Triple point numbers of surface-links and symmetric quandle cocycle invariants

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For any positive integer n, we give a 2–component surface-link $F = F_1 \cup F_2$ such that F_1 is orientable, F_2 is non-orientable and the triple point number of F is equal to $2n$. To give lower bounds of the triple point numbers, we use symmetric quandle cocycle invariants.

57Q45; 18G99, 55N99, 57Q35

1 Introduction

A *surface-link* is a closed surface smoothly embedded in $\mathbb{R}^4$. Two surface-links F and F' are assumed to be the same if and only if there exists an ambient isotopy $\{h_t\}$ of $\mathbb{R}^4$ such that $h_1(F) = F'$. When F and F' are oriented, it is assumed that $h_1|_F\colon F \to F'$ is an orientation-preserving homeomorphism. In particular, when a surface-link is connected, we call it a *surface-knot*.

The *triple point number* of a surface-link F is defined by the smallest number of the triple points among all the diagrams of F, and we denote it by $t(F)$. There are several studies on triple point numbers. For example, quandle cocycle invariants (see Carter, Jelsovsky, Kamada, Langford and Saito [1]) are used to give lower bounds of triple point numbers of orientable surface-links; for example, Satoh and Shima [14] determined the triple point number of the 2–twist-spun trefoil to be four, and Hatakenaka [4] gave a lower bound for the triple point number of the 2–twist-spun figure-eight knot. By a geometric argument about normal Euler numbers, Satoh [12] gave the following theorem:

Theorem 1.1 (Satoh [12]) *For any positive integer n, there exists a 2–component surface-link $F = F_1 \cup F_2$ such that* (i) *each F_i is a non-orientable surface-knot with the Euler characteristic $\chi(F_i) = 2 - n$, and* (ii) *$t(F) = 2n$.*

In Section 4, we show a method which gives lower bounds for the triple point numbers of surface-links by using the symmetric quandle cocycle invariants (see Kamada [7]

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and Kamada–Oshiro [8]). We remark that by the symmetric quandle cocycle invariants, we can give alternative proof of Theorem 1.1. Using new examples of surface-links, we can also prove the following theorem which is analogous to Theorem 1.1:

Theorem 1.2 *For any positive integer n, there exists a 2–component surface-link $F = F_1 \cup F_2$ such that*

(i) *F_1 is an orientable surface-knot with $\chi(F_1) = 0$,*

(ii) *F_2 is a non-orientable surface-knot with $\chi(F_2) = 2-2n$, and*

(iii) *$t(F) = 2n$.*

By a connected sum of the surface-link which Satoh used for Theorem 1.1 and an orientable surface-knot, the following was given in [8]: For any positive integers m and n with $m \equiv n \pmod 2$ and $m \geq n$, there is a surface-link $F = F_1 \cup F_2$ such that F_1 is a non-orientable surface with $\chi(F_1) = 2-m$, F_2 is a non-orientable surface with $\chi(F_2) = 2-n$ and $t(F) = 2n$. For surface-links composed of two non-orientable surfaces, we give the following theorem:

Theorem 1.3 *For any positive integer n and for any integer m with $m \geq 3$, there is a surface-link $F = F_1 \cup F_2$ such that*

(i) *F_1 is a non-orientable surface with $\chi(F_1) = 2-m$,*

(ii) *F_2 is a non-orientable surface with $\chi(F_2) = 2-2n$, and*

(iii) *$t(F) = 2n$.*

The paper is organized as follows. In Sections 2 and 3, we recall symmetric quandles, symmetric quandle 3–cocycles, and surface-link invariants with symmetric quandles introduced in [7; 8]. In Section 4, we show a method to estimate the triple point numbers of surface-links by using the symmetric quandle invariants. Theorems 1.2 and 1.3 are proved by giving new examples of surface-links in Section 5. In Section 6, we show several results which can be obtained by using our method for estimating triple point numbers.

2 Symmetric quandles and their cocycles

A *quandle* (see Fenn and Rourke [3], Joyce [5] or Matveev [10]) is a set X with a binary operation $(x, y) \mapsto x^y$ such that

(i) for any $x \in X$, it holds that $x^x = x$,

(ii) for any $x, y \in X$, there exists a unique $z \in X$ such that $z^y = x$, and

(iii) for any $x, y, z \in X$, it holds that $(x^y)^z = (x^z)^{(y^z)}$.

