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On the tunnel number and the Morse–Novikov number of knots

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Let L be a link in S^3; denote by $\mathcal{MN}(L)$ the Morse–Novikov number of L and by $t(L)$ the tunnel number of L. We prove that $\mathcal{MN}(L) \leqslant 2t(L)$ and deduce several corollaries.

57M25, 57M27, 57R35, 57R70; 57R19, 57R45

1 Introduction

1.1 Background

Let L be a link in S^3, that is, an embedding of several copies of S^1 to S^3. First off, we recall the definition of three numerical invariants of L. In the sequel $N(L)$ denotes a closed tubular neighbourhood of L.

(A) (Tunnel number) An arc γ in S^3 is called a *tunnel* for L if $\gamma \cap L$ consists of the two endpoints of γ. The tunnel number $t(L)$ is the minimal number m of disjoint tunnels $\gamma_1, \dots, \gamma_m$ such that the closure of $S^3 \setminus N(L \cup \gamma_1 \cup \dots \cup \gamma_m)$ is a handlebody. The tunnel number was introduced by B Clark [1]; this invariant was studied in the works of T Kohno [11], T Kobayashi [9], T Kobayashi and Y Rieck [10], M Lustig and Y Moriah [13], K Morimoto [15; 14; 16], K Morimoto, M Sakuma and Y Yokota [17; 18], M Scharlemann and J Schultens [23; 24] and others. M Scharlemann and J Schultens [23] proved that $t(nK) \geqslant n$ for any n (here nK stands for the connected sum of n copies of the knot K). They proved also that $t(nK) \geqslant \frac{2}{5} n t(K)$ if K is not a 2–bridge knot [24]. T Kohno [11] gave an estimate of tunnel number of knots in terms of quantum invariants. K Morimoto, M Sakuma and Y Yokota [18] computed the tunnel number of all prime knots with $\leqslant 10$ crossings.

For any two knots K_1, K_2 we have $t(K_1 \# K_2) \leqslant t(K_1) + t(K_2) + 1$. K Morimoto [15] constructed knots $K_1,\ K_2$ such that $t(K_1 \# K_2) < t(K_1) + t(K_2)$. T Kobayashi and Y Rieck [10] define the growth rate for a knot K by the formula

$$gr_t(K) = \limsup_{m \to \infty} \frac{t(mK) - m t(K)}{m-1}.$$

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It follows from results of [24] that $gr_t(K) \geqslant -1 - \frac{2}{3}t(K)$.

(B) (Bridge numbers) Let $S^3 = H_1 \cup H_2$ be a Heegaard splitting of S^3; put $\Sigma = H_1 \cap H_2$, and $g = g(\Sigma)$. We say (following H Doll [2]) that L is in an n–bridge position with respect to Σ if Σ intersects L in $2n$ points and $\Sigma \cap H_i$ is a union of n trivial arcs in H_i for $i = 1, 2$. The g–bridge number $b_g(L)$ of L is defined as the minimal number n such that L can be put in an n–bridge position with respect to a Heegaard decomposition of genus g. Thus $b_0(L)$ is the classical bridge number as defined by H Schubert [25]. We have

$$t(L) \leqslant g + b_g(L) - 1.$$

(C) (Morse–Novikov numbers) Pick an orientation preserving trivialisation of the normal bundle of L. The corresponding diffeomorphism of disc bundles $\phi\colon L \times D^2 \to N(L)$ will be called *framing* of L. Let C_L denote the closure of $S^3 \setminus N(L)$. A Morse function $f\colon C_L \to S^1$ is called *regular* if its restriction to the boundary $\partial N(L)$ is the canonical fibration over the circle: $(f \circ \phi)(l, z) = z/|z|$. The number of the critical points of index i of a regular Morse function f will be denoted by $m_i(f)$; the total number of critical points of f will be denoted by $m(f)$. The minimal value of $m(f)$ over all possible framings ϕ and Morse maps $f\colon C_L \to S^1$ is called *the Morse–Novikov number of the link* L and denoted by $\mathcal{MN}(L)$ (see Veber, Pajitnov and Rudolph [26]).

