Equivariant $sl(n)$–link homology

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For every positive integer n we construct a bigraded homology theory for links, such that the corresponding invariant of the unknot is closely related to the $U(n)$–equivariant cohomology ring of $\mathbb{CP}^{n-1}$; our construction specializes to the sl_n–homology of Khovanov and Rozansky. We are motivated by the "universal" rank two Frobenius extension studied by M Khovanov in [10] for sl_2–homology.

1 Introduction

In [7], M Khovanov introduced a bigraded homology theory of links, with Euler characteristic the Jones polynomial, now widely known as "Khovanov homology." In short, the construction begins with the Kauffman solid-state model for the Jones polynomial and associates to it a complex where the 'states' are replaced by tensor powers of a certain Frobenius algebra. In the most common variant, the Frobenius algebra in question is $\mathbb{Z}[x]/(x^2)$, a graded algebra with $\deg(1) = 1$ and $\deg(x) = -1$, that is, of quantum dimension $(q^{-1}+q)$, this being the value of the unreduced Jones polynomial of the unknot. This algebra defines a 2–dimensional TQFT which provides the maps for the complex. (A 2–dimensional TQFT is a tensor functor from oriented $(1+1)$–cobordisms to R–modules, with R a commutative ring, that assigns R to the empty 1–manifold, a ring A to the circle, where A is also a commutative ring with a map $\iota\colon R \longrightarrow A$ that is an inclusion, $A \otimes_R A$ to the disjoint union of two circles, etc.) In [9] M Khovanov extended this to an invariant of tangles by associating to a tangle a complex of bimodules and showing that the isomorphism class of this complex is an invariant in the homotopy category. The operation of "closing off" the tangles gave complexes isomorphic to the original construction for links.

Variants of this homology theory quickly followed. In [13], E S Lee deformed the algebra above to $\mathbb{Z}[x]/(x^2 - 1)$ introducing a different invariant, and constructed a spectral sequence with E^2 term Khovanov homology and E^∞ term the 'deformed' version. Even though this homology theory was no longer bigraded and was essentially trivial, it allowed Lee to prove structural properties of Khovanov homology for alternating links. J Rasmussen used Lee's construction to establish results about the slice genus

of a knot, and give a purely combinatorial proof of the Milnor conjecture [17]. In [1], D Bar-Natan introduced a series of such invariants repackaging the original construction in, what he called, the "world of topological pictures." It became quickly obvious that these theories were not only powerful invariants, but also interesting objects of study in their own right. M Khovanov unified the above constructions in [10], by studying how rank two Frobenius extensions of commutative rings lead to link homology theories. We overview these results below.

1.1 Frobenius extensions

Let $\iota\colon R \longrightarrow A$ be an inclusion of commutative rings. We say that ι is a *Frobenius extension* if there exists an A–bimodule map $\Delta\colon A \longrightarrow A \otimes_R A$ and an R–module map $\varepsilon\colon A \longrightarrow R$ such that Δ is coassociative and cocommutative, and $(\varepsilon \otimes \mathrm{Id})\Delta = \mathrm{Id}$. We refer to Δ and ε as the *comultiplication* and *trace* maps, respectively.

This can be defined in the non-commutative world as well (see Kadison [6]) but we will work with only commutative rings. We denote by $\mathscr{F} = (R, A, \varepsilon, \Delta)$ a Frobenius extension together with a choice of Δ and ε, and call $\mathscr{F}$ a *Frobenius system*. Lets look at some examples from Khovanov [10]; we'll try to be consistent with the notation.

Example 1.1 $\mathscr{F}_1 = (R_1, A_1, \varepsilon_1, \Delta_1)$ where $R_1 = \mathbb{Z}$, $A_1 = \mathbb{Z}[x]/(x^2)$ and

$$\varepsilon_1(1) = 0, \quad \varepsilon_1(x) = 1, \quad \Delta_1(1) = 1 \otimes x + x \otimes 1, \quad \Delta_1(x) = x \otimes x.$$

This is the Frobenius system used in the original construction of Khovanov homology [7].

Example 1.2 The construction due to Khovanov [7] also worked for the following system: $\mathscr{F}_2 = (R_2, A_2, \varepsilon_2, \Delta_2)$ where $R_2 = \mathbb{Z}[c]$, $A_2 = \mathbb{Z}[x, c]/(x^2)$ and

$$\varepsilon_2(1) = -c, \quad \varepsilon_2(x) = 1, \quad \Delta_2(1) = 1 \otimes x + x \otimes 1 + cx \otimes x, \quad \Delta_2(x) = x \otimes x.$$

Here $\deg(x) = 2$ and $\deg(c) = -2$.

Example 1.3 $\mathscr{F}_3 = (R_3, A_3, \varepsilon_3, \Delta_3)$ where $R_3 = \mathbb{Z}[t]$, $A_3 = \mathbb{Z}[x]$, $\iota\colon t \longmapsto x^2$ and

$$\varepsilon_3(1) = 0, \quad \varepsilon_3(x) = 1, \quad \Delta_3(1) = 1 \otimes x + x \otimes 1, \quad \Delta_3(x) = x \otimes x + t1 \otimes 1.$$

Here $\deg(x) = 2$, $\deg(t) = 4$ and the invariant becomes a complex of graded, free $\mathbb{Z}[t]$–modules (up to homotopy). This was Bar-Natan's modification found in [1], with t a formal variable equal to $\frac{1}{8}$ th of his invariant of a closed genus 3 surface. The framework of the Frobenius system $\mathscr{F}_3$ gives a nice interpretation of Rasmussen's results, allowing us to work with graded rather than filtered complexes, see [10] for a more in-depth discussion.

Example 1.4 $\mathcal{F}_5 = (R_5, A_5, \varepsilon_5, \Delta_5)$ where $R_5 = \mathbb{Z}[h, t]$, $A_5 = \mathbb{Z}[h, t][x]/(x^2 - hx - t)$ and

$$\varepsilon_5(1) = 0, \quad \varepsilon_5(x) = 1, \quad \Delta_5(1) = 1 \otimes x + x \otimes 1 - h 1 \otimes 1, \quad \Delta_5(x) = x \otimes x + t 1 \otimes 1.$$

Here $\deg(h) = 2$ and $\deg(t) = 4$.

Proposition 1.5 (M Khovanov [10]) *Any rank two Frobenius system is obtained from $\mathcal{F}_5$ by a composition of base change and twist.*

Given an invertible element $y \in A$ we can "twist" ε and Δ, defining a new comultiplication and counit by $\varepsilon'(x) = \varepsilon(yx)$, $\Delta'(x) = \Delta(y^{-1}x)$ and, hence, arriving at a new Frobenius system. For example: $\mathcal{F}_1$ and $\mathcal{F}_2$ differ by twisting with $y = 1 + x \in A_2$.

We can say $\mathcal{F}_5$ is "universal" in the sense of the proposition, and this system will be of central interest to us being the model case for the construction we embark on. For example, by sending $h \longrightarrow 0$ in $\mathcal{F}_5$ we arrive at the system $\mathcal{F}_3$. Note, if we change to a field of characteristic other than 2, h can be removed by sending $x \longrightarrow x - \frac{h}{2}$ and by modifying $t = -\frac{h^2}{4}$.

1.2 Cohomology and Frobenius extensions

There is an interpretation of rank two Frobenius systems which give rise to link homology theories via equivariant cohomology. Let us recall some definitions.

Given a topological group G that acts continuously on a space X we define the *equivariant cohomology* of X with respect to G to be

$$H_G^*(X, R) = H^*(X \times_G EG, R),$$

where $H^*(-, R)$ denotes singular cohomology with coefficients in a ring R, EG is a contractible space with a free G action such that $EG/G = BG$, the classifying space of G, and $X \times_G EG = X \times EG/(gx, e) \sim (x, eg)$ for all $g \in G$. For example, if $X = \{p\}$, a point, then $H_G^*(X, R) = H^*(BG, R)$. Returning to the Frobenius extensions encountered we have:

Example 1.6 $G = \{e\}$, the trivial group. Then $R_1 = \mathbb{Z} = H_G^*(p, \mathbb{Z})$ and $A_1 = H_G^*(\mathbb{S}^2, \mathbb{Z})$.

Example 1.7 $G = SU(2)$. This group is isomorphic to the group of unit quaternions which, up to sign, can be thought of as rotations in 3–space, that is, there is a surjective

map from $SU(2)$ to $SO(3)$ with kernel $\{I, -I\}$. This gives an action of $SU(2)$ on $\mathbb{S}^2$.

$$R_3 = \mathbb{Z}[t] \cong H^*_{SU(2)}(p, \mathbb{Z}) = H^*(BSU(2), \mathbb{Z}) = H^*(\mathbb{HP}^\infty, \mathbb{Z}),$$
$$A_3 = \mathbb{Z}[x] \cong H^*_{SU(2)}(\mathbb{S}^2, \mathbb{Z}) = H^*(\mathbb{S}^2 \times_{SU(2)} ESU(2), \mathbb{Z}) = H^*(\mathbb{CP}^\infty, \mathbb{Z}),$$
$$x^2 = t.$$

Example 1.8 $G = U(2)$. This group has an action on $\mathbb{S}^2$ with the center $U(1)$ acting trivially.

$$R_5 = \mathbb{Z}[h, t] \cong H^*_{U(2)}(p, \mathbb{Z}) = H^*(BU(2), \mathbb{Z}) = H^*(Gr(2, \infty), \mathbb{Z}),$$
$$A_5 = \mathbb{Z}[h, x] \cong H^*_{U(2)}(\mathbb{S}^2, \mathbb{Z}) = H^*(\mathbb{S}^2 \times_{U(2)} EU(2), \mathbb{Z})$$
$$\cong H^*(BU(1) \times BU(1), \mathbb{Z}).$$

$Gr(2, \infty)$ is the Grassmannian of complex 2–planes in $\mathbb{C}^\infty$; its cohomology ring is freely generated by h and t of degree 2 and 4, and $BU(1) \cong \mathbb{CP}^\infty$. Notice that A_5 is a polynomial ring in two generators x and $h - x$, and R_5 is the ring of symmetric functions in x and $h - x$, with h and $-t$ the elementary symmetric functions.