We denote by $x^{y^{-1}}$ the element z given in the condition (ii). For a quandle X, a *good involution* ρ *of* X [7; 8] means an involution of X such that

(i) for any $x, y \in X$, $\rho(x^y) = \rho(x)^y$, and

(ii) for any $x, y \in X$, $x^{\rho(y)} = x^{y^{-1}}$.

A pair of a quandle and a good involution is called a *symmetric quandle*.

Let (X, ρ) be a symmetric quandle, and A an abelian group. A homomorphism $\theta\colon \mathbb{Z}(X^3) \to A$ is a *symmetric quandle* 3*–cocycle* of (X, ρ) if the following conditions are satisfied:

(i) For any $(a, b, c, d) \in X^4$,

$$\theta(a,c,d) - \theta(a^b,c,d) - \theta(a,b,d) + \theta(a^c,b^c,d) + \theta(a,b,c) - \theta(a^d,b^d,c^d) = 0,$$

(ii) for any $(a, b) \in X^2$, $\theta(a, a, b) = 0$ and $\theta(a, b, b) = 0$, and

(iii) for any $(a, b, c) \in X^3$,

$$\theta(a,b,c) + \theta(\rho(a),b,c) = 0, \qquad \theta(a,b,c) + \theta(a^b,\rho(b),c) = 0$$
$$\text{and} \quad \theta(a,b,c) + \theta(a^c,b^c,\rho(c)) = 0.$$

Here, $\mathbb{Z}(X^3)$ is the free $\mathbb{Z}$–module generated by all the elements of $X^3 = X \times X \times X$. Notice that a symmetric quandle 3–cocycle of (X, ρ) is a 3–cocycle of the cochain complex defined for the symmetric quandle (X, ρ) in [7; 8].

For any element k in $\mathbb{Z}$, we use the same symbol k to indicate the element $[k]$ in $\mathbb{Z}_2$, and any element of $\mathbb{Z}_2 \oplus \mathbb{Z}$ is denoted by a form $\alpha \oplus \beta$, where α is the entry of $\mathbb{Z}_2$, and β is the entry of $\mathbb{Z}$.

Example 2.1 The set $\{0, 1, \cdots, n-1\}$ with the operation $x^y \equiv 2y - x \pmod{n}$ for any $x, y \in \{0, 1, \cdots, n-1\}$ is a quandle, which is called a *dihedral quandle* of order n. All of the good involutions of a dihedral quandle are determined in [8]. Let X be the dihedral quandle $\{0, 1, 2, 3\}$ of order 4. The involution $\rho\colon X \to X$ defined by $\rho(0) = 2$ and $\rho(1) = 3$, is a good involution of X. Define a map $\theta\colon X^3 \to \mathbb{Z}_2 \oplus \mathbb{Z}$ such that

$$\theta(a,b,c) = \begin{cases} 0 \oplus 1 & (a,b,c) = (0,1,0), (0,3,0), (2,1,2), (2,3,2), \\ & \qquad\qquad (1,0,3), (1,2,3), (3,0,1), (3,2,1), \\ 0 \oplus (-1) & (a,b,c) = (0,1,2), (0,3,2), (2,1,0), (2,3,0), \\ & \qquad\qquad (1,0,1), (1,2,1), (3,0,3), (3,2,3), \\ 0 \oplus 0 & \text{otherwise.} \end{cases}$$

Then the the linear extension $\theta\colon \mathbb{Z}(X^3) \to \mathbb{Z}_2 \oplus \mathbb{Z}$ is a symmetric quandle 3–cocycle of (X,ρ).

Example 2.2 Let $X = \{0,1,2\}$ be the quandle such that

$$\begin{array}{lll} 0^0 = 0, & 0^1 = 0, & 0^2 = 0, \\ 1^0 = 2, & 1^1 = 1, & 1^2 = 1, \\ 2^0 = 1, & 2^1 = 2, & 2^2 = 2. \end{array}$$

The involution $\rho\colon X \to X$ defined by $\rho(0) = 0$ and $\rho(1) = 2$, is a good involution of X. Define a map $\theta\colon X^3 \to \mathbb{Z}_2 \oplus \mathbb{Z}$ such that

$$\theta(a,b,c) = \begin{cases} 1 \oplus 0 & (a,b,c) = (0,1,0),(0,2,0) \\ 0 \oplus 1 & (a,b,c) = (1,0,2),(2,0,1) \\ 0 \oplus (-1) & (a,b,c) = (1,0,1),(2,0,2) \\ 0 \oplus 0 & \text{otherwise.} \end{cases}$$

Then the linear extension $\theta\colon \mathbb{Z}(X^3) \to \mathbb{Z}_2 \oplus \mathbb{Z}$ is a symmetric quandle 3–cocycle of (X,ρ).

