On the tunnel number and the Morse–Novikov number of knots

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Let $L$ be a link in $S^3$; denote by $\mathcal{MN}(L)$ the Morse–Novikov number of $L$ and by $t(L)$ the tunnel number of $L$. We prove that $\mathcal{MN}(L) \leq 2t(L)$ and deduce several corollaries.

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1 Introduction

1.1 Background

Let $L$ be a link in $S^3$, that is, an embedding of several copies of $S^1$ to $S^3$. First off, we recall the definition of three numerical invariants of $L$. In the sequel $N(L)$ denotes a closed tubular neighbourhood of $L$.

(A) (Tunnel number) An arc $\gamma$ in $S^3$ is called a tunnel for $L$ if $\gamma \cap L$ consists of the two endpoints of $\gamma$. The tunnel number $t(L)$ is the minimal number $m$ of disjoint tunnels $\gamma_1, \ldots, \gamma_m$ such that the closure of $S^3 \setminus N(L \cup \gamma_1 \cup \cdots \cup \gamma_m)$ is a handlebody. The tunnel number was introduced by B Clark [1]; this invariant was studied in the works of T Kohno [11], T Kobayashi [9], T Kobayashi and Y Rieck [10], M Lustig and Y Moriah [13], K Morimoto [15; 14; 16], K Morimoto, M Sakuma and Y Yokota [17; 18], M Scharlemann and J Schultens [23; 24] and others. M Scharlemann and J Schultens [23] proved that $t(nK) \geq n$ for any $n$ (here $nK$ stands for the connected sum of $n$ copies of the knot $K$). They proved also that $t(nK) \geq \frac{2}{3}nt(K)$ if $K$ is not a 2–bridge knot [24]. T Kohno [11] gave an estimate of tunnel number of knots in terms of quantum invariants. K Morimoto, M Sakuma and Y Yokota [18] computed the tunnel number of all prime knots with $\leq 10$ crossings.

For any two knots $K_1, K_2$ we have $t(K_1 \# K_2) \leq t(K_1) + t(K_2) + 1$. K Morimoto [15] constructed knots $K_1, K_2$ such that $t(K_1 \# K_2) < t(K_1) + t(K_2)$. T Kobayashi and Y Rieck [10] define the growth rate for a knot $K$ by the formula

$$gr_t(K) = \lim_{m \to \infty} \sup \frac{t(mK) - mt(K)}{m - 1}.$$
It follows from results of [24] that \( gr_t(K) \geq -1 - \frac{2}{3} t(K) \).

(B) (Bridge numbers) Let \( S^3 = H_1 \cup H_2 \) be a Heegaard splitting of \( S^3 \); put \( \Sigma = H_1 \cap H_2 \), and \( g = g(\Sigma) \). We say (following H Doll [2]) that \( L \) is in an \( n \)–bridge position with respect to \( \Sigma \) if \( \Sigma \) intersects \( L \) in \( 2n \) points and \( \Sigma \cap H_i \) is a union of \( n \) trivial arcs in \( H_i \) for \( i = 1, 2 \). The \( g \)–bridge number \( b_g(L) \) of \( L \) is defined as the minimal number \( n \) such that \( L \) can be put in an \( n \)–bridge position with respect to a Heegaard decomposition of genus \( g \). Thus \( b_0(L) \) is the classical bridge number as defined by H Schubert [25]. We have

\[
t(L) \leq g + b_g(L) - 1.
\]

(C) (Morse–Novikov numbers) Pick an orientation preserving trivialisation of the normal bundle of \( L \). The corresponding diffeomorphism of disc bundles \( \phi: L \times D^2 \to N(L) \) will be called framing of \( L \). Let \( C_L \) denote the closure of \( S^3 \setminus N(L) \). A Morse function \( f: C_L \to S^1 \) is called regular if its restriction to the boundary \( \partial N(L) \) is the canonical fibration over the circle: \( (f \circ \phi)(l, z) = z/|z| \).

