Amalgamations of Heegaard splittings in 3–manifolds without some essential surfaces

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Let M be a compact, orientable, ∂–irreducible 3–manifold and F be a connected closed essential surface in M with $g(F)\geq 1$ which cuts M into M_1 and M_2. In the present paper, we show the following theorem: Suppose that there is no essential surface with boundary $(Q_i,\partial Q_i)$ in (M_i,F) satisfying $\chi(Q_i)>2+g(F)-2g(M_i)$, $i=1,2$. Then $g(M)=g(M_1)+g(M_2)-g(F)$. As a consequence, we further show that if M_i has a Heegaard splitting $V_i\cup_{S_i}W_i$ with distance $D(S_i)\geq 2g(M_i)-g(F)$, $i=1,2$, then $g(M)=g(M_1)+g(M_2)-g(F)$.

The main results follow from a new technique which is a stronger version of Schultens' Lemma.

57M99, 57N10; 57M27

1 Introduction

Let M be a compact, orientable, ∂–irreducible 3–manifold and F be a connected closed essential surface in M with $g(F)\geq 1$ which cuts M into M_1 and M_2. Suppose $V_i\cup_{S_i}W_i$ is a Heegaard splitting of M_i, $i=1,2$. Then $V_1\cup_{S_1}W_1$ and $V_2\cup_{S_2}W_2$ induce a natural Heegaard splitting $V\cup_S W$ of M , which is called the amalgamation of $V_1\cup_{S_1}W_1$ and $V_2\cup_{S_2}W_2$ along F, and $g(S)=g(S_1)+g(S_2)-g(F)$; see Schultens [15]. Thus $g(M)\leq g(M_1)+g(M_2)-g(F)$.

There exist examples which show that an amalgamation of two minimal genus Heegaard splittings of M_1 and M_2 may be stabilized (refer to Bachman, Schleimer and Sedgwick [1], Kobayashi, Qiu, Rieck and Wang [7], Schultens and Weidmann [17] and others). On the other hand, it has been shown that under some conditions on the manifolds, or the gluing maps, the equality $g(M)=g(M_1)+g(M_2)-g(F)$ holds; see Kobayashi and Qiu [6], Lackenby [8], Lei and Yang [9], Li [10], Souto [18], Yang and Lei [19] and others.

In the present paper, we show the following result:

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Theorem 4.2 *Suppose that there is no essential surface with boundary* $(Q_i, \partial Q_i)$ *in* (M_i, F) *satisfying* $\chi(Q_i) > 2 + g(F) - 2g(M_i)$, $i = 1, 2$. *Then* $g(M) = g(M_1) + g(M_2) - g(F)$.

As a consequence, we further show:

Theorem 4.3 *If* M_i *has a Heegaard splitting* $V_i \cup_{S_i} W_i$ *with distance* $D(S_i) \geq 2g(M_i) - g(F)$, $i = 1, 2$, *then* $g(M) = g(M_1) + g(M_2) - g(F)$.

The paper is organized as follows. In Section 2, we review some preliminaries and notation which will be used later. In Section 3, we will prove two technical lemmas. The statements and proofs of the main results are given in Section 4. The lemmas in Section 3 play an important role in our proof of Theorem 4.2.

2 Preliminaries

The concepts and terminologies not defined in the paper are standard; for example, see, Hempel [3] and Jaco [5].

Suppose F is a subsurface of ∂M or a surface properly embedded in a 3–manifold M. If there is a disk $D \subset M$ such that $D \cap F = \partial D$ and ∂D is an essential loop in F, then we say that F is *compressible* in M. Such a disk D is called a *compressing disk*. We say that F is *incompressible* in M if F is not compressible in M. If ∂M is incompressible, then M is said to be *∂–irreducible*. If F is an incompressible surface in M and not parallel to a subsurface of ∂M, then F is an *essential* surface in M.

A 3–manifold C is called a *compression body* if there exists a connected closed orientable surface S such that C is obtained from $S \times I$ by attaching 2–handles along mutually disjoint loops in $S \times \{0\} \subset S \times I$ and capping off any resulting 2–sphere boundary components with 3–handles. We denote $S \times \{1\}$ by $\partial_+ C$ and $\partial C - \partial_+ C$ by $\partial_- C$. An essential disk in C is a compressing disk of $\partial_+ C$ in C.

A *Heegaard splitting* of a 3–manifold M is a triplet $(C_1, C_2; S)$, where C_1 and C_2 are compression bodies with $C_1 \cup C_2 = M$ and $C_1 \cap C_2 = \partial_+ C_1 = \partial_+ C_2 = S$. The surface S is called a *Heegaard surface* and the genus of a Heegaard splitting is defined by the genus of the Heegaard surface. We use $g(M)$ to denote the *Heegaard genus* of M, which is equal to the minimal genus of all Heegaard splittings of M. A Heegaard splitting $C_1 \cup_S C_2$ for M is *minimal* if $g(S) = g(M)$. A Heegaard splitting $C_1 \cup_S C_2$ is *trivial* if $\partial_- C_1 \cong \partial_+ C_1$ or $\partial_- C_2 \cong \partial_+ C_2$.

