin{tikzcd}
R(n) \otimes \widetilde{K}(S^{2m}) \arrow[r, "\cong"] \arrow[d, "\alpha_m \otimes 1"] & \widetilde{K}_G(S^{2m}) \arrow[r, "\rho_c^k"] & \widetilde{K}_G(S^{2m})_k \arrow[d, hook] \\
R(n)_{\mathbb{Q}} \otimes \widetilde{K}(S^{2m}) \arrow[r, "\cong"] & \widetilde{K}_G(S^{2m})_{\mathbb{Q}} \arrow[r, "\psi^k - 1"] & \widetilde{K}_G(S^{2m})_{\mathbb{Q}}
\end{tikzcd}$$

4.2 Example: $G = \mathbb{Z}/q$

Suppose $G=\mathbb{Z}/q$, where $q \neq p$ is an prime which is not congruent to 1 modulo p. Thus, $n=q$ for this example. Let k be an integer which determines a topological generator of $\mathbb{Z}_p^\wedge$, and which is also relatively prime to q. Using Proposition 4.12, we will show here that for $m \geq 1$, $\rho_c^k\colon \widetilde{W}_G(S^{2m}) \to 1+\widetilde{W}_G(S^{2m})$ is an isomorphism.

By Remark 3.8, with $Q=G$ and P being trivial, $\widetilde{W}_G(S^{2m}) \cong W(G) \otimes \widetilde{W}(S^{2m})$. Here $\widetilde{W}(S^{2m})$ is trivial unless $p-1|m$, and $W(G)$ is generated by 1 and $(z+z^2+\cdots+z^{q-1}+z^q)$. (See Examples 3.7 (2).) Thus, we may assume $p-1|m$, so that, in particular, m is even.

We have

$$\rho_c^k(x_m) = 1+(k^m-1)\alpha_{m,0}x_m.$$

Since $\alpha_{m,0,1}$ and $\alpha_{m,0,q}$ are both equal to α'_m, we may identify $\alpha_{m,0} \in R(n)_{\mathbb{Q}}$ with $\alpha'_m \in \mathbb{Q} \subseteq R(n)_{\mathbb{Q}}$. Thus,

$$\rho_c^k(x_m) = 1+(k^m-1)\alpha'_m x_m.$$

On the other hand, by Proposition 4.12, exponentiality of ρ_c^k, and triviality of x_m^2,

$$\begin{aligned}\rho_c^k((z+z^2+\cdots+z^q)x_m) &= \prod_{s=1}^{q}\left(1+(k^m\alpha_{m,ks}-\alpha_{m,s})x_m\right)\\ &= 1+\left(\sum_{s=1}^{q} k^m\alpha_{m,ks}-\alpha_{m,s}\right)x_m\\ &= 1+(k^m-1)\left(\sum_{s=1}^{q}\alpha_{m,s}\right)x_m.\end{aligned}$$

The last equality follows since $q \nmid k$. To calculate $\sum_{s=1}^q \alpha_{m,s} \in R(n)_{\mathbb{Q}}$, we must consider the images $\sum_{s=1}^q \alpha_{m,s,1} \in R^1_{\mathbb{Q}}$ and $\sum_{s=1}^q \alpha_{m,s,q} \in R^q_{\mathbb{Q}}$. Returning to Definition 4.4, we have:

$$\sum_{m=1}^{\infty}\left(\sum_{s=1}^{q}\alpha_{m,s,1}\right)\frac{x^m}{m!} = \sum_{s=1}^{q}\sum_{m=1}^{\infty}\alpha_{m,s,1}\frac{x^m}{m!} = q\log\left(\frac{e^x-1}{x}\right)$$

Thus, $\sum_{s=1}^{q} \alpha_{m,s,1} = q\alpha'_m \in R^1_{\mathbb{Q}}$. Using Definition 4.4 again, we have:

$$\sum_{m=1}^{\infty}\left(\sum_{s=1}^{q}\alpha_{m,s,q}\right)\frac{x^m}{m!} = \sum_{s=1}^{q}\sum_{m=1}^{\infty}\alpha_{m,s,q}\frac{x^m}{m!} = \sum_{s=1}^{q-1}\log\frac{z^s e^x-1}{z^s-1} + \log\frac{e^x-1}{x}$$
$$= \log\left(\left(\prod_{s=1}^{q-1}\frac{z^s e^x-1}{z^s-1}\right)\frac{e^x-1}{x}\right) = \log\left(\frac{\prod_{s=1}^{q}(z^s e^x-1)}{x\prod_{s=1}^{q-1}(z^s-1)}\right)$$

For the last equality, we used the fact that $1 = z^q$. To compute $\prod_{s=1}^{q-1}(z^s-1)$, note that the set $\{z^s-1 : 1 \le s \le q-1\}$ is the set of roots of the monic polynomial $\sum_{s=0}^{q-1}(y+1)^s$. Since the constant term of this polynomial is q, and the degree is $q-1$, the product of its roots is $(-1)^{q-1}q$. In a similar way, one can show that $\prod_{s=1}^{q}(z^s e^x-1) = (-1)^q(1-e^{qx})$. Thus,

$$\log\left(\frac{\prod_{s=1}^{q}(z^s e^x-1)}{x\prod_{s=1}^{q-1}(z^s-1)}\right) = \log\frac{e^{qx}-1}{qx} = \sum_{m=0}^{\infty}\alpha'_m\frac{(qx)^m}{m!}.$$

It follows that $\sum_{s=1}^{q}\alpha_{m,s,q} = q^m\alpha'_m \in R^q_{\mathbb{Q}}$.

Thus, $\sum_{s=1}^{q}\alpha_{m,s}$ is the element in $R(n)_{\mathbb{Q}}$ which maps to $q\alpha'_m \in R^1_{\mathbb{Q}}$ and to $q^m\alpha'_m$ in $R^q_{\mathbb{Q}}$. Therefore, this element must be

$$\alpha'_m\big(q^m + (1-q^{m-1})(z+z^2+\cdots+z^q)\big).$$

In summary, we have shown that

$$\rho^k_c((z+z^2+\cdots+z^q)x_m) = 1 + (k^m-1)\alpha'_m\big(q^m + (1-q^{m-1})(z+z^2+\cdots+z^q)\big)$$

and $$\rho^k_c(x_m) = 1 + (k^m-1)\alpha'_m x_m$$

Now, let $S\colon W(Q) \to W(Q)$ be the map whose matrix representation with respect to the basis 1 and $z+z^2+\cdots+z^q$ is given by

$$\begin{pmatrix} 1 & q^m \\ 0 & 1-q^{m-1}\end{pmatrix}.$$

As in Remark 4.14, we write ρ^k_c for the map $\widetilde{W}_G(S^{2m}) \to \widetilde{W}_G(S^{2m})$ given by composing ρ^k_c with subtraction by 1, so that by (4-2), $\rho^k_c\colon \widetilde{W}(S^{2m}) \to \widetilde{W}(S^{2m})$ takes

x_m to $(k^m-1)\alpha'_m x_m$. Thus, the following diagram commutes:

$$\begin{tikzcd} \widetilde{W}_G(S^{2m}) \arrow[r, "\cong"] \arrow[d, "\rho_c^k"] & W(G)\otimes \widetilde{W}(S^{2m}) \arrow[d, "S\otimes\rho_c^k"] \\ \widetilde{W}_G(S^{2m}) \arrow[r, "\cong"] & W(G)\otimes \widetilde{W}(S^{2m}). \end{tikzcd} \tag{4-5}$$

As Adams proved [2, Theorem 2.6 and Lemma 2.12], $(k^m-1)\alpha'_m$ is a p–adic unit, provided that m is even, p odd, and k is a generator of the units in $\mathbb{Z}_p^\wedge$ (all of which we have assumed). Thus $\rho_c^k\colon \widetilde{W}(S^{2m})\to\widetilde{W}(S^{2m})$ is an isomorphism. On the other hand, since $p-1|m$, $q^m\equiv 1 \bmod p$. Since $q\not\equiv 1 \bmod p$, $1-q^{m-1}$ is not divisible by p, so the determinant of the matrix representation for S is not divisible by p. Thus, S induces an isomorphism after completing at p, so $\rho_c^k\colon \widetilde{W}_G(S^{2m})\to\widetilde{W}_G(S^{2m})$ is an isomorphism.

4.3 Representation theory for cyclic groups

We now turn to the case of cyclic groups $G=\mathbb{Z}/n$, where n is a positive integer such that none of the prime divisors of n are congruent to 1 modulo p, and write $n=n'_p\cdot p^{r(n)}$, where $p\nmid n'_p$. By Remark 3.8, $\widetilde{W}_G(S^{2m})\cong W(\mathbb{Z}/n'_p)\otimes\widetilde{W}_P(S^{2m})^\wedge_p$, where P is the Sylow p–subgroup of $\mathbb{Z}/n$. Recall from Definition 3.6 that $W(\mathbb{Z}/n'_p)$ is the subgroup of $R(\mathbb{Z}/n'_p)$ consisting of representations fixed by ψ^k for all k relatively prime to n'_p. For brevity, we will write $W(n'_p)$ for $W(\mathbb{Z}/n'_p)$.

It is not hard to see that $W(n'_p)$ is free, with a basis consisting of elements $f_d=\sum_{d|s,1\le s\le n'_p} z^s$, where d ranges over the divisors of n'_p. It will sometimes be more convenient to use the basis consisting of elements $e_d=\sum_{d|s,1\le s\le n} z^s$, where d ranges over all the divisors of n which are divisible by $p^{r(n)}$. Note that e_d is obtained from f_d simply by substituting $z^{p^{r(n)}}$ for z.

In Section 4.2, where $n=q$, we studied $\rho_c^k(e_1\cdot x_m)$ and $\rho_c^k(e_q\cdot x_m)$. This study led us to consider $\sum_{s=1}^q \alpha_{m,s,1}$ and $\sum_{s=1}^q \alpha_{m,s,q}$. More generally, when p may divide n and n'_p may potentially have several factors, we will need to study $\rho_c^k(e_d\cdot z^c x_m)$, where d is a multiple of $p^{r(n)}$ which divides n, and z^c is in $R(P)$. We can and will assume throughout that c is a multiple of n'_p; this will be technically convenient. In any case, we need to understand certain sums of the form $\sum \alpha_{m,s,d'}$, where d' is a divisor of n'_p and the sum is taken over a certain set of values for s. After some notation for describing the indexing set for these sums, we will prove Proposition 4.16, in which we show how such sums can be simplified. In Section 4.4, we will need a corollary of this proposition, Corollary 4.19, in order to show that a certain diagram commutes.

Notation 4.15 Let $S^n_{d,c}$ denote the set of all integers s between 1 and n such that $s \equiv c \bmod p^{r(n)}$ and $s \equiv d \bmod n'_p$. Assuming $p^{r(n)} | d$, we can describe $S^n_{d,c}$ more simply as the set of integers between 1 and n of the form $c + dt$, $t \in \mathbb{Z}$.

Note that the sum of terms of the form z^s, where s ranges over $S^n_{d,c}$, is the element in $R(n)$ corresponding to $e_d \otimes z^c \in R(n'_p) \otimes R(P)$.

After two easy lemmas, we will prove:

Proposition 4.16 *Suppose $p^{r(n)} | d | n$ and $d' | n$, and let $l = \operatorname{lcm}(d, d')$. Then*

$$\sum_{s \in S^n_{d,c}} \alpha_{m,s,d'} = \frac{n}{l} \left(\frac{l}{d}\right)^r \alpha_{m,cl/d,d'}.$$

In the following, $S^l_{d,c}$ is just the set of elements in $S^n_{d,c}$ between 1 and l.

Lemma 4.17 *There is an element $s' \in S^l_{d,c}$ which is divisible by d' if and only if $c(l/d)$ is divisible by d'. Moreover, in this case, s' is unique.*

Proof If $c(l/d) = md'$, then $\gcd(d, d')$ divides c, so $c = d'u - dv$ for some integers u, v. Now there is an integer r so that the number $s' = c + dv + rl$ is in $\{1, 2, \ldots, l\}$. Since $d | l$, s' can be written in the form $c + dt$, so $s' \in S^l_{d,c}$. Since d' divides $c + dv$ and l, d' also divides s'. Conversely, if $c + dt = d'u$ for some u, then $c(l/d) = d'u(l/d) - tl$, which is divisible by d'. Uniqueness of s' is obvious – any two elements of the form $c + dt$ which are both divisible by d' must differ by a multiple of l. □

Lemma 4.18 *In $R^{d'}_{\mathbb{Q}}$, we have*

$$\prod_{s \in S^l_{d,c}} z^s e^x - 1 = (-1)^{l/d} (1 - z^{cl/d} e^{lx/d}).$$

If cl/d is divisible by d', let $s' \in S^l_{d,c}$ be the unique element divisible by d'. Then in $R^{d'}_{\mathbb{Q}}$, we have

$$\prod_{s \in S^l_{d,c}, s \neq s'} z^s - 1 = (-1)^{(l/d)-1} (l/d).$$

Proof We first claim that the set $\{z^s : s \in S^l_{d,c}\}$ is the set of roots of the polynomial $y^{l/d} - z^{cl/d}$ in $R^{d'}_{\mathbb{Q}}$. To see this, first note that $S^l_{d,c}$ consists of l/d elements. Next, note that each of these elements is a root of $y^{l/d} - z^{cl/d}$: if $s = c + dt$, then

$z^{sl/d} = z^{cl/d} z^{tl}$, and $z^l = 1$ in $R_{\mathbb{Q}}^{d'}$. Finally, since z is a primitive d'–th root of unity in $R_{\mathbb{Q}}^{d'}$, and any two elements in $S_{d,c}$ which are congruent modulo d' must be congruent modulo l, the elements in $\{z^s : s \in S_{d,c}^l\}$ are distinct.

Therefore, the set $\{z^s e^x - 1 : s \in S_{d,c}^l\}$ is the set of roots of the polynomial

$$(e^{lx/d})\left(\left(\frac{y+1}{e^x}\right)^{l/d} - z^{cl/d}\right) = (y+1)^{l/d} - z^{cl/d} e^{lx/d}.$$

This is a monic polynomial of degree l/d, with constant coefficient $1 - z^{cl/d} e^{lx/d}$. Therefore,

$$\prod_{s \in S_{d,c}^l} (z^s e^x - 1) = (-1)^{l/d}(1 - z^{cl/d} e^{lx/d}).$$

The second statement follows using a similar argument. □

We now prove Proposition 4.16.

Proof Since $\alpha_{r,s,d'} = \alpha_{r,s+l,d'}$, we have $\sum_{s \in S_{d,c}^n} \alpha_{r,s,d'} = (n/l) \sum_{s \in S_{d,c}^l} \alpha_{r,s,d'}$. It therefore suffices to show that

$$\sum_{s \in S_{d,c}^l} \alpha_{r,s,d'} = \left(\frac{l}{d}\right)^r \alpha_{r,cl/d,d'}.$$

If cl/d is not a multiple of d', then d' does not divide any $s \in S_{d,c}^l$ by Lemma 4.17, so we have

$$\begin{aligned}
\sum_{r=1}^{\infty} \sum_{s \in S_{d,c}^l} \alpha_{r,s,d'} \frac{x^r}{r!} &= \sum_{s \in S_{d,c}^l} \log \frac{z^s e^x - 1}{z^s - 1} = \log\left(\prod_{s \in S_{d,c}^l} \frac{z^s e^x - 1}{z^s - 1}\right) \\
&= \log\left(\frac{\prod_{s \in S_{d,c}^l} z^s e^x - 1}{\prod_{s \in S_{d,c}^l} z^s - 1}\right) = \log\left(\frac{z^{cl/d} e^{lx/d} - 1}{z^{cl/d} - 1}\right) \\
&= \sum_{r=1}^{\infty} \alpha_{r,cl/d,d'} \left(\frac{l}{d}\right)^r \frac{x^r}{r!}.
\end{aligned}$$

The penultimate equality above uses Lemma 4.18, and its specialization to the case $x = 0$.

