

Faithfulness of a functor of Quillen

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There exists a canonical functor from the category of fibrant objects of a model category modulo cylinder homotopy to its homotopy category. We show that this functor is faithful under certain conditions, but not in general.

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1 Introduction

We let \mathcal{M} be a model category. Quillen defines in [5, Chapter I, Section 1] a homotopy relation on the full subcategory $\mathbf{Fib}(\mathcal{M})$ of fibrant objects, using cylinders. He obtains a quotient category $\mathbf{Fib}(\mathcal{M})/\overset{\mathcal{C}}{\sim}$ and a canonical functor

$$\mathbf{Fib}(\mathcal{M})/\overset{\mathcal{C}}{\sim} \rightarrow \mathrm{Ho} \mathbf{Fib}(\mathcal{M}).$$

The question occurs whether this functor is faithful.

We show that it is faithful if \mathcal{M} is left proper and fulfills an additional technical condition. Moreover, we show by an example that it is not faithful in general.

Conventions and notation

- The composite of morphisms $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ is denoted by $fg: X \rightarrow Z$.
- Given $n \in \mathbb{N}_0$, we abbreviate $\mathbb{Z}/n := \mathbb{Z}/n\mathbb{Z}$. Given $k, m, n \in \mathbb{N}_0$, we write $k: \mathbb{Z}/m \rightarrow \mathbb{Z}/n$, $a + m\mathbb{Z} \mapsto ka + n\mathbb{Z}$, provided n divides km .
- Given a category \mathcal{C} with finite coproducts and objects $X, Y \in \mathrm{Ob} \mathcal{C}$, we denote by $X \amalg Y$ a (chosen) coproduct. The embedding $X \rightarrow X \amalg Y$ is denoted by emb_0 , the embedding $Y \rightarrow X \amalg Y$ by emb_1 . Given morphisms $f: X \rightarrow Z$ and $g: Y \rightarrow Z$ in \mathcal{C} , the induced morphism $X \amalg Y \rightarrow Z$ is denoted by $\begin{pmatrix} f \\ g \end{pmatrix}$.
- Given a category \mathcal{C} and an object $X \in \mathrm{Ob} \mathcal{C}$, the category of objects in \mathcal{C} under X will be denoted by $(X \downarrow \mathcal{C})$. The objects in $(X \downarrow \mathcal{C})$ are denoted by (Y, f) , where $Y \in \mathrm{Ob} \mathcal{C}$ and $f: X \rightarrow Y$ is a morphism in \mathcal{C} .

2 Preliminaries from homotopical algebra

We recall some basic facts from homotopical algebra. Our main reference is Quillen [5, Chapter I, Section 1].

Model categories

Throughout this note, we let \mathcal{M} be a model category [5, Chapter I, Section 1, Definition 1]. In \mathcal{M} , there are three kinds of distinguished morphisms, called *cofibrations*, *fibrations* and *weak equivalences*. Cofibrations are closed under pushouts. If weak equivalences in \mathcal{M} are closed under pushouts along cofibrations, \mathcal{M} is said to be *left proper* [3, Definition 13.1.1(1)].

An object $X \in \text{Ob } \mathcal{M}$ is said to be *fibrant* if the unique morphism $\mathcal{M} \rightarrow *$ is a fibration, where $*$ is a (chosen) terminal object in \mathcal{M} . The full subcategory of \mathcal{M} of fibrant objects is denoted by $\mathbf{Fib}(\mathcal{M})$.

The *homotopy category* of $\mathcal{C} = \mathcal{M}$ resp. $\mathcal{C} = \mathbf{Fib}(\mathcal{M})$ is a localisation of \mathcal{C} with respect to the weak equivalences in \mathcal{C} and is denoted by $\text{Ho } \mathcal{C}$. The localisation functor of $\text{Ho } \mathcal{C}$ is denoted by $\Gamma = \Gamma^{\text{Ho } \mathcal{C}}: \mathcal{C} \rightarrow \text{Ho } \mathcal{C}$.

Given an object $X \in \text{Ob } \mathcal{M}$, the category $(X \downarrow \mathcal{M})$ of objects under X obtains a model category structure where a morphism in $(X \downarrow \mathcal{M})$ is a weak equivalence resp. a cofibration resp. a fibration if and only if it is one in \mathcal{M} .