Let $C_1 \cup_S C_2$ be a Heegaard splitting for M. $C_1 \cup_S C_2$ is said to be *reducible (or weakly reducible)* if there are essential disks $D_1 \subset C_1$ and $D_2 \subset C_2$ with $\partial D_1 = \partial D_2$ (or $\partial D_1 \cap \partial D_2 = \varnothing$). The splitting $C_1 \cup_S C_2$ is said to be *irreducible* if it is not reducible; and the splitting $C_1 \cup_S C_2$ is said to be *strongly irreducible* if it is not weakly reducible.

Scharlemann and Thompson showed in [13] that any irreducible and ∂–irreducible Heegaard splitting $C_1 \cup_S C_2$ can be broken up into a series of strongly irreducible Heegaard splittings. That is ,we can begin with the handle structure determined by $C_1 \cup_S C_2$ and rearrange the order of adding the 1– and 2–handles, so that ultimately

$$M = (C_1^1 \cup_{S_1} C_2^1) \cup_{F_1} (C_1^2 \cup_{S_2} C_2^2) \cup_{F_2} \cdots \cup_{F_{m-1}} (C_1^m \cup_{S_m} C_2^m),$$

such that each $C_1^i \cup_{S_i} C_2^i$ is a strongly irreducible Heegaard splitting with intersections $\partial_- C_2^i \cap \partial_- C_1^{i+1} = F_i$, $1 \le i \le m-1$ and $\partial_- C_1^1 = \partial_- C_1$, $\partial_- C_2^m = \partial_- C_2$. For each i, each component of F_i is a closed incompressible surface of positive genus, and only one component of $M_i = C_1^i \cup_{S_i} C_2^i$ is not a product. None of the compression bodies C_1^i, C_2^i, $1 \le i \le m$ is trivial. Such a rearrangement of handles is called an *untelescoping* of the Heegaard splitting $C_1 \cup_S C_2$. Then it is easy to see $\chi(S) \le \chi(S_i), \chi(F_i)$ for each i, and when $m \ge 2$, $\chi(S) < \chi(S_i), \chi(F_i)$ for each i.

Let $C_1 \cup_S C_2$ be a Heegaard splitting, α and β two essential simple closed curves in S. The *distance* $d(\alpha, \beta)$ of α and β is the smallest integer $n \ge 0$ such that there is a sequence of essential simple closed curves $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_n = \beta$ in S with $\alpha_{i-1} \cap \alpha_i = \varnothing$, for $1 \le i \le n$. The *distance* of the Heegaard splitting $C_1 \cup_S C_2$ is defined to be $\min\{d(\alpha, \beta)\}$, where α bounds an essential disk in C_1 and β bounds an essential disk in C_2, and is denoted by $D(S)$.

The concept of Heegaard distance of a Heegaard splitting was first defined by Hempel [4]. It is clear that $C_1 \cup_S C_2$ is reducible if and only if $D(S) = 0$, $C_1 \cup_S C_2$ is weakly reducible if and only if $D(S) = 1$. Its relations to the genus of the Heegaard splitting have been discussed by Hartshorn [2], Hempel [4], Scharlemann and Tomova [14] and others.

The following Lemmas are some well-known basic facts and results:

Lemma 2.1 [12] *Suppose $(Q, \partial Q) \subset (M, \partial M)$ is an essential surface and Q' is the result of ∂–compressing Q. Then Q' is essential.*

Lemma 2.2 [16] *Let V be a compression body and F be a properly embedded incompressible surface in V with $\partial F \subset \partial_+ V$, then each component of $V \backslash F$ is a compression body.*

Lemma 2.3 [11] *Let $M = V \cup_S W$ be a strongly irreducible Heegaard splitting. If α is an essential simple loop in S which bounds a disk D in M such that D is transverse to S, then α bounds an essential disk in V or W.*

Lemma 2.4 [2] *Let $V \cup_S W$ be a Heegaard splitting of M and F be a properly embedded incompressible surface (maybe not connected) in M. Then any component of F is parallel to ∂M or $D(S) \leq 2 - \chi(F)$.*

In the rest of this paper, we use $M \backslash N$ to denote the manifold obtained by cutting M along $M \cap N$ and $N(F, M)$ the compact regular neighborhood of the submanifold F in the manifold M.

3 Two technical lemmas

Definition 3.1 Let F be a 2–sided surface properly embedded in M, $F \times [0, 1]$ a regular neighborhood of F in M with $F \times \{\frac{1}{2}\} = F$. If there are two compressing disks D and E of F such that $D \cap (F \times \{0\}) = \varnothing$ and $E \cap (F \times \{1\}) = \varnothing$, then F is called *bicompressible* and (D, E) is called a *bicompressing disk pair* of F. If F is bicompressible and any bicompressing disk pair (D, E) satisfies $\partial D \cap \partial E \neq \varnothing$, then F is called *strongly irreducible*.

