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Splittings and C–complexes

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The intersection pattern of the translates of the limit set of a quasi-convex subgroup of a hyperbolic group can be coded in a natural incidence graph, which suggests connections with the splittings of the ambient group. A similar incidence graph exists for any subgroup of a group. We show that the disconnectedness of this graph for codimension one subgroups leads to splittings. We also reprove some results of Peter Kropholler on splittings of groups over malnormal subgroups and variants of them.

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1 Introduction

Let M be a closed 3–manifold and $f\colon S \to M$ an immersed least area surface such that not all complementary regions in M are handlebodies. Thickening $f(S)$ in M and filling in all compressing disks and balls, we obtain a codimension zero submanifold with incompressible boundary F. Then $\pi_1(M)$ splits over $\pi_1(F)$. An interesting special case occurs when M admits two immersed least area surfaces which are disjoint, as the condition on complementary components of each of the surfaces is then automatically satisfied. The aim of this paper is to obtain group-theoretic analogues of these and related facts using the theory of algebraic regular neighbourhoods developed by Scott and Swarup [13].

1.1 Statement of Results

Let G be a group and H an infinite subgroup. A simplicial complex (termed C–complex) can be constructed from the incidence relations determined by the cosets of H as follows (see Mj [8]). The vertices of $C(G,H)$ are the cosets gH and the $(n-1)$–cells are n–tuples $\{g_1H,\dots,g_nH\}$ of distinct cosets such that $\bigcap_1^n g_iHg_i^{-1}$ is infinite. When G is hyperbolic and H quasiconvex, this is equivalent to the incidence complex where vertices are limit sets and $(n-1)$–cells are n–tuples of limit sets with non-empty intersection. (See Sageev [11] and Gitik–Mitra–Rips–Sageev [4] for related material.)

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Let $e(G)$ denote the number of ends of a group G, and let $e(G, H)$ denote the number of ends of a group pair (G, H). Our main Theorem states:

Theorem 2.3 *Suppose that G is a finitely generated group and H a finitely generated subgroup. Further, suppose that $e(G) = e(H) = 1$ and $e(G, H) \geq 2$. If $C(G, H)$ is disconnected, then G splits over a subgroup (that may not be finitely generated).*

Since we are only interested in the connectivity of $C(G, H)$, it is enough to consider the connectivity of its 1–skeleton $C_1(G, H)$ which has the following simple description: the vertices of $C_1(G, H)$ are the essentially distinct cosets gH of H in G and two vertices gH and kH are joined by an edge if and only if gHg^{-1} and kHk^{-1} intersect in an infinite set.

The principal technique used to prove Theorem 2.3 is the theory of algebraic regular neighbourhoods developed by Scott and Swarup [13] and a lemma on crossings (in the sense of Scott [12]) which may be of independent interest. Our results have some thematic overlap with results of Kropholler [5] and Niblo [9], and this is discussed at the end of the paper. We also prove a slight generalization of a theorem of Kropholler [5] and the following variant of that theorem:

Theorem 3.7 *Let G be a finitely generated, one-ended group and let K be a subgroup which may not be finitely generated. Suppose that $e(G, K) \geq 2$, and that K is contained in a proper subgroup H of G such that H is almost malnormal in G and $e(H) = 1$. Then G splits over a subgroup of K.*

1.2 Crossing

We recall certain basic notions from Scott [12] and Scott–Swarup [13]. We say that a subset A of G is *H–finite* if A is contained in a finite number of right cosets Hg of H in G. Two subsets X and Y of G are said to be *H–almost equal* if their symmetric difference $(X - Y) \cup (Y - X)$ is H–finite. A subset X of G is said to be *H–almost invariant* if $HX = X$, and X and Xg are H–almost equal, for all g in G. We may also say that X is *almost invariant over H*. Such a set X is said to be *nontrivial* if both X and its complement X^* are not H–finite. The number of ends, $e(G, H)$, of the pair (G, H) is ≥ 2 if and only if G has nontrivial H–almost invariant subsets.

The following simple result will be needed later.

Lemma 1.1 (Scott–Swarup [13, Lemma 2.13]) *Let G be a group with subgroups H and K. Suppose that Xg is K–almost equal to X for all g in G, and that X is H–finite. Then either X is K–finite or H has finite index in G.*

Remark 1.2 We do not assume that $KX = X$, so that X need not be K–almost invariant.

We shall use the notion of crossing of almost invariant sets in the sense of Scott [12]. Let G be a finitely generated group, let H and K be subgroups of G, and let X and Y be almost invariant subsets of G over H and K respectively. Let X^* and Y^* denote their complements.

Given two subsets X and Y of a group G, it will be convenient to use the terminology *corner* for any one of the four sets $X \cap Y$, $X^* \cap Y$, $X \cap Y^*$ and $X^* \cap Y^*$. Thus any pair (X, Y) has four corners.