3 Symmetric quandle cocycle invariants

Let D be a diagram in $\mathbb{R}^3$ of a surface-link F in $\mathbb{R}^4$, where the lower sheets are divided along double point curves to indicate crossing information. We divide over-sheets along the double point curves and we call the sheets of the result *semi-sheets* of D. Note that every semi-sheet is orientable even if F is non-orientable, see Kamada [6].

For a symmetric quandle (X,ρ), we say that an assignment of a normal orientation and an element of X to each semi-sheet of D satisfies the *coloring conditions* if it satisfies the following:

(i) Suppose that two adjacent semi-sheets coming from an over-sheet of D about a double point curve are labeled by x_1 and x_2. If the normal orientations are coherent then $x_1 = x_2$, otherwise $x_1 = \rho(x_2)$. See the top row of Figure 1.

(ii) Suppose that two adjacent semi-sheets S_1 and S_2 coming from under-sheets about a double point curve are labeled by x_1 and x_2, and that one of the two semi-sheets coming from an over-sheet of D, say S_3, is labeled by x_3. We assume that the normal orientation of S_3 points from S_1 to S_2. If the normal orientations of S_1 and S_2 are coherent, then $x_1^{x_3} = x_2$, otherwise $x_1^{x_3} = \rho(x_2)$. See the bottom row of Figure 1.

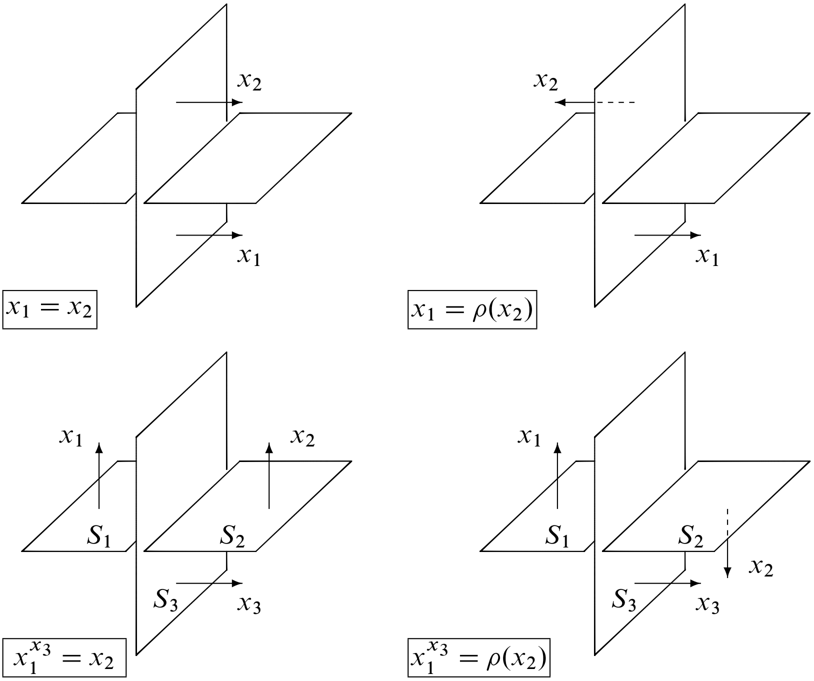

Figure 1

An (X,ρ)*–coloring* of D is the equivalence class of an assignment of normal orientations and elements of X to the semi-sheets of D satisfying the coloring conditions. Here, the equivalence relation is generated by *basic inversions*, that is, a basic inversion reverses the normal orientation of a semi-sheet and changes the element x assigned the sheet by $\rho(x)$. See Figure 2.

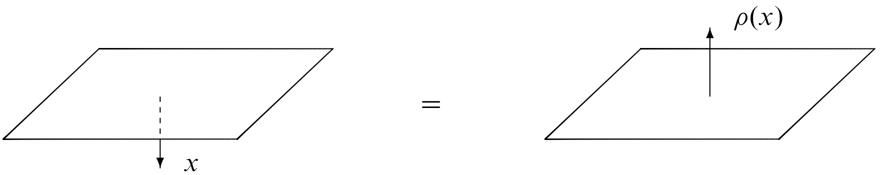

Figure 2

We call a diagram with an (X,ρ)–coloring C_D an (X,ρ)*–colored diagram* and denote it by (D, C_D).

