Permutative categories, multicategories and algebraic K–theory

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We show that the K–theory construction of our paper [7], which preserves multiplicative structure, extends to a symmetric monoidal closed bicomplete source category, with the multiplicative structure still preserved. The source category of [7], whose objects are permutative categories, maps fully and faithfully to the new source category, whose objects are (based) multicategories.

1 Introduction

In [7], we introduced a K–theory functor from permutative categories to symmetric spectra, equivalent to previous definitions, which also preserves multiplicative structure. The multiplicative structure on the category of permutative categories was captured by a *multicategory* structure, which is a simultaneous generalization of the concepts of operad and symmetric monoidal category. Since symmetric spectra support a symmetric monoidal multiplicative structure, they automatically form a multicategory, and it is this structure that our K–theory functor preserves.

Part of the added flexibility that multicategories offer is that a full subcategory of a multicategory inherits a multicategory structure. In particular, a full subcategory of a symmetric monoidal category inherits a multicategory structure, although it will no longer be monoidal unless it is closed under the monoidal product. In addition, the maps of multicategories, or *multifunctors*, between symmetric monoidal categories are simply the lax symmetric monoidal functors. Since the K–theory map produced in [7] is a multifunctor from a multicategory (the permutative categories) to a symmetric monoidal category (the symmetric spectra), it is a natural question whether the source multicategory can be embedded as a full subcategory of a symmetric monoidal category, with the K–theory map extending to it as a lax symmetric monoidal functor. We can also ask whether the new source category can be made bicomplete, to make it a convenient place to do homotopy theory. We answer both these questions in the affirmative: this is

the content of Theorem 1.1 and Theorem 1.2, with the objects of the larger category being, ironically enough, multicategories! We actually use *based* multicategories, in the sense that the objects come equipped with a preferred map from the terminal multicategory, and the embedding takes a permutative category to its underlying based multicategory.

Theorem 1.1 *Let* **P** *be the multicategory of permutative categories, and let* **Mult**$_*$ *be the symmetric monoidal category of based multicategories. Then the underlying based multicategory construction gives a full and faithful multifunctor from* **P** *to* **Mult**$_*$ *of multicategories enriched over* **Cat**.

The category **Mult**$_*$ cures many of the defects of the multicategory of permutative categories: in addition to being symmetric monoidal, it is closed, complete and cocomplete. We will derive these properties from the same ones for unbased multicategories.

Theorem 1.2 *The categories* **Mult** *of unbased multicategories and* **Mult**$_*$ *of based multicategories are both symmetric monoidal, closed and bicomplete.*

Our final main result extends the K–theory map of [7] to a lax symmetric monoidal map from the symmetric monoidal category of pointed multicategories, and here too there is an improvement. The K–theory map of [7] is actually a composite: the first piece is a multifunctor from permutative categories to what we call $\mathcal{G}_*$–categories, which form a symmetric monoidal category $\mathcal{G}_*$–**Cat**. Then the second piece is a lax symmetric monoidal functor from $\mathcal{G}_*$–**Cat** to symmetric spectra. Our extension result produces a lax symmetric monoidal functor J' from pointed multicategories to $\mathcal{G}_*$–categories. In addition, we identify $\mathcal{G}_*$–**Cat** as a category of functors from an index category $\mathcal{G}_*$ to **Cat** and J' as a representable functor.

Theorem 1.3 *There is a lax symmetric monoidal functor J' from* **Mult**$_*$ *to* $\mathcal{G}_*$–**Cat** *such that the composite with the full and faithful functor of Theorem 1.1 is naturally isomorphic to the multifunctor J constructed in [7]. Moreover, J' is the representable functor* **Mult**$_*(E^*, \_\,)$, *where* $E^* \colon \mathcal{G}_*^{\mathrm{op}} \to$ **Mult**$_*$.

The extension we seek is then the composite of this representable extension with the lax symmetric monoidal piece from [7]. In summary, we can speak meaningfully of the K–theory of a (pointed) multicategory, and the K–theory of a permutative category depends only on its underlying pointed multicategory. All the multiplicative structure captured by the K–theory map of [7] also depends only on underlying based multicategories.

The paper is organized as follows. In Section 2, we describe the multicategory structure on the category of multicategories; this is what will underlie the symmetric monoidal structure we will describe later. We also describe the enrichment present, which follows easily from the internal hom construction for multicategories. We then extend all these constructions to the based context. Section 3 discusses the various categories of permutative categories of interest to us and reviews the multicategory structure on permutative categories from [7]. We will then have sufficient tools on hand to give the proof of Theorem 1.1. Section 4 is devoted to the deeper structure of the category of multicategories that allows us to show that it is symmetric monoidal closed, complete and cocomplete, as is its pointed analogue. Section 5 then introduces the representing object for our lax symmetric monoidal functor J', and Section 6 concludes the paper by showing our represented functor is consistent with the one defined in [7] when restricted to permutative categories.

It is a pleasure to acknowledge the anonymous referees of [7] for a very interesting comment about partial permutative categories. We have, however, chosen the more drastic route all the way to multicategories in this paper, with what we hope are satisfying results. We also thank the referee of this paper for a careful reading and thorough comments that have undoubtedly improved the paper.

2 The multicategory of multicategories

The basic idea of a multicategory is very simple. Like a category, it has objects, but the essential difference is that the source of a morphism is a string of objects of a specified length (including length 0), rather than a single object. The target remains a single object. Consequently, to compose one must consider strings of strings, which are then concatenated to obtain the source of the composite. As with operads, which are simply multicategories with a single object, there are two flavors of multicategory: with or without permutations. The flavor without permutations was the original one introduced by Lambek [8]. We will be concerned almost exclusively with the flavor *with* permutations and so we will call these simply "multicategories" rather than "symmetric multicategories" as in, for example, Leinster [9, A.2]. The technicalities of the definition are now fairly straightforward, and are as follows.

Definition 2.1 A *multicategory* **M** consists of the following:

(1) a collection of objects, which may form a proper class,

(2) for each $k \geq 0$, k–tuple of objects $(a_1, \ldots, a_k)$ (the "source") and single object b (the "target"), a set $\mathbf{M}_k(a_1, \ldots, a_k; b)$ (the "k–morphisms"),

(3)   a right action of Σ_k on the collection of all k–morphisms, where for $\sigma \in \Sigma_k$,

$$\sigma^*\colon \mathbf{M}_k(a_1, \ldots, a_k; b) \to \mathbf{M}_k(a_{\sigma(1)}, \ldots, a_{\sigma(k)}; b),$$

(4)   a distinguished "unit" element $1_a \in \mathbf{M}_1(a; a)$ for each object a, and

(5)   a "composition"

$$\Gamma\colon \mathbf{M}_n(b_1, \ldots, b_n; c) \times \mathbf{M}_{k_1}(a_{11}, \ldots, a_{1k_1}; b_1) \times \cdots \times \mathbf{M}_{k_n}(a_{n1}, \ldots, a_{nk_n}; b_n)$$
$$\longrightarrow \mathbf{M}_{k_1 + \cdots + k_n}(a_{11}, \ldots, a_{nk_n}; c),$$

all subject to the identities for an operad listed on pages 1–2 in May [11], which still make perfect sense in this context. For greater detail, we refer the reader to [7, Definition 2.1]. A *multifunctor* is a structure preserving map of multicategories.

As with categories, if the objects of a multicategory form a set, we call it *small*; otherwise it is *large*. We obtain the category **Mult** whose objects are all small multicategories. We note that restricting attention to 1–morphisms gives an underlying category for each multicategory.

We may also ask that the k–morphisms $\mathbf{M}_k(a_1, \ldots, a_k; b)$ take values in a symmetric monoidal category other than sets, giving us the concept of an *enriched* multicategory. The enriched multicategories of interest to us are all large, and generally enriched over **Cat**; this will be the case for **Mult**, in particular.

A basic observation is that if we let **M** be a full subcategory of a symmetric monoidal category $(\mathcal{C}, \oplus, 0)$, then **M** becomes a multicategory by defining

$$\mathbf{M}_k(a_1, \ldots, a_k; b) := \mathcal{C}(a_1 \oplus \cdots \oplus a_k, b)$$

(for an arbitrary choice of association). The same observation also holds if **M** is merely the source of a full and faithful functor to $\mathcal{C}$. In particular, 0–morphisms in **M** are given by morphisms in $\mathcal{C}$ out of the monoidal unit 0. As a consequence, every symmetric monoidal category has an underlying multicategory, and it is an interesting exercise to check that the maps of underlying multicategories between two symmetric monoidal categories are the lax symmetric monoidal maps (see Section 3).

We begin our description of the additional structure on **Mult** by observing that the multifunctors between two multicategories are themselves the objects of a multicategory. These internal Hom multicategories will give **Mult** the structure of a closed category. It is crucial for the construction that we use multicategories *with* permutations; this construction does not extend to nonsymmetric multicategories. For notational convenience, we will often write lists such as $(c_1, \ldots, c_k)$ as $\langle c_i \rangle_{i=1}^k$, or even just $\langle c_i \rangle$ when the limits are clear.

Definition 2.2 We define an internal Hom object in **Mult** as follows. Given small multicategories M and N, we define $\mathbf{Hom}(M, N)$ to be a multicategory with objects the multifunctors from M to N. Given a source k–tuple $(f_1, \ldots, f_k)$ of multi-functors and a target multifunctor g, we define a k–*natural transformation* from $(f_1, \ldots, f_k)$ to g to be a function ξ that assigns to each object a of M a k–morphism $\xi_a \colon (f_1 a, \ldots, f_k a) \to ga$ of N, such that for any m–morphism $\phi \colon (a_1, \ldots, a_m) \to b$ in M, the following diagram commutes:

$$
\begin{array}{ccc}
\langle\langle f_j a_i \rangle_{j=1}^k \rangle_{i=1}^m & \xrightarrow{\langle \xi_{a_i} \rangle} & \langle g a_i \rangle_{i=1}^m \\
\cong \downarrow & & \\
\langle\langle f_j a_i \rangle_{i=1}^m \rangle_{j=1}^k & & \downarrow g\phi \\
\langle f_j \phi \rangle \downarrow & & \\
\langle f_j b \rangle_{j=1}^k & \xrightarrow{\xi_b} & gb.
\end{array}
$$

Here the unlabelled isomorphism is the permutation that reverses the priority of the indices i and j, ie, it shuffles m blocks of k entries each into k blocks of m entries each. The k–natural transformations are then the k–morphisms in the multicategory $\mathbf{Hom}(M, N)$. The composition is induced from the composition in N, as are the actions by the symmetric groups. The reader can now verify that the axioms for a multicategory are satisfied.

We wish to define a multicategory structure on **Mult**, and the crucial step is the definition of 2–morphisms, which we call *bilinear maps*. Given multicategories M, N and P, the idea of a bilinear map $f \colon (M, N) \to P$ is a map that is multifunctorial in each variable separately, such that the multifunctors in each variable are "multinatural" in a way that mirrors our definition of Hom multicategories. The precise definition follows.

Definition 2.3 Let M, N, and P be multicategories. A *bilinear map* $f \colon (M, N) \to P$ consists of:

(1) a function $f \colon \mathrm{Ob}\,(M) \times \mathrm{Ob}\,(N) \to \mathrm{Ob}\,(P)$,

(2) for each m–morphism $\phi \colon (a_1, \ldots, a_m) \to a'$ of M and object b of N, an m–morphism $f(\phi, b) \colon (f(a_1, b), \ldots, f(a_m, b)) \to f(a', b)$ of P,

(3) for each n–morphism $\psi \colon (b_1, \ldots, b_n) \to b'$ of N and object a of M, an n–morphism $f(a, \psi) \colon (f(a, b_1), \ldots, f(a, b_n)) \to f(a, b')$ of P,

such that

(1)   for each object a of M, $f(a, \_\ )$ is a multifunctor from N to P,

(2)   for each object b of N, $f(\_\, , b)$ is a multifunctor from M to P,

(3)   given an m–morphism $\phi\colon (a_1, \ldots, a_m) \to a'$ in M and an n–morphism $\psi\colon (b_1, \ldots, b_n) \to b'$ in N, the following diagram commutes, where the unlabelled isomorphism is given by the permutation that reverses the priority of the indices:

$$
\begin{array}{ccc}
\langle\langle f(a_i, b_j)\rangle_{i=1}^m \rangle_{j=1}^n & \xrightarrow{\ \langle f(\phi, b_j)\rangle\ } & \langle f(a', b_j)\rangle_{j=1}^n \\
\cong \downarrow & & \Big\downarrow \\
\langle\langle f(a_i, b_j)\rangle_{j=1}^n \rangle_{i=1}^m & & f(a', \psi) \\
\langle f(a_i, \psi)\rangle \Big\downarrow & & \Big\downarrow \\
\langle f(a_i, b')\rangle_{i=1}^m & \xrightarrow[\ f(\phi, b')\ ]{} & f(a', b').
\end{array}
$$

We write the set of such bilinear maps as **Bilin**$(M, N; P)$.

It is worth remarking that a bilinear map $f\colon (M, N) \to P$ gives, by restriction to 1–morphisms, a functor $f\colon M \times N \to P$ on underlying categories, but that a multifunctor $f\colon M \times N \to P$ is quite different from a bilinear map: a multifunctor $M \times N \to P$ assigns a k–morphism in P to each pair of k–morphisms from M and N, while a bilinear map assigns an mn–morphism given by the common diagonal of the above diagram to each pair consisting of an m–morphism in M and an n–morphism in N.

A key point is that **Bilin**$(M, N; P)$ gives the objects of a multicategory naturally isomorphic to both **Hom**$(M, \mathbf{Hom}(N, P))$ and **Hom**$(N, \mathbf{Hom}(M, P))$. It is routine to verify that the morphisms in this multicategory are given as follows.

Definition 2.4   Suppose given bilinear maps $f_1, \ldots, f_k$ and g in **Bilin**$(M, N; P)$. Then a k–morphism $\xi\colon (f_1, \ldots, f_k) \to g$ is a k–natural transformation in each variable: it consists of a choice of k–morphism $\xi_{(a,b)}\colon (f_1(a, b), \ldots, f_k(a, b)) \to g(a, b)$ in P for each pair of objects (a, b) in $\mathrm{Ob}(M) \times \mathrm{Ob}(N)$, such that for each morphism $\phi\colon (a_1, \ldots, a_m) \to a'$ of M and each morphism $\psi\colon (b_1, \ldots, b_n) \to b'$ of N the following pair of diagrams commutes:

$$
\begin{array}{ccc}
\langle\langle f_j(a_i, b)\rangle_{i=1}^m \rangle_{j=1}^k & \xrightarrow{\ \langle f_j(\phi, b)\rangle\ } & \langle f_j(a', b)\rangle_{j=1}^k \\
\cong \downarrow & & \Big\downarrow \\
\langle\langle f_j(a_i, b)\rangle_{j=1}^k \rangle_{i=1}^m & & \xi_{(a', b)} \\
\langle \xi_{(a_i, b)}\rangle \Big\downarrow & & \Big\downarrow \\
\langle g(a_i, b)\rangle_{i=1}^m & \xrightarrow[\ g(\phi, b)\ ]{} & g(a', b),
\end{array}
$$

$$\langle\langle f_j(a, b_i)\rangle_{i=1}^n\rangle_{j=1}^k \xrightarrow{\langle f_j(a,\psi)\rangle} \langle f_j(a, b')\rangle_{j=1}^k$$

$$\cong\Big\downarrow \qquad\qquad\qquad\qquad \Big\downarrow \xi_{(a,b')}$$

$$\langle\langle f_j(a, b_i)\rangle_{j=1}^k\rangle_{i=1}^n$$

$$\langle\xi_{(a,b_i)}\rangle\Big\downarrow$$

$$\langle g(a, b_i)\rangle_{i=1}^n \xrightarrow{\quad g(a,\psi)\quad} g(a, b').$$

Corollary 2.5 *With morphisms as in Definition 2.4, **Bilin**$(M, N; P)$ forms a multi-category with natural isomorphisms of multicategories*

$$\mathbf{Hom}(M, \mathbf{Hom}(N, P)) \cong \mathbf{Bilin}(M, N; P) \cong \mathbf{Hom}(N, \mathbf{Hom}(M, P)).$$

The definition of a k–linear map of multicategories for $k \geq 2$ presents no further difficulties, as it is merely a map that is multifunctorial in each variable separately, and bilinear in each pair of variables separately.

Definition 2.6 Let $M_1, \ldots, M_k$ and N be multicategories. A k*–linear map*

$$f\colon (M_1, \ldots, M_k) \to N$$

consists of:

(1)   a function $\mathrm{Ob}(M_1) \times \cdots \times \mathrm{Ob}(M_k) \to \mathrm{Ob}(N)$

(2)   for each m–morphism $\phi_j\colon (a_{j1}, \ldots, a_{jm}) \to a_j'$ in M_j and choices of objects b_i in M_i for $i \neq j$, an m–morphism

$$f(b_1, \ldots, b_{j-1}, \phi, b_{j+1}, \ldots, b_k)\colon$$
$$\quad (f(b_1, \ldots, b_{j-1}, a_{j1}, b_{j+1}, \ldots, b_k), \ldots, f(b_1, \ldots, b_{j-1}, a_{jm}, b_{j+1}, \ldots, b_k))$$
$$\to f(b_1, \ldots, b_{j-1}, a', b_{j+1}, \ldots, b_k)$$

such that

(1) f is multifunctorial in each variable separately, and

(2) f is bilinear in each pair of variables separately.