Definition 1.3 Let X be a H–almost invariant subset of G and let Y be a K–almost invariant subset of G. We will say that Y *crosses* X if each of the four corners of the pair (X, Y) is H–infinite. Thus each of the corners of the pair projects to an infinite subset of $H \backslash G$.

It is shown in [12] that if X and Y are nontrivial, then $X \cap Y$ is H–finite if and only if it is K–finite. It follows that crossing of nontrivial almost invariant subsets of G is symmetric, i.e. that X crosses Y if and only if Y crosses X.

Next we recall some material from [13]. Let G be a group with subgroups H and K, and let X and Y be nontrivial almost invariant subsets of G over H and K respectively. We will denote the unordered pair $\{X, X^*\}$ by $\overline{X}$, and will say that $\overline{X}$ crosses $\overline{Y}$ if X crosses Y.

Now let H_i be a subgroup of G and let X_i be a nontrivial H_i–almost invariant subset of G. Let $E = \{gX_i, gX_i^* : g \in G, 1 \leq i \leq n\}$, and let $\overline{E} = \{g\overline{X}_i : g \in G, 1 \leq i \leq n\}$. Thus G acts on the left on E and on $\overline{E}$. Define an equivalence relation on $\overline{E}$ to be generated by the relation that two elements A and B of $\overline{E}$ are related if they cross. We call an equivalence class of this relation a *cross-connected component* (CCC) of $\overline{E}$, and denote the equivalence class of A by $[A]$. We will denote the collection of all CCC's of $\overline{E}$ by P. Note that the action of G on $\overline{E}$ induces an action of G on P.

We will first introduce a partial order on E. If U and V are two elements of E such that $U \subset V$, then our partial order will have $U \leq V$. But we also want to define $U \leq V$ when U is "nearly" contained in V. If U is L–almost invariant and V is M–almost invariant, we will say that a corner of the pair (U, V) is *small* if it is L–finite (and hence M–finite). We want to define $U \leq V$ if $U \cap V^*$ is small. Clearly there will be a problem with such a definition if the pair (U, V) has two small corners, but this can be handled if we know that whenever two corners of the pair (U, V) are small, then one of them is empty. Thus we consider the following condition on E:

(∗) If U and V are in E, and two corners of the pair (U, V) are small, then one of them is empty.

If E satisfies Condition (∗), we will say that the family $X_1, \dots, X_n$ is in *good position*.

Assuming that this condition holds, we can define a relation $\leq$ on E by saying that $U \leq V$ if and only if $U \cap V^*$ is empty or is the only small set among the four corners of the pair (U, V). Then $\leq$ turns out to be a partial order on E. If $U \leq V$ and $V \leq U$, it is easy to see that we must have $U = V$, using the fact that E satisfies Condition (∗). It is proved in [13] that $\leq$ is transitive. We note here that the argument that $\leq$ is transitive does not require that the H_i's be finitely generated. Now there is a natural idea of betweenness on the set P of all CCC's of $\overline{E}$. Given three distinct elements A, B and C of P, we say that B *lies between* A and C if there are elements U, V and W of E such that $\overline{U} \in A$, $\overline{V} \in B$, $\overline{W} \in C$ and $U \leq V \leq W$. Note that the action of G on P preserves betweenness.

For the remainder of this discussion we will assume that G and the H_i's are all finitely generated.

An important point is that if one is given a family $X_1, \dots, X_n$ of almost invariant subsets of G, the family need not be in good position, but it was shown by Niblo, Sageev, Scott and Swarup [10], using the finite generation of G and the H_i's, that there is a family $Y_1, \dots, Y_n$ of almost invariant subsets of G, such that X_i and Y_i are equivalent, and the Y_i's are in good position.

A *pretree* consists of a set P together with a ternary relation on P denoted xyz which one should think of as meaning that y is strictly between x and z. The relation should satisfy the following four axioms:

(T0) If xyz, then $x \neq z$.

(T1) xyz implies zyx.

(T2) xyz implies not xzy.

(T3) If xyz and $w \neq y$, then xyw or wyz.

A pretree is said to be *discrete*, if, for any pair x and z of elements of P, the set $\{y \in P : xyz\}$ is finite. In [13], Scott and Swarup showed that if G and the H_i's are all finitely generated, then the set P of all CCC's of $\overline{E}$ with the above idea of betweenness is a discrete pretree. We say that two elements x and y of P are *adjacent* if xzy does not hold for any z in P. We define a *star* in P to be a maximal subset of P which consists of mutually adjacent elements.