Let (D, C_D) and $(D', C_{D'})$ be (X, ρ)–colored diagrams of a surface-link F. We say that (D, C_D) and $(D', C_{D'})$ (or the (X, ρ)–colorings C_D of D and $C_{D'}$ of D') are *equivalent* if they are related by a finite sequence of Roseman moves (see Roseman [11], and also Carter and Saito [2]) over which the colorings extend. We call the equivalence class of (D, C_D) an *(X, ρ)–coloring of F*. An *(X, ρ)–colored surface-link* (F, C) is a surface-link F equipped with an (X, ρ)–coloring C.

Let (D, C_D) be an (X, ρ)–colored diagram of an (X, ρ)–colored surface-link (F, C). Let $\theta\colon \mathbb{Z}(X^3) \to A$ be a symmetric quandle 3–cocycle of (X, ρ). For a triple point of D, define the *θ–weight* as follows: Choose one of eight 3–dimensional complementary regions around the triple point and call the region a *specified region*. There exist 12 semi-sheets around the triple points. Let S_T, S_M and S_B be the three of them that face the specified region, where S_T, S_M and S_B are in the top sheet, the middle sheet and the bottom sheet at the triple point, respectively. Let n_T, n_M and n_B be the normal orientations of S_T, S_M and S_B which point away from the specified region. Let x, y and z be the elements of X assigned to the semi-sheets S_T, S_M and S_B with the normal orientations n_T, n_M and n_B, respectively. The θ–weight of the triple point is defined by $\varepsilon\theta(z, y, x)$, where ε is $+1$ (or -1) if the triple of the normal orientations (n_T, n_M, n_B) does (or does not) match with the orientation of $\mathbb{R}^3$. The sign of the triple point as shown in Figure 3 is positive.

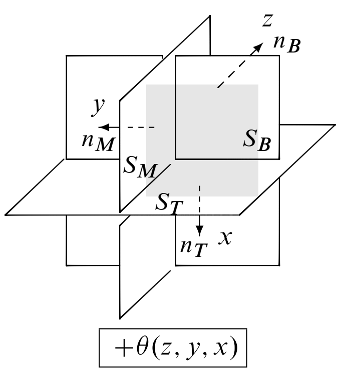

Figure 3

Define $\theta(D, C_D)$ by

$$\theta(D, C_D) = \sum_{\tau} (\theta\text{–weight of } \tau) \in A,$$

where τ runs over all the triple points of D.

Theorem 3.1 (Kamada and Oshiro [8]) *The value $\theta(D, C_D)$ is an invariant of an (X, ρ)–colored surface-link (F, C).*

We denote $\theta(D, C_D)$ by $\theta(F, C)$.

4 Estimates of triple point numbers

For non-negative integers s and t, let $A_{s,t}$ denote the direct sum of s copies of $\mathbb{Z}_2$ and t copies of $\mathbb{Z}$, that is, $A_{s,t} = (\mathbb{Z}_2)^s \oplus (\mathbb{Z})^t$. Every element of $A_{s,t}$ has a form $(\alpha_1 \oplus \cdots \oplus \alpha_s) \oplus (\beta_1 \oplus \cdots \oplus \beta_t)$, where α_i is the entry of ith $\mathbb{Z}_2$ $(1 \le i \le s)$, and β_j is the entry of jth $\mathbb{Z}$ $(1 \le j \le t)$. We denote by p_i and q_j the elements of $A_{s,t}$ whose entries are all zeros except $\alpha_i = 1$ and $\beta_j = 1$, respectively.

Let (X, ρ) be a symmetric quandle, and $\theta \colon \mathbb{Z}(X^3) \to A_{s,t}$ a 3–cocycle of (X, ρ). We consider the following condition for θ:

$(*)$ For any generator $(a, b, c) \in X^3$ of $\mathbb{Z}(X^3)$, it holds that

$$\theta(a, b, c) \in \{0, p_i, \pm q_j \mid 1 \le i \le s, 1 \le j \le t\}.$$

We remark that the symmetric quandle 3–cocycles given in Examples 2.1 and 2.2 satisfy the condition $(*)$.