Now, if cl/d is a multiple of d', we let $s' \in S^l_{d,c}$ be the unique element divisible by d'. Then, we have

$$\sum_{r=1}^{\infty} \sum_{s \in S^l_{d,c}} \alpha_{r,s,d'} \frac{x^r}{r!} = \left(\sum_{s \in S^l_{d,c}, s \neq s'} \log \frac{z^s e^x - 1}{z^s - 1} + \log \frac{e^x - 1}{x} \right)$$
$$= \log \left(\left(\prod_{s \in S^l_{d,c}, s \neq s'} \frac{z^s e^x - 1}{z^s - 1} \right) \cdot \frac{e^x - 1}{x} \right).$$

Since $z^{s'} = 1$, this is

$$\log \left(\frac{\prod_{s \in S^l_{d,c}} z^s e^x - 1}{x \prod_{s \in S^l_{d,c}, s \neq s'} z^s - 1} \right) = \log \left(\frac{e^{lx/d} - 1}{lx/d} \right) = \sum_{r=1}^{\infty} \alpha_{r,cl/d,d'} \left(\frac{l}{d} \right)^r \frac{x^r}{r!}.$$

Again, the penultimate equality uses by Lemma 4.18. □

Corollary 4.19 *Suppose $p^{r(n)} | d | n$ and $d' | n'_p$, and let $l = \operatorname{lcm}(d, d')$. Then*

$$\sum_{s \in S^n_{d,c}} \alpha_{r,s} \quad \text{and} \quad \frac{n}{l} \left(\frac{l}{d} \right)^r \alpha_{r,cl/d}$$

have the same image under the map

$$R(n)_{\mathbb{Q}} \cong R(n'_p)_{\mathbb{Q}} \otimes R(p^{r(n)})_{\mathbb{Q}} \to R^{d'}_{\mathbb{Q}} \otimes R(p^{r(n)})_{\mathbb{Q}}.$$

Proof Observe that

$$R^{d'}_{\mathbb{Q}} \otimes R(p^{r(n)})_{\mathbb{Q}} \cong \prod_{d'' | p^{r(n)}} R^{d'}_{\mathbb{Q}} \otimes R^{d''}_{\mathbb{Q}} \cong \prod_{d'' | p^{r(n)}} R^{d'd''}_{\mathbb{Q}}.$$

Therefore, it suffices to see that the cited quantities are equal in $R^{d'd''}_{\mathbb{Q}}$, which follows from Proposition 4.16. □

4.4 A commutative diagram

It is not so obvious how to define an analogue of the map S in Diagram (4-5) of Section 4.2. In this subsection, we construct a substitute diagram

$$\begin{tikzcd}
\widetilde{W}_G(S^{2m}) \arrow[r, equal] \arrow[d, "\rho^k_c"] & W(n'_p) \otimes \widetilde{W}_P(S^{2m}) \arrow[r, "T'_m"] & M(n'_p) \otimes \widetilde{W}_P(S^{2m}) \arrow[d, "1 \otimes \rho^k_c"] \\
\widetilde{W}_G(S^{2m}) \arrow[r, equal] & W(n'_p) \otimes \widetilde{W}_P(S^{2m}) \arrow[r, "\gamma \otimes 1"] & M(n'_p) \otimes \widetilde{W}_P(S^{2m}).
\end{tikzcd} \tag{4-6}$$

Once again, we have written ρ_c^k for the map that is technically given by ρ_c^k followed by subtracting 1. We will show that γ induces an isomorphism after inverting n'_p (Lemma 4.22), that T'_m is an isomorphism (Lemma 4.26), and that Diagram (4-6) commutes (Proposition 4.28). Thus, the demonstration of Theorem 4.1 for cyclic groups G reduces to the case of cyclic p–groups, which follows from our earlier paper [13].

To define $M(n'_p)$, we first need the following lemma. Recall that if d is a divisor of n'_p, then $f_d \in W(n'_p)$ is given by $\sum_{d|s,1\leq s\leq n'_p} z^s$.

Lemma 4.20 *Suppose d and d' are divisors of n'_p. Then the image of $f_d \in W(n'_p)$ in $R^{d'}$ is equal to 0 if d' does not divide d, and n'_p/d if d' does divide d.*

Proof If d' does divide d, then $z^d = 1$ in $R^{d'}$, so $f_d = n'_p/d$. If d' does not divide d, then $z^d \neq 1$ in $R^{d'}$. Since $(1+z+z^2+\cdots+z^{d-1})(z-1) = z^d - 1$, it follows that $(1+z+z^2+\cdots+z^{d-1}) \neq 0$. But $(1+z+z^2+\cdots+z^{d-1}) f_d = \sum_{k=0}^{n'_p-1} z^k = 0$ in $R^{d'}$. Since $R^{d'}$ is an integral domain, $f_d = 0$ in $R^{d'}$. □

Definition 4.21 For each divisor d' of n'_p, let $M_{d'}$ be the rank 1 subgroup of $R^{d'}$ generated by 1, and let $M(n'_p) = \bigoplus_{d'|n'_p} M_{d'}$. By Lemma 4.20, the projection $R(n'_p) \to R^{d'}$ takes each element f_d to $M_{d'}$. Let $\gamma\colon W(n'_p) \to M(n'_p)$ be the homomorphism induced by the projections $R(n'_p) \to R^{d'}$.

Lemma 4.22 *The homomorphism γ induces an isomorphism after inverting n'_p.*

Proof We order the divisors of n'_p as $d_1, d_2, d_3, \ldots, d_{\sigma(n'_p)}$ in such a way that $d_i | d_j$ implies that $i < j$. This determines an obvious basis for $M(n'_p)$, and also gives an ordering for our basis $\{f_d\}$ of $W(n'_p)$. By Lemma 4.20, the component of $\gamma(f_d)$ in $M_{d'}$ is 0 if $d' \nmid d$ and n'_p/d if $d'|d$. Thus the matrix representation of γ is upper triangular, and the determinant of the matrix representation of γ is

$$\prod_{i=1}^{\sigma(n'_p)} \frac{n'_p}{d_i},$$

which becomes a unit after inverting n'_p. □

Definition 4.23 We now define a map

$$T_m\colon W(n'_p) \otimes R(p^{r(n)}) \to M(n'_p) \otimes R(p^{r(n)}).$$

As in Lemma 4.22, we order the divisors d_i of n'_p. However, we take as our basis of $W(n'_p)$ the elements $e_{p^{r(n)} d_i}$. (Recall that if $p^{r(n)}|d|n$, then $e_d = \sum_{d|s,1\leq s\leq n} z^s$.) We can then describe T_m with a $\sigma(n'_p)\times\sigma(n'_p)$ matrix in which the entries are self-maps of $R(p^{r(n)})$. For each pair d_i, d_j of divisors of n'_p, let $l_{i,j}=\mathrm{lcm}(d_i,d_j)$. Then let the (i,j) entry of T_m be $(n'_p/l_{i,j})(l_{i,j}/d_i)^m\psi^{l_{i,j}/d_i}$.

Remark 4.24 If d and d' both divide n'_p, and $l=\mathrm{lcm}(d,d')$, then the d'–component of $T_m(e_{p^{r(n)}d}\otimes z^c)$ is $(n'_p/l)(l/d)^m z^{cl/d}$. Equivalently, if $p^{r(n)}|d|n$ and $d'|n'_p$ and $l=\mathrm{lcm}(d,d')$, then the d' component of $T_m(e_d\otimes z^c)$ is $(n/l)(l/d)^m z^{cl/d}$.

Remark 4.25 We can extend T_m to a map

$$T_m\otimes 1\colon W(n'_p)\otimes R(p^{r(n)})\otimes \widetilde{K}(S^{2m})^\wedge_p\to M(n'_p)\otimes R(p^{r(n)})\otimes \widetilde{K}(S^{2m})^\wedge_p.$$

Since $R(p^{r(n)})\otimes\widetilde{K}(S^{2m})\cong \widetilde{K}_P(S^{2m})$, and $\widetilde{W}_P(S^{2m})\subseteq \widetilde{K}_P(S^{2m})^\wedge_p$ consists of elements fixed by $\widetilde{\psi}^\alpha$, $T_m\otimes 1$ restricts to a map

$$T'_m\colon W(n'_p)\otimes \widetilde{W}_P(S^{2m})\to M(n'_p)\otimes\widetilde{W}_P(S^{2m}).$$

Since $\psi^{l_{i,j}/d_i}$ acts on $\widetilde{K}(S^{2m})^\wedge_p$ as multiplication by $(l_{i,j}/d_i)^m$, the (i,j) entry of T'_m is given by

$$\frac{n'_p}{l_{i,j}}\psi^{l_{i,j}/d_i}\colon \widetilde{W}_P(S^{2m})\to\widetilde{W}_P(S^{2m}).$$

Lemma 4.26 *T'_m is an isomorphism.*

Proof Since p does not divide n'_p or d_i, it suffices to consider the $\sigma(n'_p)\times\sigma(n'_p)$ matrix whose (i,j) entry is given by

$$\frac{\psi^{l_{i,j}/d_i}}{l_{i,j}/d_i}\colon \widetilde{W}_P(S^{2m})\to\widetilde{W}_P(S^{2m}).$$

In turn, this matrix can be viewed as a tensor product of matrices of the form

$$\begin{pmatrix}
1 & \frac{\psi^q}{q} & \frac{\psi^{q^2}}{q^2} & \frac{\psi^{q^3}}{q^3} & \cdots & \frac{\psi^{q^a}}{q^a}\\
1 & 1 & \frac{\psi^q}{q} & \frac{\psi^{q^2}}{q^2} & \cdots & \frac{\psi^{q^{a-1}}}{q^{a-1}}\\
1 & 1 & 1 & \frac{\psi^q}{q} & \cdots & \frac{\psi^{q^{a-2}}}{q^{a-2}}\\
 & & & \cdots & & \\
 & & & \cdots & & \\
1 & 1 & 1 & 1 & \cdots & 1
\end{pmatrix}$$

where q ranges over the prime divisors of n'_p, and a is the exponent of q in the prime factorization of n'_p. Subtracting each row from the next row reduces the above matrix to an upper triangular matrix with diagonal entries given by 1 (in the top left) or $1-\psi^q/q$ (everywhere else). Since none of the prime factors of n'_p are congruent to 0 or 1 modulo p, it follows by Lemma 3.13 that $1-\psi^q/q$ induces an isomorphism on $\widetilde{W}_P(S^{2m})$. This completes the proof. □

As a stepping stone to proving that diagram (4-6) commutes, we consider the following diagram:

$$\begin{tikzcd}
W(n'_p)\otimes R(p^{r(n)}) \arrow[r, "T_m"] \arrow[d, "\alpha_m"] & M(n'_p)\otimes R(p^{r(n)}) \arrow[d, "1\otimes\alpha_m"] \\
W(n'_p)\otimes R(p^{r(n)})_{\mathbb{Q}} \arrow[r, "\gamma\otimes 1"] & M(n'_p)\otimes R(p^{r(n)})_{\mathbb{Q}}
\end{tikzcd} \tag{4-7}$$

A word of explanation is needed about the left vertical map. If we identify $R(n)$ with $R(n'_p)\otimes R(p^{r(n)})$, then the submodule $W(n'_p)\otimes R(p^{r(n)})$ corresponds to the submodule of $R(n)$ consisting of elements fixed by the action of ψ^k for all k relatively prime to n'_p and such that $k\equiv 1 \bmod p^{r(n)}$. The map $\alpha_m\colon R(n)\to R(n)_{\mathbb{Q}}$ commutes with the action of ψ^k for all k relatively prime to n, and so restricts to a map $\alpha_m\colon W(n'_p)\otimes R(p^{r(n)})\to W(n'_p)\otimes R(p^{r(n)})_{\mathbb{Q}}$, which is the left vertical map in Diagram (4-7).

Proposition 4.27 *Diagram (4-7) commutes.*

Proof Suppose $n'_p|c$ and $p^{r(n)}|d|n$. Then $e_d\otimes z^c\in W(n'_p)\otimes R(p^{r(n)})$ corresponds to $\sum_{s\in S^n_{d,c}} z^s\in R(n)$, so $\alpha_m(e_d\otimes z^c)$ corresponds to $\sum_{s\in S^n_{d,c}}\alpha_{m,s}$.

If $d'|n'_p$, then by Corollary 4.19, the d'–component of $((\gamma\otimes 1)\circ\alpha_m)(e_d\otimes z^c)$ is

$$\frac{n}{l}\left(\frac{l}{d}\right)^m\alpha_m(z^{cl/d}) = (1\otimes\alpha_m)\left(\frac{n}{l}\left(\frac{l}{d}\right)^m z^{cl/d}\right) = (1\otimes\alpha_m)(T_m(e_d\otimes z^c)).$$

The last equality follows from Remark 4.24. Since $W(n'_p)\otimes R(p^{r(n)})$ is generated by elements of the form $e_d\otimes z^c$ where $n'_p|c$ and $p^{r(n)}|d|n$, the result now follows. □

Proposition 4.28 *Diagram (4-6) commutes.*

Proof By Proposition 4.27, the middle square of the diagram in Figure 5 commutes. The top and bottom squares clearly commute.

Since ψ^k acts as the identity on $W(n'_p)$, it follows from the extension of Corollary 4.13 in Remark 4.14 that the composite of the left vertical maps is the restriction of

$$\rho^k_c\colon \widetilde{K}_G(S^{2m})^\wedge_p\to\widetilde{K}_G(S^{2m})_{\mathbb{Q}^\wedge_p}.$$

$$
\begin{array}{ccc}
W(n'_p)\otimes \widetilde{W}_P(S^{2m}) & \xrightarrow{T'_m} & M(n'_p)\otimes \widetilde{W}_P(S^{2m}) \\
\big\downarrow & & \big\downarrow \\
W(n'_p)\otimes R(p^{r(n)})\otimes \widetilde{K}(S^{2m})^\wedge_p & \xrightarrow{T_m\otimes 1} & M(n'_p)\otimes R(p^{r(n)})\otimes \widetilde{K}(S^{2m})^\wedge_p \\
\big\downarrow{\scriptstyle \alpha_m\otimes 1} & & \big\downarrow{\scriptstyle 1\otimes\alpha_m\otimes 1} \\
W(n'_p)\otimes R(p^{r(n)})_{\mathbb{Q}}\otimes \widetilde{K}(S^{2m})^\wedge_p & \xrightarrow{\gamma\otimes 1\otimes 1} & M(n'_p)\otimes R(p^{r(n)})_{\mathbb{Q}}\otimes \widetilde{K}(S^{2m})^\wedge_p \\
\big\downarrow{\scriptstyle (1\otimes\psi^k\otimes\psi^k)-1} & & \big\downarrow{\scriptstyle (1\otimes\psi^k\otimes\psi^k)-1} \\
W(n'_p)\otimes R(p^{r(n)})_{\mathbb{Q}}\otimes \widetilde{K}(S^{2m})^\wedge_p & \xrightarrow{\gamma\otimes 1\otimes 1} & M(n'_p)\otimes R(p^{r(n)})_{\mathbb{Q}}\otimes \widetilde{K}(S^{2m})^\wedge_p
\end{array}
$$

Figure 5

Similarly, the composite of the right vertical maps is the restriction of

$$1\otimes\rho^k_c\colon M(n'_p)\otimes \widetilde{K}_P(S^{2m})^\wedge_p \to M(n'_p)\otimes \widetilde{K}_P(S^{2m})_{\mathbb{Q}^\wedge_p}. \qquad \square$$

4.5 Finite groups

Here, we prove Theorem 4.1.