Finally, we state explicitly that a 0–morphism to a multicategory N consists of a choice of object of N. With these definitions in place, it is now possible to verify directly that **Mult** becomes a multicategory with the evident notion of composition; however, the details are unnecessary, since we will show in Section 4 that **Mult** actually supports the structure of a symmetric monoidal closed category of which this is the underlying

multicategory. We remark that the Hom construction introduced here will give the closed part of this structure. In particular, remembering only the 1–morphisms in the Hom construction gives **Mult** an enrichment over **Cat**, which we use in our extension of the K–theory functor in Section 5.

We turn next to the multicategory of main interest in this paper, whose objects are *based* multicategories. We note first that **Mult** has a terminal object, namely the multicategory with one object and one k–morphism for each $k \geq 0$. We will denote this terminal multicategory by the usual unilluminating $*$. Since it has only one object, it is an operad. In fact, it is the operad that parametrizes commutative monoids in a multicategory; ie, a multifunctor from $*$ to a multicategory M consists of an object in M and a commutative monoid structure on that object.

Definition 2.7 A *based multicategory* consists of a multicategory M, together with a preferred multifunctor $* \to M$. A *based multifunctor* is a multifunctor preserving the basepoint structure.

We obtain a category **Mult**$_*$ of based multicategories. This will be the source of our extended K–theory functor, and one of the main purposes of this paper is to show that it is actually symmetric monoidal, closed, complete, cocomplete, and that the extended K–functor is lax symmetric monoidal. We content ourselves in this section with a description of the underlying multicategory structure.

We say that an object or morphism in a based multicategory *comes from the basepoint* if it is in the image of the given map from $*$.

Definition 2.8 Let M, N, and P be based multicategories. A *based bilinear map* $f \colon (M, N) \to P$ is a bilinear map of the underlying (unbased) multicategories such that if input data from either variable comes from the basepoint, the output also comes from the basepoint.

We also define a 0–morphism to a based multicategory to be simply a 0–morphism to the underlying unbased multicategory, that is, a choice of object. (As an analogue, think of a 0–morphism to a pointed topological space as a based map from the unit of the smash product, S^0.) It is now straightforward to extend the definition of a k–linear map of multicategories (Definition 2.6) to the based context by attaching the word "based" where appropriate. Note also that the morphism induced by a ϕ_j in M_j must come from the basepoint whenever ϕ_j or any of the b_i come from the basepoint. It is now straightforward, but tedious and unnecessary, to verify that we do get a multicategory of based multicategories – again unnecessary because we will

show that **Mult**$_*$ actually supports a symmetric monoidal structure for which this is the underlying multicategory structure.

We close this section with some remarks on enrichment. We will show in Section 4 that both **Mult** and **Mult**$_*$ are symmetric monoidal closed, and so in particular are enriched over themselves. We obtain an enrichment over **Cat** from the lax symmetric monoidal forgetful functors

$$\mathbf{Mult}_* \to \mathbf{Mult} \to \mathbf{Cat},$$

where **Cat** is given its Cartesian monoidal structure. This is the enrichment of greatest topological significance, although we will use the full enrichment of **Mult** over itself to show that **Mult**$_*$ is symmetric monoidal closed.
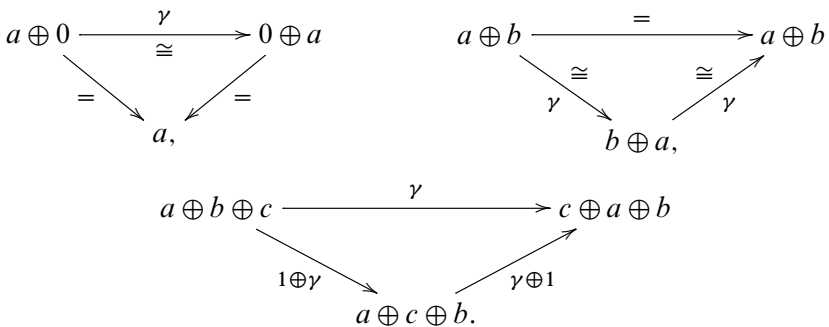
3 Permutative categories

In this section we introduce the categories of permutative categories we will need, review the multicategory structure on permutative categories from [7], and prove that this multicategory of permutative categories admits a full and faithful multifunctor to **Mult**$_*$.

A permutative category (also called a "symmetric strict monoidal category") is a more rigid version of a symmetric monoidal category: it is as rigid as possible without giving up homotopical generality. The precise definition is as follows.

Definition 3.1 A *permutative category* is a category $\mathcal{C}$ with a functor $\oplus\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an object $0 \in \mathrm{Ob}\,(\mathcal{C})$, and a natural isomorphism $\gamma\colon a \oplus b \cong b \oplus a$ satisfying:

(1) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (strict associativity),

(2) $a \oplus 0 = a = 0 \oplus a$ (strict unit), and

(3) the following three diagrams must commute:

(Conditions (1) and (2) of course hold for both objects and morphisms.)

There are several reasonable ways to define maps between permutative categories, of which we shall need to make use of two. First, there are the *strict* maps.

Definition 3.2 A *strict map* of permutative categories $f: \mathcal{C} \to \mathcal{D}$ is a functor for which $f(a \oplus b) = fa \oplus fb$ (for both objects and morphisms), $f(0) = 0$, and the following diagram commutes:

$$
\begin{array}{ccc}
f(a \oplus b) & \xrightarrow{\;=\;} & fa \oplus fb \\
\Big\downarrow{\scriptstyle f(\gamma)} & & \Big\downarrow{\scriptstyle \gamma} \\
f(b \oplus a) & \xrightarrow{\;=\;} & fb \oplus fa.
\end{array}
$$

We obtain a category **Strict** of permutative categories and strict maps. We will make use of this category in the next section in our proofs that **Mult** and **Mult**$_*$ are cocomplete.

Second, every permutative category $\mathcal{C}$ has an underlying multicategory, and we can define a *lax map* of permutative categories to be a map of the underlying multicategories. (This applies to symmetric monoidal categories as well.) A lax map $f: \mathcal{C} \to \mathcal{D}$ can be expressed explicitly as a functor together with natural maps

$$\eta: 0 \to f(0) \quad \text{and} \quad \lambda: f(a) \oplus f(b) \to f(a \oplus b)$$

subject to some coherence diagrams. However, the fact that $f(0)$ can be distinct from 0 causes problems with basepoint control, and consequently we will make no use of this sort of map.

Instead, the other sort of map of permutative categories we will use exploits the fact that a permutative category actually has an underlying *based* multicategory. The basepoint is given by 0 and all the identifications $0 \oplus \cdots \oplus 0 = 0$. We could now define a *lax$_*$ map* of permutative categories to be a map of underlying based multicategories. However, for consistency with [7], we give the explicit description, and the claim that this *is* a map of underlying based multicategories will follow from Theorem 1.1 by restriction to 1–morphisms. Note that these morphisms were erroneously called "lax" in [7].

Definition 3.3 Let $\mathcal{C}$ and $\mathcal{D}$ be permutative categories. A *lax$_*$ map* $f: \mathcal{C} \to \mathcal{D}$ is a functor such that $f(0) = 0$, together with a natural transformation

$$\lambda: f(a) \oplus f(b) \to f(a \oplus b),$$

such that

(1) $\lambda = \mathrm{id}$ if either a or b are 0, and

(2)   the following two diagrams commute:

$$
\begin{array}{ccc}
f(a) \oplus f(b) \oplus f(c) & \xrightarrow{\ 1 \oplus \lambda\ } & f(a) \oplus f(b \oplus c) \\
{\scriptstyle \lambda \oplus 1}\Big\downarrow & & \Big\downarrow{\scriptstyle \lambda} \\
f(a \oplus b) \oplus f(c) & \xrightarrow[\ \lambda\ ]{} & f(a \oplus b \oplus c),
\end{array}
\qquad
\begin{array}{ccc}
f(a) \oplus f(b) & \xrightarrow{\ \lambda\ } & f(a \oplus b) \\
{\scriptstyle \gamma}\Big\downarrow & & \Big\downarrow{\scriptstyle f(\gamma)} \\
f(b) \oplus f(a) & \xrightarrow[\ \lambda\ ]{} & f(b \oplus a).
\end{array}
$$

We obtain a category $\mathbf{P}$ of permutative categories and lax_* maps, which in fact supports the structure of a multicategory enriched over $\mathbf{Cat}$. We review the definitions from [7].

Definition 3.4 Let $\mathcal{C}_1, \ldots, \mathcal{C}_k$ and $\mathcal{D}$ be small permutative categories. We define categories $\mathbf{P}_k(\mathcal{C}_1, \ldots, \mathcal{C}_k; \mathcal{D})$ that provide the categories of k–morphisms for the multicategory $\mathbf{P}$ of permutative categories as follows. The objects of $\mathbf{P}_k(\mathcal{C}_1, \ldots, \mathcal{C}_k; \mathcal{D})$ consist of functors

$$
f \colon \mathcal{C}_1 \times \cdots \times \mathcal{C}_k \to \mathcal{D},
$$

which we think of as k–linear maps, satisfying $f(c_1, \ldots, c_k) = 0$ if any of the c_i are 0, together with natural transformations for $1 \le i \le k$, which we think of as distributivity maps,

$$
\delta_i \colon f(c_1, \ldots, c_i, \ldots, c_k) \oplus f(c_1, \ldots, c_i', \ldots, c_k) \to f(c_1, \ldots, c_i \oplus c_i', \ldots, c_k).
$$

We conventionally suppress the variables that do not change, writing

$$
\delta_i \colon f(c_i) \oplus f(c_i') \to f(c_i \oplus c_i').
$$

We require $\delta_i = \mathrm{id}$ if either c_i or c_i' is 0, or if any of the other c_j's are 0. These natural transformations are subject to the commutativity of the following diagrams:

$$
\begin{array}{ccc}
f(c_i) \oplus f(c_i') \oplus f(c_i'') & \xrightarrow{\ 1 \oplus \delta_i\ } & f(c_i) \oplus f(c_i' \oplus c_i'') \\
{\scriptstyle \delta_i \oplus 1}\Big\downarrow & & \Big\downarrow{\scriptstyle \delta_i} \\
f(c_i \oplus c_i') \oplus f(c_i'') & \xrightarrow[\ \delta_i\ ]{} & f(c_i \oplus c_i' \oplus c_i''),
\end{array}
\qquad
\begin{array}{ccc}
f(c_i) \oplus f(c_i') & \xrightarrow{\ \delta_i\ } & f(c_i \oplus c_i') \\
{\scriptstyle \gamma}\Big\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\Big\downarrow{\scriptstyle f(\gamma)} \\
f(c_i') \oplus f(c_i) & \xrightarrow[\ \delta_i\ ]{} & f(c_i' \oplus c_i),
\end{array}
$$

and for $i \neq j$,

$$f(c_i \oplus c_i', c_j) \oplus f(c_i \oplus c_i', c_j')$$

$$\delta_i \oplus \delta_i \nearrow \qquad \searrow \delta_j$$

$$f(c_i, c_j) \oplus f(c_i', c_j) \oplus f(c_i, c_j') \oplus f(c_i', c_j')$$

$$f(c_i \oplus c_i', c_j \oplus c_j').$$

$$1 \oplus \gamma \oplus 1 \Big\downarrow \cong$$

$$\delta_i$$

$$f(c_i, c_j) \oplus f(c_i, c_j') \oplus f(c_i', c_j) \oplus f(c_i', c_j')$$

$$\delta_j \oplus \delta_j \searrow \qquad \nearrow$$

$$f(c_i, c_j \oplus c_j') \oplus f(c_i', c_j \oplus c_j')$$

We explicitly define $\mathbf{P}_0(; \mathcal{D})$ to be the category $\mathcal{D}$. This completes the definition of the objects of $\mathbf{P}_k(\mathcal{C}_1, \ldots, \mathcal{C}_k; \mathcal{D})$. To specify its morphisms, given two objects f and g, a morphism $\phi \colon f \to g$ is a natural transformation commuting with all the δ_i's, in the sense that all the diagrams

$$\begin{array}{ccc}
f(c_i) \oplus f(c_i') & \xrightarrow{\ \delta_i^f\ } & f(c_i \oplus c_i') \\
{\scriptstyle \phi \oplus \phi}\Big\downarrow & & \Big\downarrow{\scriptstyle \phi} \\
g(c_i) \oplus g(c_i') & \xrightarrow[\ \delta_i^g\ ]{} & g(c_i \oplus c_i')
\end{array}$$

commute. We also require that $\phi(c_1, \ldots, c_k) = \mathrm{id}_0$ whenever any of the $c_i = 0$.

In order to make the $\mathbf{P}_k(\mathcal{C}_1, \ldots, \mathcal{C}_k; \mathcal{D})$'s the k–morphisms of a multicategory, we must specify a Σ_k action and a composition. The Σ_k action

$$\sigma^* f \colon \mathcal{C}_{\sigma(1)} \times \cdots \times \mathcal{C}_{\sigma(k)} \to \mathcal{D}$$

is specified by

$$\sigma^* f(c_{\sigma(1)}, \ldots, c_{\sigma(k)}) = f(c_1, \ldots, c_k),$$

with the structure maps δ_i inherited from f (with the appropriate permutation of the indices). We define the composition as follows: Given $f_j \colon \mathcal{C}_{j1} \times \cdots \times \mathcal{C}_{jk_j} \to \mathcal{D}_j$ for $1 \leq j \leq n$ and $g \colon \mathcal{D}_1 \times \cdots \times \mathcal{D}_n \to \mathcal{E}$, we define

$$\Gamma(g; f_1, \ldots, f_n) := g \circ (f_1 \times \cdots \times f_n).$$

To specify the structure maps, suppose $k_1 + \cdots + k_{j-1} < s \le k_1 + \cdots + k_j$, and let $i = s - (k_1 + \cdots + k_{j-1})$. Then δ_s is given by the composite

$$g(f_j(c_{ji})) \oplus g(f_j(c'_{ji})) \xrightarrow{\delta_j^g} g(f_j(c_{ji}) \oplus f_j(c'_{ji})) \xrightarrow{g(\delta_i^{f_j})} g(f_j(c_{ji} \oplus c'_{ji})).$$

Once we have verified that this structure maps fully and faithfully to the multicategory **Mult**$_*$, it will follow that this does define a multicategory structure on **P**, although this can also be done directly. We remark that in the context of multifunctors, "full and faithful" means that the multifunctor induces a bijection (or isomorphism in the enriched context) on the morphism sets for any particular choice of source and target. We can now give the proof of the first of our main theorems.

Proof of Theorem 1.1   Given a permutative category $\mathcal{C}$, let $U\mathcal{C}$ be its underlying based multicategory. Then the claim is that U extends to a full and faithful multifunctor

$$U \colon \mathbf{P} \to \mathbf{Mult}_*,$$

enriched over **Cat**. We begin by defining U on the 1–morphisms of **P**, which are the lax$_*$ morphisms. Note that by the associativity diagram for the structure map of a lax$_*$ morphism, it induces a canonical map

$$\bigoplus_{i=1}^{k} f(a_i) \xrightarrow{\lambda} f(\textstyle\bigoplus_{i=1}^{k} a_i).$$

Given such a lax$_*$ morphism, we define the induced multifunctor on the underlying based multicategories $Uf \colon U\mathcal{C} \to U\mathcal{D}$ as having the same map on objects, and given a k–morphism $\phi \colon a_1 \oplus \cdots \oplus a_k \to a'$ in $U\mathcal{C}$, we define $Uf(\phi)$ to be the composite

$$f(a_1) \oplus \cdots \oplus f(a_k) \xrightarrow{\lambda} f(a_1 \oplus \cdots \oplus a_k) \xrightarrow{f(\phi)} fa'.$$

More generally, given an n–morphism of permutative categories $f \colon (\mathcal{C}_1, \ldots, \mathcal{C}_n) \to \mathcal{D}$, we need to specify a based n–linear map $Uf \colon (U\mathcal{C}_1, \ldots, U\mathcal{C}_n) \to U\mathcal{D}$. From the map being lax$_*$ in each variable separately we get a map on underlying multicategories that is multifunctorial in each variable separately. From the map being identically 0 whenever any input is 0 we get the basepoint condition. The only issue remaining is whether the pentagonal diagram relating lax morphism structure maps generates the diagram relating the variables in a bilinear map. Since all the conditions refer to only two variables at a time, we can reduce to the case of only two variables, and the

diagram for a bilinear map reduces in this case to the following one:

$$
\begin{array}{ccc}
\displaystyle\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n} f(a_i,b_j) & \xrightarrow{\ \lambda_2\ } & \displaystyle\bigoplus_{i=1}^{n} f(a_i,\bigoplus_{j=1}^{n} b_j) \\
\Big\downarrow{\scriptstyle\cong} & & \\
\displaystyle\bigoplus_{j=1}^{n}\bigoplus_{i=1}^{m} f(a_i,b_j) & & \Big\downarrow{\lambda_1} \\
\Big\downarrow{\lambda_1} & & \\
\displaystyle\bigoplus_{j=1}^{n} f(\bigoplus_{i=1}^{m} a_i,b_j) & \xrightarrow{\ \lambda_2\ } & f(\bigoplus_{i=1}^{m} a_i,\bigoplus_{j=1}^{n} b_j).
\end{array}
$$

The diagram relating the lax structure maps in a 2–morphism of permutative categories is precisely the case $m = n = 2$ of this diagram. Note that reversing the roles of m and n does not affect the diagram. We now proceed by induction, first holding $n = 2$ and inducting on m, then reversing the roles of m and n to conclude that the diagram commutes for $m = 2$ and arbitrary n, and finally inducting on m again for a fixed, but arbitrary n. This is all accomplished by Figure 1, which displays the desired diagram for index limits $m + 1$ and n and has subdiagrams for index limits (m, n) and $(2, n)$.