Theorem 4.1 *Let θ be a 3–cocycle of a symmetric quandle (X, ρ) with the condition $(*)$. If the invariant $\theta(F, C)$ of a surface-link F with an (X, ρ)–coloring C is equal to $(\alpha_1 \oplus \cdots \oplus \alpha_s) \oplus (\beta_1 \oplus \cdots \oplus \beta_t)$, then we have $t(F) \ge \sum_{i=1}^{s} \alpha_i + \sum_{j=1}^{t} |\beta_j|$, where the sum is taken in $\mathbb{Z}$ by regarding $\alpha_k = 0$ or 1 as an element of $\mathbb{Z}$.*

Proof We take any (X, ρ)–colored diagram (D, C_D) of (F, C). Let $t(D)$ denote the number of triple points of D, and m_i $(1 \le i \le s)$, n_j and n'_j $(1 \le j \le t)$ the number(s) of triple points whose θ–weights are p_i, q_j and $-q_j$, respectively.

Since the θ–weight of any triple point of D is one of 0, p_i, q_j, and $-q_j$, it holds that

$$\begin{aligned} \theta(D, C_D) &= \textstyle\sum_{i=1}^{s} m_i p_i + \sum_{j=1}^{t} n_j q_j + \sum_{j=1}^{t} n'_j(-q_j) \\ &= (m_1 \oplus \cdots \oplus m_s) \oplus \big((n_1 - n'_1) \oplus \cdots \oplus (n_t - n'_t)\big). \end{aligned}$$

Hence, we have $\alpha_i \equiv m_i \pmod 2$ and $\beta_j = n_j - n'_j$ by assumption. Since $\alpha_i \le m_i$ and $|\beta_j| \le n_j + n'_j$, it holds that

$$\sum_{i=1}^{s} \alpha_i + \sum_{j=1}^{t} |\beta_j| \le \sum_{i=1}^{s} m_i + \sum_{j=1}^{t} (n_j + n'_j) \le t(D). \qquad \square$$

5 Proofs of Theorems 1.2 and 1.3

In this section, we give surface-links which satisfy Theorems 1.2 and 1.3.

Let F be a surface-link in $\mathbb{R}^4 = \{(x,y,z,t) \in \mathbb{R}^4\}$ whose motion picture is given in Figure 4. It is composed of an unknotted torus F_1 and an unknotted, non-orientable surface F_2 with $\chi(F_2) = 2-2n$. Notice that in Figure 4, the deformations from (i) to (ii) and from (iii) to (iv) are the isotopic deformations corresponding to n Reidemeister moves of type III, respectively. The other isotopies are obtained by Reidemeister moves I and II only.

Let D be the diagram obtained by the projection $\pi\colon \mathbb{R}^4 \to \mathbb{R}^3$ with $\pi(x,y,z,t) \mapsto (x,y,0,t)$. Instead of illustrating the whole of D, we use the one-parameter family $\{D \cap \mathbb{R}^2[t]\}_{t\in\mathbb{R}}$, where $\mathbb{R}^2[t] = \{(x,y,0,t) \mid x \in \mathbb{R}, y \in \mathbb{R}\}$.

Proof of Theorem 1.2 We will prove that the surface-link F constructed as above satisfies $t(F) = 2n$. It is not difficult to see that $\chi(F_2) = 2-2n$.

Let (X,ρ) and θ be the symmetric quandle and the symmetric quandle 3–cocycle given in Example 2.2. We define an (X,ρ)–coloring C for D such that (i) any semi-sheet of F_1 is assigned by $0 \in X$ with any normal orientation, and (ii) the semi-sheet of F_2 marked by $*$ in $\mathbb{R}^2[-2]$ is assigned by $1 \in X$ with the orientation as in the figure, which can be extended to any other semi-sheets of F_2 uniquely.