Proof The primary step in the proof will be to show that the map $\rho^k_c\colon \widetilde{W}_G(S^{2m})\to 1+\widetilde{W}_G(S^{2m})$ is an isomorphism. In the last paragraph of the proof, we will show how this implies the corresponding result for ρ^k.

We first claim that Diagram (4-6) commutes in the case when G is a p–elementary group ($p\neq 2$); ie $G=\mathbb{Z}/n'_p\times P$, where P is a Sylow p–subgroup. To see this, it suffices to show that elements in $\widetilde{W}_P(S^{2m})$ are detected by their restrictions to cyclic subgroups. That is, the map below is injective:

$$\widetilde{W}_P(S^{2m})\to \lim_{P'\leq P}\widetilde{W}_{P'}(S^{2m})$$

where P' ranges over the cyclic subgroups of P. But this is clear since

$$\widetilde{W}_P(S^{2m})\subseteq \widetilde{K}_P(S^{2m})^\wedge_p\cong R(P)\otimes\widetilde{K}(S^{2m})^\wedge_p,$$

and representations are detected by their restrictions to cyclic subgroups.

Thus, if G is p–elementary, then Diagram (4-6) commutes, T'_m is an isomorphism by Lemma 4.26, and $\gamma\otimes 1$ is an isomorphism by Lemma 4.22. Finally, it follows from

[13, Section 10] that $\rho_c^k\colon \widetilde{W}_P(S^{2m}) \to \widetilde{W}_P(S^{2m})$ is an isomorphism. (This is where we require the hypothesis that k be a topological generator of the units in $\mathbb{Z}_p^\wedge$.) This completes the proof for p–elementary groups.

Now, consider the map

$$\chi\colon R(G) \to \lim_{H \leq G} R(H),$$

where H ranges over all p–elementary subgroups of G. Then it follows from [19, Section 10.2, Theorem 18] that χ becomes an isomorphism after completing at p. Thus the corresponding map

$$\chi_K\colon \widetilde{K}_G(S^{2m})_p^\wedge \to \lim_{H \leq G} \widetilde{K}_H(S^{2m})_p^\wedge$$

is an isomorphism. Since χ_K commutes with Adams operations, it follows that the corresponding projection

$$\chi_W\colon \widetilde{W}_G(S^{2m}) \to \lim_{H \leq G} \widetilde{W}_H(S^{2m})$$

is an isomorphism. By naturality, $\rho_c^k\colon \widetilde{W}_G(S^{2m}) \to \widetilde{W}_G(S^{2m})$ is an isomorphism.

Now, we can extend this result to ρ^k. In [13, Lemma 9.1], we showed that when y is a $(G, \mathrm{SU}(V))$–bundle and V has complex dimension divisible by 4, we have $\rho_c^k(y) = c\rho^k(ry)$. Thus, $\rho_c^k = c\rho^k r$ when we restrict to $\widetilde{K\mathrm{SU}}_G(S^{2m})$. Since p is odd,

$$\widetilde{K\mathrm{SO}}_G(S^{2m})_p^\wedge \quad \text{and} \quad \widetilde{KO}_G(S^{2m})_p^\wedge$$

are equivalent, so that complexification takes $\widetilde{KO}_G(S^{2m})_p^\wedge$ to $\widetilde{K\mathrm{SU}}_G(S^{2m})_p^\wedge$. Since $rc = 2$, complexification is injective. Now, $\rho_c^k(cx) = c\rho^k(rcx) = c\rho^k(2x)$, and since ρ_c^k is injective, it follows that ρ^k is injective. Now suppose $1 + x \in 1 + \widetilde{W}_G(S^{2m})$. Since ρ_c^k is surjective, $1 + cx = \rho_c^k(y)$ for some $y \in \widetilde{W}_G(S^{2m})$. By Corollary 3.10, y is in the image of the complexification map, so $\rho_c^k(y) = c\rho^k(ry)$. Since c is injective and $1 + cx = c\rho^k(ry)$, we conclude $1 + x = \rho^k(ry)$, so ρ^k is surjective.

For the last statement of the theorem, we observe that any subgroup H of G also satisfies the hypotheses of the theorem. Thus, the map $\rho^k\colon \widetilde{W}_H(S^{2m}) \to 1 + \widetilde{W}_H(S^{2m})$ is also an isomorphism for $m \geq 1$. Since W_G is equivalent to W_H as an H–equivariant space, the map

$$\rho^k\colon \pi_{2m}(W_G^H) \cong \widetilde{W}_H(S^{2m}) \to 1 + \widetilde{W}_H(S^{2m}) \cong \pi_{2m}((W_G)_\otimes^H)$$

is an isomorphism for $m \geq 1$. Now the result follows since W_G is G–connected and the homotopy groups of W_G^H are concentrated in even dimensions for each $H \leq G$. □

4.6 The prime $p = 2$

In this subsection, we prove Theorem 4.2. In fact, this result is a simple corollary of a result by Atiyah and Tall in [9] about oriented γ–rings, so we only need a few preliminary lemmas in order to ensure that we are in the setting where this result applies.

Lemma 4.29 *Suppose X is a finite G–CW complex, where G is a any group. Then $IK\mathrm{SO}_G(X)$ is an oriented γ–ring in the sense of Atiyah and Tall [9, Section III.4].*

Proof The proof by Allard in [6] that $KO(X) \oplus KSp(X)$ is a special λ–ring generalizes immediately to the equivariant case, and since $K\mathrm{SO}_G(X) \to KO_G(X)$ is a monomorphism by Lemma 5.2, it follows that $K\mathrm{SO}_G(X)$ is a special λ–ring in the sense of Atiyah and Tall [9, Section 1.1] Now, it suffices to show that for an n–dimensional oriented bundle ξ over a G–space X, $\lambda^r(\xi) \cong \lambda^{n-r}(\xi)$. We may write the total space of ξ as $E \times_{\mathrm{SO}(V)} V$, where E is a principal $(G, \mathrm{SO}(V))$–bundle over X.

If W is a G–representation whose action map factors through $\mathrm{SO}(W)$, then W and W^* are isomorphic since they have the same characters. Also, $\lambda^n V$ is a trivial one-dimensional representation of $\mathrm{SO}(V)$. The pairing

$$\lambda^k(V) \otimes \lambda^{n-k}(V) \to \lambda^n(V) \cong \mathbb{R}$$

then exhibits an isomorphism between $\lambda^k(V)$ and $\lambda^{n-k}(V)^*$, so that $\lambda^k(V)$ and $\lambda^{n-k}(V)$ are isomorphic as $\mathrm{SO}(V)$–representations. This implies that

$$\lambda^k(\xi) = E \times_{\mathrm{SO}(V)} \lambda^k(V) \cong E \times_{\mathrm{SO}(V)} \lambda^{n-k}(V) = \lambda^{n-k}\xi. \qquad \square$$

Lemma 4.30 *If G is a p–group, then for any $m > 0$, the γ–ring $IK\mathrm{SO}_G(S^m)^\wedge_p$ is a p–adic γ–ring in the sense of Atiyah and Tall [9, Section III.2].*

Proof It is enough to show that the topology induced by the power filtration is finer than the p–adic topology. But $IK\mathrm{SO}_G(S^m)^\wedge_p$ is isomorphic as a γ–ring to $\widetilde{K\mathrm{SO}}_G(S^m)^\wedge_p \oplus IR\mathrm{SO}(G)^\wedge_p$, where $IR\mathrm{SO}(G)$ denotes the kernel of the augmentation of $R\mathrm{SO}(G)$. Now, $\widetilde{K\mathrm{SO}}_G(S^m)$ has trivial multiplication, so it suffices to show that the topology induced by the power filtration of $IR\mathrm{SO}(G)$ is finer than the p–adic topology. But the complexification homomorphism $c\colon R\mathrm{SO}(G) \to R(G)$ is a λ–ring monomorphism, since $rc = 2$ and $R\mathrm{SO}(G)$ is torsion free. Now the result follows since the p–adic topology of $IR(G)$ is equivalent to the topology induced by the power filtration by [9, Section III, Proposition 1.1]. $\square$

Lemma 4.31 *Suppose that ξ is an oriented G–bundle of dimension $2n$. Then the class $\rho^3(\xi-2n)$ as defined in Section 6 is equal to the class $\sigma_3(\xi-2n)$ as defined in [9, Section III.4].*

Proof We may view ξ as a $(G,\mathrm{SO}(V))$–bundle, where V is an inner product space of dimension $2n$. Let $i\colon T\to \mathrm{SO}(V)$ be the inclusion of a maximal torus, and suppose $z_1,z_2,\dots,z_n$ is a choice coordinates for T. For $0\le r\le 2n$, let σ_r be the r–th elementary symmetric polynomial in the $2n$ variables $z_1,z_1^{-1},z_2,z_2^{-1},\dots,z_n,z_n^{-1}$. We observe that $\sigma_r=\sigma_{2n-r}$. We will denote the standard representation of $\mathrm{SO}(V)$ given by the identity map as just V. For $0\le r\le 2n$, the character of $\lambda^r V$ restricted to T is σ_r.

Now let ζ be a primitive cube root of unity, so $1+\zeta=-\zeta^2$ and $\zeta^{-1}=\zeta^2$. Then

$$\begin{aligned} z_i^{-1}+1+z_i &= \frac{1+z_i+z_i^2}{z_i}=\frac{(z_i-\zeta)(z_i-\zeta^2)}{z_i}=-(1-\zeta z_i^{-1})(\zeta^2-z_i)\\ &=-\zeta^2(1-\zeta z_i^{-1})(1-\zeta z_i)=(1+\zeta)(1-\zeta z_i^{-1})(1-\zeta z_i). \end{aligned}$$

$$\begin{aligned} \text{So}\quad \prod_{i=1}^n(z_i^{-1}+1+z_i) &= (1+\zeta)^n\prod_{i=1}^n(1-\zeta z_i^{-1})\prod_{i=1}^n(1-\zeta z_i)=(1+\zeta)^n\sum_{r=0}^{2n}(-\zeta)^r\sigma_r\\ &=(1+\zeta)^n\left(\left(\sum_{r=0}^{n-1}(-1)^r(\zeta^r+\zeta^{2n-r})\sigma_r\right)+(-1)^n\zeta^n\sigma_n\right). \end{aligned}$$

We may write this as $\sum_{r=0}^n a_{n,r}\sigma_r$, and it is not difficult to check directly that each $a_{n,r}$ is an integer. Thus, $\prod_{i=1}^n(z_i^{-1}+1+z_i)$ is the character of the representation $\sum_{r=0}^n a_{n,r}\lambda^r(V)$, and if ξ has dimension $2n$, $\rho^3(\xi)=\sum_{r=0}^n a_{n,r}\lambda^r(\xi)$. Moreover, $\rho^3(2)=3$, so

$$\rho^3(\xi-2n)=\frac{1}{3^n}\sum_{r=0}^n a_{n,r}\lambda^r(\xi).$$

$$\begin{aligned} \text{Now}\quad \sigma_3(\xi-2n) &= \gamma_{\zeta/(\zeta-1)}(\xi-2n)=\lambda_{-\zeta}(\xi-2n)=\frac{\lambda_{-\zeta}(\xi)}{\lambda_{-\zeta}(1)^{2n}}=\frac{\lambda_{-\zeta}(\xi)}{(1-\zeta)^{2n}}\\ &=\frac{\lambda_{-\zeta}(\xi)}{3^n(-\zeta)^n}=\frac{(1+\zeta)^n}{3^n}\left(\sum_{r=0}^{2n}(-\zeta)^r\lambda^r(\xi)\right)=\frac{1}{3^n}\sum_{r=0}^n a_{n,r}\lambda^r(\xi) \end{aligned}$$

since $\lambda^r(\xi)=\lambda^{2n-r}(\xi)$. So, $\rho^3(\xi-2n)=\sigma_3(\xi-2n)$. □

We now prove Theorem 4.2.

Proof Atiyah and Tall [9, Section III, Theorem 4.5], proved that if A is an orientable p–adic γ–ring, then the map $\sigma_3\colon A \to 1 + A$ induces isomorphisms on invariants and coinvariants of the action of $\Gamma/\{\pm 1\}$, where Γ is the group of units in $\mathbb{Z}_p^\wedge$, acting via the Adams operations, and h is a generator of $\Gamma/\{\pm 1\}$. It is clear from their proof, however, that in the case when $p = 2$, σ_3 is itself an isomorphism. By Lemma 4.29 and Lemma 4.30, $IK\mathrm{SO}_G(S^m)_2^\wedge$ is an orientable 2–adic γ–ring, so the claim follows from Lemma 4.31.

The second statement follows in a similar way to the proof of the second statement in Theorem 4.1. For the last statement, suppose $H \le G$ and consider the following diagram:

$$\begin{array}{ccc}
\pi_n((\mathcal{S}_G B_G \mathrm{Spin})_2^{\wedge H}) & \longrightarrow & \pi_n((\mathcal{S}_G B_G \mathrm{SO})_2^{\wedge H}) \\
\Big\downarrow{\scriptstyle \rho^3} & & \Big\downarrow{\scriptstyle \rho^3} \\
\pi_n(((\mathcal{S}_G B_G \mathrm{Spin}_\otimes^\wedge)_2^\wedge)^H) & \longrightarrow & \pi_n(((\mathcal{S}_G B_G \mathrm{SO})_\otimes^\wedge)_2^\wedge)^H).
\end{array}$$

By Lemma 5.3, the horizontal maps are both isomorphisms for $n \ge 3$ and are one-to-one with cokernel $\mathbb{Z}/2$ when $n = 2$. Thus, for $n \ge 3$, the claim follows from the previous remarks. For $n = 2$, the claim follows if and only if the diagram induces an isomorphism on the cokernels $\mathbb{Z}/2$. But this holds if and only if it holds nonequivariantly. Since $\rho^3\colon B\mathrm{Spin}_2^\wedge \to B\mathrm{Spin}_{\otimes 2}^\wedge$ is a homotopy equivalence by [16, Chapter V, Theorem 4.4], the claim follows. □

5 Equivariant Spin structures

5.1 Comparing classifying spaces

Nonequivariantly, there are fiber sequences

$$\mathrm{SO} \to BO \to K(\mathbb{Z}/2, 1),$$
$$B\mathrm{Spin} \to B\mathrm{SO} \to K(\mathbb{Z}/2, 2).$$

These facts are useful when one wants to show that an orthogonal vector bundle is uniquely orientable, or that an orientable vector bundle has a unique Spin structure. In [13], we considered the inclusions $B_G\mathrm{SO} \to B_G O$ and $B_G\mathrm{Spin} \to B_G\mathrm{SO}$ when G is a finite group with odd order. Since most of our results there fail for groups with even order, we need to consider the general case.

First, we need to explain how we build these classifying spaces, and this requires the following definition.

Definition 5.1 A *complete sequence* of complex G–representations is a sequence $\mathbb{V} = V_1 \subseteq V_2 \subseteq V_3 \subseteq \cdots$ such that $\bigcup_i V_i$ forms a complete G–universe and the action map $\rho_i\colon G \to U(V_i)$ factors through $\mathrm{SU}(V_i)$ for each i.