We have finished describing the multifunctor structure of $U: \mathbf{P} \to \mathbf{Mult}_*$, and we leave to the reader the task of checking the necessary preservation properties. It remains to show that U is full and faithful. To see that it is faithful, we observe that the objects of $U\mathcal{C}$ are the same as those of $\mathcal{C}$, and the morphisms of $\mathcal{C}$ are simply the 1–morphisms of $U\mathcal{C}$. We can therefore recover the underlying category of $\mathcal{C}$ from $U\mathcal{C}$. If we have a lax$_*$ functor $f: \mathcal{C} \to \mathcal{D}$, we can recover the functor part of f from Uf by considering only its effect on 1–morphisms, and we recover the lax structure map $\lambda: fa \oplus fb \to f(a \oplus b)$ by looking at the image under Uf of the 2–morphism $\mathrm{id}_{a \oplus b} \in U\mathcal{C}(a, b; a \oplus b)$. Now we can apply the same argument in each variable separately for an n–morphism of $\mathbf{P}$, say $f: (\mathcal{C}_1, \ldots, \mathcal{C}_n) \to \mathcal{D}$, to recover f from Uf. Consequently, U is faithful.

To see that U is full, suppose first that $g: U\mathcal{C} \to U\mathcal{D}$ is a based multifunctor, that is, a 1–morphism in $\mathbf{Mult}_*$. Then certainly g determines a functor $f: \mathcal{C} \to \mathcal{D}$ by restriction to 1–morphisms, and $f(0) = 0$ because g is based. We define a lax structure map for f, as above, by looking at the image in $U\mathcal{D}$ of the canonical 2–morphism $\mathrm{id}_{a \oplus b} \in U\mathcal{C}(a, b; a \oplus b)$. It is now an interesting exercise to show that these data determine a lax$_*$ functor $f: \mathcal{C} \to \mathcal{D}$ for which $g = Uf$.

$$\bigoplus_{i=1}^{m+1} \bigoplus_{j=1}^{n} f(a_i, b_j) \xrightarrow{\qquad \lambda_2 \qquad} \bigoplus_{i=1}^{m+1} f(a_i, \bigoplus_{j=1}^{n} b_j)$$

$$= \Bigg| \qquad \bigoplus_{i=1}^{m} f(a_i, \bigoplus_{j=1}^{n} b_j) \oplus \bigoplus_{j=1}^{n} f(a_{m+1}, b_j) \qquad \Bigg| =$$

$$\overset{\lambda_2 \oplus 1}{\nearrow} \qquad \overset{1 \oplus \lambda_2}{\searrow}$$

$$\bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} f(a_i, b_j) \oplus \bigoplus_{j=1}^{n} f(a_{m+1}, b_j) \qquad \qquad \bigoplus_{i=1}^{m} f(a_i, \bigoplus_{j=1}^{n} b_j) \oplus f(a_{m+1}, \bigoplus_{j=1}^{n} b_j)$$

$$\overset{\lambda_1 \oplus 1}{\downarrow}$$

$$\cong \Bigg| \qquad f(\bigoplus_{i=1}^{m} a_i, \bigoplus_{j=1}^{n} b_j) \oplus \bigoplus_{j=1}^{n} f(a_{m+1}, b_j) \qquad \Bigg| \lambda_1 \oplus 1$$

$$\overset{1 \oplus \lambda_2}{\searrow}$$

$$\overset{\lambda_2 \oplus 1}{\nearrow}$$

$$\bigoplus_{j=1}^{n} \bigoplus_{i=1}^{m} f(a_i, b_j) \oplus \bigoplus_{j=1}^{n} f(a_{m+1}, b_j) \qquad f(\bigoplus_{i=1}^{m} a_i, \bigoplus_{j=1}^{n} b_j) \oplus f(a_{m+1}, \bigoplus_{j=1}^{n} b_j)$$

$$\overset{\lambda_1 \oplus 1}{\searrow} \qquad \overset{\lambda_2 \oplus \lambda_2}{\nearrow}$$

$$\cong \Bigg| \qquad \bigoplus_{j=1}^{n} f(\bigoplus_{i=1}^{m} a_i, b_j) \oplus \bigoplus_{j=1}^{n} f(a_{m+1}, b_j)$$

$$\bigoplus_{j=1}^{n} \left(\bigoplus_{i=1}^{m} f(a_i, b_j) \oplus f(a_{m+1}, b_j) \right) \qquad \cong \qquad \lambda_1$$

$$\overset{\lambda_1}{\searrow}$$

$$= \Bigg| \qquad \bigoplus_{j=1}^{n} \left(f(\bigoplus_{i=1}^{m} a_i, b_j) \oplus f(a_{m+1}, b_j) \right)$$

$$\bigoplus_{j=1}^{n} \bigoplus_{i=1}^{m+1} f(a_i, b_j) \qquad \lambda_1 \qquad f(\bigoplus_{i=1}^{m+1} a_i, \bigoplus_{j=1}^{n} b_j)$$

$$\overset{\lambda_1}{\searrow} \qquad \overset{\lambda_2}{\nearrow}$$

$$\bigoplus_{j=1}^{n} f(\bigoplus_{i=1}^{m+1} a_i, b_j)$$

Figure 1

In the general case where $g\colon (U\mathcal{C}_1, \ldots, U\mathcal{C}_k) \to U\mathcal{D}$ is a k–morphism in $\mathbf{Mult}_*$, we again recover a functor $f\colon \mathcal{C}_1 \times \cdots \times \mathcal{C}_k \to \mathcal{D}$ by restriction to 1–morphisms, and the structure maps for a k–linear map of permutative categories are again determined by the images of the identity maps on $c_i \oplus c_i'$ considered as a 2–morphism in $U\mathcal{C}_i(c_i, c_i'; c_i \oplus c_i')$. The basepoint conditions on f follow from those on g, and the coherence pentagon relating structure maps is the case $m = n = 2$ of the diagram in Definition 2.3. Seeing that f is lax_* in each variable is the same interesting exercise as before, so we obtain a k–linear map f in $\mathbf{P}$ such that $g = Uf$. Therefore U is full. $\qquad\square$

At this point, the question of whether the underlying based multicategory functor is an embedding of multicategories is a purely set-theoretic one. Our treatment above does not treat the precise point sets involved in enough detail to answer it: we have really only constructed the functor up to natural isomorphism. With a more precise construction, the functor is an embedding. (We thank the referee for pointing this out.) Specifically, the permutative product on the objects can be recovered from the identification of $U\mathcal{C}_2(a, b; c)$ as a subset of the morphisms in $\mathcal{C}$: the source is $a \oplus b$. This then can be extended to recover the product on the maps using the element of $U\mathcal{C}_2(a, b; a \oplus b)$ identified with the identity in $\mathcal{C}(a \oplus b, a \oplus b)$. On the other hand, as the following example indicates, such an observation can be somewhat misleading.

Example 3.5 Let A be a based set with six elements, including the basepoint. Then we can give A two nonisomorphic group structures, say G_1 and G_2, both having A as their underlying based set, but with $G_1 \cong \Sigma_3$ and $G_2 \cong \mathbb{Z}/6$. Let EA be the indiscrete category with objects the elements of A, so all morphism sets have exactly one element. Then the group structures G_1 and G_2 give EA two distinct structures as a permutative category, since the groups of objects are not isomorphic. However, the underlying based multicategories *are* isomorphic: all k–morphism sets again have exactly one element for all k. (This of course generalizes to any based set A with nonisomorphic monoid structures.) Consequently, the strict isomorphism class of a permutative category cannot be recovered from the isomorphism class of its underlying based multicategory.

4 Colimits and tensor products

In this section we show that $\mathbf{Mult}_*$ has all the good formal properties required for homotopy theory. To be specific, we will show that it is complete, cocomplete and supports a symmetric monoidal closed structure whose underlying multicategory is the

one specified in Section 2. Our strategy is to prove these things first in the unbased context of **Mult** (without the star), and then bootstrap from there to **Mult**$_*$. This material appears to be well-known to the experts. In particular, the symmetric monoidal structure is a special case of the tensor product of theories of Boardman and Vogt [2, Chapter 2]; the closed structure is summarized in Moerdijk and Weiss [13, Section 5.1]. Since we have not found a full treatment of these properties in their entirety, we produce one here.

Completeness in **Mult** (and **Mult**$_*$) is easy: as in **Cat**, limits are computed on objects and morphism sets within the category of sets.

Since the construction of the monoidal product in **Mult** (to which we will refer as the *tensor product*) is as a colimit, our first major goal is the cocompleteness of **Mult**. We start by establishing the analogous property for the category **Strict** of permutative categories and strict maps. We will then identify **Mult** as a category of coalgebras over a comonad in **Strict**, showing it to be cocomplete.

Lemma 4.1 *The category **Strict** of permutative categories and strict maps is cocomplete.*

Proof The forgetful functor from **Strict** to **Cat** is the right adjoint in a monadic adjunction; the monad is given explicitly by

$$\mathbb{P}\mathcal{C} := \coprod_{k \geq 0} E\Sigma_k \times_{\Sigma_k} \mathcal{C}^k,$$

where $E\Sigma_k$ is the translation category on the symmetric group Σ_k. (This was apparently first pointed out by Dunn in [5].) A slight variation on the argument given in the proof of [6, II.7.2] shows that this monad preserves reflexive coequalizers, and therefore by [6, II.7.4], **Strict** is cocomplete. $\qquad\square$

It is interesting to note that the forgetful functor from **Strict** to monoids that forgets the morphisms and remembers only the objects and their monoidal structure has a right adjoint, namely the functor that takes a monoid M to the indiscrete category EM that has objects M, single element morphism sets, and permutative product the monoidal product. The forgetful functor therefore preserves coproducts: The monoid of objects of a coproduct of permutative categories is the coproduct of the monoids of objects of the individual categories (in the category of monoids). Note that these are *not* commutative monoids in general, because the commutativity isomorphism has been forgotten.

The underlying multicategory construction gives us a forgetful functor $G\colon$ **Strict** $\to$ **Mult**, and our next goal is to show that it has a left adjoint.

Theorem 4.2 *The forgetful functor G: **Strict** $\to$ **Mult** has a left adjoint.*

Proof We construct the left adjoint F as follows. Let **M** be a multicategory. Then $F\mathbf{M}$ has as its objects the free monoid on the objects of **M**,

$$\coprod_{k \geq 0} \mathrm{Ob}\,(\mathbf{M})^k.$$

Given objects $\underline{a} = (a_1, \ldots, a_m)$ and $\underline{b} = (b_1, \ldots, b_n)$, a morphism in $F\mathbf{M}$ from $\underline{a}$ to $\underline{b}$ consists of a function $\phi \colon \{1, \ldots, m\} \to \{1, \ldots, n\}$, together with an n–tuple of morphisms $(f_1, \ldots, f_n)$, where f_j is a morphism of **M** from $\langle a_i \rangle_{\phi(i)=j}$ to b_j. Composition of morphisms is given by composition of the set functions on the indices, together with induced maps using the composition on the multicategory **M** and permutations necessary to preserve coherence. The permutative structure is given by concatenation of lists. The reader can now safely verify that this does give a left adjoint.   □

This construction is similar to the "categories of operators" used by May and Thomason in [12], but differs in that it involves the unbased sets $\{1, \ldots, n\}$ rather than the based sets $\{0, \ldots, n\}$. The basepoint in a category of operators encodes projection operators $a \oplus b \to a$ or $a \oplus b \to b$, which do not exist in a general permutative category.

Our major use of this adjunction is the following:

Theorem 4.3 *The adjunction*

$$F \colon \mathbf{Mult} \rightleftarrows \mathbf{Strict} \colon G$$

*is comonadic, ie, the canonical comparison functor from **Mult** to the category of coalgebras over the comonad FG on **Strict** is an equivalence of categories.*

Proof We use the dual form of Beck's Theorem; see Barr and Wells [1, Theorem 3.14]. We must show that F has a right adjoint, reflects isomorphisms, that **Mult** has equalizers of reflexive F–split equalizer pairs, and that F preserves them. We already have the right adjoint, namely the forgetful functor G, and we already know that **Mult** is complete, so it has all the equalizers required. We will show that F reflects isomorphisms, and that it preserves all equalizers, not just the ones required for the hypotheses of Beck's Theorem.

To see that F reflects isomorphisms, we note that for any map $\alpha \colon M \to N$ of multicategories, the diagram

commutes, where $*$ is the terminal multicategory having one object and one k–morphism for every k, and the unlabelled arrows are induced by F from the maps to this terminal object. The permutative category $F(*)$ has objects $[k] = \{1, \dots, k\}$ for $k \geq 0$, ordinary functions as morphisms, and sum operation $[k_1] \oplus [k_2] := [k_1 + k_2]$, using the obvious extension to sums of functions. Further, the image of the unit $\eta \colon M \to GFM$ is precisely the preimage of the full submulticategory of $GF(*)$ generated by the single object $\{1\}$. Now suppose $F\alpha$ is an isomorphism; we wish to show that α must itself be an isomorphism. Then $GF\alpha$ is an isomorphism, so we get the commutative diagram

$$
\begin{array}{ccc}
M & \xrightarrow{\ \alpha\ } & N \\
\downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\eta} \\
GFM & \xrightarrow[\cong]{\ GF\alpha\ } & GFN \\
& \searrow \qquad \swarrow & \\
& GF(*). &
\end{array}
$$

But since M and N are precisely the preimages of the full submulticategory generated by $\{1\}$, and $GF\alpha$ is an isomorphism, it follows that α must be an isomorphism as well.

To see that F preserves equalizers, we observe that equalizers in **Strict** are created in **Cat**, since **Strict** is monadic over **Cat**, and further, that equalizers in **Cat** are simply computed in **Set** on objects and morphisms separately. It is now straightforward to use the definition of F to see that equalizers are preserved. (Note, however, that F does not preserve products.) $\qquad\square$

We get as an immediate corollary:

Corollary 4.4 *The category* **Mult** *is cocomplete.*

Proof It is equivalent to the category of coalgebras over a comonad on the cocomplete category **Strict**. See Mac Lane [10, VI.2, exercise 2] for the dual statement. $\qquad\square$

We turn next to the construction of the tensor product in **Mult**. The composite of a bilinear map $(M, N) \to P$ with an ordinary map of multicategories $P \to Q$ is again a bilinear map, and the tensor product of multicategories that we will construct is a universal bilinear target relative to ordinary maps. This tensor product is equivalent to the tensor product of theories constructed in [2], but we give an explicit construction in terms of multicategories, rather than theories.

Theorem 4.5 *For any two multicategories M and N, there is a tensor product multi-category $M \otimes N$ and a universal bilinear map $(M, N) \to M \otimes N$. This tensor product makes* **Mult** *into a symmetric monoidal category.*

In order to prove Theorem 4.5, we must first discuss some other categories and adjunctions related to **Mult**, using a modified version of Leinster's discussion of multicategories as generalized monoids in [9]. Let $\mathbb{M}$ be the free monoid monad in **Set**,

$$\mathbb{M}A := \coprod_{k \geq 0} A^k.$$

Definition 4.6 An $\mathbb{M}$–*graph* X consists of two sets, X_0 (the *objects*) and X_1 (the *arrows*), together with two functions, the *source* $s: X_1 \to \mathbb{M}X_0$ and the *target* $t: X_1 \to X_0$. We usually display an $\mathbb{M}$–graph as a span

$$\mathbb{M}X_0 \xleftarrow{\;s\;} X_1 \xrightarrow{\;t\;} X_0.$$

A *map* of $\mathbb{M}$–graphs $f: X \to Y$ consists of functions $f_0: X_0 \to Y_0$ and $f_1: X_1 \to Y_1$ for which the obvious diagram

$$
\begin{array}{ccccc}
\mathbb{M}X_0 & \xleftarrow{\;s\;} & X_1 & \xrightarrow{\;t\;} & X_0 \\
{\scriptstyle \mathbb{M}f_0}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_0}\downarrow \\
\mathbb{M}Y_0 & \xleftarrow{\;s\;} & Y_1 & \xrightarrow{\;t\;} & Y_0
\end{array}
$$

commutes.

We get a category **Mgrph** of $\mathbb{M}$–graphs, and there is a forgetful functor $U: \textbf{Mult} \to \textbf{Mgrph}$ that remembers the objects, morphisms, sources, and targets, but forgets about the identities, permutations, and composition. We use the following theorem in our construction of the tensor product.

Theorem 4.7 *The forgetful functor*

$$U: \textbf{Mult} \to \textbf{Mgrph}$$

has a left adjoint

$$L: \textbf{Mgrph} \to \textbf{Mult}.$$

Proof We proceed in two steps, using as an intermediate stop the category **Nmult** of nonsymmetric multicategories, applying some observations of [9] (where nonsymmetric multicategories are called simply multicategories). Clearly the forgetful functor U factors through **Nmult**, and we claim that both terms in the composite have left adjoints. The desired left adjoint is then the composite of these two left adjoints.

First, consider the forgetful functor U': **Nmult** $\to$ **Mgrph**. As observed in [9, 2.3], U' has a left adjoint L', constructed as follows. Given an $\mathbb{M}$–graph X, the free nonsymmetric multicategory $L'X$ is a multicategory where the k–morphisms are the trees with k leaves generated by the arrows in X, with all nodes (including the root and the leaves) labelled by objects of X. In detail, the objects of $L'X$ are X_0, and the morphisms of $L'X$, called trees, are generated recursively by the following requirements:

(1) For each object $a \in X_0$, there is an identity tree 1_a.