Other Frobenius systems and their cohomological interpretations were studied by Khovanov [10], but $\mathscr{F}_5$ with its "universality" property will be our starting point and motivation.
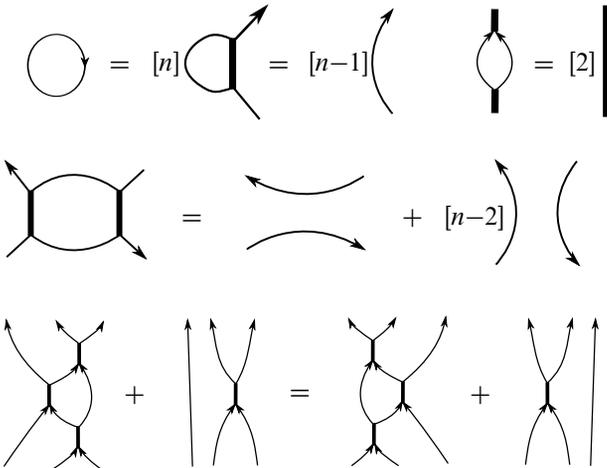

Figure 1: MOY graph skein relation $[i] := \dfrac{q^i - q^{-i}}{q - q^{-1}}$

1.3 sl_n–link homology

Following [7], M Khovanov constructed a link homology theory with Euler characteristic the quantum sl_3–link polynomial $P_3(L)$ (the Jones polynomial is the sl_2–invariant) [8]. In succession, M Khovanov and L Rozansky introduced a family of link homology theories categorifying all of the quantum sl_n–polynomials and the HOMFLY-PT polynomial, see [11; 12]. The equivalence of the specializations of the Khovanov–Rozansky theory to the original constructions were easy to see in the case of $n = 2$ and recently proved in the case of $n = 3$, see Mackaay–Vaz [15].
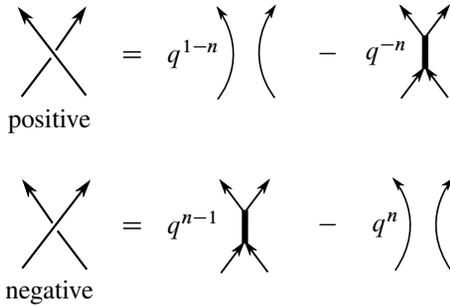

Figure 2: Skein formula for $P_n(L)$

The sl_n–polynomial $P_n(L)$ associated to a link L can be computed in the following two ways. We can resolve the crossings of L and using the rules in Figure 2, with a selected value of the unknot, arrive at a recursive formula, or we could use the Murakami, Ohtsuki, and Yamada [16] calculus of planar graphs (this is the sl_n generalization of the Kauffman solid-state model for the Jones polynomial). Given a diagram D of a link L and resolution Γ of this diagram, that is, a trivalent graph, we assign to it a polynomial $P_n(\Gamma)$ which is uniquely determined by the graph skein relations in Figure 1. Then we sum $P_n(\Gamma)$, weighted by powers of q, over all resolutions of D, that is,

$$P_n(L) = P_n(D) := \sum_{\text{resolutions}} \pm q^{\alpha(\Gamma)} P_n(\Gamma),$$

where $\alpha(\Gamma)$ is determined by the rules in Figure 2. The consistency and independence of the choice of diagram D for $P_n(\Gamma)$ are shown in [16].

To construct their homology theories, Khovanov and Rozansky first categorify the graph polynomial $P_n(\Gamma)$. They assign to each graph a 2–periodic complex whose cohomology is a graded $\mathbb{Q}$–vector space $H(\Gamma) = \oplus_{i \in \mathbb{Z}} H^i(\Gamma)$, supported only in one of the cohomological degrees, such that

$$P_n(\Gamma) = \sum_{i \in \mathbb{Z}} \dim_{\mathbb{Q}} H^i(\Gamma) q^i.$$

These complexes are made up of *matrix factorizations*, which we will discuss in detail later. They were first seen in the study of isolated hypersurface singularities in the early and mid-eighties (see Eisenbud [4]) but have since seen a number of applications. The graph skein relations for $P_n(\Gamma)$ are mirrored by isomorphisms of matrix factorizations assigned to the corresponding trivalent graphs in the homotopy category.

Nodes in the cube of resolutions of L are assigned the homology of the corresponding trivalent graph, and maps between resolutions, see Figure 3, are given by maps between matrix factorizations which further induce maps on cohomology. The resulting complex is proven to be invariant under the Reidemeister moves. The homology assigned to the unknot is the Frobenius algebra $\mathbb{Q}[x]/(x^n)$, the rational cohomology ring of $\mathbb{CP}^{n-1}$.
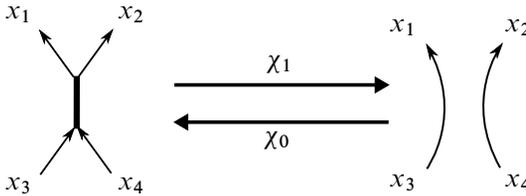

Figure 3: Maps between resolutions

The main goal of this paper is to generalize the above construction by extending the Khovanov–Rozansky homology to that of $\mathbb{Q}[a_0, \ldots, a_{n-1}]$–modules, where the a_i's are coefficients, such that

$$H_n(\varnothing) = \mathbb{Q}[a_0, \ldots, a_{n-1}],$$
$$H_n(\text{unknot}) = \mathbb{Q}[a_0, \ldots, a_{n-1}][x]/(x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0).$$

Our construction is motivated by the "universal" Frobenius system $\mathscr{F}_5$ introduced in [10] and its cohomological interpretation, that is, for every n we would like to construct a homology theory that assigns to the unknot the analogue of $\mathscr{F}_5$ for $n \geq 2$. Notice that,

$$\mathbb{Q}[a_0, \ldots, a_{n-1}] \cong H^*_{U(n)}(p, \mathbb{Q}) = H^*(BU(n), \mathbb{Q}) = H^*(Gr(n, \infty), \mathbb{Q}),$$
$$\mathbb{Q}[a_0, \ldots, a_{n-1}][x]/(x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0) \cong H^*_{U(n)}(\mathbb{CP}^{n-1}, \mathbb{Q}).$$

In practice, we will change basis as above for $\mathscr{F}_5$, getting rid of a_{n-1}, and work with the algebra

$$H_n(\text{unknot}) = \mathbb{Q}[a_0, \ldots, a_{n-2}][x]/(x^n + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0).$$

Theorem 1.9 *For every $n \in \mathbb{N}$ there exists a bigraded homology theory that is an invariant of links, such that*

$$H_n(\text{unknot}) = \mathbb{Q}[a_0, \ldots, a_{n-2}][x]/(x^n + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0),$$

where setting $a_i = 0$ for $0 \le i \le n - 2$ in the chain complex gives the Khovanov–Rozansky invariant, that is, a bigraded homology theory of links with Euler characteristic the quantum sl_n–polynomial $P_n(L)$.

The paper is organized in the following way: in Section 2 we review the basic definitions, work out the necessary statements for matrix factorizations over the ring $\mathbb{Q}[a_0, \ldots, a_{n-2}]$, assign complexes to planar trivalent graphs and prove MOY-type decompositions. Section 3 explains how to construct our invariant of links and Section 4 is devoted to the proofs of invariance under the Reidemeister moves. We conclude with a discussion of open questions and a possible generalization in Section 5.

1.4 Acknowledgements

2 Matrix factorizations

2.1 Basic definitions

Let R be a Noetherian commutative ring, and let $\omega \in R$. A *matrix factorization* with *potential* ω is a collection of two free R–modules M^0 and M^1 and R–module maps $d^0 \colon M^0 \to M^1$ and $d^1 \colon M^1 \to M^0$ such that

$$d^0 \circ d^1 = \omega \operatorname{Id} \quad \text{and} \quad d^1 \circ d^0 = \omega \operatorname{Id}.$$

The d^i's are referred to as 'differentials' and we often denote a matrix factorization by

$$M = \qquad M^0 \xrightarrow{\ d^0\ } M^1 \xrightarrow{\ d^1\ } M^0$$

Note M^0 and M^1 need not have finite rank.

A *homomorphism* $f\colon M \to N$ of two factorizations is a pair of homomorphisms $f^0\colon M^0 \to N^0$ and $f^1\colon M^1 \to N^1$ such that the following diagram is commutative:

$$
\begin{array}{ccccc}
M^0 & \xrightarrow{\ d^0\ } & M^1 & \xrightarrow{\ d^1\ } & M^0 \\
\big\downarrow{\scriptstyle f^0} & & \big\downarrow{\scriptstyle f^1} & & \big\downarrow{\scriptstyle f^0} \\
N^0 & \xrightarrow{\ d^0\ } & N^1 & \xrightarrow{\ d^1\ } & N^0.
\end{array}
$$

Let M_ω^{all} be the category with objects matrix factorizations with potential ω and morphisms homomorphisms of matrix factorizations. This category is additive with the direct sum of two factorizations taken in the obvious way. It is also equipped with a shift functor $\langle 1 \rangle$ whose square is the identity,

$$
M\langle 1 \rangle^i = M^{i+1}
$$
$$
d^i_{M\langle 1 \rangle} = -d^{i+1}_M, \quad i = 0, 1 \mod 2.
$$

We will also find the following notation useful. Given a pair of elements $b, c \in R$ we will denote by $\{b, c\}$ the factorization

$$
R \xrightarrow{\ b\ } R \xrightarrow{\ c\ } R.
$$

If $\mathbf{b} = (b_1, \ldots, b_k)$ and $\mathbf{c} = (c_1, \ldots, c_k)$ are two sequences of elements in R, we will denote by $\{\mathbf{b}, \mathbf{c}\} := \bigotimes_i \{b_i, c_i\}$ the tensor product factorization, where the tensor product is taken over R. We will call the pair $(\mathbf{b}, \mathbf{c})$ *orthogonal* if

$$
\mathbf{bc} := \sum_i b_i c_i = 0.
$$

Hence, the factorization $\{\mathbf{b}, \mathbf{c}\}$ is a complex if and only if the pair $(\mathbf{b}, \mathbf{c})$ is orthogonal. If in addition the sequence $\mathbf{c}$ is R–regular the cohomology of the complex becomes easy to determine. [Recall that a sequence $(r_1, \ldots, r_n)$ of elements of R is called R–*regular* if r_i is not a zero divisor in the quotient ring $R/(r_1, \ldots, r_{i-1})$.]

