Degree ± 1 self-maps and self-homeomorphisms on prime 3–manifolds

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We determine all closed orientable geometrizable prime 3–manifolds that admit a degree 1 or -1 self-map not homotopic to a homeomorphism.

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1 Introduction

1.1 Background

All manifolds in this paper are closed and orientable unless stated otherwise. Definitions of terminology not stated here can be found in Hempel [5] and Hatcher [2].

Given a closed oriented n–manifold M, it is natural to ask whether all the degree ± 1 self-maps of M can be homotopic to homeomorphisms.

If the property stated above holds for M, we say M has property H. In particular, if all the degree 1 (-1) self-maps of M are homotopic to homeomorphisms, we say M has property 1H (-1H). M has property H if and only if M has both property 1H and property -1H. Observe that if M admits an orientation-reversing self-homeomorphism, then M has property 1H if and only if M has property -1H. So we need only consider property 1H in most of this paper.

The first positive result on property H is the Hopf theorem: two self-maps of S^n are homotopic if and only if they have the same mapping degree. The result that every 1– and 2–dimensional manifold has property H is also well-known: since its fundamental group is Hopfian (see Hempel [4]), all automorphisms of $\pi_1(M^2)$ can be realized by a homeomorphism [5, 13.1], and every M^2 except S^2 is a $K(\pi, 1)$.

For dimension > 3, it seems difficult to get general results, since there are no classification results for manifolds of dimension $n > 3$, and the homotopy groups can be rather complicated.

Now we restrict to dimension 3. From now on, unless stated otherwise, all manifolds in the following are 3–manifolds. Thurston's geometrization conjecture [20], which

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seems to be confirmed, implies that closed oriented 3–manifolds can be classified in a reasonable sense. So we can check whether 3–manifolds have property H case-by-case.

Thurston's geometrization conjecture claims that each Jaco–Shalen–Johannson piece of a prime 3–manifold supports one of the eight geometries, E^3, H^3, S^3, $S^2 \times E^1$, $H^2 \times E^1$, $\widetilde{\mathrm{PSL}}(2, R)$, Nil, Sol (for details see Thurston [20] and Scott [19]). Call a closed orientable 3–manifold M *geometrizable* if each prime factor of M meets Thurston's geometrization conjecture. All 3–manifolds discussed in this paper are geometrizable, and we may sometimes omit "geometrizable".

In this paper, we would like to determine which prime 3–manifolds, the basic part of 3–manifolds, have property H.

Since all degree ± 1 maps f on M induce surjections on its fundamental group, and the fundamental groups of geometrizable 3–manifolds are residually finite (therefore, Hopfian) (for example, see Hempel [5, 15.13; 6, 1.3] or Kalliongis and McCullough [11, 3.22]), $f_*\colon \pi_1(M) \to \pi_1(M)$ is an isomorphism.

Hyperbolic 3–manifolds, which seem to be the most mysterious, have property H by the celebrated Mostow rigidity theorem [13]. By Waldhausen's theorem on Haken manifolds (see Hempel [5, 13.6]), all Haken manifolds also have property H.

These two theorems cover most cases of prime geometrizable 3–manifolds, including manifolds with nontrivial JSJ decomposition, hyperbolic manifolds and Seifert manifolds with incompressible surface. It is also easy to see that $S^2 \times S^1$ has property H by elementary obstruction theory. So the remaining cases are:

- (Class 1) M^3 supporting the S^3–geometry ($M = S^3/\Gamma$, where $\Gamma < O(4) \cong \mathrm{Iso}_+(S^3)$ acts freely on S^3).
- (Class 2) Seifert manifolds M^3 supporting the Nil or $\widetilde{\mathrm{PSL}(2, R)}$ geometries with orbifold $S^2(p, q, r)$.

Essentially, it is known that the manifolds in Class 2 have property H. However, the author can't find a proper reference. We can copy the proof of [19, Theorem 3.9] word-for-word to prove this result.

1.2 Main Results

Mainly, the aim of this paper is to determine which S^3–manifolds (manifolds in Class 1) have property H.