It is a standard result that a discrete pretree P can be embedded in a natural way into the vertex set of a tree T, and that an action of G on P which preserves betweenness will automatically extend to an action without inversions on T. Also T is a bipartite tree with vertex set $V(T) = V_0(T) \cup V_1(T)$, where $V_0(T)$ equals P, and $V_1(T)$ equals the collection of all stars in P. It follows that the quotient $G\backslash T$ is naturally a bipartite graph of groups Θ with V_0–vertex groups conjugate to the stabilisers of elements of P and V_1–vertex groups conjugate to the stabilisers of stars in P.

When this construction is applied to the pretree P of all CCC's of $\bar{E}$, the points of P form the V_0–vertices of the bipartite G–tree T [13, Theorem 3.8] with V_1–vertices corresponding to stars of V_0–vertices. The tree T is minimal [13, Theorem 5.2] and if T has more than one V_0–vertex, i.e. if $\bar{E}$ has more than one CCC, then $G\backslash T$ does not reduce to a point, so that edges of $G\backslash T$ correspond to splittings of G.

1.3 C–complexes

The notion of height of a subgroup was introduced by Gitik, Mitra, Rips and Sageev in [4] and further developed by Mitra in [7].

Definition 1.4 Let H be a subgroup of a group G. We say that the elements $g_1, \dots, g_n$ of G are essentially distinct if $g_i g_j^{-1} \notin H$ for $i \neq j$. Conjugates of H by essentially distinct elements are called essentially distinct conjugates.

Note that we are abusing terminology slightly here, as a conjugate of H by an element belonging to the normalizer of H but not belonging to H is still essentially distinct from H. Thus in this context a conjugate of H records (implicitly) the conjugating element.

We now proceed to define the simplicial complex $C(G,H)$ for a group G and H a subgroup.

Definition 1.5 Let G be a group with an infinite subgroup H. Then the simplicial complex $C(G,H)$ has vertices (0–cells) which are the cosets gH of H (or equivalently the conjugates gHg^{-1} of H by essentially distinct elements), and the $(n-1)$–cells of $C(G,H)$ are n–tuples $\{g_1H, \dots, g_nH\}$ of distinct cosets such that $\bigcap_1^n g_iHg_i^{-1}$ is infinite.

We shall refer to the complex $C(G,H)$ as the *C–complex* for the pair (G,H). (C stands for "coarse" or "Čech" or "cover", since $C(G,H)$ is like a coarse nerve of a cover, reminiscent of constructions in Čech cochains.)

If G is a word hyperbolic group and H is a quasiconvex subgroup, we give below two descriptions of $C(G,H)$ which are equivalent to the above definition. In this case, let ∂G denote the boundary of G, let Λ denote the limit set of H, and let J denote the 'convex hull' (or join, strictly speaking) of Λ in the Cayley graph Γ_G.

(1) Vertices (0–cells) of $C(G,H)$ are translates of Λ by essentially distinct elements, and $(n-1)$–cells are n–tuples $\{g_1\Lambda,\dots,g_n\Lambda\}$ of distinct translates such that $\bigcap_1^n g_i\Lambda \neq \varnothing$.

(2) Vertices (0–cells) are translates of J by essentially distinct elements, and $(n-1)$–cells are n–tuples $\{g_1 J,\dots,g_n J\}$ of distinct translates such that $\bigcap_1^n g_i J$ is infinite.

2 Non-crossing and splittings

The Cayley graph Γ_G of a group G with respect to a finite generating set S, such that $S=S^{-1}$, will play a key role in our arguments. The vertex set of Γ_G equals G, and elements g and h of G are joined by an edge if $g=hs$ for some s in S. Thus the action of G on itself by left multiplication extends to a free action of G on Γ_G on the left. In particular, we will regard an almost invariant subset of G as a set of vertices of Γ_G. We define the distance d between two vertices v and w of Γ_G to be the least number of edges among all paths joining v and w. For the proof of Lemma 2.2 below, instead of using the notion of coboundary as in [13] we use terminology introduced by Guirardel in a different context. Let A be a subset of G (the vertex set of Γ_G). Define

$$\partial A = \{a \in A \mid \text{there exists } a' \in A^*, d(a,a')=1\}.$$

Then

$$\partial(A\cap B) = (\partial A \cap B)\cup(A\cap \partial B).$$

By a *connected component* of A we mean a maximal subset of A whose elements (vertices of Γ_G) can be joined by edge paths of Γ_G, none of whose vertices lie in A^*. If B is finite, $G\setminus B$ has finitely many components. It is a beautiful fact that a subset X of G is H–almost invariant if and only if ∂X is H–finite. This was first proved by Cohen [2], but in the coboundary setting.