The number of the critical points of index \( i \) of a regular Morse function \( f \) will be denoted by \( m_i(f) \); the total number of critical points of \( f \) will be denoted by \( m(f) \). The minimal value of \( m(f) \) over all possible framings \( \phi \) and Morse maps \( f: C_L \to S^1 \) is called the Morse–Novikov number of the link \( L \) and denoted by \( \mathcal{M}N(L) \) (see Veber, Pajitnov and Rudolph [26]).

The Morse–Novikov theory of circle-valued maps (see Novikov [19] and Pajitnov [20; 21]) allows one to obtain homological lower bounds for \( \mathcal{M}N(L) \) as follows. Let \( \tilde{C}_L \) be the infinite cyclic covering induced by \( f \) from the covering \( \mathbb{R} \to S^1 \). Denote the ring \( \mathbb{Z}[t, t^{-1}] \) by \( \Lambda \), and the ring \( \mathbb{Z}((t)) \) by \( \hat{\Lambda} \). The \( \hat{\Lambda} \)–module

\[
\mathcal{N}_s(L) = H_*(\tilde{C}_L) \otimes_{\Lambda} \hat{\Lambda}
\]

is called the Novikov homology of the link \( L \). The rank and torsion numbers of the \( \hat{\Lambda} \)–module \( \mathcal{N}_1(L) \) are denoted respectively by \( b_1(L) \) and \( q_1(L) \). We have then [26]

\[
\mathcal{M}N(L) \geq 2(b_1(L) + q_1(L)).
\]

In case when the Novikov numbers are not sufficient to determine the \( \mathcal{M}N(L) \) the twisted Novikov numbers (introduced by H Goda and the author in [5]) are useful.

As for upper bounds for \( \mathcal{M}N(L) \), not much is known. H Goda announced in [4] that \( \mathcal{M}N(L) \leq 2 \) for every prime link \( L \) with \( \leq 10 \) crossings. M Hirasawa proved that for every \( 2 \)–bridge knot \( K \) we have \( \mathcal{M}N(K) \leq 2 \) (unpublished). In the papers [22; 7] of L Rudolph and M Hirasawa it is proved that \( \mathcal{M}N(K) \leq 4g_f(K) \) where \( g_f(K) \) is the free genus of \( K \), that is, the minimal possible genus of a Seifert surface \( \Sigma \) bounding \( K \) such that \( S^3 \setminus \Sigma \) is an open handlebody.
1.2 Main results

The main result of this work is the following theorem.

**Theorem 1.1** For every link $L$ in $S^3$ we have

$$ \mathcal{MN}(L) \leq 2t(L). $$

The following corollaries are easily deduced.

**Corollary 1.2** For every $g$ we have

$$ \mathcal{MN}(L) \leq 2(g + b_g(L) - 1). $$

**Corollary 1.3** For every tunnel number 1 knot $K$ we have $\mathcal{MN}(K) \leq 2$. In particular this holds for any $(1,1)$–knot $K$.

**Corollary 1.4** For every link $L$ we have

$$ q_1(L) + b_1(L) \leq t(L). $$

**Corollary 1.5** For every knot $K$

$$ gr_t(K) \geq -t(K) + q_1(K). $$

2 Proof of Theorem 1.1

Let $m = t(L)$. Pick a framing $\phi: L \times D^2 \to N(L)$. Then the manifold $C_L = S^3 \setminus N(L)$ is obtained from $\partial C_L$ by attaching $m$ one-handles and then attaching a handlebody of genus $m + 1$ to the resulting cobordism. So we obtain a Morse function $g: C_L \to \mathbb{R}$ which is constant on $\partial C_L$ and has the following Morse numbers: $m_0(g) = 0$, $m_1(g) = m$, $m_2(g) = m + 1$, $m_3(g) = 1$. Pick any Morse map $h: C_L \to S^1$ such that $h|\partial C_L$ is the canonical fibration: $(h \circ \phi)(l, z) = z/|z|$. The 1–form induced by $h$ from the canonical volume form on $S^1$ will be denoted by $dh$. Consider a closed 1–form $\omega_\epsilon = dg + \epsilon dh$. For $\epsilon > 0$ sufficiently small $\omega_\epsilon$ is a Morse form with the same Morse numbers as $dg$. The De Rham cohomology class of the 1–form

$$ \frac{1}{\epsilon} \omega_\epsilon = \frac{1}{\epsilon} dg + dh $$

is the same as that of $dh$; therefore this form is the differential of a Morse map $g_1: C_L \to S^1$ homotopic to $h$.  