(2) Each element $f \in X_1$ is a tree with source sf and target tf.

(3) Given an $f \in X_1$ with target c and source $(b_1, \ldots, b_k)$, and trees $R_1, \ldots, R_k$, not all identity trees, with the target of R_j being b_j and the source being $(a_{j1}, \ldots, a_{jn_j})$, there is a tree $(f; R_1, \ldots, R_k)$ with source $(a_{11}, \ldots, a_{kn_k})$ and target c.

We must define a composition on this collection of trees, and we do so by induction on the height of a tree, where we define the height of an identity tree to be 0, the height of an element of X_1 to be 1, and the height of a tree $(f; R_1, \ldots, R_k)$ to be $1 + \max_j\{\text{height}(R_j)\}$. Given trees $A; B_1, \ldots, B_n$ which are composable, we define $\Gamma(A; B_1, \ldots, B_n)$ by induction on the height of A. If A has height 0, then it is 1_a for some object a, $n = 1$, and $B = B_1$ has output a. We define $\Gamma(1_a; B) = B$, as required for a multicategory. Similarly, if $B_1, \ldots, B_n$ are all identity trees, then we require $\Gamma(A, B_1, \ldots, B_n) = A$.

If the height of A is 1, then $A = f$ for some $f \in X_1$, and we define $\Gamma(A; B_1, \ldots, B_n) = (f; B_1, \ldots, B_n)$. For taller trees, A must be of the form $(f; A_1, \ldots, A_k)$, with n partitioning into k segments so the j'th segment of B's feeds into A_j. Then we define

$$\Gamma(A; B_1, \ldots, B_n) = (f; \Gamma(A_1; B_1, \ldots, B_{n_1}), \ldots, \Gamma(A_k; B_{n-n_k+1}, \ldots, B_n)),$$

where the compositions on the right side are already defined, since the heights of the A_j's are all less than the height of A. It is now a routine exercise to show that the requirements for a composition are satisfied, and that this construction provides a left adjoint to the forgetful functor U': **Nmult** $\to$ **Mgrph**.

Next, we consider the forgetful functor $U''\colon \mathbf{Mult} \to \mathbf{Nmult}$, and construct a left adjoint L''. Given a nonsymmetric multicategory P, we construct $L''P$ as follows. First, the objects of $L''P$ are the same as the objects of P. Next, given a source string $(a_1, \dots, a_k)$ and a target b, we define

$$L''P(a_1, \dots, a_k; b) = \coprod_{\sigma \in \Sigma_k} P(a_{\sigma^{-1}(1)}, \dots, a_{\sigma^{-1}(k)}; b),$$

so a k–morphism in $L''P(a_1, \dots, a_k; b)$ consists of an ordered pair (f, σ) where $\sigma \in \Sigma_k$ and $f \in P(a_{\sigma^{-1}(1)}, \dots, a_{\sigma^{-1}(k)}; b)$. We let Σ_k act on the right via the natural group action on the symmetric group coordinate. The composition, which is forced by the equivariance requirements for a multicategory, is given by

$$\Gamma((f, \sigma); (g_1, \tau_1), \dots, (g_k, \tau_k))$$
$$= (\Gamma(f; g_{\sigma^{-1}(1)}, \dots, g_{\sigma^{-1}(k)}), \sigma\langle n_1, \dots, n_k\rangle \circ (\tau_1 \oplus \cdots \oplus \tau_k)).$$

Again, it is an exercise to show that this construction satisfies the requirements for a multicategory, and gives a left adjoint to the forgetful functor. The composite $L = L'' \circ L'$ therefore gives a left adjoint for $U = U' \circ U''$. $\qquad\square$

We will also need the following proposition in our construction of the tensor product.

Proposition 4.8 *The set-of-objects functor* $\mathrm{Ob}\,(\_)\colon \mathbf{Mult} \to \mathbf{Set}$ *has both a left and a right adjoint.*

Proof The left adjoint assigns to a set A the multicategory FA with the set A as its objects and with only identity morphisms, while the right adjoint RA also has A as its objects, but with exactly one morphism for each possible source and target. The necessary verifications are trivial. $\qquad\square$

Corollary 4.9 *The objects of a limit or colimit of multicategories are computed in* $\mathbf{Set}$.

We are now in a position to construct the tensor product and prove that it is a universal bilinear target.

Construction 4.10 Let M and N be multicategories. Then we can construct the coproducts of multicategories

$$\coprod_{a \in \mathrm{Ob}(M)} N \quad \text{and} \quad \coprod_{b \in \mathrm{Ob}(N)} M.$$

Each of these coproducts has $\mathrm{Ob}\,(M) \times \mathrm{Ob}\,(N)$ as its set of objects. Further, the first of them is universal for any map that sends the objects $\mathrm{Ob}\,(M) \times \mathrm{Ob}\,(N)$ to the objects of a multicategory P and which is a multifunctor in N; similarly, the second is universal for maps that are multifunctors in M. Any bilinear map $(M, N) \to P$ therefore induces multifunctors from each of these coproducts to P that restrict to the same map on the common set of objects, and therefore induces a map from the pushout

$$
\begin{array}{ccc}
F(\mathrm{Ob}\,(M) \times \mathrm{Ob}\,(N)) & \longrightarrow & \displaystyle\coprod_{b \in \mathrm{Ob}(N)} M \\
\downarrow & & \downarrow \\
\displaystyle\coprod_{a \in \mathrm{Ob}(M)} N & \longrightarrow & M \# N,
\end{array}
$$

where the upper left corner is the free multicategory on the set of objects $\mathrm{Ob}\,(M) \times \mathrm{Ob}\,(N)$. This pushout $M \# N$ is universal with respect to maps that are multifunctors in each variable separately, and what remains is to make the bilinearity diagrams of Definition 2.3 commute universally.

Given one morphism from each multicategory, say an m–morphism $\phi \colon (a_1, \ldots, a_m) \to a'$ in M and an n–morphism $\psi \colon (b_1, \ldots, b_n) \to b'$ in N, we define $\mathbb{M}$–graphs $X(\phi, \psi)$ and $Y(\phi, \psi)$ as follows. The objects of both will be

$$(\{a_1, \ldots, a_m\} \times \{b_1, \ldots, b_n\}) \cup \{(a', b')\},$$

and in $X(\phi, \psi)$ there are to be precisely two arrows, both with source $\langle \langle (a_i, b_j) \rangle_{i=1}^m \rangle_{j=1}^n$ and target (a', b'), while in $Y(\phi, \psi)$ there is exactly one arrow with the same source and target as the arrows in $X(\phi, \psi)$. There is an obvious map of $\mathbb{M}$–graphs $X(\phi, \psi) \to Y(\phi, \psi)$ collapsing the two arrows of X to the one arrow of Y. There is also a map of $\mathbb{M}$–graphs from $X(\phi, \psi)$ to $U(M \# N)$ sending each arrow to one way around the diagram in Definition 2.3, but without the f's. We take the adjoints of all of these maps and form the following pushout:

$$
\begin{array}{ccc}
\displaystyle\coprod_{(\phi, \psi)} LX(\phi, \psi) & \longrightarrow & M \# N \\
\downarrow & & \downarrow \\
\displaystyle\coprod_{(\phi, \psi)} LY(\phi, \psi) & \longrightarrow & M \otimes N.
\end{array}
$$

It now follows that we have universally forced the diagrams in the definition of a bilinear map to commute, so $M \otimes N$ is a universal bilinear target. As we can see by checking at each step, the objects of $M \otimes N$ are still $\mathrm{Ob}(M) \times \mathrm{Ob}(N)$.

Our next goal is the proof of the following theorem; cf [2, II.2.18].

Theorem 4.11 *The tensor product of* Construction 4.10 *and the internal Hom object of* Definition 2.2 *make* **Mult** *into a symmetric monoidal closed category.*

It is clear from the construction that the tensor product is symmetric, and it is easy to verify that it is adjoint to the internal Hom; further, the unit is easily seen to be the multicategory with one object and only the identity morphism on that object. This leaves the associativity of the tensor product to verify, and our strategy is to enrich the Hom-tensor adjunction and use the Yoneda Lemma.

Definition 4.12 Let S be a set of morphisms in a multicategory M. The multicategory $\langle S \rangle$ *generated by* S is the smallest submulticategory of M that contains all the morphisms in S. If $\langle S \rangle = M$, we say that S is a *generating set of morphisms* for M, or that M is *generated by* S.

The next proposition is clear from the construction of L in the proof of Theorem 4.7.

Proposition 4.13 *The morphisms of* $\langle S \rangle$ *consist of the identities on the objects appearing as targets or components of sources in* S, *together with those constructed recursively from* S *by means of permutations and compositions.*

Lemma 4.14 *Let A and B be multicategories, let $f_1, \ldots, f_k, g \in$ **Mult**(A, B), and suppose we have a function ξ assigning to each object a of A a k–morphism $\xi_a \colon (f_1 a, \ldots, f_k a) \to ga$ such that the diagram of* Definition 2.2 *commutes for all ϕ in a generating set for A. Then the diagram commutes for all morphisms ϕ of A, so ξ is a k–natural transformation, ie, a k–morphism in* **Hom**(A, B).

Proof We will say that ξ is natural with respect to those morphisms for which the diagram commutes. We show that the diagram commutes for compositions and permutations of elements with respect to which ξ is natural, so by Proposition 4.13, commutes for all morphisms of A.

First, suppose we are given composable elements with respect to which ξ is natural, say $\phi_1, \ldots, \phi_n$ with $\phi_i \colon (a_{i1}, \ldots, a_{im_i}) \to b_i$ and $\psi \colon (b_1, \ldots, b_n) \to c$. Then the

following diagram shows that ξ is natural with respect to $\Gamma(\psi; \phi_1, \dots, \phi_n)$:

$$\langle\langle\langle f_j(a_{is})\rangle_{s=1}^{m_i}\rangle_{i=1}^n\rangle_{j=1}^k \xrightarrow{\langle\langle f_j \phi_i\rangle_{i=1}^n\rangle_{j=1}^k} \langle\langle f_j b_i\rangle_{i=1}^n\rangle_{j=1}^k \xrightarrow{\langle f_j \psi\rangle_{j=1}^k} \langle f_j c\rangle_{j=1}^k$$

with vertical isomorphisms $\cong$ down to

$$\langle\langle\langle f_j(a_{is})\rangle_{s=1}^{m_i}\rangle_{j=1}^k\rangle_{i=1}^n \xrightarrow{\langle\langle f_j \phi_i\rangle_{j=1}^k\rangle_{i=1}^n} \langle\langle f_j b_i\rangle_{j=1}^k\rangle_{i=1}^n$$

then $\cong$ down to

$$\langle\langle\langle f_j(a_{is})\rangle_{j=1}^k\rangle_{s=1}^{m_i}\rangle_{i=1}^n$$

with map $\langle\langle \xi_{a_{is}}\rangle_{s=1}^{m_i}\rangle_{i=1}^n$ down to

$$\langle\langle g(a_{is})\rangle_{s=1}^{m_i}\rangle_{i=1}^n \xrightarrow{\langle g\phi_i\rangle_{i=1}^n} \langle gb_i\rangle_{i=1}^n \xrightarrow{g\psi} gc.$$

with middle map $\langle \xi_{b_i}\rangle_{i=1}^n$ and right map ξ_c.

Now suppose given also $\sigma \in \Sigma_n$. Then the following diagram shows that ξ is natural with respect to $\psi \cdot \sigma$:

$$\langle\langle f_j b_{\sigma(i)}\rangle_{i=1}^n\rangle_{j=1}^k \xrightarrow{\cong} \langle\langle f_j b_{\sigma(i)}\rangle_{j=1}^k\rangle_{i=1}^n \xrightarrow{\langle \xi_{b_{\sigma(i)}}\rangle_{i=1}^n} \langle gb_{\sigma(i)}\rangle_{i=1}^n$$

with vertical maps $\langle\sigma\rangle$ down to

$$\langle\langle f_j b_i\rangle_{i=1}^n\rangle_{j=1}^k \xrightarrow{\cong} \langle\langle f_j b_i\rangle_{j=1}^k\rangle_{i=1}^n \xrightarrow{\langle \xi_{b_i}\rangle_{i=1}^n} \langle gb_i\rangle_{i=1}^n$$

with left map $\langle f_j \psi\rangle_{j=1}^k$ down to $\langle f_j c\rangle_{j=1}^k$, right map $g\psi$, bottom map ξ_c to gc.

$$\langle f_j c\rangle_{j=1}^k \xrightarrow{\xi_c} gc.$$

Since we were given that ξ was natural with respect to morphisms in a generating set for A, it now follows that it is natural with respect to all morphisms in A, and therefore ξ is a k–natural transformation. $\square$

Notation 4.15 Let M and N be multicategories, ϕ a morphism of M and b an object of N. Then we write $\phi \otimes b$ for the morphism of $M \otimes N$ induced from ϕ and b by the universal bilinear map $(M, N) \to M \otimes N$. Similarly, we write $a \otimes \psi$ given an object a of M and a morphism ψ of N.

We obtain the following proposition from the universal property of the tensor product.

Proposition 4.16 *The morphisms of $M \otimes N$ of the form $a \otimes \psi$ and $\phi \otimes b$ generate the entire multicategory $M \otimes N$.*

Combining the previous proposition with Lemma 4.14, we obtain the following proposition.

Proposition 4.17 *The k–morphisms of* $\mathbf{Hom}(M \otimes N, P)$ *are precisely those functions as in Lemma 4.14 that are natural with respect to all morphisms of the form* $a \otimes \psi$ *or* $\phi \otimes b$.

The enriched adjunction we desire is now the following.

Proposition 4.18 *The adjunction*

$$\mathbf{Mult}(M \otimes N, P) \cong \mathbf{Mult}(M, \mathbf{Hom}(N, P))$$

enriches to a natural isomorphism of multicategories

$$\mathbf{Hom}(M \otimes N, P) \cong \mathbf{Hom}(M, \mathbf{Hom}(N, P)).$$

Proof Lemma 4.14 and Proposition 4.17 show that the isomorphism on objects

$$\mathbf{Mult}(M \otimes N, P) \cong \mathbf{Bilin}(M, N; P)$$

also gives an isomorphism of multicategories

$$\mathbf{Hom}(M \otimes N, P) \cong \mathbf{Bilin}(M, N; P)$$

using the morphisms on the right given in Definition 2.4. However, these morphisms are precisely those giving an isomorphism of multicategories

$$\mathbf{Bilin}(M, N; P) \cong \mathbf{Hom}(M, \mathbf{Hom}(N, P)),$$

and composing these isomorphisms gives the desired enriched adjunction. $\square$

The proof that the tensor product is associative now proceeds as follows. We have

$$\begin{aligned}
\mathbf{Mult}((M \otimes N) \otimes P, Q) &\cong \mathbf{Mult}(M \otimes N, \mathbf{Hom}(P, Q)) \\
&\cong \mathbf{Mult}(M, \mathbf{Hom}(N, \mathbf{Hom}(P, Q))) \\
&\cong \mathbf{Mult}(M, \mathbf{Hom}(N \otimes P, Q)) \\
&\cong \mathbf{Mult}(M \otimes (N \otimes P), Q).
\end{aligned}$$

The result now follows from the Yoneda Lemma. The analogous argument with four factors proves that this associativity isomorphism satisfies the pentagon law. The unit diagrams are clear, and this completes the proof that **Mult** is symmetric monoidal, closed, and bicomplete.

Next, we wish to establish the same properties for $\mathbf{Mult}_*$. We exploit the following general construction and lemma about symmetric monoidal closed bicomplete categories.