Now suppose that X and Y are nontrivial almost invariant subsets of G over subgroups H and K respectively, and that they are H–almost equal. Thus the corners $X\cap Y^*$ and $X^*\cap Y$ are both H–finite. As discussed immediately after Definition 1.3 this implies that both these corners are K–finite, so that X and Y are also K–almost equal. In this situation, we will simply say that X and Y are *equivalent*. This is indeed an equivalence relation on nontrivial almost invariant subsets of G. The following simple fact will be used in this paper.

Lemma 2.1 *Let G be a finitely generated group, let H and K be subgroups of G, and let X and Y be nontrivial almost invariant subsets of G over H and K respectively. If X and Y are equivalent, then H and K are commensurable subgroups of G, i.e. $H \cap K$ has finite index in H and in K.*

Proof As X is H–almost invariant and Y is K–almost invariant, we know that ∂X is H–finite and ∂Y is K–finite. As X is equivalent to Y, it follows that X, and hence ∂X, is contained in a bounded neighbourhood of Y. Similarly ∂X^* is contained in a bounded neighbourhood of Y^*. It follows that ∂X must be contained in a bounded neighbourhood of ∂Y. As ∂Y is K–finite, ∂X must also be K–finite. As ∂X is H–finite, it follows that ∂X is $(H \cap K)$–finite, so that $H \cap K$ must have finite index in H. By reversing the roles of X and Y, the same argument shows that $H \cap K$ must have finite index in K. Thus H and K are commensurable subgroups of G, as required. □

We also need a simple lemma on the crossings of almost invariant sets; arguments similar to those in the following lemma occur in Kropholler's paper [5]. We give a topological argument which is also used later.

2.1 A non-crossing lemma

Lemma 2.2 *Let G be a finitely generated group with finitely generated subgroups H and K. Let X and Y be nontrivial almost invariant subsets of G over H and K respectively. Suppose that $e(G) = e(H) = e(K) = 1$, and that $H \cap K$ is finite. Then X and Y do not cross.*

Proof Let Γ_G be the Cayley graph of G with respect to some finite generating set. Thus the vertex set of Γ_G equals G. Our first step is to thicken X, X^*, Y and Y^* in Γ_G to make them connected. For any subset A of Γ_G, we let $N_R(A)$ denote the R–neighbourhood of A in Γ_G.

As X is H–almost invariant, ∂X is H–finite. Thus the image of ∂X in $H \backslash \Gamma$ is finite. Hence we can choose an R–neighbourhood W of this image which is connected and such that the natural map from $\pi_1(W)$ to H is surjective. Thus the inverse image of W in Γ, which equals $N_R(\partial X)$, is also connected. Since $N_R(\partial X) \subset N_R(X)$ and since any point of X can be connected to a point of ∂X by an edge path all of whose vertices lie in X, it follows that $N_R(\partial X) \cup X = N_R(X)$ is connected. Similarly, there is S such that $N_S(\partial X^*)$, and hence $N_S(X^*)$, is also connected. Hence for any $T \geq \max\{R, S\}$, $N_T(X)$, $N_T(X^*)$, $N_T(\partial X)$ and $N_T(\partial X^*)$ are all connected. Similar arguments apply to Y and Y^*. In what follows we will consider only sets

$N_R(A)$, where A is one of the sets ∂X, ∂Y, X, X^*, Y or Y^* in G, and R is fixed so that each $N_R(A)$ is connected. Thus for notational simplicity we will denote $N_R(A)$ by $N(A)$.

Now $N(\partial X) \cap N(\partial Y)$ is the intersection of an H–finite set with a K–finite set, and is therefore $(H \cap K)$–finite. As $H \cap K$ is finite, it follows that $N(\partial X) \cap N(\partial Y)$ is finite. Let U denote this intersection. Then $N(\partial X)$ can be expressed as the union of U, $(N(\partial X) \cap N(Y)) \setminus U$ and $(N(\partial X) \cap N(Y^*)) \setminus U$. Since U is finite, $(N(\partial X) \cap N(Y)) \setminus U$ and $(N(\partial X) \cap N(Y^*)) \setminus U$ have finitely many components. As $e(H) = 1$, it follows that $N(\partial X)$ also has one end, so that only one of these components can be infinite. Thus one of $N(\partial X) \cap N(Y)$ and $N(\partial X) \cap N(Y^*)$ must be finite. Without loss of generality, we can suppose that $N(\partial X) \cap N(Y)$ is finite. Similarly, by reversing the roles of X and Y, one of $N(X) \cap N(\partial Y)$ and $N(X^*) \cap N(\partial Y)$ must be finite.