Proposition 2.1   *If $(\mathbf{b}, \mathbf{c})$ is orthogonal and $\mathbf{c}$ is R–regular then*

$$
H^0(\{\mathbf{b}, \mathbf{c}\}) \cong R/(c_1, \ldots, c_k) \quad \textit{and} \quad H^1(\mathbf{b}, \mathbf{c}) = 0.
$$

For more details we refer the reader to Khovanov–Rozansky [11, Section 2].

2.2 Homotopies of matrix factorizations

A homotopy h between maps $f, g \colon M \to N$ of factorizations is a pair of maps $h^i \colon M^i \to N^{i-1}$ such that $f - g = h \circ d_M + d_N \circ h$ where d_M and d_N are the differentials in M and N respectively.

Example 2.2 Any matrix factorization of the form

$$R \xrightarrow{\ r\ } R \xrightarrow{\ \omega\ } R \,,$$

or of the form

$$R \xrightarrow{\ \omega\ } R \xrightarrow{\ r\ } R \,,$$

with $r \in R$ invertible, is null-homotopic. Any factorization that is a direct sum of these is also null-homotopic.

Let HMF_ω^{all} be the category with the same objects as MF_ω^{all} but fewer morphisms:

$$\operatorname{Hom}_{HMF}(M, N) := \operatorname{Hom}_{MF}(M, N)/\{\text{null-homotopic morphisms}\}.$$

Consider the free R–module $\operatorname{Hom}(M, N)$ given by

$$\operatorname{Hom}^0(M, N) \xrightarrow{\ d\ } \operatorname{Hom}^1(M, N) \xrightarrow{\ d\ } \operatorname{Hom}^0(M, N) \,,$$

where

$$\operatorname{Hom}^0(M, N) = \operatorname{Hom}(M^0, N^0) \oplus \operatorname{Hom}(M^1, N^1),$$
$$\operatorname{Hom}^1(M, N) = \operatorname{Hom}(M^0, N^1) \oplus \operatorname{Hom}(M^1, N^0),$$

and the differential given in the obvious way, that is, for $f \in \operatorname{Hom}^i(M, N)$ and $m \in M$

$$(df)(m) = d_N(f(m)) + (-1)^i f(d_M(m)).$$

It is easy to see that this is a 2–periodic complex, and following the notation of [11], we denote its cohomology by

$$\operatorname{Ext}(M, N) = \operatorname{Ext}^0(M, N) \oplus \operatorname{Ext}^1(M, N).$$

Notice that

$$\operatorname{Ext}^0(M, N) \cong \operatorname{Hom}_{HMF}(M, N),$$
$$\operatorname{Ext}^1(M, N) \cong \operatorname{Hom}_{HMF}(M, N\langle 1\rangle).$$

2.3 Tensor Products

Given two matrix factorizations M_1 and M_2 with potentials ω_1 and ω_2, respectively, their tensor product is given as the tensor product of complexes, and a quick calculation shows that $M_1 \otimes M_2$ is a matrix factorization with potential $\omega_1 + \omega_2$.

To keep track of minus signs, it is convenient to assign a label to factorization M and denote it by

$$M = \qquad M(\varnothing) \xrightarrow{\ d^0\ } M(a) \xrightarrow{\ d^1\ } M(\varnothing),$$

so that the tensor product of a factorization with itself $M \otimes M$ can be written as

$$\begin{pmatrix} M(\varnothing) \\ M(ab) \end{pmatrix} \longrightarrow \begin{pmatrix} M(a) \\ M(b) \end{pmatrix} \longrightarrow \begin{pmatrix} M(\varnothing) \\ M(ab) \end{pmatrix}.$$

Here we are simply replacing M^0 by $M(\varnothing)$ and M^1 by a label such as $M(a)$; this will be useful below when we assign factorizations to planar graphs. See [11] for a more detailed treatment. Note that if $\omega_1 + \omega_2 = 0$ then $M_1 \otimes M_2$ becomes a 2–periodic complex.

An easy but useful exercise shows that if M has finite rank then $\mathrm{Hom}(M, N) \cong N \otimes_R M_-^*$, where M_-^* is the factorization

$$(M^0)^* \xrightarrow{\ -(d^1)^*\ } (M^1)^* \xrightarrow{\ (d^0)^*\ } (M^0)^*.$$

2.4 Cohomology of matrix factorizations

Suppose now that R is a local ring with maximal ideal $\mathbf{m}$ and M a factorization over R. If we impose the condition that the potential $\omega \in \mathbf{m}$ then

$$M^0/\mathbf{m}M \xrightarrow{\ d^0\ } M^1/\mathbf{m}M \xrightarrow{\ d^1\ } M^0/\mathbf{m}M \ ,$$

is a 2 periodic complex, since $d^2 = \omega \in \mathbf{m}$. Let $H(M) = H^0(M) \oplus H^1(M)$ be the cohomology of this complex.

Proposition 2.3 *Let M be a matrix factorization over a local ring R, with potential ω contained in the maximal ideal $\mathbf{m}$. The following are equivalent:*

  (1) $H(M) = 0$,

  (2) $H^0(M) = 0$,

  (3) $H^1(M) = 0$,

(4) *M is null-homotopic, and*

(5) *M is isomorphic to a (possibly infinite) direct sum of*

$$M = \quad R \xrightarrow{\ r\ } R \xrightarrow{\ \omega\ } R \,,$$

$$\text{and} \quad M = \quad R \xrightarrow{\ \omega\ } R \xrightarrow{\ r\ } R \,.$$

Proof The proof is the same as in [11], and we only need to notice that it extends to factorizations over any commutative, Noetherian, local ring. The idea is as follows: consider a matrix representing one of the differentials and suppose that it has an entry not in the maximal ideal, that is, an invertible entry; then change bases and arrive at block-diagonal matrices with blocks representing one of the two types of factorizations listed above (both of which are null-homotopic). Using Zorn's lemma we can decompose M as a direct sum of $M_{es} \oplus M_c$ where M_c is made up of the null-homotopic factorizations as above, that is, the "contractible" summand, and M_{es} the factor with corresponding submatrix containing no invertible entries, that is, the "essential" summand. Now it is easy to see that $H(M) = 0$ if and only if M_{es} is trivial. □

Proposition 2.4 *If $f\colon M \to N$ is a homomorphism of factorizations over a local ring R then the following are equivalent:*

(1) *f is an isomorphism in $HMF_\omega^{\mathrm{all}}$, and*

(2) *f induces and isomorphism on the cohomologies of M and N.*

Proof This is done in [11]. Decompose M and N as in the proposition above and notice that the cohomology of a matrix factorization is the cohomology of its essential part. Now a map of two free R–modules $L_1 \to L_2$ that induces an isomorphism on $L_1/\mathbf{m} \cong L_2/\mathbf{m}$ is an isomorphism of R–modules. □

Corollary 2.5 *Let M be a matrix factorization over a local ring R. The decomposition $M \cong M_{es} \oplus M_c$ is unique; moreover if M has finite-dimensional cohomology then it is the direct sum of a finite rank factorization and a contractible factorization.*

Let MF_ω be the category whose objects are factorizations with finite-dimensional cohomology and let HMF_ω be corresponding homotopy category.

2.5 Matrix factorizations over a graded ring

Let $R = \mathbb{Q}[a_0, \ldots, a_{n-2}][x_1, \ldots, x_k]$, a graded ring of homogeneous polynomials in variables $x_1, \ldots, x_k$ with coefficients in $\mathbb{Q}[a_0, \ldots, a_{n-2}]$. The gradings are as follows: $\deg(x_i) = 2$ and $\deg(a_i) = 2(n - i)$ with $i = 0, \ldots n - 2$. Furthermore let $\mathbf{m} = \langle a_0, \ldots, a_{n-2}, x_1, \ldots, x_k \rangle$ the maximal homogeneous ideal, and let $\mathbf{a} = \langle a_0, \ldots, a_{n-2} \rangle$ the ideal generating the ring of coefficients.

A matrix factorization M over R naturally becomes graded and we denote $\{i\}$ the grading shift up by i. Note that $\{i\}$ commutes with the shift functor $\langle 1 \rangle$. All of the categories introduced earlier have their graded counterparts which we denote with lower-case. However, here we will only deal with matrix factorizations with the grading bounded from below; this is a technical condition necessary for [Proposition 2.3](#) to hold in this case. For example hmf_ω^{all} is the homotopy category of graded matrix factorizations with the grading bounded from below.

Proposition 2.6 *Let* $f \colon M \longrightarrow N$ *a homomorphism of matrix factorizations over* $R = \mathbb{Q}[\alpha_0, \ldots, \alpha_{n-2}][x_1, \ldots, x_k]$ *and let* $\bar{f} \colon M/\mathbf{a}M \longrightarrow N/\mathbf{a}N$ *be the induced map. Then* f *is an isomorphism of factorizations if and only if* $\bar{f}$ *is.*

Proof One only needs to notice that modding out by the ideal $\mathbf{a}$ we arrive at factorizations over $\bar{R} = \mathbb{Q}[x_1, \ldots, x_k]$, the graded ring of homogeneous polynomials with coefficients in $\mathbb{Q}$ and maximal ideal $\mathbf{m}' = \langle x_1, \ldots, x_k \rangle$. Since $\mathbf{a} \subset \mathbf{m}$ and $\mathbf{m}' \subset \mathbf{m}$, $H(M) = H(M/\mathbf{a}M)$ for any factorization and, hence, the induced maps on cohomology are the same, that is, $H(f) = H(\bar{f})$. Since an isomorphism on cohomology implies an isomorphism of factorizations over R and $\bar{R}$ the proposition follows. □

The matrix factorizations used to define the original link invariants in [11] were defined over $\bar{R}$. With [Proposition 2.6](#) we will able to bypass many of the calculations necessary for MOY-type decompositions and Reidemeister moves, citing those from the original paper. This simple observation will prove to be one of the most useful.