Construction 4.19 Let $(\mathbf{C}, \otimes, \mathrm{hom})$ be a symmetric monoidal closed bicomplete category with terminal object t, and let $\mathbf{C}_*$ be the category of objects under t in $\mathbf{C}$. For objects a and b in $\mathbf{C}_*$, we define their *smash product* $a \wedge b$ to be the object of $\mathbf{C}_*$ given by the following pushout in $\mathbf{C}$:

$$
\begin{array}{ccc}
(a \otimes t) \amalg (t \otimes b) & \longrightarrow & a \otimes b \\
\downarrow & & \downarrow \\
t & \longrightarrow & a \wedge b.
\end{array}
$$

We also define the *based hom object* for a and b to be the pullback in $\mathbf{C}$ given in the following diagram:

$$
\begin{array}{ccc}
\mathrm{hom}_*(a, b) & \longrightarrow & t \\
\downarrow & & \downarrow \\
\mathrm{hom}(a, b) & \longrightarrow & \mathrm{hom}(t, b).
\end{array}
$$

The arrows in the pullback system are induced by the structure maps for a and b and the isomorphism $t \cong \mathrm{hom}(t, t)$ that comes from the fact that $\mathrm{hom}(t, -)$ preserves products (and t is the empty product). The composite

$$t \cong \mathrm{hom}(a, t) \to \mathrm{hom}(a, b) \to \mathrm{hom}(t, b)$$

coincides with the given arrow from t to $\mathrm{hom}(t, b)$, so induces a structure map for $\mathrm{hom}_*(a, b)$ as an object of $\mathbf{C}_*$.
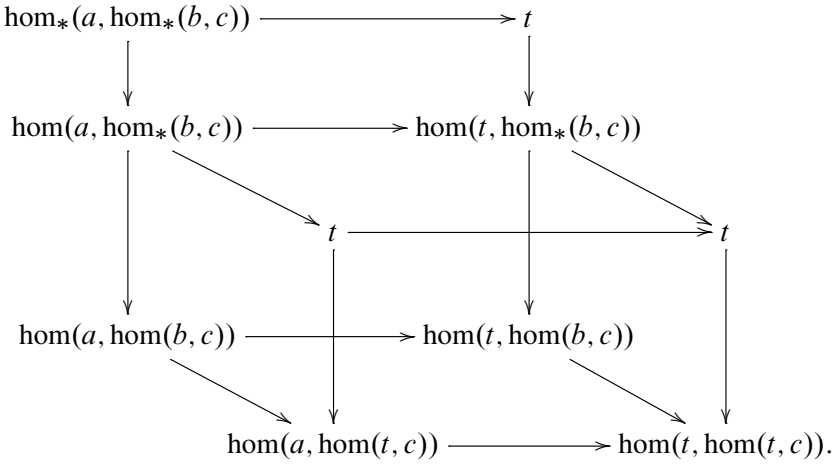
Lemma 4.20 *Construction 4.19 makes $\mathbf{C}_*$ into a symmetric monoidal closed bicomplete category.*

Proof First, $\mathbf{C}_*$ is bicomplete, being a slice category of $\mathbf{C}$. The definition makes it clear that $\wedge$ is symmetric. The rest of the claim, as in the proof of Theorem 4.11, relies on an enriched adjunction,
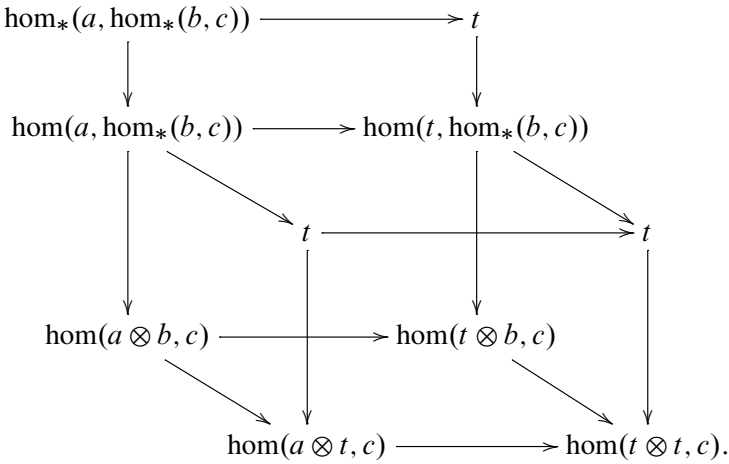
$$\mathrm{hom}_*(a, \mathrm{hom}_*(b, c)) \cong \mathrm{hom}_*(a \wedge b, c),$$

which we establish first. Since $\mathrm{hom}(a, x)$ for a constant object a is a right adjoint, it preserves limits in x, and in particular pullbacks. Consequently, we can display the left side of the adjunction we seek as part of the following diagram, in which there are

three pullbacks: the top rectangle, and the left and right sides of the cubical diagram to which it is connected:

$$
\begin{array}{ccc}
\hom_*(a, \hom_*(b,c)) & \longrightarrow & t \\
\downarrow & & \downarrow \\
\hom(a, \hom_*(b,c)) & \longrightarrow & \hom(t, \hom_*(b,c)) \\
& t \longrightarrow & t \\
\downarrow & & \downarrow \\
\hom(a, \hom(b,c)) & \longrightarrow & \hom(t, \hom(b,c)) \\
& \hom(a, \hom(t,c)) \longrightarrow & \hom(t, \hom(t,c)).
\end{array}
$$

Next, observe that $\hom(a, \hom(b,c)) \cong \hom(a \otimes b, c)$ in $\mathbf{C}$ as a consequence of the associativity of $\otimes$, so we can rewrite the bottom of our diagram to get

$$
\begin{array}{ccc}
\hom_*(a, \hom_*(b,c)) & \longrightarrow & t \\
\downarrow & & \downarrow \\
\hom(a, \hom_*(b,c)) & \longrightarrow & \hom(t, \hom_*(b,c)) \\
& t \longrightarrow & t \\
\downarrow & & \downarrow \\
\hom(a \otimes b, c) & \longrightarrow & \hom(t \otimes b, c) \\
& \hom(a \otimes t, c) \longrightarrow & \hom(t \otimes t, c).
\end{array}
$$

Now observe that on the right side of the diagram, we have the vertical composite

$$t \to \hom(t, \hom_*(b,c)) \to \hom(t \otimes b, c),$$

which coincides with

$$t \cong \hom(t \otimes b, t) \to \hom(t \otimes b, c).$$

Consequently, the diagram actually displays $\hom_*(a, \hom_*(b, c))$ as the limit of the following diagram:

$$\begin{array}{ccc}
\hom(a \otimes b, c) & & \\
& & \hom(t \otimes b, c) \\
t & \longrightarrow & \\
& \hom(a \otimes t, c) \longrightarrow \hom(t \otimes t, c).
\end{array}$$

Since both squares commute, we can remove the $\hom(t \otimes t, c)$ and consequently have $\hom_*(a, \hom_*(b, c))$ as the limit of the smaller diagram

$$\begin{array}{ccc}
\hom(a \otimes b, c) & \longrightarrow & \hom(t \otimes b, c) \\
\downarrow & & \uparrow \\
\hom(a \otimes t, c) & \longleftarrow & t.
\end{array}$$

Since $\otimes$ preserves coproducts, being a left adjoint, and is symmetric, we have

$$\hom(a \otimes t, c) \times \hom(t \otimes b, c) \cong \hom((a \otimes t) \amalg (t \otimes b), c),$$

so the pairs of arrows out of a single source can be combined, and we can display $\hom_*(a, \hom_*(b, c))$ as the pullback in the diagram

$$\begin{array}{ccc}
\hom_*(a, \hom_*(b, c)) & \longrightarrow & \hom(a \otimes b, c) \\
\downarrow & & \downarrow \\
t & \longrightarrow & \hom((a \otimes t) \amalg (t \otimes b), c).
\end{array}$$

Next, $\hom(x, c)$ sends colimits in x to limits, again by adjointness, so the pushout defining $a \wedge b$ gives us a pullback

$$\begin{array}{ccc}
\hom(a \wedge b, c) & \longrightarrow & \hom(a \otimes b, c) \\
\downarrow & & \downarrow \\
\hom(t, c) & \longrightarrow & \hom((a \otimes t) \amalg (t \otimes b), c).
\end{array}$$

This in turn pastes onto the pullback diagram defining $\hom_*$, giving us a composite pullback

$$
\begin{array}{ccccc}
\hom_*(a \wedge b, c) & \longrightarrow & \hom(a \wedge b, c) & \longrightarrow & \hom(a \otimes b, c) \\
\downarrow & & \downarrow & & \downarrow \\
t & \longrightarrow & \hom(t, c) & \longrightarrow & \hom((a \otimes t) \amalg (t \otimes b), c).
\end{array}
$$

By the uniqueness of pullbacks, we get the enriched adjunction we claimed.

Next, the same argument, but with the outer hom's replaced with $\mathbf{C}$'s and the outer $\hom_*$'s replaced with $\mathbf{C}_*$'s shows that $\hom_*$ really is right adjoint to $\wedge$, ie,

$$\mathbf{C}_*(a \wedge b, c) \cong \mathbf{C}_*(a, \hom_*(b, c))$$

natural in a, b, and c; note that by definition,

$$
\begin{array}{ccc}
\mathbf{C}_*(a, b) & \longrightarrow & * \\
\downarrow & & \downarrow \\
\mathbf{C}(a, b) & \longrightarrow & \mathbf{C}(t, b)
\end{array}
$$

is a pullback (of sets.) For associativity of $\wedge$, we can now use the Yoneda Lemma:

$$\mathbf{C}_*((a \wedge b) \wedge c, d) \cong \mathbf{C}_*(a \wedge b, \hom_*(c, d)) \cong \mathbf{C}_*(a, \hom_*(b, \hom_*(c, d)))$$
$$\cong \mathbf{C}_*(a, \hom_*(b \wedge c, d)) \cong \mathbf{C}_*(a \wedge (b \wedge c), d).$$

Consequently, $(a \wedge b) \wedge c \cong a \wedge (b \wedge c)$, naturally in a, b, and c. The unit for $\wedge$ is easily seen to be $e \amalg t$, where e is the unit for $\otimes$. This concludes the proof. $\square$

Corollary 4.21 *The category* $\mathbf{Mult}_*$ *of pointed multicategories is bicomplete and symmetric monoidal closed using this smash product construction.*

We leave to the reader the straightforward task of verifying that the underlying multicategory structure for this symmetric monoidal structure on $\mathbf{Mult}_*$ coincides with the one specified in Section 2.

5   The K–theory of multicategories

This section is devoted to the description of our lax symmetric monoidal K–theory functor from $\mathbf{Mult}_*$ to symmetric spectra, and the following section will show it is consistent with the K–theory of permutative categories described in [7].

As mentioned in the introduction, our construction is the composite of two functors, with the intermediate category being the category of $\mathcal{G}_*$–categories introduced in [7, Section 5] (and given a simplified description here), and with the functor from $\mathcal{G}_*$–**Cat** to symmetric spectra being the one described in [7, Section 7]. We are therefore left with the task of describing a lax symmetric monoidal functor J' from **Mult**$_*$ to $\mathcal{G}_*$–**Cat**, and from the construction, this functor will actually be representable in a sense that we will make clear below.

We begin with the following general categorical proposition, which we will need in two separate places. We would like to thank Mike Shulman for pointing it out to us, as well as for noting that it is a special case of [3, Proposition 4.2.3]. We provide a brief proof (due to Shulman) for the convenience of the reader.

Proposition 5.1 *Let* $(\mathcal{C}, \wedge, 1)$ *be a monoidal category,* A *an object of* $\mathcal{C}$, *and let* $\mu\colon A \wedge A \to A$ *be an isomorphism providing the multiplicative structure map for* A *as a monoid in* $\mathcal{C}$. *Then for any left* A–*module* M, *the structure map* $\xi\colon A \wedge M \to M$ *is an isomorphism, the category of left* A–*modules is a full subcategory of* $\mathcal{C}$, *and similarly for right* A–*modules. Further, given a left* A–*module* M *and a right* A–*module* N, *the two maps*

$$N \wedge A \wedge M \begin{array}{c} \xi_N \wedge 1 \\ \rightrightarrows \\ 1 \wedge \xi_M \end{array} N \wedge M$$

coincide, so the canonical map $N \wedge M \to N \wedge_A M$ *is an isomorphism.*

Proof Let $\eta\colon 1 \to A$ be the unit map for A as a monoid in $\mathcal{C}$. Then the commutativity of

$$A \xrightarrow{\eta \wedge 1} A \wedge A \xleftarrow{1 \wedge \eta} A$$

with maps $=$, μ (iso $\cong$), $=$ going to A

shows that $\eta \wedge 1 = \mu^{-1} = 1 \wedge \eta$. Now let M be a left A–module with structure map $\xi\colon A \wedge M \to M$. Then the commutative naturality diagram

$$
\begin{array}{ccc}
A \wedge M & \xrightarrow{\eta \wedge 1 \wedge 1} & A \wedge A \wedge M \\
\downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle 1 \wedge \xi} \\
M & \xrightarrow{\eta \wedge 1} & A \wedge M
\end{array}
$$

can be rewritten by replacing $\eta \wedge 1$ with $1 \wedge \eta$ in the top arrow, resulting in

$$
\begin{array}{ccc}
A \wedge M & \xrightarrow{1 \wedge \eta \wedge 1} & A \wedge A \wedge M \\
\xi \downarrow & & \downarrow 1 \wedge \xi \\
M & \xrightarrow{\eta \wedge 1} & A \wedge M.
\end{array}
$$

Now since $\xi \circ (\eta \wedge 1) = \mathrm{id}_M$, the clockwise composite is $\mathrm{id}_{A \wedge M}$, which shows that $(\eta \wedge 1) \circ \xi = \mathrm{id}_{A \wedge M}$, so $\eta \wedge 1 = \xi^{-1}$, and ξ is an isomorphism. The analogous argument holds for right A–modules.

Next, the naturality of η shows that for all $f \colon M \to N$ in $\mathcal{C}$, the following diagram commutes:

$$
\begin{array}{ccc}
A \wedge M & \xrightarrow{1 \wedge f} & A \wedge N \\
\eta \wedge 1 \uparrow \cong & & \cong \uparrow \eta \wedge 1 \\
M & \xrightarrow{f} & N.
\end{array}
$$

But since $\eta \wedge 1 = \xi^{-1}$ for left A–modules, it follows that if M and N are left A–modules,

$$
\begin{array}{ccc}
A \wedge M & \xrightarrow{1 \wedge f} & A \wedge N \\
\xi \downarrow & & \downarrow \xi \\
M & \xrightarrow{f} & N
\end{array}
$$

commutes. Consequently left A–modules form a full subcategory of $\mathcal{C}$, and similarly for right A–modules.

Finally, given a left A–module M and a right A–module N,

$$
N \wedge M \xrightarrow{1 \wedge \eta \wedge 1} N \wedge A \wedge M
$$

is inverse to both $1 \wedge \xi_M$ and $\xi_N \wedge 1$, so the two maps coincide. $\qquad \square$

Our first use of Proposition 5.1 will be to formalize some machinery involving the smash product of based categories; this in turn is used to describe the category $\mathcal{G}_*$ and the category of $\mathcal{G}_*$–categories.

From Lemma 4.20, we know the category **Cat**$_*$ of based categories is symmetric monoidal, closed, and bicomplete, where a based category is simply a category with a selected object. In particular, there is *no* property the basepoint object must satisfy. On the other hand, when we require the basepoint object to be null (initial and final), the

morphism sets become based, and we have the following straightforward description of the smash product.

Proposition 5.2 *If $\mathcal{C}$ and $\mathcal{D}$ are based categories with null basepoint objects, then:*

(1) $\mathcal{C} \wedge \mathcal{D}$ *has null basepoint object.*

(2) $\mathrm{Ob}(\mathcal{C} \wedge \mathcal{D}) \cong \mathrm{Ob}\mathcal{C} \wedge \mathrm{Ob}\mathcal{D}$.

(3) *For any objects a_1, a_2 of $\mathcal{C}$, and objects b_1, b_2 of $\mathcal{D}$,*

$$(\mathcal{C} \wedge \mathcal{D})((a_1, b_1), (a_2, b_2)) \cong \mathcal{C}(a_1, a_2) \wedge \mathcal{D}(b_1, b_2).$$

Proof We can construct a category $\mathcal{B}$ with objects $\mathrm{Ob}\mathcal{B} = \mathrm{Ob}\mathcal{C} \wedge \mathrm{Ob}\mathcal{D}$ and morphisms $\mathcal{B}((a_1, b_1), (a_2, b_2)) = \mathcal{C}(a_1, a_2) \wedge \mathcal{D}(b_1, b_2)$, with composition and identities defined by composition and identities in $\mathcal{C}$ and $\mathcal{D}$. We then have a canonical functor $\mathcal{C} \times \mathcal{D} \to \mathcal{B}$ and to see that it satisfies the universal property defining the smash product the only issue is whether, given morphisms $\phi \in \mathcal{C}(a_1, a_2)$ and $\psi \in \mathcal{D}(b_1, b_2)$, we have

$$g(\phi, *) = g(*, *) = g(*, \psi).$$

Consider the diagram

$$
\begin{array}{ccccc}
g(a_1, b_1) & \longrightarrow & g(a_1, *) & \longrightarrow & g(a_1, b_2) \\
\downarrow{\scriptstyle g(\phi,1)} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle g(\phi,1)} \\
g(a_2, b_1) & \longrightarrow & g(a_2, *) & \longrightarrow & g(a_2, b_2).
\end{array}
$$

This shows that $g(\phi, *)$ coincides with the composite

$$g(a_1, b_1) \to g(a_1, *) = g(a_2, *) \to g(a_2, b_2),$$

which is independent of ϕ. Therefore $g(\phi, *) = g(*, *)$. A similar diagram shows that $g(*, \psi) = g(*, *)$. □

The unit for the smash product in $\mathbf{Cat}_*$ is the two object discrete category S^0; it has two objects, namely a basepoint $*$ and a nonbasepoint $()$, and has only identity morphisms. Note that the basepoint object of S^0 is not null. On the other hand, we can construct a unit in the full subcategory of based categories with null basepoint object as follows.

Definition 5.3 Let e be the based category with two objects, $*$ and $()$, with $*$ a null basepoint object, and with the set of self maps of $()$ consisting of the null map and the identity.

The following theorem is essentially a corollary of Proposition 5.1 and Proposition 5.2.

Theorem 5.4 *The based category e satisfies $e \wedge e \cong e$, with the isomorphism making e a commutative monoid in $\mathbf{Cat}_*$. The category of e–modules is precisely the full subcategory of $\mathbf{Cat}_*$ of based categories with a null basepoint object, and the smash product over e is naturally isomorphic to the smash product in $\mathbf{Cat}_*$.*

Although the smash products are the same, the units are different: e is the unit for the category of e–modules, and as mentioned above, S^0 is the unit for $\mathbf{Cat}_*$.