Definition 3.2 Two surfaces F_1 and F_2 embedded in a 3–manifold are almost transverse if they have exactly one nontransverse intersection point, and it is a saddle point.

The following Lemma 3.3 is a stronger version of Schultens's lemma [16] as well as Lemma 3.3 of Bachman, Schleimer and Sedgwick [1]. Lemma 3.3 is essential in our proof of Theorem 4.2.

Lemma 3.3 *Let $M = V \cup_S W$ be a strongly irreducible Heegaard splitting and F be a 2–sided essential surface (not a disk or 2–sphere) in M. Then F can be isotoped such that at least one of the following conclusions holds:*

(1) *F is transverse to S and any component of $S \backslash F$ is incompressible in the respective submanifold of $M \backslash F$ except for exactly one strongly irreducible component;*

(2) *F is almost transverse to S and any component of $S \backslash N(F, M)$ is incompressible in the respective submanifold of $M \backslash N(F, M)$.*

Proof From Schultens's lemma in [16], we may assume that each component of $S \cap F$ is an essential loop in both F and S, and $|S \cap F|$ is minimal. If $S \backslash F$ is bicompressible in $M \backslash F$, then (1) is true. In the following arguments, we suppose (1) is not true. So $S \backslash F$ is incompressible in V or W, say V. Then F satisfies the following three conditions: (i) each component of $S \cap F$ is an essential loop in both F and S; (ii) $S \backslash F$ is incompressible in V; (iii) any component of $F \cap V$ is essential in V.

We can take F such that $-\chi(F \cap V)$ is minimal among all surfaces isotopic to F which satisfies conditions: (i), (ii) and (iii). If $F \cap V$ is not boundary compressible, then any component of $F \cap V$ is spanning annulus by the properties of compression body. This means that $S \backslash F$ is compressible in V, a contradiction. So $F \cap V$ is boundary compressible.

We claim that there exists a boundary compressing disk $\triangle$ of $F \cap V$ in V such that any component of the result of ∂–compressing $F \cap V$ along $\triangle$ is essential in V.

Let $\triangle_1$ be a boundary compressing disk of $F \cap V$ in V, α_1 the corresponding essential arc in $F \cap V$ and β_1 the corresponding essential arc in ∂V with $\alpha_1 \cup \beta_1 = \partial \triangle_1$. Denote the component of $F \cap V$ which contains α_1 by P. Obviously, $\chi(P) \leq 0$.

If $\chi(P) = 0$, then P is an essential annulus in V. By performing ∂–compression to P along $\triangle_1$, we get an essential disk E with $E \cap F = \varnothing$ in V. This means that $S \backslash F$ is compressible in V, a contradiction.

So $\chi(P) \leq -1$. Denoted the result of ∂–compressing P along $\triangle_1$ by $P^{'}$. If any component of $P^{'}$ is essential in V, we take $\triangle = \triangle_1$, $\alpha = \alpha_1$ and $\beta = \partial \triangle \backslash \alpha$. If one component P^* of $P^{'}$ is parallel to a subsurface Q of ∂V in V with $\partial Q = \partial P^*$, then α_1 is separating in P and the other component of $P^{'}$ is essential in V by Lemma 2.1. Since $\partial \beta_1 \cap \partial Q = \varnothing$, $\beta_1 \subset Q$ or $\beta_1 \cap Q = \varnothing$. If $\beta_1 \subset Q$, then P is compressible in V, a contradiction. Hence $\beta_1 \cap Q = \varnothing$ and this means that there exists a nonseparating essential arc α of both P and $P^{'}$ and an essential arc $\beta \subset \partial V$ which satisfies $\alpha \cap \beta = \partial \alpha = \partial \beta$ and $\alpha \cup \beta$ bounding a disk $\triangle$ with $\triangle \cap (F \cap V) = \alpha$. So $\triangle$ is a boundary compressing disk of $F \cap V$ in V. By Lemma 2.1, any component of the result of ∂–compressing $F \cap V$ along $\triangle$ is essential in V.

Perform ∂–compression to $F \cap V$ along $\triangle$ to get F^*, which is an isotopy of F. Then F^* satisfies conditions (i) and (iii). Obviously, $-\chi(F^* \cap V) < -\chi(F \cap V)$. Since $-\chi(F \cap V)$ is minimal among all surfaces which are isotopic to F and satisfy conditions: (i), (ii) and (iii), $S \backslash F^*$ is compressible in V. If $S \backslash F^*$ is compressible in W, then $S \backslash F$ is bicompressible in $M \backslash F$, ie (1) is true, a contradiction. Hence, $S \backslash F^*$ is incompressible in W. From $S \backslash F$ to $S \backslash F^*$, only cut the band $B_1 = N(\beta, S \backslash F)$ from $S \backslash F$ along one pair opposite edges of B_1 and paste it to $S \backslash F$ along the other pair opposite edges of B_1. Then only the component of $S \backslash F^*$ which contains β is compressible in V and any component of $S \backslash F^*$ is incompressible in W. Furthermore, only the component of $S \backslash F^*$ which contains β is compressible in $M \backslash F^*$ and the band B_1 intersects with the boundary of any compressing disk D of $S \backslash F^*$; see Figure 1 below. Otherwise, there is a compressing disk D_1 for some component of $S \backslash F^*$ with $\partial D_1 \cap B_1 = \varnothing$, by Lemma 2.3, ∂D_1 must bound a compressing disk of $S \backslash F$ in V, a contradiction. Hence, at most one of $S \backslash F$ is compressible in W. If one component of $S \backslash F$ is compressible in W, then the component must contain β. Otherwise, $S \backslash F^*$ is bicompressible, ie (1) is true, a contradiction again. So at most the component of $S \backslash F$ which contains β is compressible in $M \backslash F$ and any compressing disk E of $S \backslash F$ satisfies $\partial E \cap B_1 \neq \varnothing$; see Figure 1.

