Correction to “Topological nonrealization results via the Goodwillie tower approach to iterated loop space homology”

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Manfred Stelzer has pointed out that part of Corollary 4.5 of [1] was not sufficiently proved, and, indeed, is likely incorrect as stated. This necessitates a little more argument to finish the proof of the main theorem of [1]. The statement of this theorem, and all the examples, remain unchanged.

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In [1], the author showed that certain unstable modules over the mod 2 Steenrod algebra couldn’t be realized as the reduced mod 2 cohomology of a space. The modules have the form $\Sigma^n M$, where $M$ is an unstable module of a special sort. The method of proof was to use a 2nd quadrant spectral sequence converging to $H^*(\Omega^n X; \mathbb{Z}/2)$ to show that, were a space $X$ to exist whose cohomology realized $\Sigma^n M$, $H^*(\Omega^n X; \mathbb{Z}/2)$ could not admit a cup product compatible with Steenrod operations.

The spectral sequence for $n > 1$ is a newish one, arising from the Goodwillie tower of the functor $X \mapsto \Sigma^\infty \Omega^n X$, and Section 4 of [1] is devoted to collecting and proving some basic facts about this spectral sequence. I thank Manfred Stelzer for pointing out that part of Corollary 4.5 is likely over optimistic, and certainly was not sufficiently proved.

We assume notation as in [1].

In Corollary 4.5, it was asserted that if $\tilde{H}^*(X; \mathbb{Z}/2) \simeq \Sigma^n M$ with $M$ unstable and also has no nontrivial cup products, then in the spectral sequence, one will have $E_3^{-1,*} = E_2^{-1,*} = E_1^{-1,*}$ and $E_2^{-2,*} = E_1^{-2,*}$. My mistake was in not adequately considering possible differentials on elements in $E_1^{-3,*}$. Under the hypotheses on the cup product, the $d_1$ differential on such terms will be 0, by the same argument given explaining why $d_1$ is zero on terms of the form $L(x \otimes y)$: by comparison to the classical Eilenberg–Moore spectral sequence. But there is no apparent reason why $d_2$ need also be zero on such terms. We can only conclude that $E_2^{-1,*} = E_1^{-1,*}$, and $E_2^{-2,*} = E_1^{-2,*}$.

Corollary 4.5 is used at one critical point in the proof of the main theorem given in Section 5. Lemma 5.3 asserts that a certain element in $E_1^{-1,2d+2k+2+1}$ is not a
boundary. The argument given is that for dimension reasons, no $d_r$ for $r > 2$ could have nonzero image in this bigrading. Implicit is that Corollary 4.5 takes care of $d_1$ and $d_2$. In light of the comments above, one needs a new argument for $d_2$.

It turns out that, except for one special case, a dimension argument still works: $E_2^{3, 2d + 2k + 2}$ contains no elements of the form $\sigma^3 L_{n-1} (x \otimes y \otimes z)$. There are two extreme cases to consider: if $x$, $y$, and $z$ are all chosen from the top of $N_0$, and if $x$ and $y$ are chosen from the bottom of $N_0$ and $z$ is chosen from the bottom of $M_1$.

In the first case, $|x| = |y| = |z| = m + 2^k$, and so $\sigma^3 L_{n-1} (x \otimes y \otimes z)$ has bidegree $(-3, 3m + 3 \cdot 2^k + 2n + 1)$. In the second case, $|x| = |y| = d + 2^k$ and $|z| = l + 2^{k+1}$, and so $\sigma^3 L_{n-1} (x \otimes y \otimes z)$ has bidegree $(-3, 2d + l + 2^{k+2} + 2n + 1)$.

We are assuming inequality (5–3), which says that $2^k > 4m - 2l + 2n - 2$. One also has that $0 \leq l \leq d \leq m$ and $n \geq 1$. One can then check that, indeed,

$$3m + 3 \cdot 2^k + 2n + 1 < 2d + 2^{k+2} + 2 < 2d + l + 2^{k+2} + 2n + 1,$$

unless we are in the special case $k = 0, n = 1, l = d = m = 0$.

In this final special case, $n = 1$, so we are trying to use the classical Eilenberg–Moore spectral sequence to show that, if $M$ is a $\mathbb{Z}/2$ vector space concentrated in degree 0, there cannot exist a space $X$ with $\tilde{H}^*(X; \mathbb{Z}/2) \simeq \Sigma M \otimes \Phi(0, 2)$, if all cup products are zero. Such a space will necessarily fit into a cofibration sequence of the form

$$\bigvee S^4 \to \bigvee \Sigma \mathbb{R} P^2 \to X.$$

We leave it to the reader to check that, by appropriately including $S^4$ into the first wedge, and projecting out onto a $\Sigma \mathbb{R} P^2$ in the second wedge, one sees that $X$ will have a “subquotient” $Y$ with $\tilde{H}^*(Y; \mathbb{Z}/2) \simeq \Sigma \Phi(0, 2)$, and still with all cup products 0.

Similar to, but simpler than, arguments in Section 6 of [1] (which dealt with $\Sigma^2 \Phi(1, 3)$), our arguments show that such a $Y$ can’t exist. Repressing some suspensions from the notation, Figure 1 shows all of $E_1^{*,*}$ in total degree less than or equal to 4, in the Eilenberg–Moore spectral sequence converging to $H^*(\Omega Y; \mathbb{Z}/2)$.

As cup products are assumed zero, $E_2^{*,*} = E_1^{*,*}$. Furthermore, $d_2 (a \otimes a \otimes a) = 0$ (and thus not $c$), because $a \otimes a \otimes a = (a \otimes a) \ast a$ and $d_2$ is a derivation with respect to the shuffle product $\ast$. Thus through degree 4, $F^{-2} H^*(\Omega Y; \mathbb{Z}/2)$ would have a basis given by elements $1, \alpha, \beta, \delta, \epsilon_1, \epsilon_2, \gamma, \omega$, in respective degrees 0, 1, 2, 2, 3, 3, 4, and 4, and represented by $1, a, b, a \otimes a, a \otimes b, b \otimes a, c$, and $b \otimes b$. The structure of $\Phi(0, 2)$ ($\text{Sq}^1 a = b, \text{Sq}^2 b = c$) shows that $\gamma = \beta^2 = \alpha^4$. Furthermore,
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Figure 1: $E_1^{s,t}$ when $\tilde{H}^*(Y; \mathbb{Z}/2) \simeq \Sigma \Phi(0, 2)$

$$\begin{array}{cccccc}
a \otimes a \otimes a \otimes a & & & & & 8 \\
a \otimes a \otimes a & b \otimes b & & & & 7 \\
a \otimes b, b \otimes a & c & & & & 6 \\
a \otimes a & & & & & 5 \\
& b & & & & 4 \\
& & a & & & 3 \\
& & & b & & 2 \\
& & & & a & 1 \\
& & & & & 1 \\
& & & & & 0 \\
& & & & & -1 \\
& & & & & -2 \\
& & & & & -3 \\
& & & & & -4 \\
\end{array}$$

$\text{Sq}^1 \delta = \epsilon_1 + \epsilon_2 = \alpha \cup \beta$, as all three are represented by $a \otimes b + b \otimes a$. One then gets a contradiction, as

$$0 = \text{Sq}^1 \text{Sq}^1 \delta = \text{Sq}^1 (\alpha \cup \beta) = \beta^2 = \gamma \neq 0.$$ 

We end by observing that $\tilde{H}^*(\text{SU}(3)/\text{SO}(3); \mathbb{Z}/2) \simeq \Sigma \Phi(0, 2)$. Here, of course, cup products are not zero, due to Poincaré duality.

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References