2.6 The category hmf_ω is Krull–Schmidt

In order to prove that the homology theory we assign to links is indeed a topological invariant with Euler characteristic the quantum sl_n–polynomial, we first need to show that the algebraic objects associated to each resolution, that is, to a trivalent planar graph, satisfy the MOY relations [16]. Since the objects in question are complexes constructed from matrix factorizations, the MOY decompositions are reflected by corresponding

isomorphisms of complexes in the homotopy category. Hence, in order for these relations to make sense, we need to know that if an object in our category decomposes as a direct sum then it does so uniquely. In other words we need to show that our category is Krull–Schmidt. The next subsection establishes this fact for hmf_ω, the homotopy category of graded matrix factorizations over $R = \mathbb{Q}[a_0, \ldots, a_{n-2}][x_1, \ldots, x_k]$ with finite dimensional cohomology.

Given a homogeneous, finite rank, factorization $M \in mf_\omega$, and a degree zero idempotent $e \colon M \longrightarrow M$ we can decompose M uniquely as the kernel and cokernel of e, that is, we can write $M = eM \oplus (1-e)M$. We need to establish this fact for hmf_ω; that is, we need to know that given a degree zero idempotent $e \in \mathrm{Hom}_{hmf_\omega}(M, M)$ we can decompose M as above, and that this decomposition is unique up to homotopy.

Proposition 2.7 *The category hmf_ω has the idempotents splitting property.*

Proof We follow [11] . Let $I \subset \mathrm{Hom}_{mf_\omega}(M, M)$ be the ideal consisting of maps that induce the trivial map on cohomology. Given any such map $f \in I$, we see that every entry in the matrices representing f must be contained in $\mathbf{m}$, that is, the entries must be of non-zero degree. Since a degree zero endomorphism of graded factorizations cannot have matrix entries of arbitrarily large degree, we see that there exists an $n \in \mathbb{N}$ such that $f^n = 0$ for every $f \in I$, that is, I is nilpotent.

Let K be the kernel of the map $\mathrm{Hom}_{mf_\omega}(M, M) \longrightarrow \mathrm{Hom}_{hmf_\omega}(M, M)$. Clearly $K \subseteq I$ and, hence, K is also nilpotent. Since nilpotent ideals have the idempotents lifting property (see, for example, Benson [3, Theorem 1.7.3]) we can lift any idempotent $e \in \mathrm{Hom}_{hmf_\omega}(M, M)$ to $\mathrm{Hom}_{mf_\omega}(M, M)$ and decompose $M = eM \oplus (1-e)M$. □

Proposition 2.8 *The category hmf_ω is Krull–Schmidt.*

Proof Proposition 2.7 and the fact that any object in hmf_ω is isomorphic to one of finite rank, having finite dimensional cohomology, imply that the endomorphism ring of any indecomposable object is local. Hence, hmf_ω is Krull–Schmidt. See Benson [3] for proofs of these facts. □

2.7 Planar graphs and matrix factorizations

Our graphs are embedded in a disk and have two types of edges, unoriented and oriented. Unoriented edges are called "thick" and drawn accordingly; each vertex adjoining a thick edge has either two oriented edges leaving it or two entering. In Figure 6 left x_1, x_2 are outgoing and x_3, x_4 are incoming. Oriented edges are allowed to have

marks and we also allow closed loops; points of the boundary are also referred to as marks. See for example Figure 4. To such a graph Γ we assign a matrix factorization in the following manner:

Let

$$P(x) = \frac{1}{n+1}x^{n+1} + \frac{a_{n-2}}{n-1}x^{n-1} + \cdots + \frac{a_1}{2}x^2 + a_0 x.$$

2.8 Thick edges

To a thick edge t as in Figure 6 left we assign a factorization C_t with potential

$$\omega_t = P(x_1) + P(x_2) - P(x_3) - P(x_4)$$

over the ring

$$R_t = \mathbb{Q}[a_0, \ldots, a_{n-2}][x_1, x_2, x_3, x_4].$$

Since $x^k + y^k$ lies in the ideal generated by $x + y$ and xy we can write it as a polynomial $g_k(x + y, xy)$. More explicitly,

$$g_k(s_1, s_2) = s_1^k + k \sum_{1 \le i \le \frac{k}{2}} \frac{(-1)^i}{i}\binom{k-1-i}{i-1} s_2^i s_1^{k-2i}$$

Hence, $x_1^k + x_2^k - x_3^k - x_4^k$ can be written as

$$x_1^k + x_2^k - x_3^k - x_4^k = (x_1 + x_2 - x_3 - x_4)u_k' + (x_1 x_2 - x_3 x_4)u_k''$$

where

$$u_k' = \frac{x_1^k + x_2^k - g_k(x_3 + x_4, x_1 x_2)}{x_1 + x_2 - x_3 - x_4},$$

$$u_k'' = \frac{g_k(x_3 + x_4, x_1 x_2) - x_3^k - x_4^k}{x_1 x_2 - x_3 x_4}.$$

(Notice that our u_{n+1}' and u_{n+1}'' are the same as the u_1 and u_2 in [11], respectively.)
Let

$$\mathcal{U}_1 = \frac{1}{n+1}u_{n+1}' + \frac{a_{n-2}}{n-1}u_{n-1}' + \cdots + \frac{a_1}{2}u_2' + a_0,$$

and

$$\mathcal{U}_2 = \frac{1}{n+1}u_{n+1}'' + \frac{a_{n-2}}{n-1}u_{n-1}'' + \cdots + \frac{a_1}{2}u_2''.$$

Define C_t to be the tensor product of graded factorizations

$$R_t \xrightarrow{\mathcal{U}_1} R_t\{1-n\} \xrightarrow{x_1+x_2-x_3-x_4} R_t,$$

and

$$R_t \xrightarrow{u_2} R_t\{3-n\} \xrightarrow{x_1 x_2 - x_3 x_4} R_t,$$

with the product shifted by $\{-1\}$.

2.9 Arcs

To an arc α bounded by marks oriented from j to i we assign the factorization L_j^i

$$R_\alpha \xrightarrow{\mathcal{P}_{ij}} R_\alpha \xrightarrow{x_i - x_j} R_\alpha,$$

where $R_\alpha = \mathbb{Q}[a_0, \dots, a_{n-2}][x_i, x_j]$ and

$$\mathcal{P}_{ij} = \frac{P(x_i) - P(x_j)}{x_i - x_j}.$$

Finally, to an oriented loop with no marks we assign the complex $0 \to \mathcal{A} \to 0 = \mathcal{A}\langle 1 \rangle$ where $\mathcal{A} = \mathbb{Q}[a_0, \dots, a_{n-2}][x]/(x^n + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0)$. (Note: to a loop with marks we assign the tensor product of L_j^i's as above, but this turns out to be isomorphic to $\mathcal{A}\langle 1 \rangle$ in the homotopy category.)
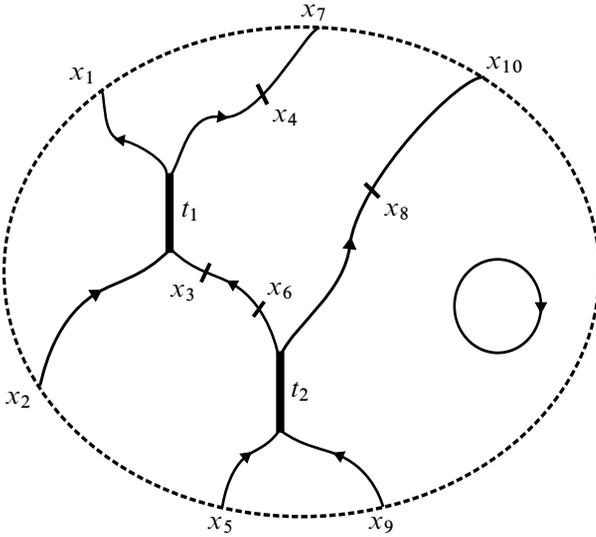
Figure 4: A planar graph

We define $C(\Gamma)$ to be the tensor product of C_t over all thick edges t, L_j^i over all edges α from j to i, and $\mathcal{A}\langle 1 \rangle$ over all oriented markless loops. This tensor product is taken over appropriate rings such that $C(\Gamma)$ is a free module over $R = \mathbb{Q}[a_0, \dots, a_{n-2}][\{x_i\}]$

where the x_i's are marks. For example, to the graph in [Figure 4](#) we assign $C(\Gamma) = L_4^7 \otimes C_{t_1} \otimes L_6^3 \otimes C_{t_2} \otimes L_8^{10} \otimes A\langle 1 \rangle$ tensored over $\mathbb{Q}[a_0, \ldots, a_{n-2}][x_4]$, $\mathbb{Q}[a_0, \ldots, a_{n-2}][x_3]$, $\mathbb{Q}[a_0, \ldots, a_{n-2}][x_6]$, $\mathbb{Q}[a_0, \ldots, a_{n-2}][x_8]$ and $\mathbb{Q}[a_0, \ldots, a_{n-2}]$ respectively. $C(\Gamma)$ becomes a $\mathbb{Z} \oplus \mathbb{Z}_2$–graded complex with the $\mathbb{Z}_2$–grading coming from the matrix factorization. It has potential $\omega = \sum_{i \in \partial \Gamma} \pm P(x_i)$, where $\partial \Gamma$ is the set of all boundary marks and the $+, -$ is determined by whether the direction of the edge corresponding to x_i is towards or away from the boundary. (Note: if Γ is a closed graph the potential is zero and we have an honest 2–complex.)