If $N(X) \cap N(\partial Y)$ is finite, then

$$\partial(X \cap Y) = (\partial X \cap Y) \cup (X \cap \partial Y) \subset (N(\partial X) \cap N(Y)) \cup (N(X) \cap N(\partial Y)),$$

which is finite. Thus $\partial(X \cap Y)$ is finite. Since $(X \cap Y)^* = X^* \cup Y^*$ is infinite and $e(G) = 1$, we see that $X \cap Y$ must itself be finite which shows that X and Y do not cross. Similarly if $N(X^*) \cap N(\partial Y)$ is finite, then $X^* \cap Y$ must be finite, which again shows that X and Y do not cross. We conclude that in all cases X and Y cannot cross, as required. □

2.2 Splitting Theorem

We will now apply the preceding non-crossing result and the material from [13] discussed in Section 1.2 to prove the following splitting results.

Theorem 2.3 *Suppose that G is a finitely generated group and H a finitely generated subgroup. Further, suppose that $e(G) = e(H) = 1$ and that $e(G, H) \geq 2$. If $C(G, H)$ is disconnected, then G splits over some subgroup (that may not be finitely generated).*

Proof Note that the assumption that $e(H) = 1$ implies that H is infinite. As $e(G, H) \geq 2$, there is a nontrivial H–almost invariant subset X of G. By Lemma 2.2 applied to X and gX, we see that if $H \cap gHg^{-1}$ is finite, then X and gX do not cross.

Hence if X and gX cross, then $H \cap gHg^{-1}$ is infinite, and H and gH must lie in the same component of the C–complex $C(G, H)$. As $C(G, H)$ is not connected, we must have more than one CCC. Thus the tree T constructed from the pretree of

CCC's does not reduce to a point, is a minimal G–tree, and each edge of T induces a non-trivial splitting of G. This completes the proof that G splits over some subgroup. Note, however, that though V_0–vertices have finitely generated stabilizers, the edges and V_1–vertices need not. Thus the splitting may be over an infinitely generated subgroup. □

In the above proof, let K denote the stabilizer of the CCC v which contains $\bar{X}$. Now any edge incident to the V_0–vertex v has stabilizer which is a subgroup of K. Thus G splits over some subgroup of K, so that we do have slightly more information than stated in the above theorem.

Essentially the same techniques show the following corollary.

Corollary 2.4 *Suppose that H and K are finitely generated subgroups of a finitely generated group G, and suppose that $e(G) = e(H) = e(K) = 1$; $e(G,H) \geq 2$; $e(G,K) \geq 2$. If all the conjugates of K intersect H in finite groups, then G admits a splitting.*

The graph considered here is reminiscent of the transversality graph considered by Niblo [9], and Corollary 2.4 is similar to his Theorem D. The transversality graph considered by Niblo is dependent on the H–almost invariant set chosen, but if one chooses a set in very good position as in [10], one obtains the regular neighbourhood graph considered above. Similarly, once we have the non-crossing lemma, by choosing almost invariant sets in very good position one can deduce Corollary 2.4 here from Niblo [9, Theorem D]. See also the discussion in [13, page 95].

3 Some other applications

For a subgroup H of a group G, and $g \in G$, we will denote the conjugate gHg^{-1} by H^g. We recall that a subgroup H of a group G is said to be *almost malnormal* if whenever $H^g \cap H$ is infinite, it follows that g lies in H. In Theorem 2.3, if we assume in addition that H is almost malnormal, then the graph $C(G,S)$ is totally disconnected, and X and gX do not cross for any g in G. Further we claim that G splits over a subgroup of H. Note that as X is H–almost invariant, gX is H^g–almost invariant. In the proof of Theorem 2.3, the CCC v of $\bar{E}$ which contains $[\bar{X}]$ consists of $[\bar{X}]$ only. Hence if g in G stabilizes v, we must have gX equal to X or to X^*. In particular, gX is equivalent to X or to X^*. Thus Lemma 2.1 tells us that H and H^g are commensurable. As H is infinite, so is $H^g \cap H$. Thus, as H is almost malnormal in G, it follows that the stabilizer of the CCC v equals H. Hence the stabilizer of the vertex v of T equals H, so that the stabilizer of any edge of T which is incident to v must be a subgroup of H. Hence G splits over a subgroup of H, as claimed.

However in this case we can do slightly better by more elementary arguments. First we recall the following criterion of Dunwoody [3]:

Theorem 3.1 *Let E be a partially ordered set with an involution $e \to \overline{e}$ where $e \neq \overline{e}$ such that:*

(D1) *If $e, f \in E$ and $e \leq f$, then $\overline{f} \leq \overline{e}$,*

(D2) *If $e, f \in E$, there are only finitely many $g \in E$ such that $e \leq g \leq f$,*

(D3) *If $e, f \in E$, then at least one of the four relations $e \leq f$, $\overline{e} \leq f$, $e \leq \overline{f}$, $\overline{e} \leq \overline{f}$ holds, and*

(D4) *If $e, f \in E$, one cannot have both $e \leq f$ and $e \leq \overline{f}$.*

Then there is an abstract tree T with edge set equal to E such that $e \leq f$ if and only if there is an oriented path in T that starts with e and ends with f.