According to [16] or [19], the fundamental group of a 3–manifold supporting the S^3–geometry belongs to one of the following eight types: $\mathbb{Z}_p$, D^*_{4n} , T^*_{24} , O^*_{48} , I^*_{120} , $T'_{8\cdot 3^q}$, $D'_{n'\cdot 2^q}$ and $\mathbb{Z}_m \times \pi_1(N)$, where N is a S^3–manifold, $\pi_1(N)$ belongs to one of the previous seven types and $|\pi_1(N)|$ is coprime to m. The cyclic group Z_p is realized by lens space $L(p,q)$. Each group in the remaining types is realized by a unique S^3–manifold.

Theorem 1.1 *For M supporting the S^3–geometry, M has property* 1H *if and only if M belongs to one of the following classes:*

(i) S^3.

(ii) $L(p,q)$ *satisfying one of the following:*
 (a) $p = 2, 4, p_1^{e_1}, 2p_1^{e_1}$.
 (b) $p = 2^s$ $(s > 2), 4p_1^{e_1}, p_1^{e_1}p_2^{e_2}, 2p_1^{e_1}p_2^{e_2}$, $q^2 \equiv 1 \bmod p$ *and* $q \neq \pm 1$.

(iii) $\pi_1(M) = \mathbb{Z}_m \times D^*_{4k}$, $(m,k) = (1, 2^k), (p_1^{e_1}, 2^k), (1, p_2^{e_2})$ *or* $(p_1^{e_1}, p_2^{e_2})$.

(iv) $\pi_1(M) = D'_{2^{k+2}p_1^{e_1}}$.

(v) $\pi_1(M) = T^*_{24}$ *or* $\mathbb{Z}_{p_1^{e_1}} \times T^*_{24}$.

(vi) $\pi_1(M) = T'_{8\cdot 3^{k+1}}$.

(vii) $\pi_1(M) = O^*_{48}$ *or* $\mathbb{Z}_{p_1^{e_1}} \times O^*_{48}$.

(viii) $\pi_1(M) = I^*_{120}$ *or* $\mathbb{Z}_{p_1^{e_1}} \times I^*_{120}$.

Here p_1, p_2 are odd prime numbers, and e_1, e_2, k, m are positive integers.

By [3] and elementary number theory, among all the S^3–manifolds, only S^3 and lens spaces admit degree -1 self-maps. So when considering property -1H, we can restrict the manifold to be $L(p,q)$.

Proposition 1.2 *$L(p,q)$ has property* -1H *if and only if $L(p,q)$ belongs to one of the following classes:*

(i) $4|p$ *or some odd prime factor of p is of the form $4k+3$.*

(ii) $q^2 \equiv -1 \bmod p$ *and* $p = 2, p_1^{e_1}, 2p_1^{e_1}$, *where p_1 is a prime number of the form* $4k+1$.

Synthesizing Mostow's theorem, Waldhausen's theorem, Theorem 1.1 and Proposition 1.2, we get the following consequence:

Theorem 1.3 *Suppose M is a prime geometrizable 3–manifold.*

(1) *M has property $1H$ if and only if M belongs to one of the following classes:*
   (i) *M does not support the S^3–geometry.*
   (ii) *M is in one of the classes stated in Theorem 1.1.*

(2) *M has property $-1H$ if and only if M belongs to one of the following classes:*
   (i) *M does not support the S^3–geometry.*
   (ii) *$M \neq L(p,q)$ and supports the S^3–geometry.*
   (iii) *M is in one of the classes stated in Proposition 1.2.*

(3) *M has property H if and only if M belongs to one of the following classes:*
   (i) *M does not support the S^3–geometry.*
   (ii) *M is in one of the classes other than* (ii) *stated in Theorem 1.1.*
   (iii) *$L(p,q)$ satisfying one of the following:*
      (a) $p = 2, 4$.
      (b) $p = p_1^{e_1}, 2p_1^{e_1}$, *where p_1 is $4k+3$ type prime number.*
      (c) $p = p_1^{e_1}, 2p_1^{e_1}$, *where p_1 is $4k+1$ type prime number and $q^2 \equiv -1 \bmod p$.*
      (d) $p = 2^s\ (s > 2), 4p_1^{e_1}$, $q^2 \equiv 1 \bmod p$, $q \neq \pm 1$.
      (e) $p = p_1^{e_1} p_2^{e_2}, 2p_1^{e_1} p_2^{e_2}$, *where one of p_1, p_2 is $4k+3$ type prime number,* $q^2 \equiv 1 \bmod p$, $q \neq \pm 1$.