Example 2.9 Let us look at the factorization assigned to an oriented loop with two marks x and y. We start out with the factorization L_x^y assigned to an arc and then "close it off," which corresponds to modding out by the ideal generated by the relation $x = y$, see [Figure 5](#). We arrive at

$$R \xrightarrow{x^n + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0} R \xrightarrow{0} R,$$

where $R = \mathbb{Q}[a_0, \ldots, a_{n-2}][x]$.
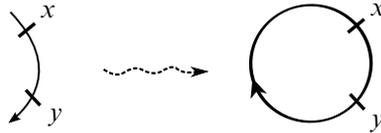
Figure 5: "Closing off" an arc

The homology of this complex is supported in degree 1, with

$$H^1(L_x^y / \langle x = y \rangle) = \mathbb{Q}[a_0, \ldots, a_{n-2}][x] / (x^n + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0).$$

This is the algebra $\mathcal{A}$ we associated to an oriented loop with no marks. As we set out to define a homology theory that assigns to the unknot the $U(n)$–equivariant cohomology of $\mathbb{CP}^{n-1}$, this example illustrates the choice of potential $P(x)$. Notice that $\mathcal{A}$ has a natural Frobenius algebra structure with trace map ε and unit map ι.

$$\varepsilon \colon \mathbb{Q}[a_0, \ldots, a_{n-2}][x] / (x^n + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0) \longrightarrow \mathbb{Q}[a_0, \ldots, a_{n-2}],$$

given by

$$\varepsilon(x^{n-1}) = 1; \ \ \varepsilon(x^i) = 0, \ i \leq n - 2,$$

and

$$\iota \colon \mathbb{Q}[a_0, \ldots, a_{n-2}] \longrightarrow \mathbb{Q}[a_0, \ldots, a_{n-2}][x] / (x^n + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0),$$

$$\iota(1) = 1.$$

Notice that $\varepsilon(x^i)$ is not equal to zero for $i \geq n$ but a homogeneous polynomial in the a_i's. Many of the calculations in [11] necessary for the proofs of invariance would fail due to this fact; Propositions 2.3 and 2.6 will be key in getting around this difference. Of course, setting $a_i = 0$, for all i, gives us the same Frobenius algebra, unit and trace maps as in [11].
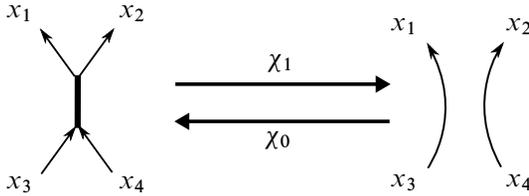

Figure 6: Maps χ_0 and χ_1

2.10 The maps χ_0 and χ_1

We now define maps between matrix factorizations associated to a thick edge and two disjoint arcs as in Figure 6. Let Γ^0 correspond to the two disjoint arcs and Γ^1 to the thick edge.

$C(\Gamma^0)$ is the tensor product of L_4^1 and L_3^2. If we assign labels a, b to L_4^1, L_3^2 respectively, the tensor product can be written as

$$\begin{pmatrix} R(\varnothing) \\ R(ab)\{2-2n\} \end{pmatrix} \xrightarrow{P_0} \begin{pmatrix} R(a)\{1-n\} \\ R(b)\{1-n\} \end{pmatrix} \xrightarrow{P_1} \begin{pmatrix} R(\varnothing) \\ R(ab)\{2-2n\} \end{pmatrix},$$

where

$$P_0 = \begin{pmatrix} \mathcal{P}_{14} & x_2 - x_3 \\ \mathcal{P}_{23} & x_4 - x_1 \end{pmatrix}, \quad P_1 = \begin{pmatrix} x_1 - x_4 & x_2 - x_3 \\ \mathcal{P}_{23} & -\mathcal{P}_{14} \end{pmatrix},$$

and $R = \mathbb{Q}[a_0, \ldots, a_{n-2}][x_1, x_2, x_3, x_4]$.

Assigning labels a' and b' to the two factorizations in $C(\Gamma^1)$, we have that $C(\Gamma^1)$ is given by

$$\begin{pmatrix} R(\varnothing)\{-1\} \\ R(a'b')\{3-2n\} \end{pmatrix} \xrightarrow{Q_0} \begin{pmatrix} R(a')\{-n\} \\ R(b')\{2-n\} \end{pmatrix} \xrightarrow{Q_1} \begin{pmatrix} R(\varnothing)\{-1\} \\ R(a'b')\{3-2n\} \end{pmatrix},$$

where

$$Q_0 = \begin{pmatrix} \mathcal{U}_1 & x_1 x_2 - x_3 x_4 \\ \mathcal{U}_2 & x_3 + x_4 - x_1 - x_2 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} x_1 + x_2 - x_3 - x_4 & x_1 x_2 - x_3 x_4 \\ \mathcal{U}_2 & -\mathcal{U}_1 \end{pmatrix}.$$

A map between $C(\Gamma^0)$ and $C(\Gamma^1)$ can be given by a pair of 2×2 matrices. Define $\chi_0 \colon C(\Gamma^0) \to C(\Gamma^1)$ by

$$U_0 = \begin{pmatrix} x_4 - x_2 + \mu(x_1 + x_2 - x_3 - x_4) & 0 \\ k_1 & 1 \end{pmatrix}, \quad U_1 = \begin{pmatrix} x_4 + \mu(x_1 - x_4) & \mu(x_2 - x_3) - x_2 \\ -1 & 1 \end{pmatrix},$$

where

$$k_1 = (\mu - 1)\mathcal{U}_2 + \frac{\mathcal{U}_1 + x_1\mathcal{U}_2 - \mathcal{P}_{23}}{x_1 - x_4}, \quad \text{for } \mu \in \mathbb{Z}$$

and $\chi_1 \colon C(\Gamma^1) \to C(\Gamma^0)$ by

$$V_0 = \begin{pmatrix} 1 & 0 \\ k_2 & k_3 \end{pmatrix}, \quad V_1 = \begin{pmatrix} 1 & x_3 + \lambda(x_2 - x_3) \\ 1 & x_1 + \lambda(x_4 - x_1) \end{pmatrix}.$$

where

$$k_2 = \lambda\mathcal{U}_2 + \frac{\mathcal{U}_1 + x_1\mathcal{U}_2 - \mathcal{P}_{23}}{x_4 - x_1}, \quad k_3 = \lambda(x_3 + x_4 - x_1 - x_2) + x_1 - x_3, \quad \text{for } \lambda \in \mathbb{Z}.$$

It is easy to see that different choices of μ and λ give homotopic maps. These maps are degree 1. We encourage the reader to compare the above factorizations and maps to that of [11], and notice the difference stemming from the fact that here we are working with new potentials.

Just like in [11] we specialize to $\lambda = 0$ and $\mu = 1$, and compute to see that the composition $\chi_1\chi_0 = (x_1 - x_3)I$, where I is the identity matrix, that is, $\chi_1\chi_0$ is multiplication by $x_1 - x_3$, which is homotopic to multiplication by $x_4 - x_2$ as an endomorphism of $C(\Gamma^0)$. Similarly $\chi_0\chi_1 = (x_1 - x_3)I$, which is also homotopic to multiplication by $x_4 - x_2$ as an endomorphism of $C(\Gamma^1)$.

2.11 Direct Sum Decomposition 0

This decomposition is in the spirit of Bar-Natan's "circle annihilation" of [2]. It takes the following form, where the maps $D_0 = \sum_{i=0}^{n-1} x^i \iota$ and $D_0^{-1} = \sum_{i=0}^{n-1} \varepsilon x^{n-1-i}$.

By the pictures above, we really mean the complexes assigned to them, that is, $\varnothing\langle 1\rangle$ is the complex with $\mathbb{Q}[a_0, \ldots, a_{n-2}]$ sitting in homological grading 1 and the unknot is the complex $\mathcal{A}\langle 1\rangle$ as before. The map εx^i is a composition of maps

$$\mathcal{A}\langle 1\rangle \xrightarrow{x^i} \mathcal{A}\langle 1\rangle \xrightarrow{\varepsilon} \varnothing\langle 1\rangle,$$

where x^i is multiplication and ε is the trace map.

The map $x^i\iota$ is analogous. It is easy to check that the above maps are grading preserving and their composition is an isomorphism in the homotopy category. $\qquad\square$
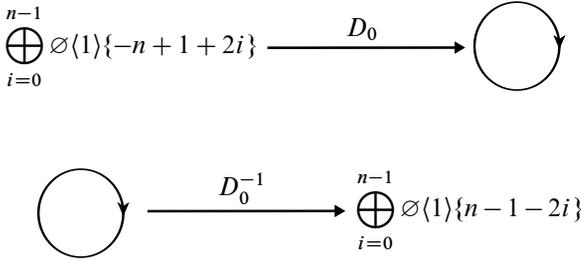
Figure 7: DSD 0

2.12 Direct Sum Decomposition I

We follow [11] closely. Recall that here matrix factorizations are over the ring $R = \mathbb{Q}[a_0, \ldots, a_{n-2}]$.
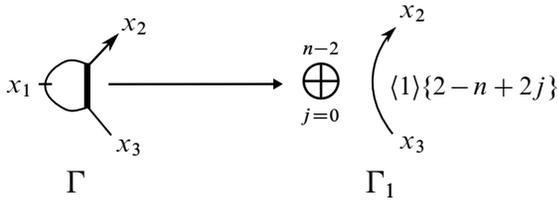

Figure 8: DSD I

Proposition 2.10 *The following two factorizations are isomorphic in hmf$_\omega$.*

$$C(\Gamma) \cong \sum_{i=0}^{n-2} C(\Gamma_1)\langle 1\rangle\{2 - n + 2i\}.$$

Proof Define grading preserving maps α_i and β_i for $0 \le i \le n-2$, as in [11],

$$\alpha_i \colon C(\Gamma_1)\langle 1\rangle \longrightarrow C(\Gamma)\{n - 2 - 2i\}, \qquad \alpha_i = \sum_{j=0}^{i} x_1^j x_2^{i-j} \alpha,$$

where $\alpha = \chi_0 \circ \iota'$ is defined to be the composition in Figure 9. ($\iota' = \iota \otimes \mathrm{Id}$ where Id corresponds to the inclusion of the arc Γ_1 into the disjoint union of the arc and circle, and ι is the unit map.)