Next we recall the following result of Kropholler. We will discuss the definition of the invariant $\widetilde{e}(G, H)$ below.

Theorem 3.2 (Kropholler [5, Theorem 4.9]) *Suppose that G is a finitely generated group with a finitely generated subgroup H, such that $e(G) = 1 = e(H)$.*

(1) *If H is malnormal in G, and $e(G, H) \geq 2$, then G splits over H.*

(2) *If H is malnormal in G, and $\widetilde{e}(G, H) \geq 2$, then G splits over a subgroup of H.*

Our methods allow us to extend this result. First we give the following slight generalization of the first part of Kropholler's theorem. The only difference is that we have replaced malnormality by the weaker condition of almost malnormality. Later we will slightly generalize the second part in the same way, and will also prove a variant of Kropholler's result.

Theorem 3.3 *Suppose that G is a finitely generated group with a finitely generated subgroup H, such that $e(G) = 1 = e(H)$. If H is almost malnormal in G, and $e(G, H) \geq 2$, then G splits over H.*

Proof As $e(G,H)\geq 2$, there is a nontrivial H–almost invariant subset X of G. To prove this result, we will apply Dunwoody's criterion to the set $E=\{gX, gX^*, g\in G\}$, with the partial order $\leq$ discussed in Section 1.2. Recall that this partial order can only be defined if X is in good position. We will show that this is automatic in the present setting.

Let g be an element of G such that two corners of the pair (X, gX) are finite. Thus gX must be equivalent to X or to X^*. Again Lemma 2.1 tells us that H and H^g are commensurable subgroups of G. As H is infinite and almost malnormal in G, this can only occur if g lies in H, so that gX equals X or X^*, and the two small corners are both empty. Thus X is in good position, as required.

Next we observe that with this partial order on E, conditions (D1) and (D4) of Dunwoody's criterion are trivial. Condition (D3) holds, because our non-crossing lemma implies that for any $e,f\in E$ one of the corners of the pair (e,f) is finite. Finally, as in the proof of [13, Lemma B.1.15], condition (D2) holds because the set of $g\in G$ for which X and gX are not nested is contained in a finite number of double cosets HgH. This crucially uses the fact that H is finitely generated and will be discussed in more detail in the proofs of the next theorems. Now Dunwoody's criterion gives us a tree T on which G acts and which is minimal. Since the stabilizer of X is H, we see that G splits over H. This completes the proof of Theorem 3.3. □

Even though, the condition $e(G)=1$ in the above result is generic, the hypotheses of almost malnormality and having one end are not generic for the subgroup H and we would like to slightly weaken this condition.

The statement of the second part of Kropholler's theorem involves the notion of the number of relative ends $\widetilde{e}(G,H)$ of a pair of groups (G,H), due to Kropholler and Roller [6]. As discussed in [13, pages 31–33], this is the same as the number of coends of the pair, as defined by Bowditch [1]. The following lemma contains the only facts we will need about relative ends.

Lemma 3.4 (Scott–Swarup [13, Lemma 2.40]) *Let G be a finitely generated group and let H be a finitely generated subgroup of infinite index in G. Then $\widetilde{e}(G,H)\geq 2$ if and only if there is a subgroup K of H with $e(G,K)\geq 2$. The subgroup K need not be finitely generated.*

Let Γ be the Cayley graph of G with respect to a finite system of generators. The number of coends of the pair (G,H) can be defined in terms of the number of H–infinite components of $\Gamma - A$ for a connected H–finite subset A of Γ. So we have

Lemma 3.5 *Let G be a finitely generated group and H a finitely generated subgroup of G. Then $\widetilde{e}(G,H) \geq 2$ if and only if there is a connected H–finite subcomplex A of Γ such that $\Gamma - A$ has at least two H–infinite components. Moreover, we may assume that A is H–invariant.*

We now proceed to the statement and proof of a slight generalization of the second part of Kropholler's Theorem 3.2, in which malnormal is again replaced by almost malnormal.