Homotopies

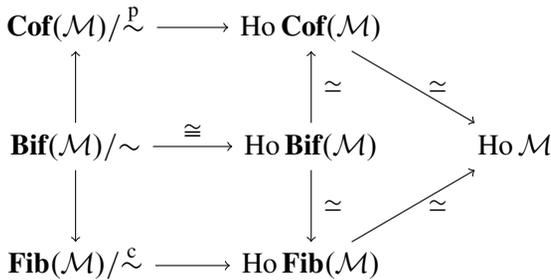
A *cylinder* for an object $X \in \text{Ob } \mathcal{M}$ consists of an object $Z \in \text{Ob } \mathcal{M}$, a cofibration $\begin{pmatrix} \text{ins}_0 \\ \text{ins}_1 \end{pmatrix} = \text{ins} = \text{ins}^Z: X \amalg X \rightarrow Z$ and a weak equivalence $s = s^Z: Z \rightarrow X$ such that $\text{ins } s = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Given parallel morphisms $f, g: X \rightarrow Y$ in \mathcal{M} , we say that f is *cylinder homotopic* to g , written $f \stackrel{\sim}{\sim} g$, if there exists a cylinder Z for X and a morphism $H: Z \rightarrow Y$ with $\text{ins}_0 H = f$ and $\text{ins}_1 H = g$. In this case, H is said to be a *cylinder homotopy* from f to g . (In the literature, cylinder homotopy is also called left homotopy; see Quillen [5, Chapter I, Section 1, Definitions 3–4, Lemma 1].) The relation $\stackrel{\sim}{\sim}$ is reflexive and symmetric, but in general not transitive. Moreover, $\stackrel{\sim}{\sim}$ is compatible with composition in $\mathbf{Fib}(\mathcal{M})$. We denote by $\mathbf{Fib}(\mathcal{M})/\stackrel{\sim}{\sim}$ the quotient category of $\mathbf{Fib}(\mathcal{M})$ with respect to the congruence generated by $\stackrel{\sim}{\sim}$.

Quillen’s homotopy category theorem

There are dual notions to fibrant objects, cylinders, cylinder homotopic $\overset{\mathcal{C}}{\sim}$, the full subcategory of fibrant objects $\mathbf{Fib}(\mathcal{M})$, its quotient category $\mathbf{Fib}(\mathcal{M})/\overset{\mathcal{C}}{\sim}$ and its homotopy category $\mathbf{Ho} \mathbf{Fib}(\mathcal{M})$, namely *cofibrant objects*, *path objects*, *path homotopic* $\overset{\mathcal{P}}{\sim}$, the full subcategory of cofibrant objects $\mathbf{Cof}(\mathcal{M})$, its quotient category $\mathbf{Cof}(\mathcal{M})/\overset{\mathcal{P}}{\sim}$ and its homotopy category $\mathbf{Ho} \mathbf{Cof}(\mathcal{M})$, respectively. Moreover, an object $X \in \text{Ob } \mathcal{M}$ is said to be bifibrant if it is cofibrant and fibrant. On the full subcategory of bifibrant objects $\mathbf{Bif}(\mathcal{M})$, the relations $\overset{\mathcal{C}}{\sim}$ and $\overset{\mathcal{P}}{\sim}$ coincide and yield a congruence. One writes $\sim := \overset{\mathcal{C}}{\sim} = \overset{\mathcal{P}}{\sim}$ in this case, and the quotient category is denoted by $\mathbf{Bif}(\mathcal{M})/\sim$. Moreover, $\mathbf{Ho} \mathbf{Bif}(\mathcal{M})$ is a localisation of $\mathbf{Bif}(\mathcal{M})$ with respect to the weak equivalences in $\mathbf{Bif}(\mathcal{M})$.

Quillen’s homotopy category theorem [5, Chapter I, Section 1, Theorem 1] (cf Hovey [4, Corollary 1.2.9, Theorem 1.2.10]) states that the various inclusion and localisation functors induce the following commutative diagram, where the functors labeled by \simeq are equivalences and the functor labeled by \cong is an isofunctor.



In this note, we treat the question whether the functors $\mathbf{Fib}(\mathcal{M})/\overset{\mathcal{C}}{\sim} \rightarrow \mathbf{Ho} \mathbf{Fib}(\mathcal{M})$ and $\mathbf{Cof}(\mathcal{M})/\overset{\mathcal{P}}{\sim} \rightarrow \mathbf{Ho} \mathbf{Cof}(\mathcal{M})$ are faithful. By duality, it suffices to consider the first functor.