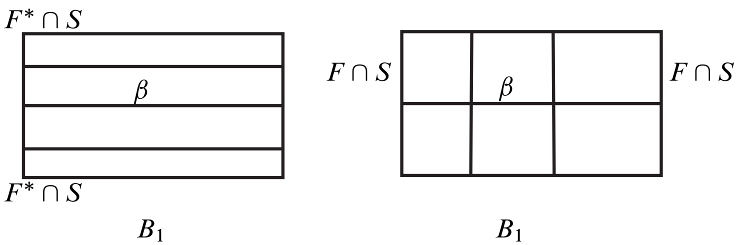

Figure 1: $\partial E \cap B_1$ and $\partial D \cap B_1$

Push F^* slightly off both F and F^* to get F', which is a parallel copy of F^* such that the disk $\triangle$ lies in the parallelism N bounded by F and F'; see Figure 2.

Figure 2: F and F^*

Then $B_1 \subset N$ and any component of $S \backslash N$ is incompressible in the respective submanifold of $M \backslash N$; see Figure 3.

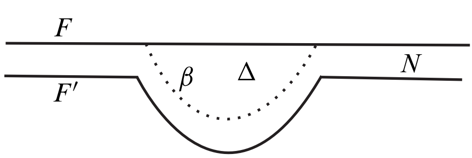

Figure 3: F, F' and N

So F can be isotoped into F_1 which is almost transverse to S with a saddle point belong to both α and β with $N(F_1, M) = N$; see Figure 4.

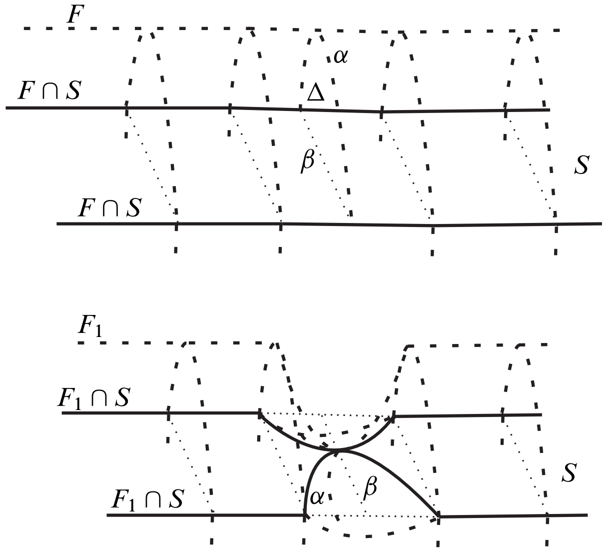

Figure 4: $F \cap S$ and $F_1 \cap S$

Thus (2) is true.

This completes the proof. □

Lemma 3.4 *Let N be a compact orientable 3–manifold and not a compression body, F a component of ∂N. Suppose Q is a properly embedded connected separating surface in N with $\partial Q \subset F$ and any component of ∂Q is essential in F, and Q cuts N into two compression bodies N_1 and N_2 with $Q = \partial_+ N_1 \cap \partial_+ N_2$. If Q can be compressed to Q_i^* in some N_i such that any component of Q_i^* is parallel to a subsurface of ∂N, then $g(N) \le 1 - \frac{1}{2}\chi(F \cap N_{3-i}) - \frac{1}{2}\chi(Q)$.*

Proof By assumption, Q^* and $F \backslash (Q \cap F)$ have no disk components, and each component of Q^* is parallel into either a subsurface of F or a component of ∂N.