The Morse–Novikov theory of circle-valued maps (see Novikov [19] and Pajitnov [20; 21]) allows one to obtain homological lower bounds for $\mathcal{MN}(L)$ as follows. Let $\bar{C}_L$ be the infinite cyclic covering induced by f from the covering $\mathbb{R} \to S^1$. Denote the ring $\mathbb{Z}[t, t^{-1}]$ by Λ, and the ring $\mathbb{Z}((t))$ by $\widehat{\Lambda}$. The $\widehat{\Lambda}$–module

$$\mathcal{N}_*(L) = H_*(\bar{C}_L) \otimes_\Lambda \widehat{\Lambda}$$

is called *the Novikov homology* of the link L. The rank and torsion numbers of the $\widehat{\Lambda}$–module $\mathcal{N}_1(L)$ are denoted respectively by $b_1(L)$ and $q_1(L)$. We have then [26]

$$\mathcal{MN}(L) \geqslant 2(b_1(L) + q_1(L)).$$

In case when the Novikov numbers are not sufficient to determine the $\mathcal{MN}(L)$ the twisted Novikov numbers (introduced by H Goda and the author in [5]) are useful.

As for upper bounds for $\mathcal{MN}(L)$, not much is known. H Goda announced in [4] that $\mathcal{MN}(L) \leqslant 2$ for every prime link L with $\leqslant 10$ crossings. M Hirasawa proved that for every 2–bridge knot K we have $\mathcal{MN}(K) \leqslant 2$ (unpublished). In the papers [22; 7] of L Rudolph and M Hirasawa it is proved that $\mathcal{MN}(K) \leqslant 4g_f(K)$ where $g_f(K)$ is the *free genus* of K, that is, the minimal possible genus of a Seifert surface Σ bounding K such that $S^3 \setminus \Sigma$ is an open handlebody.

1.2 Main results

The main result of this work is the following theorem.

Theorem 1.1 *For every link L in S^3 we have*

$$\mathcal{MN}(L) \leqslant 2t(L). \tag{1}$$

The following corollaries are easily deduced.

Corollary 1.2 *For every g we have*

$$\mathcal{MN}(L) \leqslant 2(g + b_g(L) - 1).$$

Corollary 1.3 *For every tunnel number 1 knot K we have $\mathcal{MN}(K) \leqslant 2$. In particular this holds for any $(1,1)$–knot K.*

Corollary 1.4 *For every link L we have*

$$q_1(L) + b_1(L) \leqslant t(L).$$

Corollary 1.5 *For every knot K*

$$gr_t(K) \geqslant -t(K) + q_1(K).$$

2 Proof of Theorem 1.1

Let $m = t(L)$. Pick a framing $\phi\colon L \times D^2 \to N(L)$. Then the manifold $C_L = S^3 \setminus N(L)$ is obtained from ∂C_L by attaching m one-handles and then attaching a handlebody of genus $m+1$ to the resulting cobordism. So we obtain a Morse function $g\colon C_L \to \mathbb{R}$ which is constant on ∂C_L and has the following Morse numbers: $m_0(g) = 0$, $m_1(g) = m$, $m_2(g) = m+1$, $m_3(g) = 1$. Pick any Morse map $h\colon C_L \to S^1$ such that $h|\partial C_L$ is the canonical fibration: $(h \circ \phi)(l, z) = z/|z|$. The 1–form induced by h from the canonical volume form on S^1 will be denoted by dh. Consider a closed 1–form $\omega_\epsilon = dg + \epsilon dh$. For $\epsilon > 0$ sufficiently small ω_ϵ is a Morse form with the same Morse numbers as dg. The De Rham cohomology class of the 1–form

$$\frac{1}{\epsilon}\omega_\epsilon = \frac{1}{\epsilon} dg + dh$$

is the same as that of dh; therefore this form is the differential of a Morse map $g_1\colon C_L \to S^1$ homotopic to h. [1] Observe that the map g_1 is a regular Morse map;

[1] A similar perturbation argument was used by J C Sikorav in another context; see Pajitnov [20].

it has one local maximum, and the standard elimination procedure (see for example Lemmas 3.1 and 3.2 of [26] for details) gives us a regular Morse function $f\colon C_L \to S^1$ with $m_0(f) = 0,\ m_1(f) \leqslant m,\ m_2(f) \leqslant m,\ m_3(f) = 0$. Thus $\mathcal{MN}(L) \leqslant 2m$.