Theorem 3.6 *Suppose that G is a finitely generated group with a finitely generated subgroup H, such that $e(G) = 1 = e(H)$, and suppose that $\widetilde{e}(G,H) \geq 2$. If H is almost malnormal in G, then G splits over a subgroup of H.*

Proof As $\widetilde{e}(G,H) \geq 2$, there is a H–invariant, connected subcomplex B of Γ which is also H–finite, and such that $\Gamma - B$ has at least two H–infinite components. Since H is almost malnormal in G, this implies that the stabilizer of B is equal to H. Denote one of the H–infinite components of $\Gamma - B$ by Q and let K be the stabilizer of Q. Thus K is a subgroup of H. We will denote by X the set of vertices in Q. Thus K is also the stabilizer of X. The frontier of Q and the set ∂X are in a 1–neighbourhood of each other. Since the frontier of Q is contained in B, we see that ∂X is contained in the 1–neighbourhood of B. We denote this 1–neighbourhood by A. Note that A is also H–invariant, connected and H–finite. We will show that $E = \{gX, gX^* : g \in G\}$, equipped with the partial order $\leq$ described earlier, satisfies the four conditions of Dunwoody's Criterion (Theorem 3.1) and thus G splits over K.

First we observe that $\Gamma - Q$ must be connected, since B is connected. As H preserves B it must also preserve the components of $\Gamma - B$, so that, for all h in H, we have $hX = X$ or $hX \cap X = \varnothing$. Thus the pair (hX, X) is nested, for each h in H. Now suppose that g is an element of G such that the pair (gX, X) is not nested, so that g must lie in $G - H$. Thus each of the four corners of the pair (gX, X) is non-empty. We note that ∂X must intersect both gX and gX^*, and that ∂gX must intersect both X and X^*. As ∂X and ∂gX are contained in A and gA respectively, we see that A and gA must also intersect. As A is H–finite, gA must be H^g–finite, and $A \cap gA$ must be $H \cap H^g$–finite. As H is almost malnormal in G, and $g \in G - H$, it follows that $A \cap gA$ is finite. Now recall that $e(H) = 1$. As A is H–finite, it follows that A, and hence also gA, is one-ended. Thus one of $A \cap gX$ and $A \cap gX^*$ is finite, and one of $X \cap gA$ and $X^* \cap gA$ is finite.

If the first of each pair is finite, we have

$$\partial(X \cap gX) = (\partial X \cap gX) \cup (X \cap \partial gX) \subseteq (A \cap gX) \cup (X \cap gA)$$

is finite. As $e(G)=1$, and the complement of $X\cap gX$ in G is clearly infinite, it follows that $X\cap gX$ is finite. Thus one of the corners of the pair (gX,X) is finite, and two of them cannot be finite since H is almost malnormal in G, and $g\notin H$. Similarly if one of the three other possibilities holds, then a different corner of the pair (gX,X) will be finite and will be the only finite corner. Hence X is in good position, and we have the partial order $\leq$ on the set $E=\{gX,gX^*;g\in G\}$. All the conditions in Dunwoody's Criterion (Theorem 3.1) are immediate except the finiteness condition (D2).

Let L denote $\{g\in G:\text{the pair }(gX,X)\text{ is not nested}\}$. We saw above that if $g\in L$, then gA and A have nonempty intersection. As A is H–finite, it follows that L is contained in a finite number of double cosets HgH. We want to show that L is actually contained in a finite number of double cosets KgK. To see this, consider $l\in L$. The preceding argument shows that lA and A have nonempty finite intersection. Since $A\cap lA$ is finite, $lA-A$ is contained in a finite number of components of $\Gamma-B$. Thus lA meets only finitely many translates hX of X with $h\in H$. Since ∂lX is contained in lA it follows that lX and hX can be not nested, for only finitely many translates hX of X with $h\in H$, and hence that hlX and X are not nested, for only finitely many translates hlX of X with $h\in H$. As l^{-1} also lies in L, the same argument shows that $hl^{-1}X$ and X are not nested, for only finitely many translates $hl^{-1}X$ of X with $h\in H$, and hence that X and lhX are not nested, for only finitely many translates lhX of X with $h\in H$. As the stabilizer of X is K, it follows that the intersection $L\cap HlH$ consists of finitely many double cosets KgK. Hence L itself is contained in finitely many double cosets KgK.

Choose $g_1,\dots,g_n$ such that L is contained in $\cup Kg_iK$. Consider Y in E with $Y\leq X$, so that $Y\cap X^*$ is K–finite. If $Y\cap X^*$ is not empty, so that X and Y are not nested, then Y must be of the form $kg_ik'X$ or $kg_ik'X^*$. Now $kg_ik'X^{(*)}\cap X^*=kg_iX^{(*)}\cap X^*=k(g_iX^{(*)}\cap X^*)$. Choose D such that the finite number of finite sets $(g_iX^{(*)}\cap X^*)$ all lie in a D–neighbourhood of X. Then Y also must lie in a D–neighbourhood of X. Thus every element Y of E such that $Y\leq X$ lies in a D–neighbourhood of X. Similarly every element Y of E such that $Y\leq X^*$ lies in a bounded neighbourhood of X^*. By increasing D if necessary, we can assume that this neighbourhood is also of radius D.