Between the stills (i) and (ii) in Figure 4, the Reidemeister moves of type III arise n times and each move is depicted in Figure 5. Each Reidemeister move of type III corresponds to a triple point whose θ–weight is $-\theta(2,0,2) = 0 \oplus 1$. Between the stills (iii) and (iv) in Figure 4, the Reidemeister moves of type III arise n times and each move is depicted in Figure 6. Each Reidemeister move of type III corresponds to a triple point whose θ–weight is $\theta(1,0,2) = 0 \oplus 1$. Therefore, $\theta(E^{(n)}, C)$ is equal to $0 \oplus 2n$. By Theorem 4.1, $t(E^{(n)}) \geq 2n$. □

Proof of Theorem 1.3 Let $F = F_1 \cup F_2$ be the surface-link as above, and K an unknotted non-orientable surface-knot with $\chi(K) = 4-m$ $(m \geq 3)$. We denote by $F \sharp K = (F_1 \sharp K) \cup F_2$ the connected sum of $F_1 \subset F$ and K. It follows by definition that $\chi(F_1 \sharp K) = 2-m$ and $t(F \sharp K) \leq 2n$.

On the other hand, the (X,ρ)–coloring C for F in the proof of Theorem 1.2 is extended to that for $F \sharp K$ with the same θ–weight. Hence, we have $t(F \sharp K) = 2n$ by a similar argument to the previous proof. □