Each V_i can be thought of as a complex representation, a real G–representation with Spin structure, a real orientable G–representation, and so on, and the action maps determine basepoints for the spaces

$$B_G O(V_i), \quad B_G\mathrm{SO}(V_i), \quad B_G\mathrm{Spin}(V_i), \quad B_G U(V_i), \quad B_G\mathrm{SU}(V_i).$$

We can then define the spaces $B_G O$, $B_G\mathrm{SO}$ etc. as the colimit over i of the corresponding spaces $B_G O(V_i)$, $B_G\mathrm{SO}(V_i)$ and so on. We proved in [13] that each of these classifying spaces has a weak G–Hopf space structure. Weak, here, means that the associativity and unit diagrams commute up to homotopy when restricted to finite G–CW complexes.

The group $O(1) \cong \mathbb{Z}/2$, together with the trivial map $G \to O(1)$ determines a based G–space $B_G O(1)$. Tensor product of line bundles induces a weak G–Hopf space structure on $B_G O(1)$. If V is a real G–representation whose action map ρ factors through $\mathrm{SO}(V)$, then the determinant $O(V) \to O(1)$ induces a based G–map $B_G O(V) \to B_G O(1)$. If $\mathbb{V}$ is a complete sequence of G–representations, then each action map $\rho\colon G \to O(V_i)$ does factor through $\mathrm{SO}(V_i)$, and the G–maps $B_G O(V_i) \to B_G O(1)$ assemble to a based G–map $\det_*\colon B_G O \to B_G O(1)$. Moreover, $\det_*$ is a weak map of weak G–Hopf spaces.

Lemma 5.2 *For any compact Lie group G, the sequence*

$$B_G\mathrm{SO} \xrightarrow{j_*} B_G O \xrightarrow{\det_*} B_G O(1)$$

is a split fibration sequence.

Proof We may assume $V_1 = \mathbb{C}$ is a trivial representation. Then if

$$i_*\colon B_G O(1) \subseteq B_G O(2) = B_G O(V_1) \to B_G O$$

denotes the inclusion, we have $\det_* \circ i_* \simeq \mathrm{id}$. Now let $\mu\colon B_G O \times B_G O \to B_G O$ be the addition map, obtained from the G–Hopf space structure of $B_G O$. Since $\det_*$ is a weak G–Hopf map, the following diagram commutes up to weak homotopy:

$$\begin{tikzcd}
B_G\mathrm{SO} \arrow[r] \arrow[d, equal] & B_G\mathrm{SO} \times B_G O(1) \arrow[r] \arrow[d, "\mu\circ(j_*\times i_*)"] & B_G O(1) \arrow[d, equal] \\
B_G\mathrm{SO} \arrow[r, "j_*"] & B_G O \arrow[r, "\det_*"] & B_G O(1).
\end{tikzcd}$$

It follows from the five-lemma that $\mu \circ (j_* \times i_*)$ is a weak equivariant equivalence, and therefore an equivariant homotopy equivalence. □

Lemma 5.3 *The G–connected cover of the fiber of the map $B_G\mathrm{Spin} \to B_G\mathrm{SO}$ is $BO(1) \simeq K(\mathbb{Z}/2,1)$ with trivial G–action. The corresponding inclusion $BO(1) \to B_G\mathrm{Spin}$ is null-homotopic.*

Proof Let V_i run over a complete sequence of complete G–representations $\mathbb{V}$. Note that for each V_i, there is a canonical homomorphism $\rho\colon G \to \mathrm{SU}(V_i)$, and thus canonical homomorphisms from G to $\mathrm{Spin}(V_i)$ and $\mathrm{SO}(V_i)$, endowing $B\mathrm{Spin}(V_i)$ and $B\mathrm{SO}(V_i)$ with G–actions as described in the comment preceding Proposition 2.2. By Proposition 2.2, $\mathcal{S}_G B_G\mathrm{Spin} = \operatorname{colim} B\mathrm{Spin}(V_i)$ and $\mathcal{S}_G B_G\mathrm{SO} = \operatorname{colim} B\mathrm{SO}(V_i)$. In order to establish the first claim, it suffices to show that the G–connected cover of the fiber of $B\mathrm{Spin}(V_i) \to B\mathrm{SO}(V_i)$ is $K(\mathbb{Z}/2,1)$ with trivial G–action. To show this, we will show that for any $H \le G$, the map $\mathrm{Spin}(V_i)^H \to \mathrm{SO}(V_i)^H$ induces an injection on π_0, and that if V_i contains a two-dimensional trivial representation, then for any $H \le G$, the component of the identity in $\mathrm{Spin}(V_i)^H$ surjects onto the component of the identity in $\mathrm{SO}(V_i)^H$, with kernel $\mathbb{Z}/2$.

The kernel of the map $\mathrm{Spin}(V_i)^H \to \mathrm{SO}(V_i)^H$ consists of two points t and e, where $e \in \mathrm{Spin}(V_i)$ is the identity. If V_i has a two-dimensional trivial subrepresentation, then there is a path in $\mathrm{Spin}(V_i)^G$ connecting e to t. Also, if γ is a path in $\mathrm{SO}(V_i)^H$, and $\gamma(0)$ is in the image of $\mathrm{Spin}(V_i)^H$, then it is easy to check that either lift of γ to $\mathrm{Spin}(V_i)$ factors through $\mathrm{Spin}(V_i)^H$. These facts imply that the map $\mathrm{Spin}(V_i)^H \to \mathrm{SO}(V_i)^H$ induces an injection on π_0, and that the projection from the identity component of $\mathrm{Spin}(V_i)^H$ to the identity component of $\mathrm{SO}(V_i)^H$ is surjective with kernel $\mathbb{Z}/2$.

To show that the inclusion of $K(\mathbb{Z}/2,1)$ in $B_G\mathrm{Spin}^G \simeq B\mathrm{Spin}^G$ is null-homotopic, consider the fiber sequence

$$\mathrm{SO}^G \to K(\mathbb{Z}/2,1) \to B\mathrm{Spin}^G.$$

It suffices to find a map $K(\mathbb{Z}/2,1) \to \mathrm{SO}^G$ inducing the nontrivial map on π_1. But $\mathrm{SO}^G = \operatorname{colim} \mathrm{SO}(V_i)^G$, which contains a copy of $\operatorname{colim} \mathrm{SO}(V_i^G) = \mathrm{SO}$, and the composite $\mathrm{SO} \to \mathrm{SO}^G \subseteq \mathrm{SO}$ is homotopic to the identity. It is well-known that nonequivariantly, there is a map $K(\mathbb{Z}/2,1) \to \mathrm{SO}$ inducing a nontrivial map on π_1. □

The following corollary now follows from the previous two lemmas.

Corollary 5.4 *Suppose X is G–connected. Then if a G–map $X \to B_G O$ lifts to $B_G\mathrm{Spin}$, that lift is unique up to homotopy.*

5.2 Lifting $\psi^k - 1$

In [13], we proved that the map $\psi^k - 1\colon B_G O \to B_G O$ lifts to $B_G\mathrm{Spin}$ (as in the nonequivariant case), provided that G has odd order. We have since realized that one can prove this for arbitrary compact Lie groups G. We now give the argument for this claim. We first need the following elementary lemma, in which A denotes a compact Lie group with two components (eg $O(n)$), and $a \in A$ is an element of order 2 in the nontrivial component of A (eg a reflection).

Lemma 5.5 *Suppose given a homomorphism $\rho\colon A \to O(n)$ such that the number of negative eigenvalues of $\rho(a)$ is divisible by 4, and such that ρ induces the trivial map on π_1. Then ρ lifts to* $\mathrm{Spin}(n)$.

Proof Clearly $\rho(a) \in \mathrm{SO}(n)$, so by continuity ρ factors through $\mathrm{SO}(n)$. Since $\rho(a)^2 = \rho(a^2) = 1$, the eigenvalues of $\rho(a)$ are all ± 1. Suppose $e_1, e_2, \dots, e_{4j}$ are unit eigenvectors with eigenvalue -1, and $e_{4j+1}, e_{4j+2}, \dots, e_n$ are unit eigenvectors with eigenvalue 1. Then the preimage of $\rho(a)$ in $\mathrm{Spin}(n)$ consists of the two elements $\pm e_1 e_2 \cdots e_{4j}$. Here, we have used the Clifford algebra description of $\mathrm{Spin}(n)$ (see Atiyah, Bott, and Shapiro [8]). We may let $\widetilde{\rho}(a)$ be $e_1 e_2 \cdots e_{4j}$.

Now, let $A_0 \subseteq A$ be the component of the identity, and let $A_1 = A - A_0$. Let $\rho_0\colon A_0 \to \mathrm{SO}(n)$ be the restriction of ρ to A_0. Since ρ_0 induces the trivial map on π_1, the composite

$$A_0 \xrightarrow{\rho_0} \mathrm{SO}(n) \xrightarrow{k_1} K(\mathbb{Z}/2, 1)$$

is trivial, where k_1 denotes the first k–invariant of $\mathrm{SO}(n)$. Since $\mathrm{Spin}(n)$ is equivalent to the homotopy fiber of k_1, it follows that ρ_0 lifts up to homotopy to $\mathrm{Spin}(n)$. Since the projection $\pi\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is a fibration, it follows that ρ_0 lifts continuously to a map $\widetilde{\rho}_0\colon A_0 \to \mathrm{Spin}(n)$, and we may choose $\widetilde{\rho}_0$ to take the identity to the identity. Now, the map sending $x \in A_1$ to $\widetilde{\rho(a)}\widetilde{\rho}_0(a^{-1}x)$ is a continuous lift of $\rho|_{A_1}$ to $\mathrm{Spin}(n)$. Thus, we have a continuous lift $\widetilde{\rho}\colon A \to \mathrm{Spin}(n)$ taking a to $\widetilde{\rho(a)}$.

The lift $\widetilde{\rho}$ induces a map $\widetilde{\rho}^2\colon A \times A \to \mathbb{Z}/2 \subseteq \mathrm{Spin}(n)$:

$$(a_1, a_2) \to \widetilde{\rho}(a_1)\widetilde{\rho}(a_2)\widetilde{\rho}(a_1 a_2)^{-1}.$$

Moreover, $\widetilde{\rho}^2$ takes the four pairs $(1,1)$, $(1,a)$, $(a,1)$, and (a,a) to the identity. It follows by continuity that $\widetilde{\rho}^2$ is trivial, so $\widetilde{\rho}$ is a homomorphism. □

Definition 5.6 We will say that a virtual orthogonal A–representation $W_1 - W_2$ of virtual dimension 0 *lifts to* Spin if there exists an A–representation V such that the action maps of $W_1 + V$ and $W_2 + V$ both lift to $\mathrm{Spin}(n)$, where

$$n = \dim(W_1 + V) = \dim(W_2 + V).$$

Remark 5.7 To see that Definition 5.6 makes sense, we need to show that if $W_1 - W_2$ and $W_1' - W_2'$ represent the same virtual A–representation, and $W_1 - W_2$ lifts to Spin, then so does $W_1' - W_2'$. To see this, we may suppose there are A–representations Z and Z' so that $W_1' + Z' = W_1 + Z$ and $W_2' + Z' = W_2 + Z$ as actual A–representations, and that for some A–representation V, $W_1 + V$ and $W_2 + V$ both lift to Spin as actual representations. It is easy to show that the actual A–representation $4Z$ always lifts to Spin. Thus, the actual representations

$$W_1' + (Z' + V + 3Z) = W_1 + V + 4Z, \text{ and } W_2' + (Z' + V + 3Z) = W_2 + V + 4Z$$

both lift to Spin, so that the virtual representation $W_1' - W_2'$ lifts to Spin, as needed.

Lemma 5.8 *Let ρ_i denote the action map of W_i for $i = 1, 2$. Suppose ρ_1 and ρ_2 each induce the same map on π_1, and that $\rho_1(a)$ and $\rho_2(a)$ have the same number of negative eigenvalues modulo 4. Then $W_1 - W_2$ lifts to* Spin.

Proof Given the condition, the action map of $W_1 + 3W_2$ induces the trivial maps on π_1, and the number of negative eigenvalues is divisible by 4. The same holds for $4W_2$, so we may let $V = 3W_2$ and apply Lemma 5.5. □

Since the number of negative eigenvalues of $\rho(a)$ and the map on π_1 induced by ρ are both additive in the representations, we can refer to the number of negative eigenvalues or the induced map on π_1 of a virtual representation. Also, we can refer to the number of negative eigenvalues of a virtual representation. The above lemma thus implies that a virtual A–representation lifts to Spin provided that the number of negative eigenvalues of the virtual representation is divisible by 4, and the induced map on π_1 of the virtual representation is trivial. Now let $A = O(n)$, let $a \in O(n)$ be a reflection, and let V_n be the standard A–representation, with action map given by the identity.

Lemma 5.9 *Let k be an odd integer. Then $\psi^k V_n - V_n$ lifts to* Spin.

Proof The restriction of $\psi^k V_n - V_n$ to the subgroup generated by $a \in O(n)$ is the trivial representation when k is odd, since ψ^k takes the sign representation to itself. Thus, the number of negative eigenvalues of $\psi^k V_n - V_n$ is zero. Moreover, the restriction of $\psi^k V_n - V_n$ to $\mathrm{SO}(2)$ takes the generator of $\pi_1(\mathrm{SO}(2))$ to $k-1$ times the generator. Since $k-1$ is even, the map on π_1 induced by $\psi^k V_n - V_n$ is trivial. The claim follows from Lemma 5.8. □

Remark 5.10 Suppose given a homomorphism $\psi\colon G \to A$, endowing $B_G A$ with a basepoint. If W is an A–representation with action map $\rho\colon A \to O(W)$, then we may view W as a G–representation with action map $\rho\circ\psi$. Then ρ induces a based G–map $\rho_*\colon B_G A\to B_G O(W)$, which represents taking a (G,A)–bundle E to $E\times_A W_\rho$. If ρ lifts to $\mathrm{Spin}(W)$, then ρ_* lifts to $B_G\mathrm{Spin}(W)$. Using the Hopf-space structure and homotopy inverse maps, a virtual A–representation ζ of virtual dimension 0 determines a based-map $\zeta_*\colon B_G A\to B_G O$, and if ζ lifts to Spin in the sense of Definition 5.6, then ζ_* lifts to $B_G\mathrm{Spin}$. Finally, the inclusion $B_G\mathrm{Spin}(V)\to B_G\mathrm{Spin}$ represents taking a $(G,\mathrm{Spin}(V))$–bundle ξ to the virtual bundle $\xi-\mathbf{V}$, where $\mathbf{V}$ denotes the product bundle. Thus, if $\zeta=W_1-W_2$, ζ_* represents taking a (G,A)–bundle E to

$$(E\times_A W_1-E\times_A W_2)-(\mathbf{W_1}-\mathbf{W_2}).$$

Now it follows from Lemma 5.9 that we can lift the virtual $O(V)$–representation ψ^kV-V to Spin, so that the map $\psi^k-1\colon B_G O(V)\to B_G O$ induced by ψ^kV-V lifts to $B_G\mathrm{Spin}$. It is also straightforward to show that if W is a G–representation containing V as a subrepresentation, then we can lift ψ^kW-W to Spin compatibly with a chosen lift of ψ^kV-V. Thus, we obtain a lift of $\psi^k-1\colon B_G O\to B_G O$ to $B_G\mathrm{Spin}$. By Corollary 5.4, the restriction of this lift to $\mathcal{S}_G B_G O$ is unique.

6 The Adams–Bott cannibalistic Class

6.1 Construction of ρ^k

Suppose V is an inner product space of dimension $8n$, and let C_V be the associated Clifford algebra [8]. Then, as we discussed in [13] following the ideas of Atiyah, Bott and Shapiro [8], there is a canonical graded C_V–module λ_V which gives rise to a class $b^V\in\widetilde{KO}_{\mathrm{Spin}(V)}(S^V)$. Given any homomorphism $\lambda\colon H\to\mathrm{Spin}(V)$, we denote the corresponding H–representation by V_λ. We may restrict b^V to a class b^{V_λ} in $\widetilde{KO}_H(S^{V_\lambda})$. These are the equivariant Bott classes.