$$\beta_i \colon C(\Gamma)\{n - 2 - 2i\} \longrightarrow C(\Gamma_1)\langle 1\rangle, \qquad \beta_i = \beta x_1^{n-i-2},$$

where $\beta = \varepsilon' \circ \chi_1$, see Figure 10. (Similarly, $\varepsilon' = \varepsilon \otimes \mathrm{Id}$.)
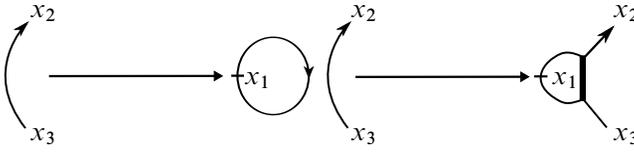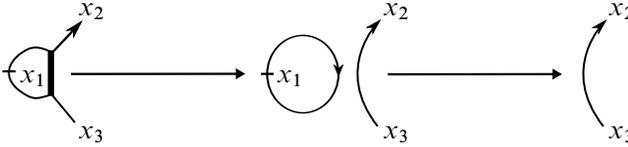
Figure 9: The map α

Figure 10: The map β

Define maps

$$\alpha' = \sum_{i=0}^{n-2} \alpha_i \colon \sum_{i=0}^{n-2} C(\Gamma_1)\langle 1\rangle\{2-n+2i\} \longrightarrow C(\Gamma)$$

$$\text{and} \quad \beta' = \sum_{i=0}^{n-2} \beta_i \colon C(\Gamma) \longrightarrow C(\Gamma_1)\langle 1\rangle\{2-n+2i\}.$$

In [11] it was shown that these maps are isomorphisms of factorizations over the ring $\bar{R} = \mathbb{Q}[x_1, x_2, x_3, x_4]$. By Proposition 2.6 we are done. $\qquad\square$

2.13 Direct Sum Decomposition II
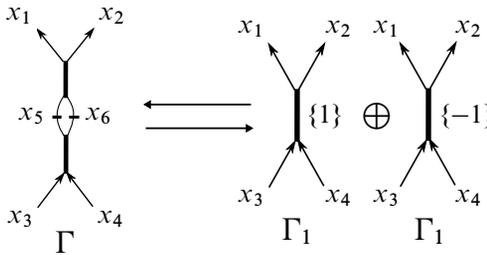
Figure 11: DSD II

Proposition 2.11 *There is an isomorphism of factorizations in hmf$_\omega$*

$$C(\Gamma) \cong C(\Gamma_1)\{1\} \oplus C(\Gamma_1)\{-1\}.$$

Proof See [11]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

2.14 Direct Sum Decomposition III
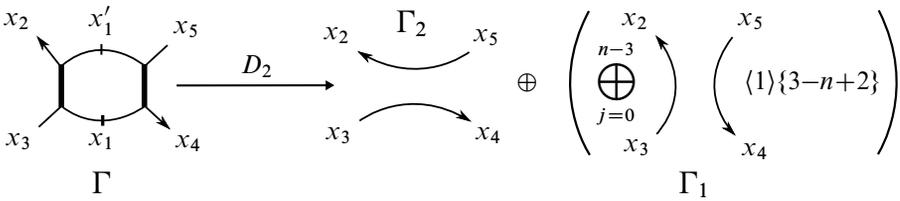
Figure 12: DSD III

Proposition 2.12 *There is an isomorphism of factorizations in hmf* $_\omega$

$$C(\Gamma) \cong C(\Gamma_2) \oplus \left(\oplus_{i=0}^{n-3} C(\Gamma_1)\langle 1\rangle\{3-n+2i\} \right).$$

Proof Define grading preserving maps α_i, β_i for $0 \le i \le n-3$

$$\alpha_i \colon C(\Gamma_1)\langle 1\rangle\{3-n+2i\} \longrightarrow C(\Gamma), \qquad \alpha_i = x_5\alpha,$$

where $\alpha = \chi_0' \circ \iota'$, $\iota' = \mathrm{Id} \otimes \iota \otimes \mathrm{Id}$ with identity maps on the two arcs, and χ_0' the composition of two χ_0's corresponding to merging the two arcs into the circle, see Figure 13.

Figure 13: The map α

Let
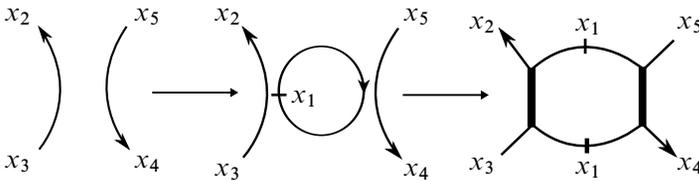
$$\beta_i \colon C(\Gamma) \longrightarrow C(\Gamma_1)\langle 1\rangle\{3-n+2i\}, \qquad \beta_i = \sum_{i=0}^{n-3} \beta \sum_{a+b+c=n-3-i} x_2^a x_4^b x_1^c,$$

where β is defined as in Figure 14.

In addition, let $S \colon C(\Gamma) \longrightarrow C(\Gamma_2)$ be the map gotten by "merging" the thick edges together to form two disjoint horizontal arcs, as in the top right-hand corner above; an exact description of S won't really matter so we will not go into details and refer the interested reader to [11].
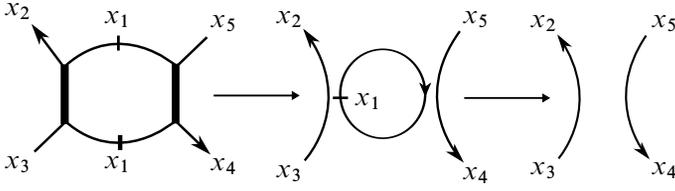
Figure 14: The map β

Let $\alpha' = \sum_{i=0}^{n-3} \alpha_i$ and $\beta' = \sum_{i=0}^{n-3} \beta_i$. In [11] it shown that $S \oplus \beta'$ is an isomorphism in hmf_ω, with inverse $S^{-1} \oplus \alpha'$, so by Proposition 2.6 we are done. $\square$

(Note: we abuse notation throughout by using a direct sum of maps to indicate a map to or from a direct summand.)
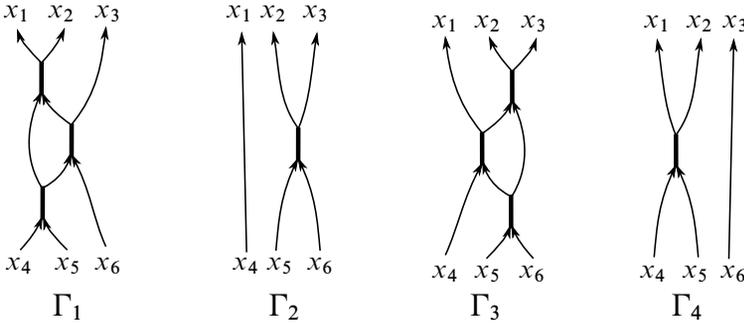
2.15 Direct Sum Decomposition IV

Figure 15: The factorizations in DSD IV

Proposition 2.13 *There is an isomorphism in hmf $_\omega$*

$$C(\Gamma_1) \oplus C(\Gamma_2) \cong C(\Gamma_3) \oplus C(\Gamma_4).$$

Proof   Notice that $C(\Gamma_1)$ turns into $C(\Gamma_3)$ if we permute x_1 with x_3, and $C(\Gamma_2)$ turns into $C(\Gamma_4)$ if we permute x_2 and x_4. The proposition is proved by introducing a new factorization Υ that is invariant under these permutations and showing that $C(\Gamma_1) \cong \Upsilon \oplus C(\Gamma_4)$, and $C(\Gamma_3) \cong \Upsilon \oplus C(\Gamma_2)$. Since these decompositions hold for matrix factorizations over the ring $\bar{R} = \mathbb{Q}[x_1, \ldots, x_6]$, they hold here as well. We refer the reader to [11] for details. $\square$

3 Tangles and complexes

By a tangle T we mean an oriented, closed one manifold embedded in the unit ball $\mathbb{B}^3$, with boundary points of T lying on the equator of the bounding sphere $\mathbb{S}^2$. An isotopy of tangles preserves the boundary points. A diagram D for T is a generic projection of T onto the plane of the equator.
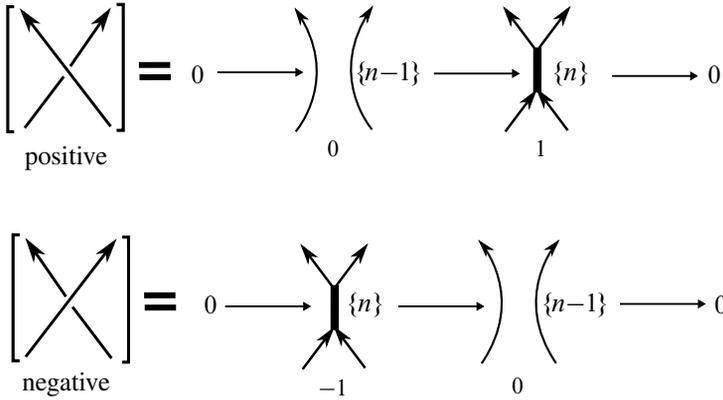

Figure 16: Complexes associated to pos/neg crossings; the numbers below the diagrams are cohomological degrees.

Given such a diagram D and a crossing p of D we resolve it in two ways, depending on whether the crossing is positive or negative, and assign to p the corresponding complex C^p, see Figure 16 . We define $C(D)$ to be the complex of matrix factorizations which is the tensor product of C^p, over all crossings p, of L_j^i over arcs $j \to i$, and of $\mathcal{A}\langle 1 \rangle$ over all crossingless markless circles in D. The tensor product is taken over appropriate polynomial rings, so that $C(D)$ is free and of finite rank as an R–module, where $R = \mathbb{Q}[a_0, \ldots, a_{n-2}][x_1, \ldots, x_k]$, and the x_i's are on the boundary of D. This complex is $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$ graded.

For example, the complex associated to the tangle in Figure 17 is gotten by first tensoring C^{p_1} with C^{p_2} over the ring $\mathbb{Q}[a_0, \ldots, a_{n-2}][x_3, x_4]$, then tensoring $C^{p_1} \otimes C^{p_2}$ with L_1^2 over $\mathbb{Q}[a_0, \ldots, a_{n-2}][x_2]$, and finally tensoring $C^{p_1} \otimes C^{p_2} \otimes L_1^2$ with $\mathcal{A}\langle 1 \rangle$ over $\mathbb{Q}[a_0, \ldots, a_{n-2}]$.