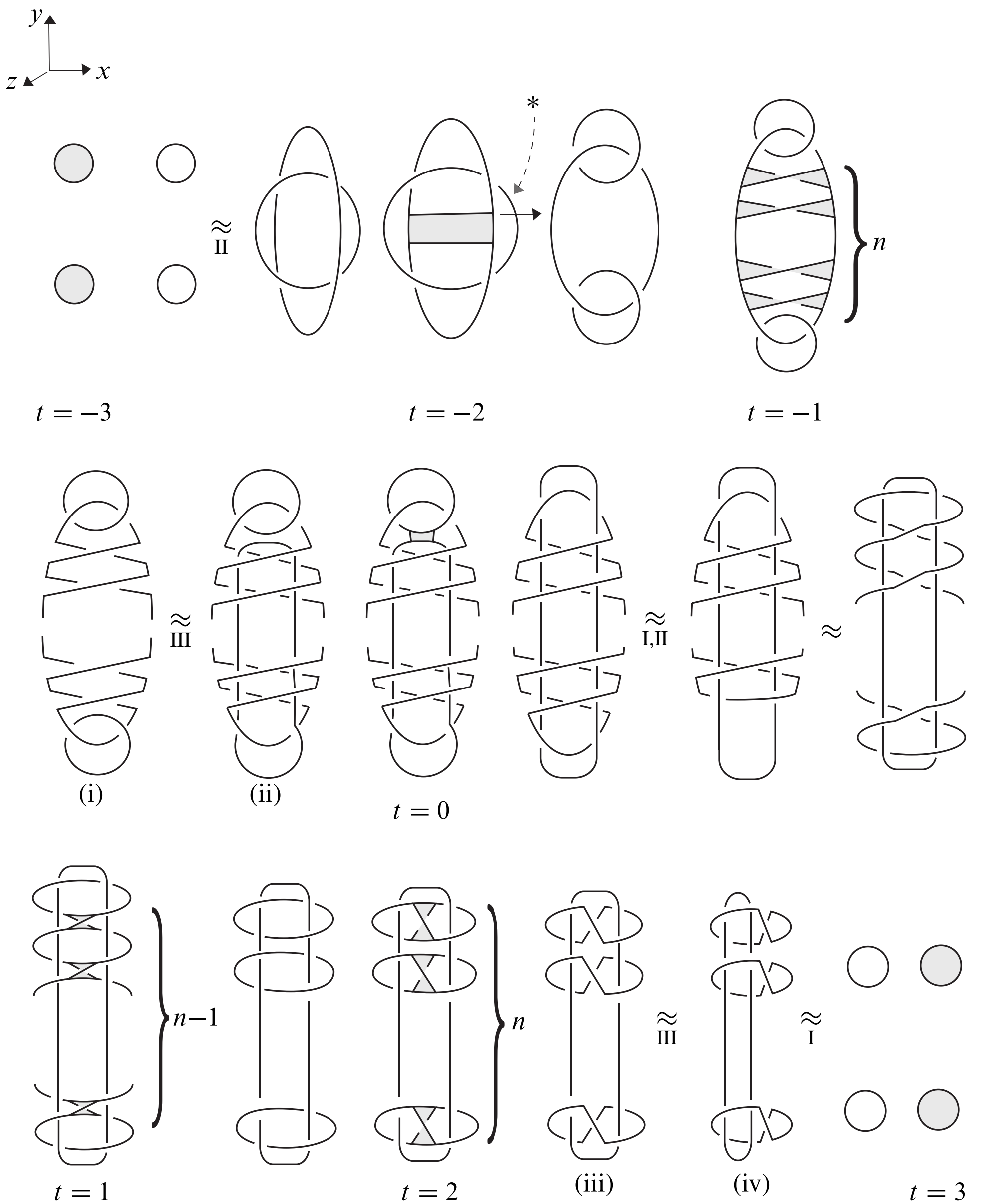

Figure 4

Remark 5.1 For the surface-link F as above, we can also use Satoh's method [12] to prove that $t(F) = 2n$. However, for the surface-link $F\sharp K$ which is constructed in the proof of Theorem 1.3, we can not prove $t(F\sharp K) = 2n$ by his method since the surface-link is P^2–reducible.

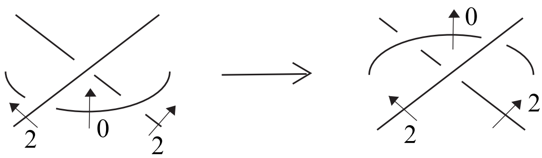

Figure 5

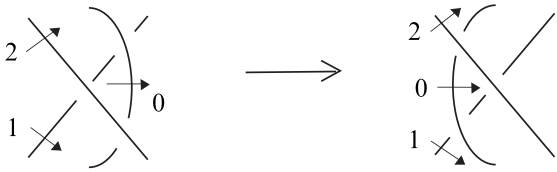

Figure 6

6 Other results by Theorem 4.1

In this section, we show some results which can be obtained as an application of Theorem 4.1.

For the positive integer n, let $G = G_1 \cup G_2$ be a surface-link in $\mathbb{R}^4 = \{(x, y, z, t) \in \mathbb{R}^4\}$ whose motion picture is given in Figure 7. Each component of G_i is a non-orientable surface with $\chi(G_i) = 2 - n$. This is the surface-link which Satoh used for proving Theorem 1.1.

The following theorem is a generalization of Theorem 1.1. We can give alternative proofs by a symmetric quandle 3–cocycle similarly to the proof of Theorem 1.2, or by a geometric argument used in [12]. We say that a surface-link is *pseudo-ribbon* if it has a diagram without triple points (see Kawauchi [9]).

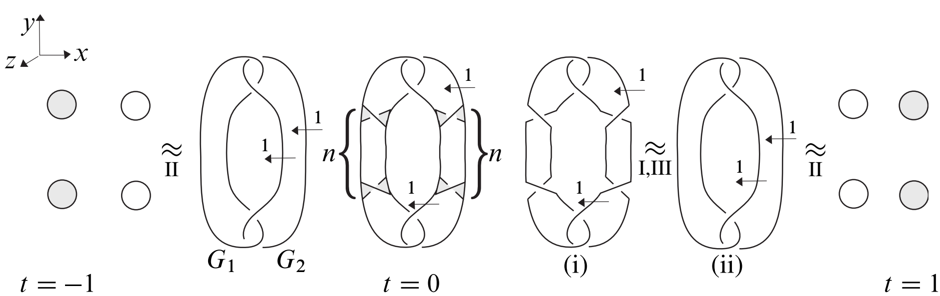

Figure 7

Theorem 6.1 (Kamada and Oshiro [8]) *Let G be the surface-link as above. For any orientable surface-knot K, the connected sum $G \sharp K = (G_1 \sharp K) \cup G_2$ satisfies*

$$t(G \sharp K) \geq 2n.$$

In particular, if K is pseudo-ribbon, then the equality holds.

For a non-orientable surface-knot K, the connected sum $G \sharp K$ is not necessarily P^2–irreducible. Hence we can not apply the Satoh's argument to the surface-link. In this case, we have the following.

Theorem 6.2 *Let G be the surface-link as above. For any non-orientable surface-knot K, it holds that*

$$t(G \sharp K) \geq \begin{cases} n+1 & \text{if } n \text{ is an odd number,} \\ n & \text{if } n \text{ is an even number.} \end{cases}$$

Proof Let (X, ρ) and θ be the symmetric quandle and the symmetric quandle 3–cocycle given in Example 2.2, respectively. By the definition,

$$\theta(a, b, c) \in \{0 \oplus 0, 1 \oplus 0, 0 \oplus 1, 0 \oplus (-1)\}$$

for any $(a, b, c) \in X^3$.