3 Examples

Theorem 1.1 can be used in two ways. A lot of information is available about the tunnel numbers, and this implies new estimates for the Morse–Novikov numbers of knots. On the other hand, the Novikov torsion number $q_1(K)$ is an invariant which is easy to compute, and in many cases this gives new information about the sequence of tunnel numbers $t(nK)$ for a given knot K. Let us consider two examples:

(A) (Pretzel knots) Let q, r be positive integers; denote by $\mathcal{P}$ the $(2r+1)$–stranded pretzel knot $P(2q+1, -2q-1, 2q+1, \dots, 2q+1)$. The knot $\mathcal{P}$ for $q = 1,\ r = 2$ is depicted below.

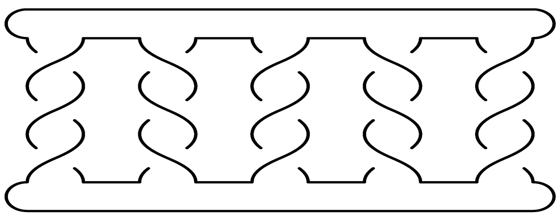

Figure 1: Pretzel knot

It is clear that $t(\mathcal{P}) \leqslant 2r$. An easy computation of the Alexander module via the Seifert matrix gives

$$\mathcal{N}_1(\mathcal{P}) \approx \big(\widehat{\Lambda}/XY\widehat{\Lambda}\big)^r$$

where $X = qt-(q+1),\ Y = (q+1)t-q$. Thus $q_1(\mathcal{P}) = r$. Since $q_1(mK) = mq_1(K)$ for any knot K, we deduce that

$$\frac{1}{2}nt(\mathcal{P}) \leqslant nq_1(\mathcal{P}) \leqslant t(n\mathcal{P}).$$

In particular the growth rate of the knot satisfies $gr_t(K) \geqslant -\frac{1}{2}t(K)$.

(B) (A twisted $5_2\,\#\,5_2$) Let K be the knot obtained from the connected sum $5_2\,\#\,5_2$ by twisting (see Figure 2).

An easy computation shows that

$$\mathcal{N}_1(K) \approx \big(\widehat{\Lambda}/S\widehat{\Lambda}\big)^2$$

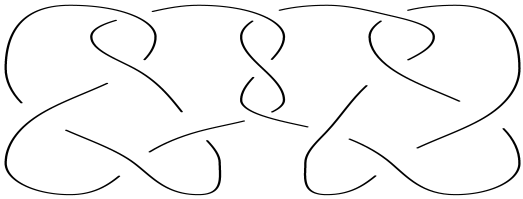

Figure 2: The twisted $5_2 \# 5_2$ knot

where $S = 2t^2 - 3t + 2$ is the Alexander polynomial of the knot 5_2. Thus $q_1(K) = 2$. Since $t(K) \leqslant 3$ we obtain

$$\frac{2}{3} n t(K) \leqslant n q_1(K) \leqslant t(nK).$$

We have therefore $gr_t(K) \geqslant -\frac{1}{3} t(K)$.

4 Relations with previously known results

A theorem of M Hirasawa says that $\mathcal{MN}(K) \leqslant 2$ if K is a two-bridge knot. Since $t(K) \leqslant b(K) - 1$ our theorem implies this result. Observe that M Hirasawa's proof uses H Schubert's presentation of 2–bridge knots, and can not be generalized to the case of arbitrary bridge number.

The inequality (1) implies also the upper bound

$$\mathcal{MN}(K) \leqslant 4 g_f(K)$$

obtained by L Rudolph and M Hirasawa [22; 7]. Indeed, J H Lee [12] has shown that $t(K) \leqslant 2 g_f(K)$.

In many cases the estimate of Theorem 1.1 is better than the free genus estimate. For example, let K be the pretzel knot $K = P(-2l, q, r)$ where $l \geqslant 2$ and $q, r \geqslant 3$ are odd numbers. Then $t(K) \leqslant 2$, and the Alexander polynomial of the knot equals

$$A(t) = l t^{q+r} - (2l - 1) t^{q+r-1} + \cdots - (2l - 1) t + l$$

(see the work [8] of D Kim and J Lee). Therefore K is not fibred, and $4 \geqslant \mathcal{MN}(K) \geqslant 2$. As for the genus of K, we have $g(K) \geqslant \deg A(t)/2 = (q + r)/2$, therefore the free genus of K is not less than $(q + r)/2$.