Proof The only claim that does not follow immediately from Propositions 5.1 and 5.2 is the identification of e–modules with based categories having null basepoints. If $\mathcal{C}$ has a null basepoint, then it is easy to produce a unique e–module structure map. Conversely, suppose $\mathcal{C}$ is an e–module. Then $\mathcal{C}$ supports a based functor $\xi\colon e \times \mathcal{C} \to \mathcal{C}$, which we claim is split epi. This follows from the fact that the induced map on smash products $e \wedge \mathcal{C} \to \mathcal{C}$ is unital, using the following diagram:

$$
\begin{array}{ccc}
S^0 \times \mathcal{C} & \xrightarrow{\quad\longleftarrow\cdots\cdots\quad} & S^0 \wedge \mathcal{C} \\
\downarrow & & \downarrow \qquad \searrow{\scriptstyle\cong} \\
e \times \mathcal{C} & \longrightarrow & e \wedge \mathcal{C} \longrightarrow \mathcal{C}.
\end{array}
$$

The vertical arrows are induced by the inclusion $S^0 \to e$, and the top rightward arrow is split by observing $S^0 \wedge \mathcal{C} \cong \mathcal{C} \cong \{()\} \times \mathcal{C}$ and including $\{()\}$ into S^0. (Here, of course, $\{()\}$ is a one point category with object $()$.) Now it follows that the bottom composite, which is ξ, splits, and that the splitting is the composite

$$
\mathcal{C} \cong \{()\} \times \mathcal{C} \to e \times \mathcal{C}.
$$

Now suppose $*$ is the basepoint object in $\mathcal{C}$, and let $\phi\colon * \to a$ be any map from the basepoint. Then we can consider the morphism $\xi(* \to (), \phi)\colon \xi(*, *) \to \xi((), a)$. Since ξ is a bifunctor, we have the commutative square with both composites being $\xi(* \to (), \phi)$:

$$
\begin{array}{ccc}
\xi(*, *) & \xrightarrow{\ \xi(* \to (),*)\ } & \xi((), *) \\
{\scriptstyle \xi(*,\phi)}\downarrow & & \downarrow{\scriptstyle \xi((),\phi)} \\
\xi(*, a) & \xrightarrow{\ \xi(* \to (),a)\ } & \xi((), a).
\end{array}
$$

But since ξ is based, both of the arrows out of the top left entry are the identity on $*$, and $\xi((), \phi) = \phi$ and $\xi((), a) = a$ follow from the splitting. The diagram then tells us that

$$
\phi = \xi((), \phi) = \xi(* \to (), a),
$$

which is independent of ϕ. Therefore the only morphism from $*$ to a is $\xi(* \to (), a)$, so $*$ is initial. Similarly, $*$ is terminal, and therefore null. □

We are now ready to describe our indexing based category $\mathcal{G}_*$; it is constructed as a Grothendieck construction in the category of e–modules. Since this may be somewhat exotic for some readers, we provide the details.

Let $\mathcal{F}$ be the skeleton of the category of finite based sets consisting of the objects $\mathbf{n} = \{0, 1, \ldots, n\}$ with basepoint 0. We observe that $\mathcal{F}$ is a category with a null basepoint, and therefore an e–module. We write $\mathcal{F}^{(r)}$ for the r–th smash power of $\mathcal{F}$ as an e–module, so in particular $\mathcal{F}^{(0)} = e$. (All other smash powers are formed in **Cat**$_*$, by Theorem 5.4.) We write objects of $\mathcal{F}^{(r)}$ as $\langle \mathbf{m} \rangle = (\mathbf{m}_1, \ldots, \mathbf{m}_r)$, with the understanding that $\langle \mathbf{m} \rangle = *$, the basepoint, if any $\mathbf{m}_i = 0$.

Now let **Inj** be the category with objects the unbased sets $\underline{r} = \{1, \ldots, r\}$ for $r = 0, 1, 2, 3, \ldots$, and morphisms the injections. The categories $\mathcal{F}^{(r)}$ are the target objects of a functor

$$\mathcal{F}^{(-)} \colon \mathbf{Inj} \to e\text{–}\mathbf{mod}$$

taking $\underline{r}$ to $\mathcal{F}^{(r)}$ on objects, so in particular $\underline{0}$ gets sent to $\mathcal{F}^{(0)} = e$. On morphisms, $\mathcal{F}^{(-)}$ rearranges the coordinates according to the given injection and, most crucially, inserts the object $\mathbf{1}$ in the slots that are missed: the intuition is that the objects of $\mathcal{F}^{(r)}$ are lists of objects of $\mathcal{F}$ that are waiting to be smashed together, and the injections merely rearrange the lists without affecting the size of the smash product. Formally, if we are given an injection $q \colon \underline{r} \to \underline{s}$, then $\mathcal{F}^{(q)}$ is the functor from $\mathcal{F}^{(r)}$ to $\mathcal{F}^{(s)}$ that takes a nonbasepoint object $\langle \mathbf{m} \rangle = (\mathbf{m}_1, \ldots, \mathbf{m}_r)$ to the s–tuple $q_* \langle \mathbf{m} \rangle = (\mathbf{m}'_1, \ldots, \mathbf{m}'_s)$ in which

$$\mathbf{m}'_j = \begin{cases} \mathbf{m}_i & \text{if } q^{-1}(j) = \{i\}, \\ \mathbf{1} & \text{if } q^{-1}(j) = \varnothing, \end{cases}$$

and takes a morphism $(\alpha_1, \ldots, \alpha_r)$ to the s–tuple $(\alpha'_1, \ldots, \alpha'_s)$ where

$$\alpha'_j = \begin{cases} \alpha_i & \text{if } q^{-1}(j) = \{i\}, \\ \mathrm{id}_{\mathbf{1}} & \text{if } q^{-1}(j) = \varnothing. \end{cases}$$

In particular, the object $()$ of $\mathcal{F}^{(0)}$ gets sent to the constant string $(\mathbf{1}, \ldots, \mathbf{1})$. We define $\mathcal{G}_*$ to be the Grothendieck construction **Inj** $\int \mathcal{F}^{(-)}$ formed in e–modules: this is formally the same as the ordinary Grothendieck construction, except we use the coproduct in e–modules (which is a wedge) instead of the coproduct in **Cat** (which is a disjoint union), and we use the smash product of based sets and categories instead of

the Cartesian product. Specifically, the set of objects of $\mathcal{G}_*$ is

$$\bigvee_{\underline{r} \in \mathrm{Ob}(\mathbf{Inj})} (\mathcal{F}^{(r)})$$

and the maps from $\langle \mathbf{m} \rangle$ to $\langle \mathbf{n} \rangle$ in $\mathcal{G}_*$ form the based set

$$\bigvee_{q:\underline{r} \to \underline{s}} \left(\bigwedge_{j=1}^{s} \mathcal{F}(\mathbf{m}_{q^{-1}(j)}, \mathbf{n}_j) \right),$$

where we agree that if $q^{-1}(j) = \varnothing$, then $\mathbf{m}_{q^{-1}(j)} = \mathbf{1}$. The empty wedge is of course the one point set, and the empty smash is $\mathbf{1}$. Note that the basepoint object $*$ of $\mathcal{G}_*$ is a null object, and the basepoint in each mapping set is the unique map that factors through $*$.

For readers of [7], we note that this is a based version of the category $\mathcal{G}$ introduced there, and that we have the following relation between $\mathcal{G}$ and $\mathcal{G}_*$. First, there is a canonical functor $\mathcal{G} \to \mathcal{G}_*$. More specifically, we can identify the category $\mathcal{G}_*$ as the category obtained from $\mathcal{G}$ by attaching a new null object $*$ and identifying $\langle \mathbf{m} \rangle$ with $*$ whenever any $\mathbf{m}_i = \mathbf{0}$. In particular, every map in $\mathcal{G}_*(\langle \mathbf{m} \rangle, \langle \mathbf{n} \rangle)$ is either the trivial morphism (factoring through $*$) or in the image of $\mathcal{G}(\langle \mathbf{m} \rangle, \langle \mathbf{n} \rangle)$. Whenever none of the entries in $\langle \mathbf{m} \rangle$ or $\langle \mathbf{n} \rangle$ are $\mathbf{0}$, the function $\mathcal{G}(\langle \mathbf{m} \rangle, \langle \mathbf{n} \rangle) \to \mathcal{G}_*(\langle \mathbf{m} \rangle, \langle \mathbf{n} \rangle)$ is in fact one-to-one onto the subset of $\mathcal{G}_*(\langle \mathbf{m} \rangle, \langle \mathbf{n} \rangle)$ that excludes the trivial morphism.

In order to avoid possible confusion as to the meaning of "functor" and "natural transformation" where they occur below, we define $\mathcal{G}_*$–objects in any category $\mathbf{C}$ with a final object, which we always denote as $*$. Let $\mathbf{C}_*$ be the category of objects under $*$, so $\mathbf{C}_*$ has $*$ as a null object. A *based functor* $\mathcal{G}_* \to \mathbf{C}_*$ is a functor that takes the null object $*$ of $\mathcal{G}_*$ to the null object $*$ of $\mathbf{C}_*$. Our intermediate category $\mathcal{G}_*$–**Cat** is a special case of the following definition:

Definition 5.5 The category $\mathcal{G}_*$–$\mathbf{C}$ is the category of based functors $\mathcal{G}_* \to \mathbf{C}_*$.

Concatenation of lists makes $\mathcal{G}_*$ into a permutative category, where concatenation with $*$ on either side yields $*$; we denote this operation by $\odot$. It follows from theorems of Day [4, Theorems 3.3 and 3.6] that when $\mathbf{C}_*$ is a bicomplete closed symmetric monoidal category, the category of based functors from $\mathcal{G}_*$ to $\mathbf{C}_*$ has a closed symmetric monoidal structure, enriched over $\mathbf{C}_*$, in which the product of functors F_1 and F_2 is given by the left Kan extension $F_1 \wedge F_2$ in the diagram on the left below. The universal property of the Kan extension is that maps from $F_1 \wedge F_2$ to G

are in one-to-one correspondence with natural transformations f as in the diagram on the right below:

$$
\begin{array}{ccc}
\mathcal{G}_* \times \mathcal{G}_* & \xrightarrow{F_1 \times F_2} & \mathbf{C}_* \times \mathbf{C}_* \xrightarrow{\wedge} \mathbf{C}_* \\
\odot \downarrow & \nearrow^{F_1 \wedge F_2} & \\
\mathcal{G}_* & &
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{G}_* \times \mathcal{G}_* & \xrightarrow{F_1 \times F_2} & \mathbf{C}_* \times \mathbf{C}_* \\
\odot \downarrow & \Downarrow f & \downarrow \wedge \\
\mathcal{G}_* & \xrightarrow{G} & \mathbf{C}_*.
\end{array}
$$

This then gives us the following theorem.

Theorem 5.6 *Let* **C** *be a bicomplete closed symmetric monoidal category. Then* $\mathcal{G}_*\text{--}\mathbf{C}$ *is a closed symmetric monoidal category enriched over* $\mathbf{C}_*$.

We are now ready to build our representable lax symmetric monoidal functor J' from **Mult**$_*$ to $\mathcal{G}_*$–**Cat**, and we do so by producing its representing object E^*. The correct formal structure E^* must exhibit is firstly that of a based $\mathcal{G}_*^{\mathrm{op}}$–multicategory, that is, a based contravariant functor from $\mathcal{G}_*$ to **Mult**$_*$. Since **Mult**$_*$ is symmetric monoidal closed, and therefore enriched over itself, the lax symmetric monoidal forgetful functor **Mult**$_* \to$ **Cat**$_*$ gives **Mult**$_*$ an enrichment over **Cat**$_*$, and therefore

$$\mathbf{Mult}_*(E^*, M)$$

the structure of a $\mathcal{G}_*$–category for any based multicategory M. Secondly, E^* needs additional structure to ensure that the functor it represents is lax symmetric monoidal; we will address this issue as well.

Our construction of E^* is based on that of a very small based multicategory E with excellent formal properties reminiscent of the based category e.

Definition 5.7 The multicategory E has two objects, 0 and 1, with morphisms given by

$$
E(a_1, \ldots, a_k; a') = \begin{cases} * & \text{if } a_1 + \cdots + a_k = a', \\ \varnothing & \text{otherwise}, \end{cases}
$$

so in particular there are no morphisms when there is more than one input with value 1. The object 0 is the basepoint object, given by the unique multifunctor $* \to E$.

We remark that E is the terminal parameter multicategory for modules [7, Definition 2.4], so a multifunctor from E gives the image of 0 the structure of a commutative monoid, and the image of 1 the structure of a module over this monoid. In the case where the target is a based multicategory, we already have a selected commutative

monoid structure on the basepoint object, and a based multifunctor from E is then the choice of a module structure over this commutative monoid.

The formal properties we need for E are the following.

Theorem 5.8 *The based multicategory E satisfies $E \wedge E \cong E$, with the isomorphism making E a commutative monoid in $\mathbf{Mult}_*$. The category of E–modules is therefore a full subcategory of $\mathbf{Mult}_*$, and has the same smash product as in $\mathbf{Mult}_*$.*

The Theorem is a special case of Proposition 5.1, so we need only show that E satisfies the hypotheses of Proposition 5.1. We begin with the following lemma.

Lemma 5.9 *Let M be a based multicategory, $f: (E, E) \to M$ a based bilinear map. Then for any object a of E and morphism ϕ of E,*

$$f(a, \phi) = f(\phi, a).$$

Proof If $a = 0$ or ϕ is in the image of $* \to E$, the lemma follows from f being based. We consider next the first case in which neither is true. Let $\phi_2 \in E_2(0, 1; 1)$ be the unique element; we wish to show that

$$f(1, \phi_2) = f(\phi_2, 1).$$

The key to the argument is to observe that bilinearity means in particular that the following diagram commutes:

$$
\begin{array}{ccc}
(f(0,0), f(0,1), f(1,0), f(1,1)) & \xrightarrow{(f(0,\phi_2), f(1,\phi_2))} & (f(0,1), f(1,1)) \\
\Big\downarrow{\scriptstyle\cong} & & \Big\downarrow{} \\
(f(0,0), f(1,0), f(0,1), f(1,1)) & & \Big\downarrow{f(\phi_2,1)} \\
\Big\downarrow{(f(\phi_2,0), f(\phi_2,1))} & & \\
(f(1,0), f(1,1)) & \xrightarrow{\quad f(1,\phi_2) \quad} & f(1,1).
\end{array}
$$

We now precompose with the ordered quadruple of morphisms $(\mathrm{id}_{f(0,0)}, \varepsilon_0, \varepsilon_0, \mathrm{id}_{f(1,1)})$, where ε_0 is the image in M of the canonical 0–morphism in $*$. Notice that $f(0,0) = f(0,1) = f(1,0)$ since all must be the basepoint of M, and that the composite

$$f(0,0) \xrightarrow{(\mathrm{id}_{f(0,0)}, \varepsilon_0)} (f(0,0), f(1,0)) \xrightarrow{f(\phi_2,0)} f(1,0),$$

being in the image of $* \to M$, must be the identity 1–morphism. Similarly, the composite

$$f(0,0) \xrightarrow{(\mathrm{id}_{f(0,0)},\varepsilon_0)} (f(0,0), f(0,1)) \xrightarrow{f(0,\phi_2)} f(0,1)$$

is the identity. Furthermore, the composite

$$f(1,1) \xrightarrow{(\varepsilon_0,\mathrm{id}_{f(1,1)})} (f(1,0), f(1,1)) \xrightarrow{f(1,\phi_2)} f(1,1)$$

arises from applying the based multifunctor $f(1,\_)$ to the composite

$$1 \xrightarrow{(\varepsilon_0,\mathrm{id})} (0,1) \xrightarrow{\phi_2} 1,$$

in E, which is id_1, and therefore the previous composite is $\mathrm{id}_{f(1,1)}$. Similarly, the composite

$$f(1,1) \xrightarrow{(\varepsilon_0,\mathrm{id})} (f(0,1), f(1,1)) \xrightarrow{f(\phi_2,1)} f(1,1)$$

is also $\mathrm{id}_{f(1,1)}$. We now have the total diagram

$$
\begin{array}{ccc}
(f(0,0), f(1,1)) & \xrightarrow{\ =\ } & (f(0,1), f(1,1)) \\
\Big\downarrow{\scriptstyle =} & & \Big\downarrow{\scriptstyle f(\phi_2,1)} \\
(f(1,0), f(1,1)) & \xrightarrow{\ f(1,\phi_2)\ } & f(1,1)
\end{array}
$$

which establishes the claim.

We next consider the unique element $\phi_n \in E_n(0^{n-1}, 1; 1)$, and claim that

$$f(1,\phi_n) = f(\phi_n, 1).$$

This follows by induction from the case $n = 2$ by use of the multifunctoriality of $f(1,\_)$ and $f(\_,1)$, together with the formula

$$\phi_n = \Gamma(\phi_2; \varepsilon_{n-1}, \phi_2),$$

where ε_{n-1} is the canonical $(n-1)$–morphism $0^{n-1} \to 0$. The general case now follows, since all morphisms in E are either part of the basepoint structure or else arise from a permutation action on one of the ϕ_n's. $\qquad\square$

Corollary 5.10 *There is a natural isomorphism $E \wedge E \cong E$ which is the product map for a commutative monoid structure on E in $\mathbf{Mult}_*$.*

Proof The isomorphism is induced from the obvious based bilinear map

$$\beta\colon (E, E) \to E$$

which sends $(1, 1)$ to 1 and both $(1, \phi)$ and $(\phi, 1)$ to ϕ for any morphism ϕ. If $f\colon (E, E) \to M$ is any other based bilinear map, then Lemma 5.9 shows that f factors uniquely through β, giving β the universal property of the map to the smash product. The isomorphism now follows from the uniqueness of universal objects. The axioms for the commutative monoid structure are trivial to verify, and follow from the fact that E has no nontrivial automorphisms.   □

The proof of Theorem 5.8 now consists of Proposition 5.1 applied to Corollary 5.10.