We may assume that Q is compressed to Q^* in N_1 by cutting Q open along a collection $\mathcal{D} = \{D_1, \ldots, D_n\}$ of pairwise disjoint compressing disks in N_1. $N_1' = N_1 - \eta(\mathcal{D})$ where $\eta(\mathcal{D})$ is the open regular neighborhood of $\mathcal{D}$. Let $A_1, \ldots, A_r$ and $A_{r+1}, \ldots, A_{r+s}$ be all the components of Q^*, where A_i is a component with $\partial A_i \subset F$ for $1 \le i \le r$ and A_{r+i} is a closed surface for $1 \le i \le s$. Suppose that each A_i is parallel to a subsurface A_i' of ∂N, where $i = 1, 2, \ldots, r+s$. We divide it into two cases to discuss:

If there exist two components A'_{i_0} and A'_{j_0} with $A'_{i_0} \subset A'_{j_0}$. Then we set $\mathcal{A}_1 = \{A_i : A_i' \subset A'_{j_0}, 1 \le i \le r+s, i \ne j_0\}$ and $\mathcal{A}_2 = \{A_i : A_i' \cap A'_{j_0} = \varnothing, 1 \le i \le r+s\}$. We claim that $\mathcal{A}_2 = \varnothing$. Otherwise, since Q is connected, there must exist $A_{i_1} \in \mathcal{A}_1$, $A_{i_2} \in \mathcal{A}_2$, and $D_{p_1}, D_{p_2} \in \mathcal{D}$ such that in the compression, the two copies (obtained by compressing Q along $\mathcal{D}$) of D_{p_k} lie in A_{i_k} and A_{j_0} respectively, $k = 1, 2$. But this contradicts the assumption that Q is separating in N. Thus $\mathcal{A}_2 = \varnothing$. Let N_1'' be the component of N_1' which contain A_{j_0}, then $A_{j_0} \subset \partial_+ N_1''$. Since A_{j_0} is parallel to A'_{j_0}, N_1'' is a handlebody , the other components of N_1' are $A_{r+i} \times I$. N is homeomorphic to $N_2 \cup N(\mathcal{D}) \cup N_1'$. So N is a compression body with $F = \partial_+ N$, a contradiction.

So for any two components A_i, A_j of Q^*, $A_i' \cap A_j' = \varnothing$. Then $N_1' = \bigcup_{i=1}^{r+s} A_i \times I$, where $A_i \times \{0\} = A_i'$ and $A_i \times \{1\} = A_i$, $i = 1, 2, \ldots, r+s$. Let $B_1, \ldots, B_t$ be the components of $F \backslash \bigcup_{i=1}^{r} A_i'$. Take a small regular neighborhood $B_i \times I$ of B_i in N_2, where $B_i \times \{0\} = B_i$ and $i = 1, 2, \ldots, t$. Set $C_1 = N_1 \cup \bigcup_{i=1}^{t} B_i \times I$ and $C_2 = N \backslash C_1$. Then C_1 is obtained from $F \times I$ and $\bigcup_{i=1}^{s} A_{r+i} \times I$ by adding 1–handles whose cocores are $\mathcal{D}$, so C_1 is a compression body. Note that $C_2 = N_2 \backslash \bigcup_{i=1}^{t} B_i \times I \cong N_2$, C_2 is a compression body. Let $S = \partial_+ C_1$, then by assumption $Q \subset \partial_+ N_2$ and $S = \partial_+ C_2$. Thus, S is a Heegaard surface of N. Now $2 - \chi(S) = 2 - \chi(F \cap N_2) - \chi(Q)$, so we have $g(N) \le g(S) = \frac{1}{2}(2 - \chi(S)) \le 1 - \frac{1}{2}\chi(F \cap N_2) - \frac{1}{2}\chi(Q)$, as required. □

4 The essential surfaces and amalgamations

Proposition 4.1 *Let M be a compact, orientable 3–manifold and F be an essential closed surface which cuts M into M_1 and M_2. Suppose that there is no essential surface with boundary $(Q_i, \partial Q_i)$ in (M_i, F) satisfying $\chi(Q_i) > 2 + g(F) - 2g(M_i)$, $i = 1, 2$. Then for any closed incompressible surface F^* in M with $g(F^*) < g(M_1) + g(M_2) - g(F) - 1$, we can isotope F in M such that after isotopy, $F \cap F^* = \varnothing$.*

Proof Since F and F^* are both incompressible, we can isotope F in M such that each component of $F \cap F^*$ is essential in both F and F^*, and $|F \cap F^*|$ is minimal.

If $|F \cap F^*| > 0$, by the minimality of $|F \cap F^*|$, each component of $F^* \cap M_i$ is essential in M_i, $i = 1, 2$. By assumption, we have $\chi(F^* \cap M_1) \le 2 + g(F) - 2g(M_1)$ and $\chi(F^* \cap M_2) \le 2 + g(F) - 2g(M_2)$. So

$$\chi(F^*) = \chi(F^* \cap M_1) + \chi(F^* \cap M_2) \le 4 + 2g(F) - 2g(M_1) - 2g(M_2).$$

Then $g(F^*) = \frac{1}{2}(2 - \chi(F^*)) \ge g(M_1) + g(M_2) - g(F) - 1$, a contradiction to the assumption.

Thus $F \cap F^* = \varnothing$. □

Now we come to the main result of the paper.