The model category $\mathbf{mod}(\mathbb{Z}/4)$

The category $\mathbf{mod}(\mathbb{Z}/4)$ of finitely generated modules over $\mathbb{Z}/4$ is a Frobenius category (with respect to all short exact sequences), that is, there are enough projective and injective objects in $\mathbf{mod}(\mathbb{Z}/4)$ and, moreover, these objects coincide (we call such objects bijective). Therefore $\mathbf{mod}(\mathbb{Z}/4)$ carries a canonical model category structure (cf Hovey [4, Section 2.2]): The cofibrations are the monomorphisms and the fibrations are the epimorphisms in $\mathbf{mod}(\mathbb{Z}/4)$. Every object in $\mathbf{mod}(\mathbb{Z}/4)$ is bifibrant, and the weak equivalences are precisely the homotopy equivalences, where parallel morphisms f and g are homotopic if $g - f$ factors over a bijective object in $\mathbf{mod}(\mathbb{Z}/4)$. That is,

the weak equivalences in $\mathbf{mod}(\mathbb{Z}/4)$ are the stable isomorphisms and the homotopy category of $\mathbf{mod}(\mathbb{Z}/4)$ is isomorphic to the stable category [2, Chapter I, Section 2.2] of $\mathbf{mod}(\mathbb{Z}/4)$.

We remark that every object in $\mathbf{mod}(\mathbb{Z}/4)$ is isomorphic to $(\mathbb{Z}/4)^{\oplus k} \oplus (\mathbb{Z}/2)^{\oplus l}$ for some $k, l \in \mathbb{N}_0$, and every bijective object is isomorphic to $(\mathbb{Z}/4)^{\oplus k}$ for some $k \in \mathbb{N}_0$.

3 Faithfulness of the functor $\mathbf{Fib}(\mathcal{M})/\overset{\sim}{\sim} \rightarrow \mathbf{Ho Fib}(\mathcal{M})$

We give a sufficient criterion for the functor under consideration to be faithful.

Proposition *If the model category \mathcal{M} is left proper and if $w \amalg w$ is a weak equivalence for every weak equivalence w in \mathcal{M} , then $\overset{\sim}{\sim}$ is a congruence on $\mathbf{Fib}(\mathcal{M})$ and the canonical functor $\mathbf{Fib}(\mathcal{M})/\overset{\sim}{\sim} \rightarrow \mathbf{Ho Fib}(\mathcal{M})$ is faithful.*

Proof We suppose given fibrant objects X and Y and morphisms $f, g: X \rightarrow Y$ with $\Gamma f = \Gamma g$ in $\mathbf{Ho Fib}(\mathcal{M})$. By [1, Theorem 1(ii)], there exists a weak equivalence $w: X' \rightarrow X$ such that $wf \overset{p}{\sim} wg$. It follows that $wf \overset{\sim}{\sim} wg$ by [5, Chapter I, Section 1, dual of Lemma 5], that is, there exists a cylinder Z' for X' and a cylinder homotopy $H': Z' \rightarrow Y$ from wf to wg . We let

$$\begin{array}{ccc} X' \amalg X' & \xrightarrow[\approx]{w \amalg w} & X \amalg X \\ \text{ins}^{Z'} \downarrow & & \downarrow i \\ Z' & \xrightarrow[\approx]{w'} & Z \end{array}$$

be a pushout of $w \amalg w$ along $\text{ins}^{Z'}$. By assumption, $w \amalg w$ and w' are weak equivalences. Since

$$(w \amalg w)\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) = \text{ins}^{Z'} s^{Z'} w,$$

there exists a unique morphism $s: Z \rightarrow X$ with

$$\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) = i s \quad \text{and} \quad s^{Z'} w = w' s.$$

Then s is a weak equivalence since $s^{Z'}$, w and w' are weak equivalences and therefore Z becomes a cylinder for X with $\text{ins}^Z := i$ and $s^Z := s$. Moreover,

$$(w \amalg w)\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) = \text{ins}^{Z'} H'$$

implies that there exists a unique morphism $H: Z \rightarrow Y$ with

$$\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) = \text{ins}^Z H \quad \text{and} \quad H' = w' H.$$

So in particular $f \simeq g$.

$$\begin{array}{ccccc}
 X' \amalg X' & \xrightarrow[w \approx]{w \amalg w} & X \amalg X & \xrightarrow{(f/g)} & Y \\
 \text{ins}^{Z'} \downarrow & & \downarrow \text{ins}^Z & & \parallel \\
 Z' & \xrightarrow[w \approx]{} & Z & \xrightarrow{H} & Y \\
 s^{Z'} \downarrow \cong & & \cong \downarrow s^Z & & \\
 X' & \xrightarrow[w \approx]{} & X & &
 \end{array}$$

Altogether, we have shown that morphisms in $\mathbf{Fib}(\mathcal{M})$ represent the same morphism in $\text{Ho } \mathbf{Fib}(\mathcal{M})$ if and only if they are cylinder homotopic. In particular, \simeq is a congruence on $\mathbf{Fib}(\mathcal{M})$. □

The following counterexample shows that the canonical functor

$$\mathbf{Fib}(\mathcal{M})/\simeq \rightarrow \text{Ho } \mathbf{Fib}(\mathcal{M})$$

is not faithful in general.