\footnote{A similar perturbation argument was used by J C Sikorav in another context; see Pajitnov \cite{Pajitnov}.} Observe that the map $g_1$ is a regular Morse map;
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it has one local maximum, and the standard elimination procedure (see for example Lemmas 3.1 and 3.2 of [26] for details) gives us a regular Morse function \( f : C_L \to S^1 \) with \( m_0(f) = 0 \), \( m_1(f) \leq m \), \( m_2(f) \leq m \), \( m_3(f) = 0 \). Thus \( MN(L) \leq 2m \).

### 3 Examples

Theorem 1.1 can be used in two ways. A lot of information is available about the tunnel numbers, and this implies new estimates for the Morse–Novikov numbers of knots. On the other hand, the Novikov torsion number \( q_1(K) \) is an invariant which is easy to compute, and in many cases this gives new information about the sequence of tunnel numbers \( t(nK) \) for a given knot \( K \). Let us consider two examples:

(A) (Pretzel knots) Let \( q, r \) be positive integers; denote by \( \mathcal{P} \) the \((2r+1)\)-stranded pretzel knot \( P(2q+1, -2q-1, 2q+1, \ldots, 2q+1) \). The knot \( \mathcal{P} \) for \( q = 1, r = 2 \) is depicted below.

![Figure 1: Pretzel knot](image)

It is clear that \( t(\mathcal{P}) \leq 2r \). An easy computation of the Alexander module via the Seifert matrix gives

\[
\mathcal{N}_1(\mathcal{P}) \approx (\hat{\Lambda}/XY\hat{\Lambda})^r
\]

where \( X = qt-(q+1) \), \( Y = (q+1)t-q \). Thus \( q_1(\mathcal{P}) = r \). Since \( q_1(mK) = mq_1(K) \) for any knot \( K \), we deduce that

\[
\frac{1}{2}n_1(K) \leq nq_1(\mathcal{P}) \leq t(n) - q(K).
\]

In particular the growth rate of the knot satisfies \( gr(K) \geq -\frac{1}{2}K \).

(B) (A twisted \( 5_2 \# 5_2 \)) Let \( K \) be the knot obtained from the connected sum \( 5_2 \# 5_2 \) by twisting (see Figure 2).

An easy computation shows that

\[
\mathcal{N}_1(K) \approx (\hat{\Lambda}/S\hat{\Lambda})^2
\]
where \( S = 2t^2 - 3t + 2 \) is the Alexander polynomial of the knot \( 5_2 \). Thus \( q_1(K) = 2 \).

Since \( t(K) \leq 3 \) we obtain

\[
\frac{2}{3}nt(K) \leq nq_1(K) \leq t(nK).
\]

We have therefore \( gr_1(K) \geq -\frac{1}{2}t(K) \).

### 4 Relations with previously known results

A theorem of M Hirasawa says that \( MN(K) \leq 2 \) if \( K \) is a two-bridge knot. Since \( t(K) \leq b(K) - 1 \) our theorem implies this result. Observe that M Hirasawa’s proof uses H Schubert’s presentation of 2–bridge knots, and can not be generalized to the case of arbitrary bridge number.

The inequality (1) implies also the upper bound

\[
MN(K) \leq 4g_f(K)
\]

obtained by L Rudolph and M Hirasawa [22; 7]. Indeed, JH Lee [12] has shown that \( t(K) \leq 2g_f(K) \).