Theorem 4.2 *Let M be a compact, orientable, ∂–irreducible 3–manifold and F be an essential closed surface which cuts M into M_1 and M_2. Suppose that there is no essential surface with boundary $(Q_i, \partial Q_i)$ in (M_i, F) satisfying $\chi(Q_i) > 2 + g(F) - 2g(M_i)$, $i = 1, 2$. Then $g(M) = g(M_1) + g(M_2) - g(F)$.*

Proof We may assume that M is irreducible. If M is reducible, we have that at least one of M_1 and M_2, say M_i, is reducible. By the Prime Decomposing Theorem of 3–manifold, we can put $M_1 = M^{'} \# M_1^{'}$ such that $F \subset \partial M_1^{'}$ and $M_1^{'}$ is irreducible. By Haken's Lemma, $g(M_1^{'}) = g(M_1) - g(M^{'})$ and $g(M_1^{'} \cup_F M_2) = g(M) - g(M^{'})$. Since any essential surface with boundary $(Q_1, \partial Q_1)$ in $(M_1^{'}, F)$ is an essential surface with boundary $(Q_1, \partial Q_1)$ in (M_1, F) and by our assumption, $\chi(Q_1) \leq 2 + g(F) - 2g(M_1) \leq 2 + g(F) - 2g(M_1^{'})$. Then the conclusion follows immediately from the irreducible case.

So M is irreducible and ∂–irreducible.

It is easy to see that $g(M) \leq g(M_1) + g(M_2) - g(F)$. Suppose that $g(M) = g(M_1) + g(M_2) - g(F)$ does not hold. Thus there exists a minimal Heegaard splitting $V \cup_S W$ of M with

$$g(S) \leq g(M_1) + g(M_2) - g(F) - 1. \tag{1}$$

We divide it into the following two cases to discuss.

Case 1 The Heegaard splitting $V \cup_S W$ is strongly irreducible.

In this case, by Lemma 3.3, there are only the following two subcases.

Subcase 1 F is transverse to S and any component of $S \setminus F$ is incompressible in the respective submanifold of $M \setminus F$ except for exactly one strongly irreducible component.

If one incompressible component of $S \cap M_1$ and $S \cap M_2$ is inessential. Then we can isotope S to $S^{'}$ such that $|S \cap F| > |S^{'} \cap F|$, one component of $S^{'} \cap M_1$ and $S^{'} \cap M_2$ is bicompressible and any other component is incompressible. So we may assume one of $S \cap M_1$, say P is bicompressible and any incompressible component of $S \cap M_1$ and $S \cap M_2$ is essential in M_1 or M_2 without loss generality.

Since F is essential in M and there is no closed essential surface in a compression body, $F \cap S \neq \varnothing$. Then $\chi(P) \leq -2$. If otherwise, $\chi(P) \geq -1$, P is either a disk, an annulus, twice-punctured disk, or a once-punctured torus, in each case we can conclude that a component of ∂P bounds a disk in M_1, therefore F is compressible in M_1, a contradiction. By assumption, $\chi(S \cap M_2) \leq 2 + g(F) - 2g(M_2)$.

If $S \cap M_1 \neq P$, then at least one component of $S \cap M_1$ is essential in M_1, by assumption $\chi(S \cap M_1) \leq 2 + g(F) - 2g(M_1) + \chi(P) \leq g(F) - 2g(M_1)$. Thus

$$\begin{aligned}\chi(S) &= \chi(S \cap M_1) + \chi(S \cap M_2) \\ &\leq g(F) - 2g(M_1) + 2 + g(F) - 2g(M_2) \\ &\leq 2 - 2(g(M_1) + g(M_2) - g(F))\end{aligned}$$

and $g(S) \geq g(M_1) + g(M_2) - g(F)$, a contradiction.

So $S \cap M_1 = P$. By Lemma 2.2, P divides M_1 into two compression bodies $V \cap M_1$ and $W \cap M_1$. Since P is bicompressible in M_1, we compress P into P_V (P_W, resp.) in $V \cap M_1$ ($W \cap M_1$, resp.) as possible as. Since $V \cup_S W$ is strongly irreducible and by Lemma 2.3, any component of P_V and P_W is incompressible in M_1. If one component of P_V or P_W, say P_V, is essential in M_1, then by assumption, $\chi(S \cap M_1) \leq \chi(P_V) - 2 \leq 2 + g(F) - 2g(M_1) - 2 \leq g(F) - 2g(M_1)$. By the same arguments as above, $g(S) \geq g(M_1) + g(M_2) - g(F)$, a contradiction again. So any component of P_V or P_W is ∂–parallel in M_1. Then by Lemma 3.4, $g(M_1) \leq 1 - \frac{1}{2}\chi(F \cap W) - \frac{1}{2}\chi(P)$ and $g(M_1) \leq 1 - \frac{1}{2}\chi(F \cap V) - \frac{1}{2}\chi(P)$. Note that $\chi(F) = \chi(F \cap V) + \chi(F \cap W)$, so $2g(M_1) \leq 2 - \frac{1}{2}\chi(F) - \chi(P) = 1 + g(F) - \chi(P)$. Then $\chi(S \cap M_1) = \chi(P) \leq 1 + g(F) - 2g(M_1)$ and

$$\begin{aligned}\chi(S) &= \chi(S \cap M_1) + \chi(S \cap M_2) \\ &\leq 1 + g(F) - 2g(M_1) + 2 + g(F) - 2g(M_2) \\ &\leq 3 - 2(g(M_1) + g(M_2) - g(F)),\end{aligned}$$

so $g(S) \geq g(M_1) + g(M_2) - g(F) - \frac{1}{2}$, a contradiction to (1).