If $\xi\colon P\to B$ is a principal $(G,\mathrm{Spin}(V))$–bundle, then just as in the nonequivariant context, there are a variety of ways to construct an equivariant Thom class of ξ. In [13], we cited equivariant Bott periodicity [7, Theorem 6.1] to define a Thom isomorphism $\Phi^\xi\colon KO_G(B)\to\widetilde{KO}_G(T\xi)$, and hence a Thom class $\mu^\xi=\Phi^\xi(1)$. One could alternatively use an evident equivariant generalization of the construction in Atiyah–Bott–Shapiro [8, Section 11], and it is not difficult to see that one gets the same Thom class using this method. One can also construct the equivariant Thom class using the methods of Bott in [10], and the arguments in [8, Sections 13–14] easily generalize to the equivariant context to show that this construction is also equivalent.

The Adams–Bott Cannibalistic class $\rho^k(\xi)$ is the element in $KO_G(B)$ which satisfies the equation $\rho^k(\xi)\mu^\xi=\psi^k(\mu^\xi)$. The arguments of Bott in [10, Section 13] show that $\rho^k(\xi)$ is induced from ξ by a virtual real $\mathrm{Spin}(V)$–representation. To describe this representation, we first recall from [10, Section 9] a little about the representation theory of $\mathrm{Spin}(V)$. Let $T \subseteq \mathrm{SO}(V)$ be a maximal torus, and let $y_1, \dots, y_{4n}$ denote a choice of coordinates. Let $\widetilde{T}$ be the preimage of T in $\mathrm{Spin}(V)$. Then $R(T)$ is isomorphic to the ring of finite Laurent polynomials in $y_1, y_2, \dots, y_{4n}$, and $R(\widetilde{T})$ is a quadratic extension of $R(T)$ obtained by adjoining a square root of $y_1 y_2 \cdots y_{4n}$. Moreover, the subrings $R(\mathrm{SO}(8n))$ and $R(\mathrm{Spin}(8n))$ of $R(T)$ and $R(\widetilde{T})$ correspond to the rings of elements invariant under permutations of the y_i and transformations $y_i \to y_i^{\varepsilon_i}$ where $\varepsilon_i = \pm 1$ for each i and $\prod \varepsilon_i = 1$.

Now, there is a virtual complex representation $\theta^k(V)$ of $\mathrm{Spin}(V)$ whose character is given by

$$\prod_{i=1}^{4n} \left(y_i^{(k-1)/2} + y_i^{(k-3)/2} + \cdots + y_i^{(3-k)/2} + y_i^{(1-k)/2}\right).$$

Since the dimension of V is divisible by 8, $R(\mathrm{Spin}(V)) \cong RO(\mathrm{Spin}(V))$ [10, Equation 10.9], so $\theta^k(V)$ is the complexification of a virtual real bundle, which we also denote $\theta^k(V)$. This is the representation inducing $\rho^k(\xi)$. Moreover, just as in [10, Section 13], we have $\psi^k(b^V) = \theta^k(V) b^V$.

Suppose $\lambda\colon H \to \mathrm{Spin}(V)$ is a given homomorphism, and $b^{V_\lambda} \in \widetilde{KO}_H(S^{V_\lambda})$ is the associated Bott class. Let $\theta^k(V_\lambda) = \lambda^*(\theta_k(V))$. Then $\psi^k(b^{V_\lambda}) = \theta^k(V_\lambda) b^{V_\lambda}$. Now write $\mathbf{V}_\lambda$ for the trivial G–bundle $B \times V_\lambda \to B$. Since the Bott class b^{V_λ} is also the orientation class of the trivial bundle $V_\lambda \to *$, we have $\rho^k(\mathbf{V}_\lambda) = \theta^k(V_\lambda) = \lambda^* \theta^k(V)$.

Remark 6.1 When k is odd, the character above defines a representation $\theta^K(V)$ of $\mathrm{SO}(V)$, which must be real since the dimension of V is divisible by 8, and virtual representations of $\mathrm{SO}(4m)$ are all real. Thus, we can also define ρ^k on $(G, \mathrm{SO}(V))$–bundles.

To construct ρ^k on stable bundles, one must know that $\rho^k(\mathbf{V})$ is a unit, where $\mathbf{V}$ is the trivial bundle $B \times V \to B$ for some G–representation V. In [13], we cited Hirata and Kono [14, Corollary 2.5], which asserts that a complex analogue of $\rho^k(\mathbf{V})$ becomes a unit after inverting k. We used this to show that $\rho^k(\mathbf{V})$ becomes a unit after inverting $2k$. In this paper, we wish to avoid inverting 2, so we need a stronger result. We must prepare for this with the following technical lemma.

Lemma 6.2 *Suppose that k and n are relatively prime, with n even and k odd. Let $t = (n+k-1)/2$. Suppose that in $(\mathbb{Z}/2)[x]/(x^n - 1)$, we have*

$$(1 + x + x^2 + \cdots + x^{k-1})(a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}) = x^t.$$

Then $a_0 = 0$.

Proof Let r be the smallest positive integer such that $rk \equiv 1 \bmod n$. Then r is odd, and $1 \le r \le n-1$. It follows that $r \le (r+n-1)/2 \le n-1$.

Suppose first that $rk \equiv 1 \bmod 2n$. Then $(rk-1)/n$ is even, so in $(\mathbb{Z}/2)[x]/(x^n-1)$, we have

$$\begin{aligned}(1+x+x^2+\cdots+x^{k-1})&(1+x^k+x^{2k}+\cdots+x^{(r-1)k})x^t \\ &= (1+x+x^2+\cdots+x^{rk-1})x^t \\ &= \left(\frac{rk-1}{n}(1+x+x^2+\cdots+x^{n-1})+1\right)x^t = x^t.\end{aligned}$$

Thus, $$a_0+a_1x+\cdots+a_{n-1}x^{n-1} = x^t+x^{t+k}+x^{t+2k}+\cdots+x^{t+(r-1)k}.$$

Now, $$t+\frac{(r+n-1)}{2}\cdot k = t+\frac{rk-1}{2}+\frac{nk}{2}-\frac{k-1}{2} \equiv 0 \bmod n$$

since k is odd and $rk \equiv 1 \bmod 2n$. Now, since $r \le (r+n-1)/2 \le n-1$, none of the numbers $t, t+k, t+2k, \dots, t+(r-1)k$ are divisible by n. So, $a_0 = 0$.

If, on the other hand, $rk \equiv n+1 \bmod 2n$, then $((n-r)k+1)/n$ is even, so in $\mathbb{Z}/2[x]/(x^n-1)$, we have

$$\begin{aligned}(1+x+x^2+\cdots+x^{k-1})&(x^{rk}+x^{(r+1)k}+\cdots+x^{(n-1)k})x^t \\ &= (x^{rk}+x^{rk+1}+\cdots+x^{nk-1})x^t \\ &= \left(-1+\frac{(n-r)k+1}{n}(1+x+x^2+\cdots+x^{n-1})\right)x^t = -x^t = x^t.\end{aligned}$$

Thus, $$a_0+a_1x+\cdots+a_{n-1}x^{n-1} = x^{t+rk}+x^{t+(r+1)k}+\cdots+x^{t+(n-1)k}.$$

Now, $$t+\left(\frac{r-1}{2}\cdot k\right) = \frac{n+k-1+rk-k}{2} = \frac{rk-n-1}{2}+n \equiv 0 \bmod n.$$

Since $0 \le (r-1)/2 < r$, none of the numbers $t+rk, t+(r+1)k, \dots, t+(n-1)k$ are congruent to $(n-(k-1))/2$ modulo n. So, again, $a_0 = 0$. □

Lemma 6.3 *Suppose that k is odd and G is a finite group whose order n is relatively prime to k. Then there is an element $h_{n,k}(V)$ in $RO(\mathrm{Spin}(V))[1/k]$ such that the product $\theta^k(V)\cdot h_{n,k}(V)$ restricts to the one-dimensional trivial representation for any homomorphism $\lambda\colon G \to \mathrm{Spin}(V)$.*

Proof Let $R = R(n)_k$. Just as in the proof of Lemma 4.8, or the proof of Hirata and Kono in [14], there are polynomials $f_{n,k}(x) \in \mathbb{Z}[1/k][x]$ such that in R,

$$(1+x+x^2+\cdots+x^{k-1})f_{n,k}(x) = 1.$$

Now suppose n is even. We wish to produce a Laurent polynomial $g_{n,k}(x)$ which is invariant under $x \to x^{-1}$ and which satisfies

$$\left(x^{(1-k)/2} + x^{(3-k)/2} + \cdots + x^{(k-3)/2} + x^{(k-1)/2}\right) \cdot g_{n,k}(x) = 1.$$

We can use essentially the same argument as above to find a polynomial

$$\widetilde{g}_{n,k}(x) = b_0 + b_1 x + b_2 x^2 + \cdots b_{n-1} x^{n-1}$$

such that the formula below holds in R:

$$(1 + x + x^2 + \cdots + x^{k-1})\widetilde{g}_{n,k}(x) = x^{(n+k-1)/2}$$

By Lemma 6.2, b_0 is even. Dividing both sides by $x^{(k-1)/2}$ yields

$$\left(x^{(1-k)/2} + x^{(3-k)/2} + \cdots + x^{(k-3)/2} + x^{(k-1)/2}\right) \cdot \widetilde{g}_{n,k}(x) = x^{n/2}.$$

Since $(x^{(1-k)/2} + x^{(3-k)/2} + \cdots + x^{(k-3)/2} + x^{(k-1)/2})$ and $x^{n/2}$ are both invariant under the transformation $x \to x^{-1}$, it follows that in R, $\widetilde{g}_{n,k}(x)$ must be invariant under the transformation $x \to x^{-1}$. Therefore, $b_i = b_{n-i}$ for $i = 1, 2, \dots, n-1$. Thus, we can write the Laurent polynomial $g_{n,k}(x) := \widetilde{g}_{n,k}(x) \cdot x^{-n/2}$ in the form

$$\frac{b_0}{2} x^{-n/2} + b_1 x^{1-(n/2)} + \cdots + b_1 x^{(n/2)-1} + \frac{b_0}{2} x^{n/2}$$

which is invariant under the transformation $x \to x^{-1}$. The case when n is odd follows similarly, but does not require Lemma 6.2.

Now, $\prod_{i=1}^{4n} g_{n,k}(y_i)$ is invariant under permutations of y_i and under transformations $y_i \to y_i^{\varepsilon_i}$, and therefore defines a character of $\mathrm{Spin}(V)$. Since the dimension of V is divisible by 8, $RO(\mathrm{Spin}(V))$ is isomorphic to $R(\mathrm{Spin}(V))$ [10, Equation 10.9], so that $\prod_{i=1}^{4n} g_{n,k}(y_i)$ is the character of a real virtual $\mathrm{Spin}(V)$–representation. We denote this representation by $h_{n,k}(V)$.

For each $g \in G$, we may assume by conjugation in $\mathrm{Spin}(V)$ that $\lambda(g) \in \widetilde{T} \subseteq \mathrm{Spin}(V)$. We write the image of $\lambda(g)$ in T as $(\lambda_1(g), \lambda_2(g), \dots, \lambda_{4n}(g))$. Then for each i, $\lambda_i(g)^n = \lambda_i(g^n) = 1$. Therefore, the character of $\theta^k(V) \cdot h_{n,k}(V)$ at $\lambda(g)$ is 1. Since this holds for all $g \in G$, it follows that $\lambda^*(\theta^k(V) \cdot h_{n,k}(V))$ is the one-dimensional trivial representation. □

We write a stable Spin bundle as a difference $\xi - \mathbf{V}_\lambda$, where ξ is a $(G, \mathrm{Spin}(V))$–bundle. Therefore, we may define $\rho^k(\xi - \mathbf{V}_\lambda)$ in $KO_G(B)[1/k]$ as $\rho^k(\xi)/\theta^k(V_\lambda)$.

Remark 6.4 It is clear from the proof of Lemma 6.3 that $h_{n,k}(V)$ may be viewed as an element in $RO(\mathrm{SO}(V))[1/k]$. Thus, by Remark 6.1, if ξ is a $(G,\mathrm{SO}(V))$–bundle and $\lambda\colon G \to \mathrm{SO}(V)$ is a given homomorphism, we can define $\rho^k(\xi - \mathbf{V}_\lambda)$ as $\rho^k(\xi)/\theta^k(V_\lambda)$.

We can define ρ^k on classifying spaces in two different ways, both of which play a role in our work. On the one hand, we can use a definition analogous to that of May [16], just as described in [13]. First, recall from Section 2 that $Bj_p^\wedge$ is equal to the composite below:

$$B_G\mathrm{Spin} \xrightarrow{g_p^\wedge} B_G(\mathbb{S}_p^\wedge; KO_p^\wedge) \xrightarrow{q} B_G(\mathbb{S}_p^\wedge)$$

As shown in [13, Section 9], there is a map $\iota\colon \mathrm{Fib}(q) \to \Omega KO_p^{\wedge\times}$. Indeed, a $KO_p^\wedge$–orientation of a trivial $G\mathcal{F}(\mathbb{S}_p^\wedge)$–fibration over a based G–space X is just given by a class in $KO_G(X)_p^\wedge$ which maps to a unit upon restriction to the basepoint of X.

If k is prime to p and $|G|$, then for a G–representation V_λ (with dimension $8n$) with Spin structure $\lambda\colon G \to \mathrm{Spin}(V)$ as in Lemma 6.3, the operation $\psi^k/\theta^k(V_\lambda)$ on $KO_p^\wedge$–theory induces a self-map $\psi^k_{V_\lambda}$ of $\Omega^\infty KO_p^\wedge$ which preserves the Bott class b^{V_λ}. The collection $\Psi^k = \{\psi^k_{V_\lambda}\}$ then induces a map $c(\Psi^k)$ from $B_G(\mathbb{S}_p^\wedge; KO_p^\wedge)$ to the homotopy fiber of q, or by inclusion to $\Omega^\infty KO_p^{\wedge\times}$. Since the basepoint component of $\Omega^\infty KO_p^{\wedge\times}$ is homotopy equivalent to $B_G O_{\otimes}{}_p^\wedge$, $c(\Psi^k)\circ g_p^\wedge$ factors through a map $\rho^k\colon B_G\mathrm{Spin} \to B_G O_{\otimes}{}_p^\wedge$, which represents the operation ρ^k described above on (G,Spin)–bundles. Also, if $\tau\colon B_G O_{\otimes}{}_p^\wedge \to B_G(\mathbb{S}_p^\wedge; KO_p^\wedge)$ is the inclusion of the basepoint component of the homotopy fiber of q, then $c(\psi^k)\circ\tau \simeq \psi^k/1$ (see Remark 4.13 in [13].)