Theorem 3.1 *If D and D' are two diagrams representing the same tangle T, then $C(D)$ and $C(D')$ are isomorphic modulo homotopy in the homotopy category hmf_ω, that is, the isomorphism class of $C(D)$ is an invariant of T.*

The proof of this statement involves checking the invariance under the Reidemeister moves to which the next section is devoted.
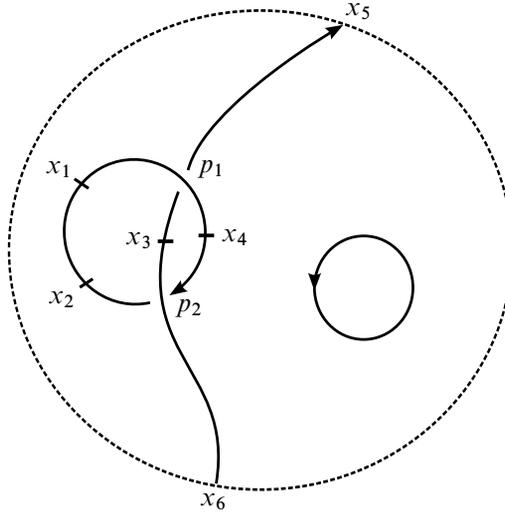
Figure 17: Diagram of a tangle

3.1 Link homology

When the tangle in question is a link L, that is, there are no boundary points and $R = \mathbb{Q}[a_0, \ldots, a_{n-2}]$, complexes of matrix factorizations associated to each resolution have non-trivial cohomology only in one degree (in the cyclic degree which is the number of components of L modulo 2). The grading of the cohomology of $C(L)$ reduces to $\mathbb{Z} \oplus \mathbb{Z}$. We denote the resulting cohomology groups of the complex $C(L)$ by

$$H_n(L) = \oplus_{i,j \in \mathbb{Z}} H_n^{i,j}(L),$$

and the Euler characteristic by

$$P_n(L) = \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \dim_R H_n^{i,j}(L).$$

It is clear from the construction that:

Corollary 3.2 *Setting the a_i's to zero in the chain complex we arrive at the Khovanov–Rozansky homology, with Euler characteristic the quantum sl_n–polynomial of L.*
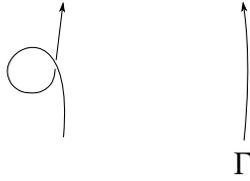
Figure 18

4 Invariance under the Reidemeister moves

4.1 R1

To the tangle in Figure 18 left we associate the following complex

$$0 \longrightarrow C(\Gamma_1)\{1-n\} \xrightarrow{\chi_0} C(\Gamma_2)\{-n\} \longrightarrow 0.$$

Using direct decompositions 0 and I, and for a moment forgoing the overall grading
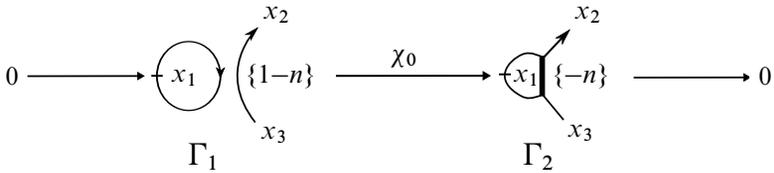

Figure 19: Reidemeister 1 complex

shifts, we see that this complex is isomorphic to

$$0 \longrightarrow \bigoplus_{i=0}^{n-1} C(\Gamma)\{1-n+2i\} \xrightarrow{\Phi} \bigoplus_{j=0}^{n-2} C(\Gamma)\{1+n-2j\} \longrightarrow 0,$$

where

$$\Phi = \beta' \circ \chi_0 \circ \sum_{i=0}^{n-1} x_1^i \iota'$$

$$= \left(\sum_{j=0}^{n-2} \varepsilon' \circ \chi_1 x_1^{n-j-2} \right) \circ \chi_0 \circ \sum_{i=0}^{n-1} x_1^i \iota'$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{n-2} \varepsilon' \circ \chi_1 \circ \chi_0 x_1^{n-j+i-2} \circ \iota'$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{n-2} \varepsilon' (x_1 - x_2) x_1^{n-j+i-2} \circ \iota'$$

$$= \varepsilon' \left(\sum_{i=0}^{n-1} \sum_{j=0}^{n-2} (x_1^{n-j+i-1} - x_2 x_1^{n-j+i-2}) \right) \iota'.$$

Hence, Φ is an upper triangular matrix with 1's on the diagonal, which implies that up to homotopy the above complexes are isomorphic to

$$0 \longrightarrow C(\Gamma)\{n-1\} \longrightarrow 0.$$

Recalling that we left out the overall grading shift of $\{-n+1\}$ we arrive at the desired conclusion:

$$0 \longrightarrow C(\Gamma_1)\{1-n\} \xrightarrow{\chi_0} C(\Gamma_2)\{-n\} \longrightarrow 0$$

is homotopic to

$$0 \longrightarrow C(\Gamma)\{n-1\} \longrightarrow 0.$$

The other Reidemeister 1 move is proved analogously. $\square$

4.2 R2

Figure 20

The complex associated to the tangle in Figure 20 left is

$$0 \longrightarrow C(\Gamma_{00})\{1\} \xrightarrow{(f_1,f_3)^t} \begin{matrix} C(\Gamma_{01}) \\ \oplus \\ C(\Gamma_{10}) \end{matrix} \xrightarrow{(f_2,-f_4)} C(\Gamma_{11})\{-1\} \longrightarrow 0.$$

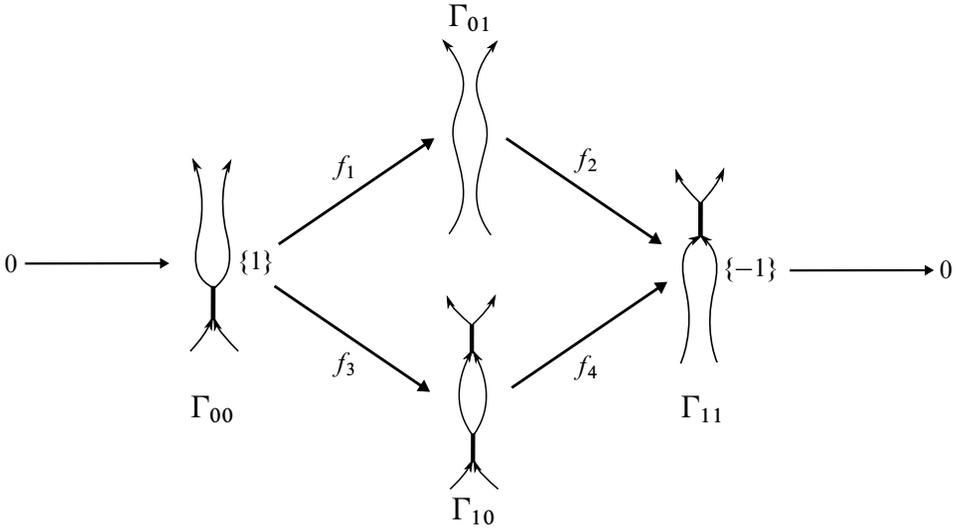Using direct decomposition II we know that

Figure 21: Reidemeister 2a complex

$$C(\Gamma_{10}) \cong C(\Gamma_1)\{1\} \oplus C(\Gamma_1)\{-1\}.$$

Hence, the above complex is isomorphic to

$$0 \longrightarrow C(\Gamma_{00})\{1\} \xrightarrow{(f_1,f_{03},f_{13})^t} \begin{array}{c} C(\Gamma_{01}) \\ \oplus \\ C(\Gamma_1)\{1\} \\ \oplus \\ C(\Gamma_1)\{-1\} \end{array} \xrightarrow{(f_2,-f_{04},-f_{14})} C(\Gamma_{11})\{-1\} \longrightarrow 0,$$

where f_{03}, f_{13}, f_{04}, f_{14} are the degree 0 maps that give the isomorphism of decomposition II. If we know that both f_{14} and f_{03} are isomorphisms then the subcomplex containing $C(\Gamma_{00})$, $C(\Gamma_{10})$, and $C(\Gamma_{11})$ is acyclic; modding out produces a complex homotopic to

$$0 \longrightarrow C(\Gamma_0) \longrightarrow 0.$$

The next two lemmas establish the fact that f_{14} and f_{03} are indeed isomorphisms.

Lemma 4.1 *The space of degree 0 endomorphisms of $C(\Gamma_1)$ is isomorphic to $\mathbb{Q}$. The space of degree 2 endomorphism is 3–dimensional spanned by x_1, x_2, x_3, x_4 with only relation being $x_1 + x_2 - x_3 - x_4 = 0$ for $n > 2$, and 2–dimensional with the relations $x_1 + x_2 = 0$ and $x_3 + x_4 = 0$ for $n = 2$.*

Proof   The complex $\mathrm{Hom}(C(\Gamma_1), C(\Gamma_1))$ is isomorphic to the factorization of the pair $(\mathbf{b}, \mathbf{c})$ where

$$\mathbf{b} = (x_1 + x_2 + x_3 + x_4, x_1 x_2 - x_3 x_4, -\mathcal{U}_1, -\mathcal{U}_2),$$
$$\mathbf{c} = (\mathcal{U}_1, \mathcal{U}_2, x_1 + x_2 + x_3 + x_4, x_1 x_2 - x_3 x_4).$$

The pair $(\mathbf{b}, \mathbf{c})$ is orthogonal, since this is a complex, and it is easy to see that the sequence $\mathbf{c}$ is regular ($\mathbf{c}$ is certainly regular when we set the a_i's equal to zero) and hence the cohomology of this 2–complex is

$$\mathbb{Q}[a_0, \ldots, a_{n-2}][x_1, x_2, x_3, x_4]/(x_1 + x_2 + x_3 + x_4, x_1 x_2 - x_3 x_4, \mathcal{U}_1, \mathcal{U}_2).$$

For $n > 2$ the last three terms of the above sequence are at least quadratic and, hence, have degree at least 4 (recall that $\deg a_i \geq 4$ for all i). For $n = 2$, $\mathcal{U}_2 = u_2''$ which is linear and we get the relations $x_1 + x_2 = 0$, $x_3 + x_4 = 0$.   □

Lemma 4.2 $f_{14} \neq 0$ and $f_{03} \neq 0$.