This finishes the proof of Subcase 1.

Subcase 2 F is almost transverse to S and any component of $S \setminus N(F)$ is incompressible in the respective submanifold of $M \setminus N(F)$.

For $i = 1, 2$, it is easy to see that $S \cap M_i$ is essential in M_i, then by assumption, $\chi(S \cap M_i) \leq 2 + g(F) - 2g(M_i)$. Clearly, $\chi(S \cap N(F)) = -1$. So

$$\begin{aligned}\chi(S) &= \chi(S \cap M_1) + \chi(S \cap M_2) - 1 \\ &\leq 2 + g(F) - 2g(M_1) + 2 + g(F) - 2g(M_2) - 1 \\ &\leq 3 - 2(g(M_1) + g(M_2) - g(F)),\end{aligned}$$

hence $g(S) \geq g(M_1) + g(M_2) - g(F) - \frac{1}{2}$, a contradiction again.

This finishes the proof of Subcase 2.

Case 2 The Heegaard splitting $V \cup_S W$ is weakly reducible.

Since M is irreducible, by Haken's lemma, $V \cup_S W$ is irreducible. By the result of Scharlemann and Thompson [13], $V \cup_S W$ is an amalgamation of n strongly irreducible Heegaard splitting $V \cup_S W = (V_1 \cup_{S_1} W_1) \cup_{F_1} (V_2 \cup_{S_2} W_2) \cup_{F_2} \cdots \cup_{F_{n-1}} (V_n \cup_{S_n} W_n)$. We may further assume that no component of F_i, $1 \le i \le n-1$, is ∂–parallel in M. Obviously, $g(F_i) < g(S_i) < g(S)$ and $g(F_i) \le g(S) - 2$. Then by (1) we have

$$\begin{aligned} g(F_i) &\le g(S) - 2 \\ &\le g(M_1) + g(M_2) - g(F) - 3 \\ &< g(M_1) + g(M_2) - g(F) - 1. \end{aligned}$$

By Proposition 4.1, we can isotope F such that $(\cup F_i) \cap F = \varnothing$. So F lies in the nontrivial component $V_j^* \cup_{S_j^*} W_j^*$ of $V_j \cup_{S_j} W_j$ for some $1 \le j \le n$.

If F is parallel to some component, say F^*, of $\bigcup F_i$, then F^* cuts M into two parts M_1 and M_2. $V \cup_S W$ is an amalgamation of two Heegaard splittings of M_1 and M_2 and $g(S) \ge g(M_1) + g(M_2) - g(F)$, a contradiction. Hence F can not be parallel to any component of $\bigcup F_i$. So F is an essential closed surface in $V_j^* \cup_{S_j^*} W_j^*$. Denote $M^* = V_j^* \cup_{S_j^*} W_j^*$. Then there are the following two subcases.

Subcase 3 F is almost transverse to S_j^* and any component of $S_j^* \backslash N(F)$ is incompressible in the respective submanifold of $M^* \backslash N(F)$;

Then any component of $S_j^* \backslash N(F)$ is incompressible in the respective submanifold of $M \backslash N(F)$. Furthermore, $S_j^* \backslash N(F)$ is essential in $M \backslash N(F)$ since there is no essential closed surface in a compression body. Then $\chi(S_j^* \cap N(F)) = -1$ and by assumption, $\chi(S_j^* \cap M_i) \le 2 + g(F) - 2g(M_i)$ for $i = 1, 2$. So

$$\begin{aligned} \chi(S) &\le \chi(S_j^*) - 2 \\ &= \chi(S_j^* \cap M_1) + \chi(S_j^* \cap M_2) + \chi(S_j^* \cap N(F)) - 2 \\ &\le 2 + g(F) - 2g(M_1) + 2 + g(F) - 2g(M_2) - 3 \\ &\le 1 - 2(g(M_1) + g(M_2) - g(F)). \end{aligned}$$

Hence $g(S) \ge g(S_j^*) \ge g(M_1) + g(M_2) - g(F) + \frac{1}{2}$, a contradiction.

This finishes the proof of Subcase 3.

Subcase 4 F is transverse to S_j^* and any component of $S_j^* \backslash F$ is incompressible in the respective submanifold of $M^* \backslash F$ except for exactly one strongly irreducible component.

Since $(\bigcup F_i)$ is incompressible in M, any component of $S_j^*\backslash F$ is incompressible in the respective submanifold of $M\backslash F$ except for exactly one strongly irreducible component. We may assume that any component of $S_j^*\backslash F$ is incompressible in M_1, and denote the compressible component of $S_j^* \cap M_2$ by Q'. Then $S_j^* \cap M_1$ is essential in M_1 and by assumption $\chi(S_j^* \cap M_1) \le 2 + g(F) - 2g(M_1)$. It is easy to see that $\chi(Q') \le -2$.