On the other hand, the representation $\theta^k(V)$ of $\mathrm{Spin}(V)$ clearly lifts to Spin in the sense of Definition 5.6, since $\mathrm{Spin}(V)$ is itself simply connected. Thus, if $\lambda\colon G \to \mathrm{Spin}(V)$ is a distinguished Spin–structure for V (giving $B_G\mathrm{Spin}(V)$ a basepoint), then $\theta^k(V)$ induces a based G–map $B_G\mathrm{Spin}(V) \to B_G\mathrm{Spin}$ by Remark 5.10. This map represents replacing a $\mathrm{Spin}(V)$–bundle ξ with the virtual 0–dimensional bundle $\rho^k(\xi) - \theta^k(\mathbf{V}_\lambda)$. If $|G| = n$, then the element $h_{n,k}(V) \in RO(\mathrm{Spin}(V))[1/k]$ of Lemma 6.3 lifts to Spin, and multiplication by the element $\lambda^* h_{n,k}(V) \in R\mathrm{Spin}(G)[1/k] \subseteq R\mathrm{Spin}(G)_p^\wedge$ induces a map $B_G\mathrm{Spin} \to B_G\mathrm{Spin}_p^\wedge$. Adding a 1–dimensional trivial bundle induces a map $B_G\mathrm{Spin}_p^\wedge \to B_G\mathrm{Spin}_{\otimes}{}_p^\wedge$. Then the composite

$$B_G\mathrm{Spin}(V) \xrightarrow{\theta^k(V)} B_G\mathrm{Spin} \xrightarrow{\cdot\lambda^* h_{n,k}(V)} B_G\mathrm{Spin}_p^\wedge \xrightarrow{+1} B_G\mathrm{Spin}_{\otimes}{}_p^\wedge$$

represents the functor taking ξ to $\rho^k(\xi)/\theta^k(\mathbf{V}_\lambda) = \rho^k(\xi - \mathbf{V}_\lambda)$. If $\mathbb{V}$ is a complete sequence of complex G–representations, then each V_i comes equipped with a map

$\lambda\colon G \to \mathrm{Spin}(V_i)$, and by taking a colimit over the spaces $B_G\mathrm{Spin}(V_i)$, we get a map $B_G\mathrm{Spin} \to B_G\mathrm{Spin}_{\otimes}{}^{\wedge}_{p}$ representing ρ^k.

Remark 6.5 Using this method, we could define a map $\rho^k\colon B_G\mathrm{SO} \to B_G\mathrm{SO}_{\otimes}{}^{\wedge}_{p}$ representing ρ^k as in Remark 6.4.

6.2 Comparison of σ^k and ρ^k

By Remark 6.5, when k is odd, the Adams–Bott cannibalistic class ρ^k can also be viewed as a map from $\mathcal{S}_G B_G\mathrm{SO}$ to $\mathcal{S}_G B_G\mathrm{SO}_{\otimes}{}^{\wedge}_{p}$. We next show that σ^k is homotopic to ρ^k. The basic idea of the argument below is the same as that in [13], when we proved the statement assuming G has odd order and p is odd.

First, in the complex case, it is easy to show using the splitting principle that the Adams–Bott cannibalistic class commutes with Adams operations. Since we do not know of a splitting principle for KO–theory, we give a proof for the following lemma.

Lemma 6.6 *The map $\rho^k\colon B_G\mathrm{SO} \to B_G\mathrm{SO}_{\otimes}{}^{\wedge}_{p}$ commutes with ψ^k.*

Proof The map ρ^k is induced by a natural exponential map $R\mathrm{SO}(G) \to R\mathrm{SO}(G)$ for connected compact Lie groups G. Now, if T is a torus, then $\psi^k\colon R(T) \to R(T)$ coincides with the map p_k^*, where $p_k\colon T \to T$ is given by $p_k(y_1, y_2, \dots, y_n) = (y_1^k, y_2^k, \dots, y_n^k)$. Since $c\colon R\mathrm{SO}(T) \to R(T)$ is an injection and commutes with both ψ^k and p_k^*, it follows that $\psi^k\colon R\mathrm{SO}(T) \to R\mathrm{SO}(T)$ also coincides with p_k^*. Now, since ρ^k is natural, $\rho^k\colon R\mathrm{SO}(T) \to R\mathrm{SO}(T)$ must commute with p_k^* and hence ψ^k. Finally, if $i\colon T \to G$ is the inclusion of a maximal torus in G, then since $c\colon R\mathrm{SO}(G) \to R(G)$ and $i^*\colon R(G) \to R(T)$ are both injections, $i^*\colon R\mathrm{SO}(G) \to R\mathrm{SO}(T)$ must also be an injection. It follows that ρ^k commutes with ψ^k. $\square$

It follows that the diagram below commutes whether the left vertical map is ρ^k or σ^k:

$$\begin{tikzcd}
\mathcal{S}_G B_G\mathrm{SO} \arrow[r, "\psi^k - 1"] \arrow[d, shift right, "\sigma^k"'] \arrow[d, shift left, "\rho^k"] & \mathcal{S}_G B_G\mathrm{SO} \arrow[d, "\rho^k"] \\
\mathcal{S}_G B_G\mathrm{SO}_{\otimes}{}^{\wedge}_{p} \arrow[r, "\psi^k/1"] & \mathcal{S}_G B_G\mathrm{SO}_{\otimes}{}^{\wedge}_{p}
\end{tikzcd}$$

Lemma 6.7 *The map*

$$\psi^k/1\colon \mathcal{S}_G B_G\mathrm{SO}_{\otimes}{}^{\wedge}_{p}[1/p] \to \mathcal{S}_G B_G\mathrm{SO}_{\otimes}{}^{\wedge}_{p}[1/p]$$

is an equivalence.

Proof We briefly sketch the argument (see Proposition 10.3 in [13]). It suffices to show that for any subgroup $H \leq G$, and any $n \geq 1$, the map

$$1 + \widetilde{KSO}_H(S^n) \otimes \mathbb{Q}_p^\wedge \xrightarrow{\psi^k/1} 1 + \widetilde{KSO}_H(S^n) \otimes \mathbb{Q}_p^\wedge$$

is an isomorphism. Since the additive and multiplicative structures on $\widetilde{KSO}_H(S^n)$ agree for $n \geq 1$, and complexification is injective on $\widetilde{KSO}_H(S^n) \otimes \mathbb{Q}_p^\wedge$ (where 2 is invertible regardless of p), it suffices to show that

$$\widetilde{K}_H(S^n) \otimes \mathbb{Q}_p^\wedge \xrightarrow{\psi^k - 1} \widetilde{K}_H(S^n) \otimes \mathbb{Q}_p^\wedge$$

is one-to-one (both sides being finite dimensional vector spaces of the same dimension). But $\widetilde{K}_H(S^n)$ is isomorphic to $R(H) \otimes \widetilde{K}(S^n)$, which is in turn isomorphic to $R(H)$ when n is even, and 0 when n is odd. Moreover, the action of ψ^k on $\widetilde{K}_H(S^n)$ corresponds to the action of $k^{n/2}\psi^k$ on $R(H)$. Using the periodicity of ψ^k on $R(G)$, it is straightforward to show that $k^{n/2}\psi^k - 1$ is injective. □

Thus, the two composites below are homotopic.

$$\mathcal{S}_G B_G \mathrm{SO} \underset{\sigma^k}{\overset{\rho^k}{\rightrightarrows}} \mathcal{S}_G B_G \mathrm{SO}_{\otimes}{}_p^\wedge \longrightarrow \mathcal{S}_G B_G \mathrm{SO}_{\otimes}{}_p^\wedge[1/p]$$

Lemma 6.8 *The map below is injective.*

$$\left[\mathcal{S}_G B_G \mathrm{SO}, \mathcal{S}_G B_G \mathrm{SO}_{\otimes}{}_p^\wedge\right]_G \to \left[\mathcal{S}_G B_G \mathrm{SO}, \mathcal{S}_G B_G \mathrm{SO}_{\otimes}{}_p^\wedge[1/p]\right]_G$$

Proof We again sketch the argument (see Proposition 10.6 of [13]). By Proposition 2.2, $\mathcal{S}_G B_G \mathrm{SO} \simeq \mathrm{colim}_V\, B\mathrm{SO}(V)$, where V runs over the G–representations of a complete sequence (recall Definition 5.1). It therefore suffices to show that

$$K\mathrm{SO}_G(BO(V)) \otimes \mathbb{Z}_p^\wedge \to K\mathrm{SO}_G(BO(V)) \otimes \mathbb{Q}_p^\wedge$$

is injective. Thus, it suffices to show that $K\mathrm{SO}_G(BO(V))$ has no torsion. Since $K\mathrm{SO}_G(X) \cong [X, B_G\mathrm{SO}]_G$, and $B_G\mathrm{SO}$ is a factor of $B_G O$ by Lemma 5.2, it suffices to show that $KO_G(BO(V))$ has no torsion.

Now the G–action on $O(V)$ determines a group $G \rtimes O(V)$ which has a canonical right action on $O(V)$. If we let $EO(V) = B(*, O(V), O(V))$, then the G–action on $O(V)$ through conjugation and the right $O(V)$ action on $B(*, O(V), O(V))$ determine an action of $G \rtimes O(V)$ on $EO(V)$, with $O(V)$ acting freely, and $EO(V)/O(V) = BO(V)$. Therefore,

$$KO_G(BO(V)) \cong KO_{G \rtimes O(V)}(EO(V)).$$

Now, if $\Lambda \leq G \rtimes O(V)$, and $H(\Lambda)$ is the image of Λ in G, then $EO(V)^{\Lambda} = B(*, O(V)^{H(\Lambda)}, O(V)^{\Lambda})$. In particular, if H is a subgroup of $G \leq G \rtimes O(V)$, then $EO(V)^H = B(*, O(V)^H, O(V)^H) \simeq *$. Conversely, if $EO(V)^{\Lambda}$ is nonempty, then $O(V)^{\Lambda}$ must be nonempty, so for some $u \in O(V)$, $g^{-1}ugs = u$ for all $(g,s) \in \Lambda$. But this implies that $u^{-1}gu = gs$ in $G \rtimes O(V)$, so Λ is conjugate to $H(\Lambda)$. Thus, if we let $\mathcal{F}$ denote the family of subgroups of $G \rtimes O(V)$ consisting of subgroups which are conjugate to a subgroup of G, then $EO(V)^{\Lambda} \simeq *$ for $\Lambda \in \mathcal{F}$ and $EO(V)^{\Lambda} = \varnothing$ for $\Lambda \notin \mathcal{F}$. Thus, $EO(V) \simeq E\mathcal{F}$.

Now, Adams, Haeberly, Jackowski, and May [5] proved that $\widetilde{KO}_{G \rtimes O(V)}(E\mathcal{F})$ is the completion of $RO(G \rtimes O(V))$ with respect to products of powers of the ideals $I_{\Lambda} = \mathrm{Ker}(RO(G \rtimes O(V)) \to RO(\Lambda))$ for $\Lambda \in \mathcal{F}$. Since $I_G \leq I_{\Lambda}$ for all $\Lambda \in \mathcal{F}$, this is $RO(G \rtimes O(V))\widehat{_{I_G}}$. Since $RO(G \rtimes O(V))$ is Noetherian, $RO(G \rtimes O(V))\widehat{_{I_G}}$ is flat over $RO(G \rtimes O(V))$. Since $RO(G \rtimes O(V))$ is itself torsion free, $RO(G \rtimes O(V))\widehat{_{I_G}}$ is also torsion free. □

Corollary 6.9 *The restrictions of the maps σ^k and ρ^k to $\mathcal{S}_G B_G \mathrm{SO}$ are homotopic.*

Remark 6.10 As we noted in Remark 2.7, we have some flexibility in defining γ^k. We now explain how to define γ^k in a way that will allow us to prove Lemma 6.11 below, concerning σ^k on π_1. By Lemma 5.2 we may define γ^k separately on the factor $\mathcal{S}_G B_G O(1) \subseteq \mathcal{S}_G B_G O$. By Proposition 2.2, $\mathcal{S}_G B_G O(1) \simeq BO(1)$. Now, since k is odd, $\psi^k - 1$ induces the trivial map on $BO(1)$, as can be seen by considering the restriction of the virtual representation inducing $\psi^k - 1$ to $O(1)$. Therefore, it suffices to define a map from $BO(1)$ to $\mathcal{S}_G \Omega B_G(\mathbb{S}^{\wedge}_p)$, which is equivalent by Corollary 2.3 to $(\mathcal{S}_G F_G)^{\wedge}_p$.

When p is odd, we choose the trivial map. When $p = 2$, $k = 3$, and we define γ^3 on $BO(1)$ as follows. By Lemma 5.3, there is a G–map from $BO(1)$ to $\mathcal{S}_G \Omega B_G \mathrm{SO} \simeq \mathcal{S}_G O$ inducing a nontrivial map on π_1. The inclusion of $\mathcal{S}_G O$ in $\mathcal{S}_G F_G$ induces an isomorphism on π_1 nonequivariantly (this is just the inclusion of SO in SF). We define γ^3 on $BO(1)$ to be the composite

$$BO(1) \to \mathcal{S}_G O \to \mathcal{S}_G F_G \to (\mathcal{S}_G F_G)^{\wedge}_p \to \mathcal{S}_G \mathrm{Fib}(Bj^{\wedge}_p).$$

Then γ^3 induces an injective map on π_1 nonequivariantly, since $\pi_2(B\mathrm{Spin}) = 0$. Since $BO(1)$ has trivial G–action, the map $BO(1)^H \to \mathrm{Fib}(Bj^{\wedge}_p)^H$ induces an injective map on π_1 for any $H \leq G$.

Lemma 6.11 *With γ^3 chosen as in Remark 6.10, the composite*

$$BO(1) \longrightarrow \mathcal{S}_G B_G O \xrightarrow{\sigma^3} \mathcal{S}_G B_G O_{\otimes}{}^{\wedge}_2 \xrightarrow{\det_*} \mathcal{S}_G B_G O(1)^{\wedge}_2$$

induces an isomorphism on π_1 when restricted to H fixed points for any $H \leq G$.

Proof Again, by Proposition 2.2, $\mathcal{S}_G B_G O(1) \simeq BO(1)$. Since the G–action on $BO(1)$ is trivial, and since $\pi_1(BO(1)) \cong \mathbb{Z}/2$, the claim is reduced to showing that the displayed map induces a nontrivial map on π_1 nonequivariantly. The inclusion $BO(1) \to BO$ and the map from $BO_\otimes$ to $BO(1)$ induced by the determinant both induce isomorphisms on π_1. Now, $\sigma^3 = i \circ f \circ \gamma^3$. By Remark 6.10, $\gamma^3 \colon \mathcal{S}_G B_G O \to \mathcal{S}_G \mathrm{Fib}(Bj_p^\wedge)$ induces a nontrivial map on π_1. Finally, it follows from May [16, Chapter V, Corollary 3.3] that the map

$$i \circ f \colon \mathcal{S}_G \mathrm{Fib}(Bj_p^\wedge) \to \mathcal{S}_G \mathrm{Fib}(q) \subseteq \mathcal{S}_G B_G O_{\otimes}{}_p^\wedge$$

induces an isomorphism on π_1 nonequivariantly. □

7 Appendix

7.1 The zeroth space of an equivariant sphere spectrum

Given a G–space X, let $\widetilde{\mathcal{A}}(X)$ denote the monoid of self-maps of X which are nonequivariant equivalences, with G acting by conjugation. We showed in [13, Section 8] that if G is a p–group (p odd), and V is a G–representation large enough that $[S^V, S^V]_G \cong A(G)$, then $\Omega B_G(\mathbb{S}^V{}_p^\wedge) \sim_G \widetilde{\mathcal{A}}(S^V{}_p^\wedge)$. This does not generally hold when G is not a p–group. For example, consider the case $G = \mathbb{Z}/q$, where q is an odd prime distinct from p. Then $A(G) \cong \mathbb{Z}[x]/(x^2 - qx)$. After completing at p, the element x, though not a unit, maps to a unit by the augmentation homomorphism $A(G)_p^\wedge \to \mathbb{Z}_p^\wedge$. Thus, there exists an equivariant self-map of $S^V{}_p^\wedge$ which is a nonequivariant equivalence, but not an equivariant equivalence. Therefore, $\pi_0(\widetilde{\mathcal{A}}(S^V{}_p^\wedge)^G)$ is not a group, so $\widetilde{\mathcal{A}}(S^V{}_p^\wedge)$ cannot be a loop space.