Theorem 1.1 leads to quick proofs of results about the Morse–Novikov numbers already known. The simplest cases are: the link A_n (the boundary of n–twisted unknotted annulus) and the twist knots K_n. See Figures 3 and 4. We shall assume that $n \geqslant 2$.

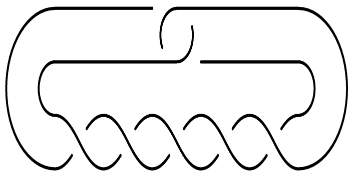

Figure 3: The knot K_2

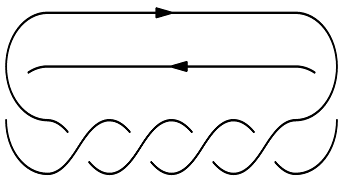

Figure 4: The link A_2

Since the tunnel number of these links equals 1 we have $\mathcal{MN}(A_n) \leqslant 2$, $\mathcal{MN}(K_n) \leqslant 2$. It is easy to show that $q_1(K_n) = q_1(A_n) = 1$ [26; 6], thus

$$\mathcal{MN}(A_n) = 2, \ \mathcal{MN}(K_n) = 2.$$

In the paper [4] H Goda announced the computation of the Morse–Novikov numbers of all prime knots and links with $\leqslant 10$ crossings. His theorem (which is based on the results of [3]) says that for every nonfibred prime link L with $\leqslant 10$ crossings we have $\mathcal{MN}(L) = 2$.

Since the tunnel numbers of prime knots with $\leqslant 10$ crossings are known from the work of K Morimoto, M Sakuma and S Yokota [18], our Theorem 1.1 provides a quick proof of H Goda's results at least for knots with $\leqslant 8$ crossings. Indeed, it is proved in [18] that among the prime knots with $\leqslant 8$ crossings only the knots 8_{16}, 8_{17}, 8_{18} have the tunnel number 2; the tunnel number of all the others equals 1. Since these three knots are fibred, we deduce that every nonfibred prime knot with $\leqslant 8$ crossings has the tunnel number equal to 1 and therefore its Morse–Novikov number is equal to 2.

5 Open questions and further remarks

(1) One of the main conjectures in the Morse–Novikov theory of knots and links is the following (M Boileau, C Weber):

(2) $$\mathcal{MN}(K_1 \# K_2) = \mathcal{MN}(K_1) + \mathcal{MN}(K_2).$$

The example of K Morimoto [15] shows that there are knots K_1, K_2 with $t(K_1 \# K_2) < t(K_1) + t(K_2)$. Moreover, T Kobayashi [9] proved that for every N there are knots K_1 and K_2 such that $t(K_1 \# K_2) \leqslant t(K_1) + t(K_2) - N$. In view of the relations between the tunnel and the Morse–Novikov numbers established in the present paper, these results provide a number of potential counterexamples to the conjecture (2).

(2) The Novikov homology $\mathcal{N}_*(K)$ can be considered as homology with local coefficients with respect to the representation

$$\mu\colon \pi_1(C_K) \to \mathbb{Z}[\mathbb{Z}]^\times = \Lambda^\times \subset \widehat{\Lambda}^\times = \mathrm{GL}(1, \widehat{\Lambda}),$$

where the first arrow is the meridian homomorphism $\pi_1(C_K) \to \mathbb{Z} \subset \mathbb{Z}[\mathbb{Z}]^\times$. Thus Corollary 1.4 can be reformulated as follows:

$$t(K) \geqslant \mathrm{m}_{\widehat{\Lambda}}(H_1(C_K, \mu))$$

where $\mathrm{m}_{\widehat{\Lambda}}(N)$ stands for the minimal number of generators over $\widehat{\Lambda}$ of the module N. For an arbitrary representation we have a weaker (obvious) inequality:

Proposition 5.1 *For every representation $\rho\colon \pi_1(C_K) \to \mathrm{GL}(n, R)$ (where R is a principal ring) we have*

$$t(K) \geqslant \frac{1}{n}\big(\mathrm{m}_R\big(H_1(C_K, \rho)\big)\big) - 1.$$

Question Is it true that

$$t(K) = \max_\rho \left(\frac{1}{n}\big(\mathrm{m}_R(H_1(C_K, \rho))\big) - 1\right) ?$$

In other words, is the information deduced from the twisted homology sufficient to determine the tunnel number of any knot?

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