We compress Q' as much as possible in V_j^* (W_j^*, resp.) to obtain subsurfaces Q'_V (Q'_W, resp.), then any component of Q'_V and Q'_W is incompressible in $V_j^* \cup_{S_j^*} W_j^*$. Furthermore, Q'_V and Q'_W is incompressible in M_2 since $\bigcup F_i$ is incompressible in M. If there is one component of $S_j^* \cap M_2$, Q'_V and Q'_W which is essential in $V_j^* \cup_{S_j^*} W_j^*$, then by assumption $\chi(S_j^* \cap M_2) \le 2 + g(F) - 2g(M_2) - 2$. Hence $g(S) \ge g(S_j^*) \ge g(M_1) + g(M_2) - g(F)$, a contradiction.

So any component of Q'_V and Q'_W is parallel to a subsurface of F or a component of $\partial_- V_j^*$ or $\partial_- W_j^*$. Any component of $F \cap V_j^*$ is incompressible in V_j^*, by Lemma 2.2 any component of $V_j^*\backslash F$ is a compression body. By the same reasons as above, any component of $W_j^*\backslash F$ is a compression body. We amalgamate the Heegaard splitting $V_j^* \cup_{S_j^*} W_j^*$ and the Heegaard splittings contained in M_2 of the Heegaard sequence $V_1 \cup_{S_1} W_1, V_2 \cup_{S_2} W_2, \dots, V_{n-1} \cup_{S_{n-1}} W_{n-1}, V_n \cup_{S_n} W_n$ along the components contained in M_2 of $\bigcup F_i$ to obtain a Heegaard splitting $V' \cup_{S'} W'$ such that the following conditions are satisfied:

(1) $V' \cap M_1 = V_j^* \cap M_1$ and $W' \cap M_1 = W_j^* \cap M_1$.

(2) $S' \cap F = S_j^* \cap F$ and $S' \cap M_1 = S_j^* \cap M_1$.

(3) Only one component of $S' \cap M_2$ is compressible in V', denoted by Q'' and other incompressible components are just the components of $S_j^* \cap M_2$.

(4) Q'' can be compressed into Q''_V(Q''_W, resp.) resp.) in V' (W', resp.) such that the incompressible components of Q''_V (Q''_W, resp.) with boundary are the same as the incompressible components of Q'_V (Q'_W, resp.) with boundary.

Then Q'' satisfies the conditions of Lemma 3.4, so $g(M_2) \le 1 - \frac{1}{2}\chi(F \cap W') - \frac{1}{2}\chi(Q'')$ and $g(M_2) \le 1 - \frac{1}{2}\chi(F \cap V') - \frac{1}{2}\chi(Q'')$. Note that $\chi(F) = \chi(F \cap V') + \chi(F \cap W')$, so $2g(M_2) \le 2 - \frac{1}{2}\chi(F) - \chi(Q'') = 1 + g(F) - \chi(Q'')$. Then $\chi(S' \cap M_2) \le \chi(Q'') \le 1 + g(F) - 2g(M_2)$. Notice that $\chi(S' \cap M_1) = \chi(S_j^* \cap M_1) \le 2 + g(F) - 2g(M_1)$. Hence,

$$\begin{aligned}\chi(S') &= \chi(S' \cap M_1) + \chi(S' \cap M_2) \\ &\le 1 + g(F) - 2g(M_1) + 2 + g(F) - 2g(M_2) \\ &\le 3 - 2(g(M_1) + g(M_2) - g(F)).\end{aligned}$$

So $g(S^{'}) \geq g(M_1)+g(M_2)-g(F)-\frac{1}{2}$ and $g(S) \geq g(S^{'}) \geq g(M_1)+g(M_2)-g(F)$, a contradiction.

This finishes the proof of Subcase 4.

Therefore, $g(M) = g(M_1) + g(M_2) - g(F)$ holds. This completes the proof of Theorem 4.2. □

In the following, we give an application of Theorem 4.2.

Theorem 4.3 *Let M be a compact, orientable 3–manifold and F be an essential closed surface which cuts M into M_1 and M_2. If M_i has a Heegaard splitting $V_i \cup_{S_i} W_i$ with distance $D(S_i) \geq 2g(M_i) - g(F)$, $i = 1, 2$. Then $g(M) = g(M_1) + g(M_2) - g(F)$.*

Proof For $i = 1, 2$, if Q_i is an essential surface in M_i. By assumptions and Lemma 2.4, $2 - \chi(Q_i) \geq D(S_i) \geq 2g(M_i) - g(F)$. Then $\chi(Q_i) \leq 2 + 2g(M_i) - g(F)$, $i = 1, 2$. And by Theorem 4.2, we have $g(M) = g(M_1) + g(M_2) - g(F)$. □

Remark The condition in Theorem 4.2 is weaker than that in the main results of Kobayashi and Qiu [6] and Yang and Lei [19].

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