Example We consider the category $(\mathbb{Z}/4 \downarrow \mathbf{mod}(\mathbb{Z}/4))$ of finitely generated $\mathbb{Z}/4$ -modules under $\mathbb{Z}/4$ with the model category structure inherited from $\mathbf{mod}(\mathbb{Z}/4)$; see Section 2. All objects in $(\mathbb{Z}/4 \downarrow \mathbf{mod}(\mathbb{Z}/4))$ are fibrant since all objects in $\mathbf{mod}(\mathbb{Z}/4)$ are fibrant.

We study morphisms $(\mathbb{Z}/4, 2) \rightarrow (\mathbb{Z}/4 \oplus \mathbb{Z}/2, (2 \ 0))$ in $(\mathbb{Z}/4 \downarrow \mathbf{mod}(\mathbb{Z}/4))$. We let (Z, t) be a cylinder of $(\mathbb{Z}/4, 2)$ and we let $H: (Z, t) \rightarrow (\mathbb{Z}/4 \oplus \mathbb{Z}/2, (2 \ 0))$ be a cylinder homotopy (from $\text{ins}_0 H$ to $\text{ins}_1 H$). Then we have a weak equivalence $(Z, t) \rightarrow (\mathbb{Z}/4, 2)$ in $(\mathbb{Z}/4 \downarrow \mathbf{mod}(\mathbb{Z}/4))$ and hence a weak equivalence $Z \rightarrow \mathbb{Z}/4$ in $\mathbf{mod}(\mathbb{Z}/4)$. Thus Z is bijective and therefore we may assume that $Z = (\mathbb{Z}/4)^{\oplus k}$. Since ins_0 and ins_1 are morphisms from $(\mathbb{Z}/4, 2)$ to (Z, t) , we have $2\text{ins}_0 = t = 2\text{ins}_1$ and hence $\text{ins}_0 \equiv_2 \text{ins}_1$ as morphisms from $\mathbb{Z}/4$ to Z . But this implies that the second components of $\text{ins}_0 H$ and $\text{ins}_1 H$ are the same. In other words, we have shown that cylinder homotopic morphisms from $(\mathbb{Z}/4, 2)$ to $(\mathbb{Z}/4 \oplus \mathbb{Z}/2, (2 \ 0))$ coincide in the second component. It follows that the morphisms $(1 \ 0): (\mathbb{Z}/4, 2) \rightarrow (\mathbb{Z}/4 \oplus \mathbb{Z}/2, (2 \ 0))$ and $(1 \ 1): (\mathbb{Z}/4, 2) \rightarrow (\mathbb{Z}/4 \oplus \mathbb{Z}/2, (2 \ 0))$ in $(\mathbb{Z}/4 \downarrow \mathbf{mod}(\mathbb{Z}/4))$ represent different morphisms in the quotient category $\mathbf{Fib}((\mathbb{Z}/4 \downarrow \mathbf{mod}(\mathbb{Z}/4)))/\simeq$.

On the other hand, since $\mathbb{Z}/4$ is bijective, the morphism $2: \mathbb{Z}/4 \rightarrow \mathbb{Z}/4$ is a weak equivalence in $\mathbf{mod}(\mathbb{Z}/4)$, and therefore $2: (\mathbb{Z}/4, 1) \rightarrow (\mathbb{Z}/4, 2)$ is a weak equivalence in $(\mathbb{Z}/4 \downarrow \mathbf{mod}(\mathbb{Z}/4))$. But $2(1 \ 0) = 2(1 \ 1)$ as morphisms from $(\mathbb{Z}/4, 1)$ to $(\mathbb{Z}/4 \oplus \mathbb{Z}/2, (2 \ 0))$ in $(\mathbb{Z}/4 \downarrow \mathbf{mod}(\mathbb{Z}/4))$, so in particular $\Gamma(2(1 \ 0)) = \Gamma(2(1 \ 1))$ and hence $\Gamma(1 \ 0) = \Gamma(1 \ 1)$.

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