In many cases the estimate of Theorem 1.1 is better than the free genus estimate. For example, let \( K \) be the pretzel knot \( K = P(-2l, q, r) \) where \( l \geq 2 \) and \( q, r \geq 3 \) are odd numbers. Then \( t(K) \leq 2 \), and the Alexander polynomial of the knot equals

\[
A(t) = lt^q + r - (2l - 1)t^q + r - 1 + \cdots - (2l - 1)t + l
\]

(see the work [8] of D Kim and J Lee). Therefore \( K \) is not fibred, and \( 4 \geq MN(K) \geq 2 \). As for the genus of \( K \), we have \( g(K) \geq \deg A(t)/2 = (q + r)/2 \), therefore the free genus of \( K \) is not less than \( (q + r)/2 \).
Theorem 1.1 leads to quick proofs of results about the Morse–Novikov numbers already known. The simplest cases are: the link $A_n$ (the boundary of $n$–twisted unknotted annulus) and the twist knots $K_n$. See Figures 3 and 4. We shall assume that $n \geq 2$.

Since the tunnel number of these links equals 1 we have $\mathcal{MN}(A_n) \leq 2$, $\mathcal{MN}(K_n) \leq 2$. It is easy to show that $q_1(K_n) = q_1(A_n) = 1$ [26; 6], thus

$$\mathcal{MN}(A_n) = 2, \mathcal{MN}(K_n) = 2.$$

In the paper [4] H Goda announced the computation of the Morse–Novikov numbers of all prime knots and links with $\leq 10$ crossings. His theorem (which is based on the results of [3]) says that for every nonfibred prime link $L$ with $\leq 10$ crossings we have $\mathcal{MN}(L) = 2$.

Since the tunnel numbers of prime knots with $\leq 10$ crossings are known from the work of K Morimoto, M Sakuma and S Yokota [18], our Theorem 1.1 provides a quick proof of H Goda’s results at least for knots with $\leq 8$ crossings. Indeed, it is proved in [18] that among the prime knots with $\leq 8$ crossings only the knots $8_{16}, 8_{17}, 8_{18}$ have the tunnel number 2; the tunnel number of all the others equals 1. Since these three knots are fibred, we deduce that every nonfibred prime knot with $\leq 8$ crossings has the tunnel number equal to 1 and therefore its Morse–Novikov number is equal to 2.
5 Open questions and further remarks

(1) One of the main conjectures in the Morse–Novikov theory of knots and links is the following (M Boileau, C Weber):

$$\mathcal{M}\mathcal{N}(K_1 \# K_2) = \mathcal{M}\mathcal{N}(K_1) + \mathcal{M}\mathcal{N}(K_2).$$

The example of K Morimoto [15] shows that there are knots $K_1, K_2$ with $t(K_1 \# K_2) < t(K_1) + t(K_2)$. Moreover, T Kobayashi [9] proved that for every $N$ there are knots $K_1$ and $K_2$ such that $t(K_1 \# K_2) \leq t(K_1) + t(K_2) - N$. In view of the relations between the tunnel and the Morse–Novikov numbers established in the present paper, these results provide a number of potential counterexamples to the conjecture (2).

(2) The Novikov homology $\mathcal{N}_\ast(K)$ can be considered as homology with local coefficients with respect to the representation

$$\mu: \pi_1(C_K) \to \mathbb{Z}[\mathbb{Z}]^\times = \Lambda^\times \subset \hat{\Lambda}^\times = \text{GL}(1, \hat{\Lambda}),$$

where the first arrow is the meridian homomorphism $\pi_1(C_K) \to \mathbb{Z} \subset \mathbb{Z}[\mathbb{Z}]^\times$. Thus Corollary 1.4 can be reformulated as follows:

$$t(K) \geq m_{\hat{\Lambda}}(H_1(C_K, \mu))$$

where $m_{\hat{\Lambda}}(N)$ stands for the minimal number of generators over $\hat{\Lambda}$ of the module $N$. For an arbitrary representation we have a weaker (obvious) inequality:

**Proposition 5.1** For every representation $\rho: \pi_1(C_K) \to \text{GL}(n, R)$ (where $R$ is a principal ring) we have

$$t(K) \geq \frac{1}{n} \left( m_R(H_1(C_K, \rho)) \right) - 1.$$

**Question** Is it true that

$$t(K) = \max_\rho \left( \frac{1}{n} \left( m_R(H_1(C_K, \rho)) \right) - 1 \right)?$$

In other words, is the information deduced from the twisted homology sufficient to determine the tunnel number of any knot?

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