In order to state what we can show, we must first recall some notation from Elmendorf [12]. Let $G\mathcal{T}$ be the category of based G–spaces, let $\mathcal{O}_G$ denote the orbit category of G, and let $\mathcal{O}_G\mathcal{T}$ denote the category of $\mathcal{O}_G$–spaces, or $\mathcal{O}_G$–shaped diagrams in $\mathcal{T}$. Let $\Phi\colon G\mathcal{T} \to \mathcal{O}_G\mathcal{T}$ be the functor given by $\Phi(X)(G/H) = X^H$. There is a functor $C\colon \mathcal{O}_G\mathcal{T} \to G\mathcal{T}$, along with natural transformations $\Phi C \to \mathrm{id}$ and $C\Phi \to \mathrm{id}$. The first of these natural transformations induces a levelwise homotopy equivalence for each object in $\mathcal{O}_G\mathcal{T}$, and the second induces an equivariant weak equivalence for each G–space. Our goal in this section is to prove the following, where G is an arbitrary finite group.

Proposition 7.1 *If $\mathbb{F}$ is a set of fibers with distinguished fiber F_ρ, then the spaces $\Omega B_G(\mathbb{F})$ and $C(\mathcal{A}(F_\rho, -))$ are weakly G–equivalent.*

Lemma 7.2 *If T is an $\mathcal{O}_G$–space and ΩT is defined by $(\Omega T)(G/H)=\Omega(T(G/H))$, then $C\Omega T$ is weakly G–equivalent to ΩCT.*

Proof It is not hard to check that a map $T_1 \to T_2$ of $\mathcal{O}_G$–spaces is a levelwise weak equivalence if and only if $CT_1 \to CT_2$ is a weak G–equivalence. Therefore, the levelwise weak equivalence $\Phi CT \to T$ induces a weak G–equivalence $C\Omega\Phi CT \to C\Omega T$. Clearly, Φ commutes with Ω. Finally, there is a weak G–equivalence $C\Phi\Omega CT \to \Omega CT$. □

We will mostly be interested in the following $\mathcal{O}_G$–spaces.

Definition 7.3 Given a G–space F, let $\mathcal{A}(F,H)\subseteq\widetilde{\mathcal{A}}(F)^H$ be the submonoid consisting of H–equivariant self-maps which are H–equivariant equivalences. Let $\mathcal{A}(F,-)$ denote the corresponding $\mathcal{O}_G$–space.

We now prove Proposition 7.1.

Proof Let $\mathcal{A}(\mathbb{F})_0$ denote the full subcategory of $\mathcal{A}(\mathbb{F})$ whose objects are of the form $G\times_H F_{\rho|_H}\to G/H$. Then by Lemma 3.15 in [13], the inclusion $B(*,\mathcal{A}(\mathbb{F})_0,\mathcal{O})\to B(*,\mathcal{A}(\mathbb{F}),\mathcal{O})$ is a G–connected cover, so that the induced map on loop spaces is an equivalence. Therefore, it suffices to prove the proposition after replacing $B_G(\mathbb{F})=B(*,\mathcal{A}(\mathbb{F}),\mathcal{O})$ with $B(*,\mathcal{A}(\mathbb{F})_0,\mathcal{O})$.

Given a category $\mathcal{C}$ and a covariant functor $F\colon\mathcal{C}\to\mathcal{U}$, we have a category $\mathcal{C}_F$ whose objects are pairs (c,x) with $c\in\mathrm{obj}(\mathcal{C})$ and $x\in F(c)$, and whose morphisms $(c,x)\to(c',x')$ are maps $f\colon c\to c'$ such that $F(f)(x)=x'$. A natural transformation of functors $F\to F'$ induces a functor $\mathcal{C}_F\to\mathcal{C}_{F'}$. Let $\mathcal{O}^H\colon\mathcal{A}(\mathbb{F})_0\to G\mathcal{U}$ be the functor taking the object $G\times_K F_{\rho|_K}\to G/K$ to the set of G–maps $G/H\to G/K$. In particular, we have a category $(\mathcal{A}(\mathbb{F})_0)_{\mathcal{O}^H}$, and it is easy to check that

$$B\left((\mathcal{A}(\mathbb{F})_0)_{\mathcal{O}^H}\right)\cong B(*,\mathcal{A}(\mathbb{F})_0,\mathcal{O}^H)\cong B(*,\mathcal{A}(\mathbb{F})_0,\mathcal{O})^H.$$

We can describe a map in $(\mathcal{A}(\mathbb{F})_0)_{\mathcal{O}^H}$ by a commutative diagram

$$\begin{tikzcd}
 & G\times_K F_{\rho|_K} \arrow[r, "\overline{\theta}"] \arrow[d] & G\times_{K'} F_{\rho|_{K'}} \arrow[d] \\
G/H \arrow[r, "\alpha"] & G/K \arrow[r, "\widetilde{\theta}"] & G/K'.
\end{tikzcd}$$

We will denote this map simply by $(\alpha,\widetilde{\theta},\overline{\theta})$.

Viewing $\mathcal{A}(F_\rho, H)$ as a category with one object, let $l\colon \mathcal{A}(F_\rho, H) \to (\mathcal{A}(\mathbb{F})_0)_{\mathcal{O}^H}$ be the functor taking this unique object to $\big((G \times_H F_\rho \to G/H), \mathrm{id}_{G/H}\big)$, and taking an H–map $\phi\colon F_\rho \to F_\rho$ to the map $(1, 1, 1\times\phi)$. Let $r\colon (\mathcal{A}(\mathbb{F})_0)_{\mathcal{O}^H} \to \mathcal{A}(F_\rho, H)$ be the functor taking each object to the unique object of $\mathcal{A}(F_\rho, H)$, and taking a map $(\alpha, \widetilde{\theta}, \overline{\theta})$ to the map ϕ determined by $\overline{\theta}(g, g^{-1}f) = (gg', (gg')^{-1}\phi(f))$, where $\alpha(eH) = gK$ and $\widetilde{\theta}(eK) = g'K'$. It is easy to check that this definition does not depend on our choices of g or g'. Now, l and r form an adjoint pair, so r induces an equivalence

$$B(*, \mathcal{A}(\mathbb{F})_0, \mathcal{O})^H \simeq B(\mathcal{A}(F_\rho, H)).$$

A map $\widetilde{\theta}\colon G/H \to G/K$ with $\widetilde{\theta}(eH) = gK$ determines a functor $\widetilde{\theta}^*\colon \mathcal{A}(F_\rho, K) \to \mathcal{A}(F_\rho, H)$ defined by $\widetilde{\theta}^*(f) = l_g \circ f \circ l_{g^{-1}}$, as well as a functor $\widetilde{\theta}^*\colon (\mathcal{A}(\mathbb{F})_0)_{\mathcal{O}^K} \to (\mathcal{A}(\mathbb{F})_0)_{\mathcal{O}^H}$, induced by the natural transformation $\mathcal{O}^K \to \mathcal{O}^H$ determined by $\widetilde{\theta}$. Moreover, the following diagram commutes:

$$\begin{array}{ccc} (\mathcal{A}(\mathbb{F})_0)_{\mathcal{O}^K} & \xrightarrow{r} & \mathcal{A}(F_\rho, K) \\ \Big\downarrow{\scriptstyle \widetilde{\theta}^*} & & \Big\downarrow{\scriptstyle \widetilde{\theta}^*} \\ (\mathcal{A}(\mathbb{F})_0)_{\mathcal{O}^H} & \xrightarrow{r} & \mathcal{A}(F_\rho, H) \end{array}$$

Taking classifying spaces, we obtain have a levelwise homotopy equivalence of $\mathcal{O}_G$–spaces

$$\Phi B(*, \mathcal{A}(\mathbb{F})_0, \mathcal{O}) \to B\mathcal{A}(F_\rho, -)$$

inducing a weak equivalence of G–spaces

$$B(*, \mathcal{A}(\mathbb{F})_0, \mathcal{O}) \xleftarrow{\simeq} C\Phi B(*, \mathcal{A}(\mathbb{F})_0, \mathcal{O}) \xrightarrow{\simeq} CB\mathcal{A}(F_\rho, -).$$

By Lemma 7.2, we get a weak G–equivalence $\Omega B(*, \mathcal{A}(\mathbb{F})_0, \mathcal{O}) \simeq C\Omega B\mathcal{A}(F_\rho, -)$. But the map of $\mathcal{O}_G$–spaces $\mathcal{A}(F_\rho, -) \to \Omega B\mathcal{A}(F_\rho, -)$ is a levelwise weak equivalence, since $\mathcal{A}(F_\rho, H)$ is grouplike for each $H \le G$. Thus $\Omega B(*, \mathcal{A}(\mathbb{F})_0, \mathcal{O})$ is equivalent to $C\mathcal{A}(F_\rho, -)$. □

Corollary 7.4 *The space $\mathcal{S}_G \Omega B_G(\mathbb{F})$ is G–equivalent to $\mathcal{S}_G \widetilde{A}(F_\rho)$.*

Proof For any $H \le G$, the basepoint component of $\mathcal{A}(F_\rho, H)$ and the basepoint component of $\widetilde{A}(F_\rho)^H$ each coincide with the set of H equivariant self-maps of F_ρ which are H–equivariantly homotopic to the identity. It follows that the map

$$\mathcal{S}_G \Omega B_G(\mathbb{F}) \simeq \mathcal{S}_G C\mathcal{A}(F_\rho, -) \to \mathcal{S}_G C\Phi(\widetilde{A}(F_\rho)) \simeq \mathcal{S}_G(\widetilde{A}(F_\rho))$$

is an equivalence. □

We now prove Corollary 2.3.

Proof For a given G–representation V (with nontrivial G–fixed points), we have that $\mathcal{S}_G \Omega B_G(\mathbb{S}^V{}^\wedge_p)$ is G–equivalent to $\mathcal{S}_G \widetilde{\mathcal{A}}(S^V{}^\wedge_p)$ by Corollary 7.4. If we let $F(S^V{}^\wedge_p, S^V{}^\wedge_p)$ denote the G–space of self-maps of $S^V{}^\wedge_p$, based at the identity, then the inclusion of $\widetilde{\mathcal{A}}(S^V{}^\wedge_p)$ in $F(S^V{}^\wedge_p, S^V{}^\wedge_p)$ induces a weak G–equivalence

$$\mathcal{S}_G \widetilde{\mathcal{A}}(S^V{}^\wedge_p) \to \mathcal{S}_G F(S^V{}^\wedge_p, S^V{}^\wedge_p),$$

since an H–equivariant map homotopic to the identity is necessarily an H–equivariant equivalence. Moreover, $\mathcal{S}_G F(S^V{}^\wedge_p, S^V{}^\wedge_p)$ is the p–completion of $\mathcal{S}_G \Omega^V S^V$. After taking a colimit over a complete sequence of G–representations, we see that $\mathcal{S}_G \Omega B_G(\mathbb{S}^\wedge_p) \sim_G (\mathcal{S}_G F_G)^\wedge_p$.

For $n \geq 2$, the second statement follows from the first. For $n = 1$, Proposition 7.1 implies that $\Omega B_G(\mathbb{S}^V{}^\wedge_p)$ and $C(\mathcal{A}(S^V{}^\wedge_p, -))$ are weakly G–equivalent, so $\pi_1(B_G(\mathbb{S}^V{}^\wedge_p)^H) = \pi_0(\mathcal{A}(S^V{}^\wedge_p, H))$, which is isomorphic to the units in the p–completed Burnside ring $A(H)^\wedge_p$ for V sufficiently large. But the units in $A(H)^\wedge_p$ are closed and therefore p–complete. □

7.2 Proof of Theorem 2.11

In this subsection, we prove the following proposition, which implies Theorem 2.11.

Proposition 7.5 *Suppose p is prime and G a finite group such that one of the prime divisors q of $|G|$ is congruent to 1 modulo p. Then there is a G–space X such that $\widetilde{KO}_G(X)$ has trivial multiplication, and there are elements $\xi_1, \xi_2, \zeta_1, \zeta_2 \in \widetilde{KO}_G(X)^\wedge_p$ such that*

(1) $\rho^k(\xi_1) = (\psi^k/1)(1+\zeta_1)$,

(2) $\rho^k(\xi_2) = (\psi^k/1)(1+\zeta_2) + \xi_1$,

(3) *ξ_2 is not in $T_G(X)^\wedge_p$.*

If $p = 2$, then in fact, we may take ξ_1, ζ_1 and ζ_2 all to be 0.

Before turning to the proof, we show how Proposition 7.5 implies Theorem 2.11. Given the proposition and assuming the hypotheses of Theorem 2.11, we may write $\xi_1 = a_1 + b_1$ and $\xi_2 = a_2 + b_2$ for unique elements $a_1, a_2 \in A(X)$ and $b_1, b_2 \in B(X)$. If a_1 is not in the image of $\psi^k - 1$, then projecting the first equation into $A(X)$ yields the claim. Otherwise, we may write $a_1 = (\psi^k - 1)(a_1')$. Then from the second equation above, since the multiplication in $\widetilde{KO}_G(X)^\wedge_p$ is trivial, we have

$$\rho^k(\xi_2) = (\psi^k/1)(1+\zeta_2) + (\psi^k - 1)(a_1') + b_1 = (\psi^k/1)(1+\zeta_2+a_1') + b_1.$$

Projecting this equation into $A(X)$, we find that $\rho^k(a_2)$ is in the image of $\psi^k/1$. If a_2 is in the image of $\psi^k - 1$, then $a_2 \in T_G(X)^\wedge_p$. Since $b_2 \in T_G(X)^\wedge_p$, this would imply $\xi_2 \in T_G(X)^\wedge_p$, contradicting our third statement above. Thus, a_2 cannot be in the image of $\psi^k - 1$, yielding the claim.

It is easy to reduce Proposition 7.5 to the case in which G is cyclic of order q. Indeed, if q is a prime divisor of $|G|$, then G must contain a subgroup C_q which is cyclic of order q. Then, given a C_q–space X and letting $X' = G_+ \wedge_{C_q} X$, we have $\widetilde{KO}_G(X') \cong \widetilde{KO}_{C_q}(X)$. This isomorphism is compatible with Adams operations and ρ^k. Thus, if Proposition 7.5 holds for C_q, then it holds for G. We now, therefore, assume that G is cyclic of order q.

We will first consider the case when p is odd. The case when $p=2$ is actually simpler, and is described at the end of the section. Let $X = S^{2(p-1)}$ with trivial G–action. Then we have a decomposition $\widetilde{K}_G(X) \cong R(q) \otimes \widetilde{K}(X)$. Let $\xi_{2,c} \in \widetilde{K}_G(X)$ be the element corresponding to $(z + z^{-1} - 2) \otimes y^{p-1}$. We will use the following remark to show that there is a unique element $\xi_2 \in \widetilde{KO}_G(X)^\wedge_p$ such that $\xi_{2,c} = c(\xi_2)$.

Remark 7.6 Since $cr = 1 + \psi^{-1}$ and 2 is invertible in $\widetilde{K}_G(X)^\wedge_p$, it follows that any element $x \in \widetilde{K}_G(X)^\wedge_p$ which is invariant under ψ^{-1} is in the image of the complexification map. To see this, just note that if $\psi^{-1}x = x$, then $cr(x) = (1+\psi^{-1})x = 2x$, so $x = c(rx/2)$. The converse is also true, since for any $y \in \widetilde{KO}_G(X)^\wedge_p$, we have $\psi^{-1}(cy) = \psi^{-1}(cr(cy/2))$ and $\psi^{-1} \circ (cr) = \psi^{-1} \circ (1+\psi^{-1}) = cr$. Finally, note that c is injective since $rc = 2$.