Now we can verify condition (D2) of Dunwoody's criterion. Suppose that U and V are elements of E. We claim that there are only finitely many $W\in E$ with $U\leq W\leq V$. The first inequality implies that $W^*\leq U^*$, so that W^* lies in a D–neighbourhood of U^*. Hence we can choose $x\in U$ which does not belong to any such W^*. Similarly the inequality $W\leq V$ implies that W lies in a D–neighbourhood of V, so that we can choose $y\in V^*$ which does not belong to any such W. If ω is a path from x

to y, then ω should intersect ∂W. Since G is finitely generated, there can be only finitely many such W. This completes the verification of Dunwoody's Criterion and thus completes the proof of the theorem. □

Finally we give our variant of Kropholler's Theorem 3.2.

Theorem 3.7 *Let G be a finitely generated, one-ended group and let K be a subgroup which may not be finitely generated. Suppose that $e(G,K) \geq 2$, and that K is contained in a proper subgroup H of G such that H is almost malnormal in G and $e(H) = 1$. Then G splits over a subgroup of K.*

Remark 3.8 Lemma 3.4 shows that the hypotheses imply that $\widetilde{e}(G,H) \geq 2$. So we regard this result as a refinement of the second part of Kropholler's theorem 3.2.

Proof We start by observing that the assumptions that H is proper and almost malnormal in G imply that H has infinite index in G.

As $e(G,K) \geq 2$, there is a nontrivial K–almost invariant subset Y of G. As usual, we let Γ denote a Cayley graph for G with respect to some finite generating set. As Y is K–almost invariant, ∂Y is K–finite. Thus the image of ∂Y in $H\backslash\Gamma$ must be finite. As H is finitely generated, we can find a finite connected subgraph W of $H\backslash\Gamma$ such that W contains the image of ∂Y and the natural map from $\pi_1(W)$ to H is surjective. Thus the pre-image A of W in Γ is connected, H–invariant and H–finite, and contains ∂Y. As W is finite, the complement of W in $H\backslash\Gamma$ has only a finite number of components. In particular it has only a finite number of infinite components. We consider the components of their inverse images in Γ. Each such component has vertex set contained in Y or Y^*, since ∂Y is contained in A. As Y is K–infinite and K–almost invariant, and H has infinite index in G, Lemma 1.1 implies that Y must also be H–infinite. Hence at least one component of $\Gamma - A$ is H–infinite and has vertex set X contained in Y. The stabilizer of X is a subgroup of K since $Y - A$ is preserved by K. Now we have the set up in the proof of Theorem 3.6. The stabilizer of X is a subgroup K' of the group K in the hypotheses of this theorem. Thus K' replaces K in the proof of Theorem 3.6. In that proof we used only the almost malnormality of H, and that K is contained in H. Thus nesting with respect to $H - K'$ is automatic as before. Almost nesting with respect to elements of $G - H$ and verification of Dunwoody's second condition follow exactly as in the previous theorem. □

In many of the above proofs, the hypotheses are used in two steps. The hypotheses on the subgroup H ensure that one of the corners of the pair (X, gX) has very small boundary and then the hypotheses on G ensure that the corner set is small. Another hypothesis which ensures one of the corner sets has a relatively small boundary is formulated in a conjecture of Kropholler and Roller (discussed in [13, pages 224–225]). We give our formulation of the conjecture:

Conjecture 3.9 *Let X be a H–almost invariant subset of G with both G and H finitely generated. Suppose that $g\partial X$ is contained in a bounded neighbourhood of X or X^* for every $g \in G$. Then G splits over a subgroup commensurable with a subgroup of H.*

This time the hypotheses ensure that if g does not commensurise H, then, one of the corners of the pair (X, gX) is an almost invariant set over a subgroup of infinite index in H. Dunwoody and Roller showed that one can get almost nesting with respect to the elements that commensurise H by changing the almost invariant set, and changing the subgroup up to commensurability. (See [13, Theorem B.3.10]. Note that almost nesting can be improved to nesting by using almost invariant sets in very good position.) This proof is one of the key steps in the proof of the algebraic torus theorem. Thus the obstructions to splitting G over H lie in almost invariant sets over subgroups of infinite index in H. One can wish away such sets by hypothesis, or can try to repeat the construction and look for conditions under which such repetitions must stop. A useful fact is that the corners obtained are invariant under the right action of H. This was originally used by Kropholler in the proof of Theorem 3.2 when H is malnormal in G, to obtain nesting. Nesting ensures the finiteness property required in the use of Dunwoody's Criterion. In our proofs, we obtained almost nesting first and had to use the finiteness of double cosets to prove the finiteness property required in Dunwoody's criterion. It is possible that a combination of these different techniques will give a bit more information about splittings.

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