Note in particular that the unit for the smash product of based multicategories, which we will call u, is the coproduct $U \coprod *$, where $*$ is the terminal multicategory and U is the unit for the tensor product in **Mult**, which has one object and only its identity morphism. We will think of u as having two objects, 0 and 1, with 0 the basepoint object, and with the only morphism involving 1 being id_1. It is now clear what the unit map $u \to E$ is.

The formal properties of our representing object E^*, which is still to be defined, rely on those of the Cartesian power multicategories E^n, for which we first need some notation. The multicategories E^n are the powers using the Cartesian product of multicategories, which provides the categorical product in both the based and the unbased settings. It is formed using the Cartesian product of sets on both objects and k–morphisms for each k. We will find it convenient to think of an object of E^n, which is merely a string of 0's and 1's of length n, as being given by the subset $T \subset \{1, \ldots, n\}$ of indices at which the string takes on the value 1. With this in mind, it is easy to verify the following proposition.

Proposition 5.11 *Given objects $T_1, \ldots, T_k$ and T' of E^n, the set of k–morphisms $E^n(T_1, \ldots, T_k; T')$ is empty unless the T_i's are mutually disjoint and $T_1 \cup \cdots \cup T_k = T'$, in which case it consists of a single k–morphism.*

Our next step in deriving the formal properties of E^* is the following structure theorem about Cartesian powers of E.

Theorem 5.12 *The Cartesian powers E^m are modules over the commutative monoid E in **Mult**$_*$.*

Proof   We define the module structure map by giving its associated bilinear map; on objects we do the only possible thing: given an object S of E^m, we send $(1, S)$ to S. On morphisms, we send $(1, \phi)$ to ϕ, and given the k–morphism $\phi_k \colon (0^{k-1}, 1) \to 1$ in E, we send (ϕ_k, S) to the single k–morphism $\phi_k^S \colon (\varnothing^{k-1}, S) \to S$ in E^m. All other assignments are now forced by equivariance. It is easy to verify the requirements for a module structure.   □

We are now ready to define our $\mathcal{G}_*^{\mathrm{op}}$–multicategory E^*.

Definition 5.13   Given an object $\langle \mathbf{m} \rangle = (\mathbf{m}_1, \ldots, \mathbf{m}_k)$ of $\mathcal{G}_*$, we define $E^*\langle \mathbf{m} \rangle$ to be $E^{m_1} \wedge \cdots \wedge E^{m_k}$, where E^m is the m'th Cartesian power of E. In particular, the 0–th Cartesian power E^0 is $*$, the null multicategory in $\mathbf{Mult}_*$, which also acts as a 0 object for the smash product in $\mathbf{Mult}_*$. We define $E^*() = E$.

Theorem 5.14 E^* *supports the structure of a $\mathcal{G}_*^{\mathrm{op}}$–multicategory.*

Proof   Given a nonbasepoint morphism $(\alpha, q) \colon (\mathbf{m}_1, \ldots, \mathbf{m}_r) \to (\mathbf{n}_1, \ldots, \mathbf{n}_s)$ in $\mathcal{G}_*$, we must define $E^*(\alpha, q) \colon E^*\langle \mathbf{n} \rangle \to E^*\langle \mathbf{m} \rangle$ compatible with the composition in $\mathcal{G}_*$.

For each j with $1 \le j \le s$, we have a given morphism in $\mathcal{F}$

$$\alpha_j \colon \mathbf{m}_{q^{-1}(j)} \to \mathbf{n}_j,$$

where we have $\mathbf{m}_{q^{-1}(j)} = \mathbf{1}$ if $q^{-1}(j) = \varnothing$. These induce maps of based multicategories

$$\alpha_j^* \colon E^{n_j} \to E^{m_{q^{-1}(j)}}$$

by requiring the maps to fit into commutative diagrams with the product projection maps for $1 \le t \le m_{q^{-1}(j)}$:

$$
\begin{array}{ccc}
E^{n_j} & \xrightarrow{\ \alpha_j^*\ } & E^{m_{q^{-1}(j)}} \\
& \searrow{\scriptstyle \pi_{\alpha_j(t)}} & \downarrow{\scriptstyle \pi_t} \\
& & E,
\end{array}
$$

where $\pi_{\alpha_j(t)} = *$, the null map, if $\alpha_j(t) = 0$.

Now smashing the α_j^*'s together gives us a map

$$\alpha^* \colon E\langle \mathbf{n} \rangle = E^{n_1} \wedge \cdots \wedge E^{n_s} \to E^{m_{q^{-1}(1)}} \wedge \cdots \wedge E^{m_{q^{-1}(s)}} = E(q_* \langle \mathbf{m} \rangle).$$

Further, Theorem 5.12 gives E–module structure maps for the Cartesian powers E^{m_i}, while Proposition 5.1 and Theorem 5.8 show that the order in which the factors

$E = E^{m_{q^{-1}(j)}}$ for $q^{-1}(j) = \varnothing$ are absorbed is immaterial. Consequently, we get a canonical isomorphism

$$q^*\colon E^*(q_*\langle \mathbf{m}\rangle) \xrightarrow{\;\cong\;} E^*\langle \mathbf{m}\rangle,$$

and we define $E^*(\alpha, q) = q^* \circ \alpha^*$. The verification that this definition is compatible with composition in $\mathcal{G}$ is left to the reader. $\qquad\square$

This Theorem now justifies the following definition.

Definition 5.15 Define $J'\colon \mathbf{Mult}_* \to \mathcal{G}_*\text{–}\mathbf{Cat}$ by letting $J'M\langle \mathbf{m}\rangle$ be the underlying based category of $\mathbf{Mult}_*(E^*\langle \mathbf{m}\rangle, M)$.

Finally, we must show that J' is lax symmetric monoidal. The existence of the lax structure map for the product follows from two observations: first, given objects $\langle \mathbf{m}\rangle$ and $\langle \mathbf{n}\rangle$ of $\mathcal{G}_*$, we have

$$E^*\langle \mathbf{m}\rangle \wedge E^*\langle \mathbf{n}\rangle = E^*(\langle \mathbf{m}\rangle \odot \langle \mathbf{n}\rangle),$$

and second, the definition of the smash product of $\mathcal{G}_*$–categories as a Kan extension makes it only necessary to observe that we have a natural map

$$\mathbf{Mult}_*(E^*\langle \mathbf{m}\rangle, M) \times \mathbf{Mult}_*(E^*\langle \mathbf{n}\rangle, N) \to \mathbf{Mult}_*(E^*\langle \mathbf{m}\rangle \wedge E^*\langle \mathbf{n}\rangle, M \wedge N)$$
$$= \mathbf{Mult}_*(E^*(\langle \mathbf{m}\rangle \odot \langle \mathbf{n}\rangle), M \wedge N).$$

The structure map for the unit follows from the observation that $\mathbf{Mult}_*(E, u) = *$, the terminal based multicategory, and it follows that $J'(u) = *$. The lax structure map for the unit is then given by the unique map to the terminal object. The necessary coherence properties for a lax symmetric monoidal functor are now easily verified.

6 Proof of consistency

This section completes the proof of Theorem 1.3 by showing that composing our forgetful multifunctor $U\colon \mathbf{P} \to \mathbf{Mult}_*$ with the represented lax symmetric monoidal functor

$$J'\colon \mathbf{Mult}_* \to \mathcal{G}_*\text{–}\mathbf{Cat}$$

results in the multifunctor J described in [7], up to natural isomorphism. Let $\mathcal{C}$ be a permutative category. We begin by recalling the definition of $J\mathcal{C}$, which assigns to each object $(\mathbf{n}_1, \ldots, \mathbf{n}_k) = \langle \mathbf{n}\rangle$ of $\mathcal{G}_*$ a category $J\mathcal{C}\langle \mathbf{n}\rangle$, which has as its objects systems of objects of $\mathcal{C}$ indexed by k–tuples $\langle S\rangle = (S_1, \ldots, S_k)$ of subsets $S_i \subset \{1, \ldots, n_i\}$. Of

course, J assigns the null based category $*$ to the base object of $\mathcal{G}_*$. (The description of $J\mathcal{C}$ in [7] is in terms of the category $\mathcal{G}$ which we have not defined here, but the descriptions are equivalent.) In order to explain the properties we require for these systems, we need some notation: given a subset $T \subset \{1, \dots, n_i\}$ for some $1 \leq i \leq k$, we write $\langle S \lceil_i T \rangle$ for the n–tuple $(S_1, \dots, S_{i-1}, T, S_{i+1}, \dots, S_k)$ obtained by substituting T in the i–th position. We can now make sense of the following definition.

Definition 6.1 Let $\mathcal{C}$ be a permutative category and $\langle \mathbf{n} \rangle = (\mathbf{n}_1, \dots, \mathbf{n}_k)$ a nonbasepoint object of $\mathcal{G}_*$. The category $J\mathcal{C}\langle \mathbf{n} \rangle$ has objects the systems $\{C_{\langle S \rangle}, \rho_{\langle S \rangle; i, T, U}\}$, where

   (1) $\langle S \rangle = (S_1, \dots, S_k)$ runs through all k–tuples of subsets $S_i \subset \{1, \dots, n_i\}$,

   (2) for $\rho_{\langle S \rangle; i, T, U}$, i runs through $1, \dots, k$, and T, U run through the subsets of S_i with $T \cap U = \varnothing$ and $T \cup U = S_i$,

   (3) the $C_{\langle S \rangle}$ are objects of $\mathcal{C}$, and

   (4) the $\rho_{\langle S \rangle; i, T, U}$ are morphisms $C_{\langle S \lceil_i T \rangle} \oplus C_{\langle S \lceil_i U \rangle} \to C_{\langle S \rangle}$ in $\mathcal{C}$,

such that

   (1) $C_{\langle S \rangle} = 0$ if $S_i = \varnothing$ for any i,

   (2) $\rho_{\langle S \rangle; i, T, U} = \mathrm{id}$ if any of the S_j (for any j), T, or U are empty,

   (3) for all $\rho_{\langle S \rangle; i, T, U}$ the following diagram commutes:

$$
\begin{array}{ccc}
C_{\langle S \lceil_i T \rangle} \oplus C_{\langle S \lceil_i U \rangle} & \xrightarrow{\ \rho_{\langle S \rangle; i, T, U}\ } & C_{\langle S \rangle} \\
{\scriptstyle \gamma} \downarrow & & \Vert \\
C_{\langle S \lceil_i U \rangle} \oplus C_{\langle S \lceil_i T \rangle} & \xrightarrow[\ \rho_{\langle S \rangle; i, U, T}\ ]{} & C_{\langle S \rangle},
\end{array}
$$

   (4) for all $\langle S \rangle$, i, and $T, U, V \subset \{1, \dots, n_i\}$ with $T \cup U \cup V = S_i$ and T, U, and V mutually disjoint, the following diagram commutes:

$$
\begin{array}{ccc}
C_{\langle S \lceil_i T \rangle} \oplus C_{\langle S \lceil_i U \rangle} \oplus C_{\langle S \lceil_i V \rangle} & \xrightarrow{\ \rho_{\langle S \lceil_i (T \cup U) \rangle; i, T, U} \oplus \mathrm{id}\ } & C_{\langle S \lceil_i (T \cup U) \rangle} \oplus C_{\langle S \lceil_i V \rangle} \\
{\scriptstyle \mathrm{id} \oplus \rho_{\langle S \lceil_i (U \cup V) \rangle; i, U, V}} \downarrow & & \downarrow {\scriptstyle \rho_{\langle S \rangle; i, T \cup U, V}} \\
C_{\langle S \lceil_i T \rangle} \oplus C_{\langle S \lceil_i (U \cup V) \rangle} & \xrightarrow[\ \rho_{\langle S \rangle; i, T, U \cup V}\ ]{} & C_{\langle S \rangle},
\end{array}
$$

(5) for all $\rho_{\langle S \rangle;i,T,U}$ and $\rho_{\langle S \rangle;j,V,W}$ with $i \neq j$, the following diagram commutes:

$$
\begin{array}{c}
 C_{\langle S \lceil_j V \rangle} \oplus C_{\langle S \lceil_j W \rangle}
\end{array}
$$

with arrows:

$C_{\langle S \lceil_i T \lceil_j V \rangle} \oplus C_{\langle S \lceil_i U \lceil_j V \rangle} \oplus C_{\langle S \lceil_i T \lceil_j W \rangle} \oplus C_{\langle S \lceil_i U \lceil_j W \rangle}$ $\xrightarrow{(\rho_{\langle S \lceil_j V \rangle;i,T,U}) \oplus (\rho_{\langle S \lceil_j W \rangle;i,T,U})}$ top

$\rho_{\langle S \rangle;j,V,W} \to C_{\langle S \rangle}.$

$\mathrm{id} \oplus \gamma \oplus \mathrm{id}$ (left vertical)

$C_{\langle S \lceil_i T \lceil_j V \rangle} \oplus C_{\langle S \lceil_i T \lceil_j W \rangle} \oplus C_{\langle S \lceil_i U \lceil_j V \rangle} \oplus C_{\langle S \lceil_i U \lceil_j W \rangle}$

$\xrightarrow{(\rho_{\langle S \lceil_i T \rangle;j,V,W}) \oplus (\rho_{\langle S \lceil_i U \rangle;j,V,W})}$ $C_{\langle S \lceil_i T \rangle} \oplus C_{\langle S \lceil_i U \rangle}$

$\rho_{\langle S \rangle;i,T,U}$

A morphism $f \colon \{C_{\langle S \rangle}, \rho_{\langle S \rangle;i,T,U}\} \to \{C'_{\langle S \rangle}, \rho'_{\langle S \rangle;i,T,U}\}$ in $J\mathcal{C}\langle \mathbf{n} \rangle$ consists of morphisms $f_S \colon C_{\langle S \rangle} \to C'_{\langle S \rangle}$ in $\mathcal{C}$ for all $\langle S \rangle$ such that $f_{\langle S \rangle}$ is the identity id_0 when $S_i = \varnothing$ for any i, and the following diagram commutes for all $\rho_{\langle S \rangle;i,T,U}$:

$$
\begin{array}{ccc}
C_{\langle S \lceil_i T \rangle} \oplus C_{\langle S \lceil_i U \rangle} & \xrightarrow{\rho_{\langle S \rangle;i,T,U}} & C_{\langle S \rangle} \\
{\scriptstyle f_{\langle S \lceil_i T \rangle} \oplus f_{\langle S \lceil_i U \rangle}} \downarrow & & \downarrow {\scriptstyle f_{\langle S \rangle}} \\
C'_{\langle S \lceil_i T \rangle} \oplus C'_{\langle S \lceil_i U \rangle} & \xrightarrow[\rho'_{\langle S \rangle;i,T,U}]{} & C'_{\langle S \rangle}.
\end{array}
$$

Note that if any of the $\mathbf{n}_i = \mathbf{0}$ in the definition above, then $\langle \mathbf{n} \rangle = *$, so $J\mathcal{C}\langle \mathbf{n} \rangle$ must be the terminal category with one object and one morphism.

The following theorem is [7, Theorem 6.1].

Theorem 6.2 *The categories $J\mathcal{C}\langle \mathbf{n} \rangle$ support the structure of a $\mathcal{G}_*$–category.*

The $\mathcal{G}_*$–category structure is constructed as follows. First, for a fixed string length k, so $\langle \mathbf{n} \rangle = (\mathbf{n}_1, \ldots, \mathbf{n}_k)$, $J\mathcal{C}\langle \mathbf{n} \rangle$ is functorial in morphisms of $\mathcal{F}^k$, as follows. Given maps $\alpha_i \colon \mathbf{m}_i \to \mathbf{n}_i$ of based sets for $1 \leq i \leq k$, we define

$$J\mathcal{C}\langle \alpha \rangle \colon J\mathcal{C}\langle \mathbf{m} \rangle \to J\mathcal{C}\langle \mathbf{n} \rangle$$

on objects by

$$J\mathcal{C}\langle \alpha \rangle \{C_{\langle S \rangle}, \rho_{\langle S \rangle;i,T,U}\} := \{C^{\alpha}_{\langle S \rangle}, \rho^{\alpha}_{\langle S \rangle;i,T,U}\},$$

where
$$C^{\alpha}_{\langle S \rangle} = C_{(\alpha_1^{-1} S_1, \ldots, \alpha_k^{-1} S_k)}$$
and
$$\rho^{\alpha}_{\langle S \rangle; i, T, U} = \rho_{\langle \alpha^{-1} S \rangle; i, \alpha^{-1} T, \alpha^{-1} U},$$

and similarly on morphisms. Note that since the α_i are based maps, $\alpha_i^{-1} S_i$ is a subset of $\{1, \ldots, m_i\}$ for all i.

Next, a permutation $\sigma \in \Sigma_k$ induces a functor

$$\sigma_!: J\mathcal{C}(\mathbf{n}_1, \ldots, \mathbf{n}_k) \to J\mathcal{C}(\mathbf{n}_{\sigma^{-1}(1)}, \ldots, \mathbf{n}_{\sigma^{-1}(k)}),$$

which is an isomorphism of categories, as follows: The object $\{C_{\langle S \rangle}, \rho_{\langle S \rangle; i, T, U}\}$ is sent to the object $\{C^{\sigma}_{\langle S' \rangle}, \rho^{\sigma}_{\langle S' \rangle; i, T}\}$ where

$$C^{\sigma}_{\langle S' \rangle} = C_{\sigma \langle S' \rangle}, \qquad \rho^{\sigma}_{\langle S' \rangle; i, T, U} = \rho_{\sigma \langle S' \rangle; \sigma(i), T, U}, \qquad \sigma \langle S' \rangle = (S'_{\sigma(1)}, \ldots, S'_{\sigma(k)}),$$

so if $S'_i = S_{\sigma^{-1}(i)} \subset \{1, \ldots, n_{\sigma^{-1}(i)}\}$, then $\sigma \langle S' \rangle = \langle S \rangle$. The morphism $\{f_{\langle S \rangle}\}$ is sent to the morphism $\{f^{\sigma}_{\langle S' \rangle}\}$ where $f^{\sigma}_{\langle S' \rangle} = f_{\sigma \langle S' \rangle}$. It is straightforward to verify that $(\sigma \tau)_! = \sigma_! \tau_!$.