Now, since $\psi^{-1}(\xi_{2,c}) = (z^{-1}+z-2) \otimes (-1)^{p-1} y^{p-1} = \xi_{2,c}$, it follows that there is an element ξ_2 such that $\xi_{2,c} = c(\xi_2)$. We next consider $\rho^k_c(c\xi_2) = \rho^k_c((z+z^{-1}-2)y^{p-1})$.

Remark 7.7 It follows as in the proof of Theorem 4.1 that

$$\rho^k_c(c\xi_2) = c\rho^k(rc\xi_2) = c\rho^k(\xi_2)^2.$$

By Corollary 4.13, we have the following equation in $1 + \widetilde{K}_G(X)_{\mathbb{Q}}$:

$$\rho^k_c(z^s y^{p-1}) = (\psi^k/1)(1 + \alpha_{p-1}(z^s) y^{p-1})) = 1 + (\psi^k - 1)(\alpha_{p-1}(z^s) y^{p-1})$$

Thus, $$\rho_c^k(c\xi_2) = \frac{\rho_c^k(zy^{p-1})\rho_c^k(z^{-1}y^{p-1})}{\rho_c^k(y^{p-1})^2}$$
$$= 1 + (\psi^k - 1)\big(\big(\alpha_{p-1,1} + \alpha_{p-1,-1} - 2\alpha_{p-1,0}\big)\, y^{p-1}\big).$$

Since q is invertible in $\mathbb{Z}^\wedge_p$, it follows from Lemma 4.3 that $R(q)^\wedge_p \cong R^1{}^\wedge_p \times R^q{}^\wedge_p$. Thus, we can define an element $\beta \in R(q)^\wedge_p$ by specifying a pair $(\beta_1, \beta_q) \in R^1{}^\wedge_p \times R^q{}^\wedge_p$. Recall that $\alpha_{p-1,0,q} = \alpha'_{p-1}$, which, Adams proved [2, Theorem 2.6], has p–adic valuation -1. Thus, since p divides $q-1$, $(q-1)\alpha_{p-1,0,1} \in \mathbb{Z}^\wedge_p = R^1{}^\wedge_p$. We let $\beta_1 = 2(1-q)\alpha_{p-1,0,1}$.

We now claim that $\alpha_{p-1,1,q}$ and $\alpha_{p-1,-1,q}$ are contained in $R^q{}^\wedge_p$. To see this, we will show that $z-1$ and $z^{-1}-1$ are invertible after inverting q. In fact,
$$(z-1)(1+2z+3z^2+\cdots+qz^{q-1}) = -1 - z - z^2 - \cdots - z^{q-1} + q = q.$$

A similar argument would show that $\alpha_{p-1,i,q}$ is contained in $R^q{}^\wedge_p$ for each i, and in particular for $i=-1$.

So we may let $\beta_q = \alpha_{p-1,1,q} + \alpha_{p-1,-1,q} \in R^q{}^\wedge_p$. Similarly, let $\gamma_1 = 2(q-1)\alpha_{p-1,0,1} \in R^1{}^\wedge_p$ and let $\gamma_q = -2\alpha_{p-1,0,q} \in R^q_{\mathbb{Q}}$. These define an element $\gamma \in R(q)_{\mathbb{Q}}$.

Next, we observe that $z + z^2 + z^3 + \cdots + z^{q-1} \in R(q)_q$ corresponds to $(q-1,-1) \in R^1_q \times R^q_q$. Therefore, in $R(q)_{\mathbb{Q}}$, we have $\gamma = 2\alpha_{p-1,0}(z+z^2+\cdots+z^{q-1})$. Since $\psi^k\gamma = \gamma$ and $\psi^k(y^{p-1}) = k^{p-1}y^{p-1}$, we have
$$(\psi^k - 1)(\gamma y^{p-1}) = (k^{p-1}-1)2\alpha_{p-1,0}(z+z^2+\cdots+z^{q-1})y^{p-1}$$
in $\widetilde{K}_G(X)_{\mathbb{Q}}$. Since $k^{p-1}-1$ is divisible by p, and $\alpha_{p-1,0,q} = \alpha_{p-1,0,1}$ has p–adic valuation -1, the same equation holds in $\widetilde{K}_G(X)^\wedge_p$.

Now, $\alpha_{p-1,1,1} = \alpha_{p-1,0,1} = \alpha_{p-1,-1,1}$, so in $R^1_{\mathbb{Q}}$, $\alpha_{p-1,1} + \alpha_{p-1,-1} - 2\alpha_{p-1,0} = 0$. It follows that
$$\beta + \gamma = \alpha_{p-1,1} + \alpha_{p-1,-1} - 2\alpha_{p-1,0}.$$
So, in $1 + \widetilde{K}_G(X)^\wedge_p$, we have
$$\rho_c^k(c\xi_2) = 1 + (\psi^k - 1)(\beta y^{p-1}) + (k^{p-1}-1)2\alpha_{p-1,0}(z+z^2+\cdots+z^{q-1})y^{p-1}.$$

Now let $\zeta_{2,c} = \beta y^{p-1}$, and let $\xi_{1,c} = (k^{p-1}-1)2\alpha_{p-1,0}(z+z^2+\cdots+z^{q-1})y^{p-1}$. By Remark 7.6 and since 2 is invertible, there are unique elements $\zeta_2, \xi_1 \in \widetilde{KO}_G(X)^\wedge_p$ such that $c(2\zeta_2) = \zeta_{2,c}$ and $c(2\xi_1) = \xi_{1,c}$. So, by Remark 7.7, we have
$$c\rho^k(\xi_2)^2 = 1 + (\psi^k - 1)(c(2\zeta_2)) + c(2\xi_1),$$
so $$\rho^k(\xi_2) = 1 + (\psi^k - 1)(\zeta_2) + \xi_1 = (\psi^k/1)(1+\zeta_2) + \xi_1.$$

This is Equation (2) of Proposition 7.5.

We next consider $\rho_c^k(c\xi_1)$. Using Corollary 4.13 as above,

$$\rho_c^k((z+z^2+\cdots+z^{q-1})y^{p-1}) = 1+(\psi^k-1)\left(\left(\sum_{i=1}^{q-1}\alpha_{p-1,i}\right)y^{p-1}\right).$$

But $\sum_{i=1}^{q-1}\alpha_{p-1,i,1} = (q-1)\alpha_{p-1,0,1}$, and this is contained in $R^1{}_p^\wedge$, since $\alpha_{p-1,0,1}$ has p–adic valuation -1 and p divides $q-1$. Now, $\alpha_{p-1,i,q}$ is contained in $R^q{}_p^\wedge$ for each i, so $\sum_{i=1}^{q-1}\alpha_{p-1,i} \in R(q)_p^\wedge$.

Let $\zeta_{1,c}$ be equal to

$$(k^{p-1}-1)\alpha_{p-1,0}\cdot\left(\sum_{i=1}^{q-1}\alpha_{p-1,i}\right)y^{p-1}.$$

Thus, $\rho_c^k(c\xi_1) = 1+(\psi^k-1)(\zeta_{1,c})$. As above, there is a unique element $\zeta_1 \in \widetilde{KO}_G(X)_p^\wedge$ such that $c(2\zeta_1) = \zeta_{1,c}$. Moreover,

$$\rho_c^k(c\xi_1) = c\rho^k(rc\xi_1) = c\rho^k(\xi_1)^2.$$

These equations then imply that

$$\rho^k(\xi_1) = 1+(\psi^k-1)(\zeta_1) = (\psi^k/1)(1+\zeta_1),$$

which is Equation (1) of Proposition 7.5.

Finally, suppose $\xi_2 \in T_G(X)_p^\wedge$. This would imply that we have an equation

$$\xi_{2,c} = c\xi_2 = \sum(\psi^{k_i}-1)(\eta_i\otimes y^{p-1}) = \left(\sum k^{p-1}\psi^{k_i}\eta_i-\eta_i\right)\otimes y^{p-1}$$

where $\eta_i \in R(q)$ for each i and k_i is relatively prime to p and q for each i. Thus, in $R(q)/p$, we have

$$z+z^{-1}-2 = \sum(\psi^{k_i}\eta_i-\eta_i).$$

Since k_i is relatively prime to q, the coefficient of z^0 in $\psi^{k_i}\eta_i-\eta_i$ must be 0 for each i. But the coefficient of z^0 in $z+z^{-1}-2$ is $2\neq 0$. Thus, ξ_2 is not in $T_G(X)_p^\wedge$, which completes the proof of Proposition 7.5 for p odd.

Now suppose $p=2$. Let $X=S^6$, with trivial G–action. (Here $G=C_q$, where q is an odd prime.) Then $\widetilde{K}_G(X)\cong R(G)\otimes\widetilde{K}(X)$, while $\widetilde{KO}_G(X)\cong R(G;\mathbb{C})\otimes\widetilde{K}(S^6)$. (Since G is cyclic, $R(G;\mathbb{H})$ is trivial, and since $X=S^6$, $\widetilde{KO}(X)$ is trivial.) Here, $R(G;\mathbb{C})$ is generated by representations of the form z^t+z^{-t}, where $1\le t\le (q-1)/2$. Moreover, the complexification map $c\colon \widetilde{KO}_G(X)\to\widetilde{K}_G(X)$ corresponds under these isomorphisms to the inclusion of $R(G;\mathbb{C})$ in $R(G)$ tensored with the identity on $\widetilde{K}(X)$.

Let $\xi_2 \in \widetilde{KO}_G(X)$ be the element corresponding to $(z+z^{-1})\otimes y^3$. Note that $c(\xi_2)\in \widetilde{K}_G(X)$ likewise corresponds to $(z+z^{-1})\otimes y^3$. As above, using Corollary 4.13, we find that

$$\rho_c^k(c\xi_2) = \rho_c^k(zy^3)\rho_c^k(z^{-1}y^3) = 1 + (\psi^k - 1)((\alpha_{3,1}+\alpha_{3,-1})y^3).$$

Now, $\alpha_{3,1,q}+\alpha_{3,-1,q}$ is the coefficient of $x^3/3!$ in

$$\begin{aligned}
\log\frac{ze^x-1}{z-1} + \log\frac{z^{-1}e^x-1}{z^{-1}-1} &= \log\left(\frac{ze^x-1}{z-1}\cdot\frac{z^{-1}e^x-1}{z^{-1}-1}\right)\\
&= \log\left(e^x\cdot\frac{ze^{x/2}-e^{-x/2}}{z-1}\cdot\frac{e^{x/2}-ze^{-x/2}}{1-z}\right)\\
&= x+\log\left(\frac{ze^{x/2}-e^{-x/2}}{z-1}\cdot\frac{ze^{-x/2}-e^{x/2}}{z-1}\right).
\end{aligned}$$

Since
$$\frac{ze^{x/2}-e^{-x/2}}{z-1}\cdot\frac{ze^{-x/2}-e^{x/2}}{z-1}$$
is an even function in x, the coefficient mentioned above is 0. Similarly, $\alpha_{3,1,0}+\alpha_{3,-1,0}=0$, since the coefficient of x^3 in $\log((e^x-1)/x)$ is 0. Together, these imply that

$$\alpha_{3,1}+\alpha_{3,-1}=0.$$

Thus, $\rho_c^k(c\xi_2)=1$. This implies $c\rho^k(rc\xi_2)=c(1)$, so $\rho^k(\xi_2)^2=1$, so $\rho^k(\xi_2)=1$.

Finally, suppose $\xi_2\in T_G(X)_2^\wedge$. This would imply that we have an equation

$$c\xi_2 = \sum(\psi^{k_i}-1)(\eta_i\otimes y^3) = \left(\sum k^3\psi^{k_i}\eta_i - \eta_i\right)\otimes y^3$$

where each η_i is of the form $z^{t_i}+z^{-t_i}$ for some t_i, and k_i is relatively prime to 2 and q for each i. Thus, in $R(q)/4$, we have

$$z+z^{-1} = \sum(k_i^3 z^{k_i t_i} + k_i^3 z^{-k_i t_i} - z^{t_i} - z^{-t_i}).$$

For each i, the sum of the coefficients of $k_i^3 z^{k_i t_i} + k_i^3 z^{-k_i t_i} - z^{t_i} - z^{-t_i}$ is $2(k_i^3-1)$, which is divisible by 4, since each k_i is odd. Thus, the sum of the coefficients of $z+z^{-1}$, which is 2, would have to be divisible by 4, a contradiction. This completes the proof of Proposition 7.5 for $p=2$.

References

[1] **J F Adams**, *On the groups $J(X)$. I*, Topology 2 (1963) 181–195 MR0159336

[2] **J F Adams**, *On the groups $J(X)$. II*, Topology 3 (1965) 137–171 MR0198468

[3] **J F Adams**, *On the groups $J(X)$. III*, Topology 3 (1965) 193–222 MR0198469

[4] **J F Adams**, *On the groups $J(X)$. IV*, Topology 5 (1966) 21–71 MR0198470

[5] **J F Adams**, **J-P Haeberly**, **S Jackowski**, **J P May**, *A generalization of the Atiyah–Segal completion theorem*, Topology 27 (1988) 1–6 MR935523

[6] **J Allard**, *Adams operations in $KO(X) \oplus KSp(X)$*, Bol. Soc. Brasil. Mat. 5 (1974) 85–96 MR0372857

[7] **M F Atiyah**, *Bott periodicity and the index of elliptic operators*, Quart. J. Math. Oxford Ser. (2) 19 (1968) 113–140 MR0228000

[8] **M F Atiyah**, **R Bott**, **A Shapiro**, *Clifford modules*, Topology 3 (1964) 3–38 MR0167985

[9] **M F Atiyah**, **D O Tall**, *Group representations, λ–rings and the J–homomorphism*, Topology 8 (1969) 253–297 MR0244387

[10] **R Bott**, *Lectures on $K(X)$*, Math. Lecture Note Ser., W. A. Benjamin, New York-Amsterdam (1969) MR0258020

[11] **T tom Dieck**, *Transformation groups and representation theory*, Lecture Notes in Math. 766, Springer, Berlin (1979) MR551743

[12] **A D Elmendorf**, *Systems of fixed point sets*, Trans. Amer. Math. Soc. 277 (1983) 275–284 MR690052

[13] **C French**, *The equivariant J–homomorphism*, Homology Homotopy Appl. 5 (2003) 161–212 MR1989617

[14] **K Hirata**, **A Kono**, *On the Bott cannibalistic classes*, Publ. Res. Inst. Math. Sci. 18 (1982) 1187–1191 MR688953

[15] **J P May**, *Classifying spaces and fibrations*, Mem. Amer. Math. Soc. 1 (1975) xiii+98 MR0370579

[16] **J P May**, *E_∞ ring spaces and E_∞ ring spectra*, Lecture Notes in Math. 577, Springer, Berlin (1977) MR0494077 With contributions by F Quinn, N Ray, and J Tornehave

[17] **J P May**, *Equivariant homotopy and cohomology theory*, from: "Symposium on Algebraic Topology in honor of José Adem (Oaxtepec, 1981)", (S Gitler, editor), Contemp. Math. 12, Amer. Math. Soc. (1982) 209–217 MR676330

[18] **D Quillen**, *The Adams conjecture*, Topology 10 (1971) 67–80 MR0279804

[19] **J-P Serre**, *Linear representations of finite groups*, Graduate Texts in Math. 42, Springer, New York (1977) MR0450380 Translated from the second French edition by L L Scott

[20] **S Waner**, *Equivariant classifying spaces and fibrations*, Trans. Amer. Math. Soc. 258 (1980) 385–405 MR558180

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