Let D be the diagram of G corresponding to the motion picture and C the (X, ρ)–coloring for G as shown in Figure 7. Between the stills (i) and (ii), the Reidemeister moves III arise $2n$ times. More precisely, a pair of moves III is depicted in Figure 8. The sum of the θ–weights is equal to

$$-\theta(1, 0, 1) + \theta(0, 2, 0) = 0 \oplus 1 + 1 \oplus 0 = 1 \oplus 1,$$

and hence, we have $\theta(G, C) = n \oplus n$.

For any non-orientable surface-knot K, we extend the (X, ρ)–coloring C for G to that for $G \sharp K$ such that K is colored trivially. Then it follows by definition that $\theta(G \sharp K, C) = \theta(G, C) = n \oplus n$, and we have the conclusion by Theorem 4.1. □

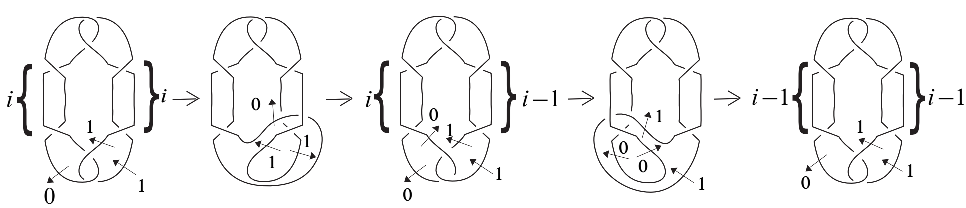

Figure 8

The equality given in Theorem 6.2 holds for $n = 1$.

Question 6.3 Does the equality in Theorem 6.2 hold for any $n \geq 2$?

We remark that the triple point number is generally not additive with respect to the connected sum (see Satoh [13]).

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References

[1] **J S Carter**, **D Jelsovsky**, **S Kamada**, **L Langford**, **M Saito**, *Quandle cohomology and state-sum invariants of knotted curves and surfaces*, Trans. Amer. Math. Soc. 355 (2003) 3947–3989 MR1990571

[2] **J S Carter**, **M Saito**, *Knotted surfaces and their diagrams*, Mathematical Surveys and Monographs 55, American Mathematical Society, Providence, RI (1998) MR1487374

[3] **R Fenn**, **C Rourke**, *Racks and links in codimension two*, J. Knot Theory Ramifications 1 (1992) 343–406 MR1194995

[4] **E Hatakenaka**, *An estimate of the triple point numbers of surface-knots by quandle cocycle invariants*, Topology Appl. 139 (2004) 129–144 MR2051101

[5] **D Joyce**, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra 23 (1982) 37–65 MR638121

[6] **S Kamada**, *Wirtinger presentations for higher-dimensional manifold knots obtained from diagrams*, Fund. Math. 168 (2001) 105–112 MR1852735

[7] **S Kamada**, *Quandles with good involutions, their homologies and knot invariants*, from: "Intelligence of low dimensional topology 2006", Ser. Knots Everything 40, World Sci. Publ., Hackensack, NJ (2007) 101–108 MR2371714

[8] **S Kamada**, **K Oshiro**, *Homology groups of symmetric quandles and cocycle invariants of links and surface-links*, Trans. Amer. Math. Soc. (to appear)

[9] **A Kawauchi**, *On pseudo-ribbon surface-links*, J. Knot Theory Ramifications 11 (2002) 1043–1062 MR1941684

[10] **S V Matveev**, *Distributive groupoids in knot theory*, Mat. Sb. (N.S.) 119(161) (1982) 78–88, 160 MR672410

[11] **D Roseman**, *Reidemeister-type moves for surfaces in four-dimensional space*, from: "Knot theory (Warsaw, 1995)", Banach Center Publ. 42, Polish Acad. Sci., Warsaw (1998) 347–380 MR1634466

[12] **S Satoh**, *Minimal triple point numbers of some non-orientable surface-links*, Pacific J. Math. 197 (2001) 213–221 MR1810216

[13] **S Satoh**, *Non-additivity for triple point numbers on the connected sum of surface-knots*, Proc. Amer. Math. Soc. 133 (2005) 613–616 MR2093086

[14] **S Satoh**, **A Shima**, *The 2–twist-spun trefoil has the triple point number four*, Trans. Amer. Math. Soc. 356 (2004) 1007–1024 MR1984465

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