Finally, we have isomorphisms of categories

$$e: J\mathcal{C}(\mathbf{n}_1, \ldots, \mathbf{n}_k) \to J\mathcal{C}(\mathbf{n}_1, \ldots, \mathbf{n}_k, \mathbf{1})$$

defined as follows: the object $\{C_{\langle S \rangle}, \rho_{\langle S \rangle; i, T, U}\}$ is sent to the object $\{C^{e}_{\langle S' \rangle}, \rho^{e}_{\langle S' \rangle; i, T, U}\}$, where

$$C^{e}_{(S_1, \ldots, S_k, \{1\})} = C_{\langle S \rangle}, \quad \rho^{e}_{(S_1, \ldots, S_k, \{1\}); i, T, U} = \rho_{\langle S \rangle; i, T, U} \qquad \text{for } i < k+1,$$
$$C^{e}_{(S_1, \ldots, S_k, \varnothing)} = 0, \quad \rho^{e}_{(S_1, \ldots, S_k, \varnothing); i, T, U} = \mathrm{id}, \qquad \rho^{e}_{(S_1, \ldots, S_k, \{1\}); k+1, T, U} = \mathrm{id}.$$

The morphism $\{f_{\langle S \rangle}\}$ is sent to the morphism $\{f^{e}_{\langle S' \rangle}\}$ where

$$f^{e}_{(S_1, \ldots, S_k, \{1\})} = f_{\langle S \rangle}, \qquad f^{e}_{(S_1, \ldots, S_k, \varnothing)} = \mathrm{id}.$$

This description of the components of the objects and morphisms is complete since the only two subsets of $\{1\}$ are $\{1\}$ and $\varnothing$. The inverse of this isomorphism is induced by dropping the $\{1\}$ from $(k+1)$–tuples of the form $(S_1, \ldots, S_k, \{1\})$. This describes image functors for a generating set of morphisms of $\mathcal{G}_*$, and since $J\mathcal{C}\langle \mathbf{n} \rangle = *$ if any of the $\mathbf{n}_i = \mathbf{0}$, it is now easy to verify that we do in fact get a $\mathcal{G}_*$–category $J\mathcal{C}$.

We now begin the construction of the natural isomorphism $J'U\mathcal{C} \cong J\mathcal{C}$, and we proceed objectwise in $\mathcal{G}_*$, so we need to produce isomorphisms of categories

$$J'U\mathcal{C}\langle \mathbf{m} \rangle \cong J\mathcal{C}\langle \mathbf{m} \rangle$$

for each object $\langle \mathbf{m} \rangle = (\mathbf{m}_1, \ldots, \mathbf{m}_k)$ of $\mathcal{G}_*$. The bulk of the construction is concerned with the bijection on objects. Suppose given an object of $J'U\mathcal{C}\langle \mathbf{m} \rangle$, that is, of

$\mathbf{Mult}_*(E^*\langle\mathbf{m}\rangle, U\mathcal{C})$, say $F\colon E^*\langle\mathbf{m}\rangle \to U\mathcal{C}$, with $\langle\mathbf{m}\rangle \neq *$. We need to produce an object of $J\mathcal{C}\langle\mathbf{m}\rangle$. But the objects of $E^*\langle\mathbf{m}\rangle$ can be considered as k–tuples $(S_1, \ldots, S_k)$ where $S_i \subset \{1, \ldots, m_i\}$, so we get the part of an object of $J\mathcal{C}$ given by a system $C_{\langle S\rangle}$ by defining

$$C_{\langle S\rangle} := F\langle S\rangle.$$

We also need to produce the structure maps in the system, so suppose given subsets T and U of $\{1, \ldots, m_i\}$ with $T \cap U = \varnothing$ and $T \cup U = S_i$. We define the associated structure map $\rho_{\langle S\rangle;i,T,U}$ to be the image under F of the 2–morphism in E^{m_i} given by

$$(T, U) \to T \cup U = S_i,$$

together with the objects in the other slots in $\langle S\rangle$. Now the coherence properties (1) and (2) follow from F being a based multifunctor, (3) follows from the commutative diagram

$$
\begin{array}{ccc}
(T, U) & \longrightarrow & T \cup U \\
\downarrow & & \Big\| = \\
(U, T) & \longrightarrow & T \cup U
\end{array}
$$

in E^{m_i}, (4) follows from the commutative diagram

$$
\begin{array}{ccc}
(T, U, V) & \longrightarrow & (T, U \cup V) \\
\downarrow & & \downarrow \\
(T \cup U, V) & \longrightarrow & T \cup U \cup V
\end{array}
$$

in E^{m_i}, and (5) follows from bilinearity.

The reverse direction is the most significant part of the proof: given an object $(C_{\langle S\rangle}, \rho_{\langle S\rangle;i,T,U})$ of $J\mathcal{C}\langle\mathbf{m}\rangle$, we need to construct a multifunctor $F\colon E^*\langle\mathbf{m}\rangle \to U\mathcal{C}$. The map is clear on objects: $F\langle S\rangle := C_{\langle S\rangle}$. Now suppose given an n–morphism in E^{m_i}, say $(T_1, \ldots, T_n) \to S_i$, so $T_r \cap T_s = \varnothing$ unless $r = s$, and $T_1 \cup \cdots \cup T_n = S_i$. We need to construct the image n–morphism in $U\mathcal{C}$ under our multifunctor F, which will be a morphism in $\mathcal{C}$

$$C_{\langle S\lceil_i T_1\rangle} \oplus \cdots \oplus C_{\langle S\lceil_i T_n\rangle} \to C_{\langle S\rangle}.$$

We define this inductively, requiring the morphism to be id_0 if $n = 0$ and $\mathrm{id}_{C_{\langle S\rangle}}$ if $n = 1$. For larger n's, we define the image n–morphism by induction to be the

composite

$$C_{\langle S\lceil_i T_1\rangle} \oplus \cdots \oplus C_{\langle S\lceil_i T_{n-1}\rangle} \oplus C_{\langle S\lceil_i T_n\rangle} \longrightarrow C_{\langle S\lceil_i (S\setminus T_n)\rangle} \oplus C_{\langle S\lceil_i T_n\rangle}$$

$$\xrightarrow{\rho_{\langle S\rangle ; i, S_i\setminus T_n, T_n}} C_{\langle S\rangle},$$

where the first map is given by induction on the first $n-1$ terms, and the second is the structure map given by the object of $JC\langle\mathbf{m}\rangle$.

We must verify that this definition actually gives a multifunctor $F\colon E^*\langle\mathbf{m}\rangle \to U\mathcal{C}$, so we must show that it respects the composition Γ and the action of Σ_n on the set of n–morphisms. By the definition of $E^*\langle\mathbf{m}\rangle$, this reduces to checking multifunctoriality in each $\mathbf{m}_i$ separately, and then bilinearity in each pair, using the based concepts in both cases. For notational convenience, we assume without loss of generality that the list $\langle\mathbf{m}\rangle = (\mathbf{m}_1, \ldots, \mathbf{m}_k)$ has length 1 for the first part of this check, so $\langle\mathbf{m}\rangle = \mathbf{m}$ is an object of $\mathcal{F}$.

Our first step is the following lemma.

Lemma 6.3 *Let $T_1, \ldots, T_i, U_1, \ldots, U_j$ be a collection of mutually disjoint subsets of $\{1, \ldots, m\}$. Let $T = T_1 \cup \cdots \cup T_i$ and $U = U_1 \cup \cdots \cup U_j$. Then the morphism induced by the $i + j$–morphism*

$$(T_1, \ldots, T_i, U_1, \ldots, U_j) \to T \cup U$$

in E^m factors through maps induced by morphisms in E^m as indicated in the following:

$$C_{T_1} \oplus \cdots \oplus C_{T_i} \oplus C_{U_1} \oplus C_{U_j} \to C_T \oplus C_U \to C_{T\cup U}.$$

Proof We induct on j, and the claim is trivially true if $j = 0$ or $j = 1$. For the general case, we examine the following diagram, in which all arrows are induced from morphisms in E^m:
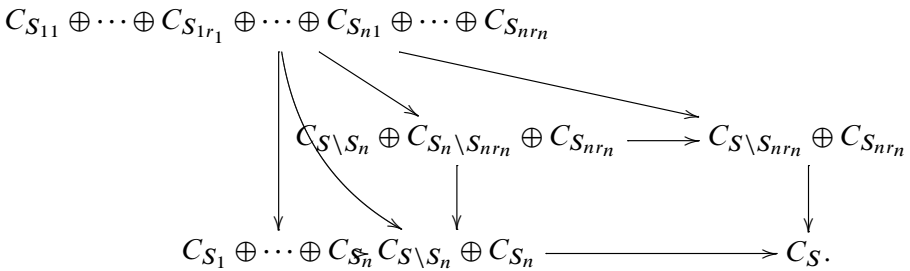

The top triangle commutes by induction, the left triangle commutes by definition of the induced maps, and the square is the associativity condition for the structure maps

of an object of $JC\langle\mathbf{m}\rangle$, property (4). The clockwise composite is the definition of the induced map, and the conclusion follows. $\qquad\square$

We can now show that our construction respects the composition Γ. Suppose we have mutually disjoint objects $S_{11},\ldots,S_{1r_1},\ldots,S_{n1},\ldots,S_{nr_n}$ of E^m, so we have S_{ij}'s for $1 \le i \le n$ and $1 \le j \le r_i$. We write $S_i = S_{i1} \cup \cdots \cup S_{ir_i}$ and $S = S_1 \cup \cdots \cup S_n$. To show our construction preserves composition, we must show that the composite of induced maps
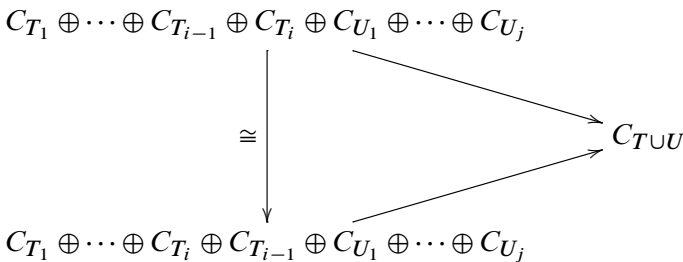
$$C_{S_{11}} \oplus \cdots \oplus C_{S_{1r_1}} \oplus \cdots \oplus C_{S_{n1}} \oplus \cdots \oplus C_{S_{nr_n}} \to C_{S_1} \oplus \cdots \oplus C_{S_n} \to C_S$$

is the induced map. We examine the following diagram, in which all maps are induced from morphisms in E^m, and proceed by induction on n:

$$
\begin{array}{l}
C_{S_{11}} \oplus \cdots \oplus C_{S_{1r_1}} \oplus \cdots \oplus C_{S_{n1}} \oplus \cdots \oplus C_{S_{nr_n}} \\
\hspace{4cm} \searrow \qquad\qquad \searrow \\
\hspace{3cm} C_{S \setminus S_n} \oplus C_{S_n \setminus S_{nr_n}} \oplus C_{S_{nr_n}} \longrightarrow C_{S \setminus S_{nr_n}} \oplus C_{S_{nr_n}} \\
\hspace{2cm} \downarrow \qquad\qquad\qquad\qquad \downarrow \qquad\qquad\qquad\qquad \downarrow \\
C_{S_1} \oplus \cdots \oplus C_{S_n} \qquad C_{S \setminus S_n} \oplus C_{S_n} \longrightarrow C_S.
\end{array}
$$

Reading from left to right, the triangles out of the top left entry commute by induction, by definition, and by Lemma 6.3, while the square is another instance of the associativity property. Now the clockwise composite defines the total induced map, while the counterclockwise composite is the given one. The conclusion follows.

In order to check that our construction respects the permutation actions, it suffices to check preservation of transposition of adjacent letters, since these generate Σ_n. Suppose given mutually disjoint subsets $T_1,\ldots,T_i,U_1,\ldots,U_j$ of $\{1,\ldots,m\}$, and we write as before $T = T_1 \cup \cdots \cup T_i$ and $U = U_1 \cup \cdots \cup U_j$. We need to show that

$$
\begin{array}{c}
C_{T_1} \oplus \cdots \oplus C_{T_{i-1}} \oplus C_{T_i} \oplus C_{U_1} \oplus \cdots \oplus C_{U_j} \\
\cong \downarrow \hspace{4cm} \searrow \\
\hspace{6cm} C_{T \cup U} \\
\hspace{3cm} \nearrow \\
C_{T_1} \oplus \cdots \oplus C_{T_i} \oplus C_{T_{i-1}} \oplus C_{U_1} \oplus \cdots \oplus C_{U_j}
\end{array}
$$

commutes. This follows from the case $j = 0$, however, by the following diagram, in which all maps are induced from morphisms in E^m:

$$C_{T_1} \oplus \cdots \oplus C_{T_{i-1}} \oplus C_{T_i} \oplus C_{U_1} \oplus \cdots \oplus C_{U_j}$$

$$C_T \oplus C_U \longrightarrow C_{T \cup U}.$$

$$C_{T_1} \oplus \cdots \oplus C_{T_i} \oplus C_{T_{i-1}} \oplus C_{U_1} \oplus \cdots \oplus C_{U_j}$$

The left triangle follows from the case $j = 0$, and the other two triangles are instances of Lemma 6.3. The case $j = 0$ follows by examining the diagram

$$C_{T_1} \oplus \cdots \oplus C_{T_{i-1}} \oplus C_{T_i}$$

$$C_{T \setminus (T_{i-1} \cup T_i)} \oplus C_{T_{i-1} \cup T_i} \longrightarrow C_T.$$

$$C_{T_1} \oplus \cdots \oplus C_{T_i} \oplus C_{T_{i-1}}$$

The left triangle follows from property (3), the transposition axiom for the structure maps of objects in $JC\langle \mathbf{m} \rangle$, and the other two triangles are further instances of Lemma 6.3. The construction is therefore multifunctorial.

We now return to the full generality of objects of $E^*\langle \mathbf{m} \rangle$, and must verify that our construction is based bilinear. Bilinearity follows from property (5) of an object of $JC\langle \mathbf{m} \rangle$ by the same argument used in the proof of Theorem 1.1 to prove that bilinearity follows from a pentagon diagram, using Figure 1. Finally, basedness follows from properties (1) and (2) of an object of $JC\langle \mathbf{m} \rangle$, requiring that $C_{\langle S \rangle} = 0$ whenever any $S_i = \varnothing$, and that the structure map be the identity whenever any S_i, T, or U is empty. This completes the verification that our construction produces a multifunctor $E^*\langle \mathbf{m} \rangle \to UC$ from an object of JC.

We must show that these correspondences are inverse to each other. Given $F: E^*\langle \mathbf{m} \rangle \to UC$, we produce the object $(C_{\langle S \rangle}, \rho)$ of $JC\langle \mathbf{m} \rangle$ with the system of objects of C given by

$$C_{\langle S \rangle} := F\langle S \rangle.$$

From this we redefine a multifunctor from $E^*\langle \mathbf{m} \rangle$ to $U\mathcal{C}$, where given an n–morphism $(T_1, \ldots, T_n) \to S_i$ in E^{m_i}, we use the inductive definition given by the composite

$$C_{\langle S \lceil_i T_1 \rangle} \oplus \cdots \oplus C_{\langle S \lceil_i T_{n-1} \rangle} \oplus C_{\langle S \lceil_i T_n \rangle} \longrightarrow C_{\langle S \lceil_i (S \setminus T_n) \rangle} \oplus C_{\langle S \lceil_i T_n \rangle}$$

$$\xrightarrow{\rho_{\langle S \rangle : i, S \setminus T_n, T_n}} C_{\langle S \rangle}$$

to define the image n–morphism in $U\mathcal{C}$. However, this must coincide with the image n–morphism given by our original F by induction and the commutativity in E^{m_i} of the diagram

$$(T_1, \ldots, T_{n-1}, T_n) \longrightarrow (S_i \setminus T_n, T_n)$$
$$\searrow \qquad\qquad \downarrow$$
$$S_i.$$

Conversely, suppose given an object $(C_{\langle S \rangle}, \rho)$ of $J\mathcal{C}$. Then we define the corresponding multifunctor F by setting $F\langle S \rangle := C_{\langle S \rangle}$ and using induction to define the correspondence on n–morphisms. But now taking that multifunctor and recovering the corresponding object of $J\mathcal{C}$ takes us back to the system of objects $C_{\langle S \rangle}$, and the structure maps ρ are all induced by maps given by F from the original object of $J\mathcal{C}$. We therefore recover the original object, and we have shown that our correspondences give inverse bijections between the objects of $J\mathcal{C}$ and of $\mathbf{Mult}_*(E^*, U\mathcal{C})$.

In order to show that we also get inverse bijections on morphisms, and therefore isomorphisms of categories, we just note that the morphisms in both $J\mathcal{C}$ and $\mathbf{Mult}_*(E^*, U\mathcal{C})$ are given by natural transformations, and that our constructions in each direction give inverse correspondences of natural transformations. Preservation of composition and naturality in $\mathcal{G}$ are easy exercises left to the reader. We have finished showing that our construction extends that of [7], and therefore the proof of Theorem 1.3.

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