Proof   With the above lemma the proof follows the lines of [11].   □

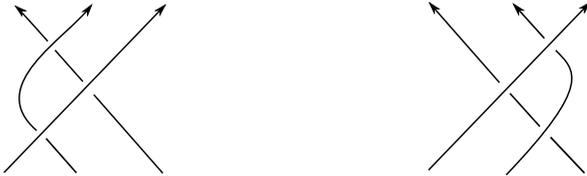Hence f_{14} and f_{03} are isomorphisms and we arrive at the desired conclusion.   □

4.3   R3

Figure 22

The complex assigned to the tangle on the left-hand side of Figure 22 is

$$
\begin{array}{ccccc}
 & C(\Gamma_{011})\{-1\} & & C(\Gamma_{100})\{-2\} & \\
 & \oplus & & \oplus & \\
0 \longrightarrow C(\Gamma_{111}) \xrightarrow{d^{-3}} C(\Gamma_{101})\{-1\} \xrightarrow{d^{-2}} C(\Gamma_{010})\{-2\} \xrightarrow{d^{-1}} C(\Gamma_{000})\{-3\} \longrightarrow 0. \\
 & \oplus & & \oplus & \\
 & C(\Gamma_{110})\{-1\} & & C(\Gamma_{001})\{-2\} &
\end{array}
$$

Direct sum decompositions II and III show that

$$C(\Gamma_{101}) \cong C(\Gamma_{100})\{1\} \oplus C(\Gamma_{100})\{-1\},$$
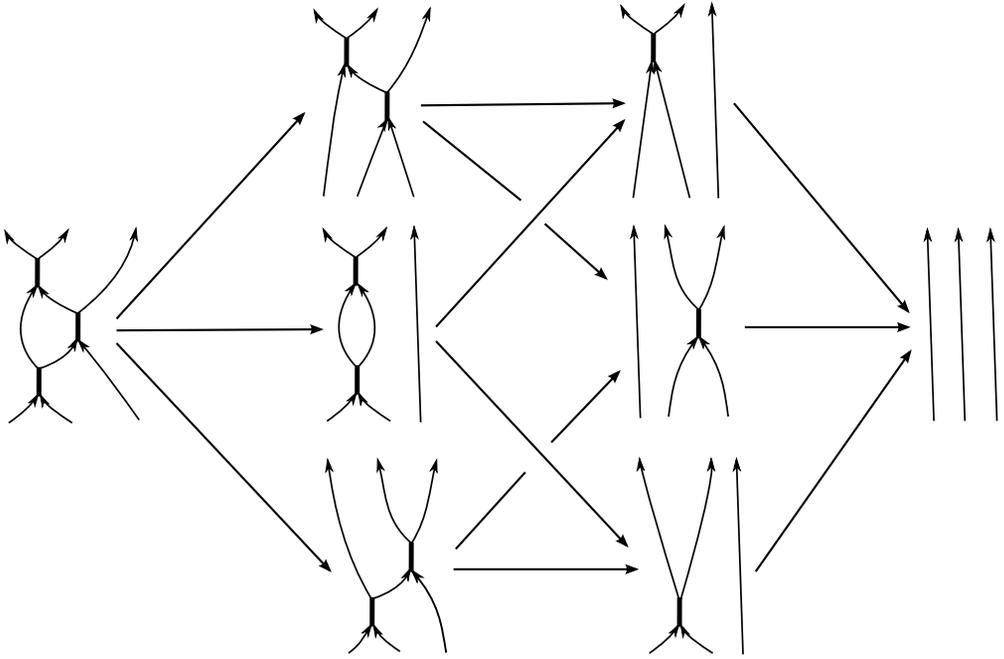
Figure 23: Reidemeister 3 complex

and

$$C(\Gamma_{111}) \cong C(\Gamma_{100}) \oplus \Upsilon.$$

Inserting these and using arguments analogous to those used in the decomposition proofs we reduce the original complex to

$$0 \longrightarrow \Upsilon \xrightarrow{d^{-3}} \begin{matrix} C(\Gamma_{011})\{-1\} \\ \oplus \\ C(\Gamma_{110})\{-1\} \end{matrix} \xrightarrow{d^{-2}} \begin{matrix} C(\Gamma_{010})\{-2\} \\ \oplus \\ C(\Gamma_{100})\{-2\} \end{matrix} \xrightarrow{d^{-1}} C(\Gamma_{000})\{-3\} \longrightarrow 0.$$

Proposition 4.3 *Assume $n > 2$, then for every arrow in* Figure 24 *from object A to B the space of grading-preserving morphisms*

$$\mathrm{Hom}_{hmf}(C(A), C(B)\{-1\})$$

is one dimensional. Moreover, the composition of any two arrows

$$C(A) \longrightarrow C(B)\{-1\} \longrightarrow C(C)\{-2\}$$
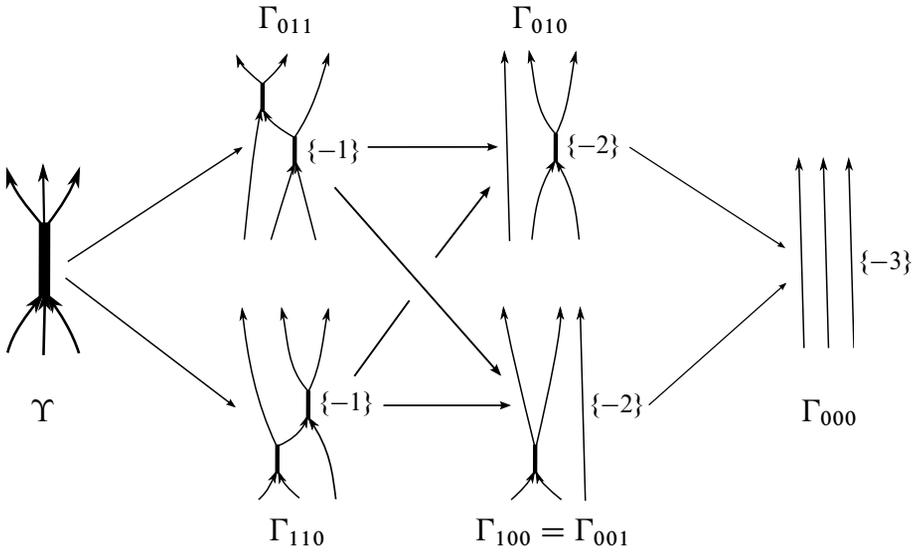
is nonzero.

Figure 24: Reidemeister 3 complex reduced

Proof Once again the maps in question are all of degree ≤ 2, and noticing that these remain nonzero when we work over the ring $\mathbb{Q}[a_0, \ldots, a_{n-2}]$, we can revert to the calculations in [11]. □

Hence, this complex is invariant under the "flip" which takes x_1 to x_3 and x_4 to x_6. This flip takes the complex associated to the braid on the left-hand side of Figure 22 to the one on the right-hand side. □

5 Remarks

Given a diagram D of a link L let $C_n(D)$ be the equivariant sl_n chain complex constructed above. The homotopy class of $C_n(D)$ is an invariant of L and consists of free $\mathbb{Q}[a_0, \ldots, a_{n-2}]$–modules where the a_i's are coefficients with $\deg(a_i) = 2(n-i)$. The cohomology of this complex $H_n(D)$ is a graded $\mathbb{Q}[a_0, \ldots, a_{n-2}]$–module. For a moment, let us consider the case where all the $a_i = 0$ for $1 \leq i \leq n-2$, and denote by $C_{n,a}(D)$ and $H_{n,a}(D)$ the corresponding complex and cohomology groups with $a = a_0$. Here the cohomology $H_{n,a}(D)$ is a finitely generated $\mathbb{Q}[a]$–module and we can decompose it as direct sum of torsion modules $\mathbb{Q}[a]/(a^k)$ for various k and free modules $\mathbb{Q}[a]$. Let $H'_{n,a}(D) = H_{n,a}(D)/\operatorname{Tor}_{n,a}(D)$, where $\operatorname{Tor}_{n,a}(D)$ is the torsion submodule. Just like in the sl_2 case in [10] we have:

Proposition 5.1 $H'_{n,a}(D)$ *is a free* $\mathbb{Q}[a]$–*module of rank* n^m, *where* m *is the number of components of* L.

Proof If we quotient $C_{n,a}(D)$ by the subcomplex $(a-1)C_{n,a}(D)$ we arrive at the complex studied by Gornik in [5], where he showed that its rank is n^m. The ranks of our complex and his are the same. □

In some sense this specialization is isomorphic to n copies of the trivial link homology which assigns to each link a copy of $\mathbb{Q}$ for each component, modulo grading shifts. In [14], M Mackaay and P Vaz studied similar variants of the sl_3–theory working over the Frobenius algebra $\mathbb{C}[x]/(x^3 + ax^2 + bx + c)$ with $a, b, c \in \mathbb{C}$ and arrived at three isomorphism classes of homological complexes depending on the number of distinct roots of the polynomial $x^3 + ax^2 + bx + c$. They showed that multiplicity three corresponds to the sl_3–homology of [8], one root of multiplicity two is a modified version of the original sl_2 or Khovanov homology, and distinct roots correspond to the "Lee-type" deformation. We expect an interpretation of their results in the equivariant version. Moreover, it would be interesting to understand these specialization for higher n and we foresee similar decompositions, that is, we expect the homology theories to break up into isomorphism classes corresponding to the number of distinct "roots" in the decomposition of the polynomial $x^n + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$.

The sl_2–homology and sl_3–homology for links, as well as their deformations, are defined over $\mathbb{Z}$; so far no such construction exists for $n > 3$.

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