In Section 2 we give some definitions which will be used later and transform our main question to the computation of $\mathrm{Out}(\pi_1(M))$ and the mapping class group of M. In Section 3, we determine which lens spaces have property H. In Section 4, we compute $\mathrm{Out}(\pi_1(M))$ by combinatorial methods. Mapping class groups of S^3–manifolds are computed in Section 5. Although the mapping class groups of S^3–manifolds are determined by Boileau and Otal [1] and McCullough [12] and some partial results are given by Hodgson and Rubinstein [8], Rubinstein [17] and Rubinstein and Birman [18], we give a complete computation based on the fact that all self-homeomorphisms on an S^3–manifold $M \neq L(p,q)$ can be isotopic to a fiber-preserving homeomorphism. In Section 6, Table 1 shows the computation results.

2 Definitions and preliminaries

Definition 2.1 Suppose an oriented 3–manifold M' is a circle bundle with a given section F, where F is a compact surface with boundary components $c_1, \dots, c_n, c_{n+1}, \dots, c_{n+m}$ with $n > 0$. On each boundary component of M', orient c_i and the circle fiber h_i so that the product of their orientation on $c_i \times S^1$ matches with the induced orientation of M'. Now attach n solid tori N_i to the first n boundary tori of M' so that the meridian of N_i is identified with slope $l_i = \alpha_i c_i + \beta_i h_i$ with $\alpha_i > 0$. Denote the resulting manifold by M, which has the Seifert fiber structure (foliated by circles) extended from the circle bundle structure of M', and the core of N_i is a "singular fiber" for $\alpha_i > 1$.

We will denote this Seifert fiber structure of M by $\{(\pm g, m); r_1, \dots, r_n\}$ where g is the genus of the section F of M, where the sign is $+$ if F is orientable and $-$ if F is nonorientable. Here "genus" of nonorientable surfaces means the number of $\mathbb{R}P^2$ connected summands and $r_i = \beta_i/\alpha_i$, while (α_i, β_i) is the index of the corresponding singular fiber.

Almost all Seifert manifolds we consider in this paper have structure $\{(0,0); r_1, \dots, r_n\}$ with $n \le 3$. For simplicity, we denote the structure $\{(0,0); r_1, \dots, r_n\}$ by $\{b; r_1', \dots, r_n'\}$, where $0 < r_i' < 1, r_i' \equiv r_i \mod 1$, and $\sum_{i=1}^n r_i = b + \sum_{i=1}^n r_i'$. This does not bring about confusion since $\{(0,0); r_1, \dots, r_n\}$ is fiber-preserving, orientation-preserving homeomorphic to $\{(0,0); r_1', \dots, r_n', b\}$, and the form $\{b; r_1', \dots, r_n'\}$ is unique.

When we identify every S^1 fiber of M to a point, we get a "2–manifold" $\mathcal{O}(M)$ with singular points corresponding to the singular fibers, which is called an orbifold. Although there is a standard definition for orbifold (see Scott [19]), we do not state it here, but just think of an orbifold as a Hausdorff space that is locally isomorphic to quotient space of R^n by a finite group action. More simply, the orbifolds we consider in this paper are just surfaces with singular points, where every neighborhood of a singular point is isomorphic to $D^2/\mathbb{Z}_n$ (the action is $2\pi/n$ rotation). When we delete a neighborhood of all singular points, the remaining part of $\mathcal{O}(M)$ can be identified with the section F in Definition 2.1. An orientation on $\mathcal{O}(M)$ is induced by an orientation on the section F.

According to Orlik [16] or Scott [19], the fundamental group of a 3–manifold with the S^3–geometry structure belong to one of the following eight types: $\mathbb{Z}_p$, D^*_{4n}, T^*_{24}, O^*_{48}, I^*_{120}, $T'_{8\cdot 3^q}$, $D'_{n'\cdot 2^q}$ and $\mathbb{Z}_m \times G$ where G belongs to one of the previous seven types and $|G|$ is coprime to m. All the manifolds are uniquely determined by the fundamental group except when $\pi_1(M) = \mathbb{Z}_p$, in this case $M = L(p,q)$ for some q. The fundamental groups and Seifert structures of these manifolds are given by Orlik [16]:

Theorem 2.2 *The manifolds supporting S^3–geometry are classified as follows:*

(1) *$M = \{b; \beta_1/\alpha_1, \beta_2/\alpha_2\}$, here we allow $\alpha_i = 1$, $\beta_i = 0$, are lens spaces with $\pi_1(M) \cong \mathbb{Z}_p$, where $p = |b\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1|$.*

(2) $M=\{b;1/2,1/2,\beta_3/\alpha_3\}$ *are called prism manifolds, let* $m=(b+1)\alpha_3+\beta_3$ *; if* $(m,2\alpha_3)=1$ *, then* $\pi_1(M)\cong\mathbb{Z}_m\times D^*_{4\alpha_3}\cong\mathbb{Z}_m\times\{x,y\mid x^2=(xy)^2=y^{\alpha_3}\}$*; if* $(m,2\alpha_3)\neq 1$*, then* $m=2^k m'$*, we have* $\pi_1(M)\cong\mathbb{Z}_{m'}\times D'_{2^{k+2}\alpha_3}\cong\mathbb{Z}_{m'}\times\{x,y\mid x^{2^{k+2}}=1, y^{\alpha_3}=1, xy=y^{-1}x\}$.

(3) $M=\{b;1/2,\beta_2/3,\beta_3/3\}$*, let* $m=6b+3+2(\beta_2+\beta_3)$*; if* $(m,12)=1$*, then* $\pi_1(M)\cong\mathbb{Z}_m\times T^*\cong\mathbb{Z}_m\times\{x,y\mid x^2=(xy)^3=y^3, x^4=1\}$*; if* $(m,12)\neq 1$*,* $m=3^k m'$*, then* $\pi_1(M)\cong\mathbb{Z}_{m'}\times T'_{8\cdot 3^k}\cong\mathbb{Z}_{m'}\times\{x,y,z\mid x^2=(xy)^2=y^2, z^{3^{k+1}}=1, zxz^{-1}=y, zyz^{-1}=xy\}$.

(4) $M=\{b;1/2,\beta_2/3,\beta_3/4\}$*, let* $m=12b+6+4\beta_2+3\beta_3$*, then* $(m,24)=1$*,* $\pi_1(M)\cong\mathbb{Z}_m\times O^*\cong\mathbb{Z}_m\times\{x,y\mid x^2=(xy)^3=y^4, x^4=1\}$.

(5) $M=\{b;1/2,\beta_2/3,\beta_3/5\}$*, let* $m=30b+15+10\beta_2+6\beta_3$*, then* $(m,60)=1$*,* $\pi_1(M)\cong\mathbb{Z}_m\times I^*\cong\mathbb{Z}_m\times\{x,y\mid x^2=(xy)^3=y^5, x^4=1\}$.

Remark The Seifert structures on lens spaces are not unique, while the orbifolds are all S^2 with at most two singular points. The prism manifolds also have another Seifert structure with orbifold P^2 with one singular point. The other S^3–manifolds have unique Seifert structures.

Since a degree ± 1 self-map f of M is surjective on fundamental group and $\pi_1(M)$ is finite, f induces an isomorphism on $\pi_1(M)$. Therefore we need only consider self-maps that induce isomorphism on $\pi_1(M)$.

All the degrees of self-maps that induce isomorphisms on $\pi_1(M)$ are given in Hayat-Legrand et al [3]:

Proposition 2.3 *For a* 3*–manifold* M *supporting the* S^3 *geometry,*

$$D_{\mathrm{iso}}(M)=\big\{k^2+l|\pi_1(M)|\;\big|\;\gcd(k,|\pi_1(M)|)=1\big\}.$$

Here $D_{\mathrm{iso}}(M)=\{\deg(f)\mid f\colon M\to M,\ f \text{ induces isomorphism on } \pi_1(M)\}$.

Any S^3–manifold $M\neq L(p,q)$ satisfies $|\pi_1(M)|=4k$, and any odd square number has form $4l+1$. So M does not admit any degree -1 self-map, and we only consider property 1H in most of this paper.

Since the second homotopy group of a S^3–manifold is trivial, the existence of self-mappings can be detected by obstruction theory. P Olum showed in [15] the first and in [14] the second part of the following proposition.

Proposition 2.4 *Let M be an orientable 3–manifold with finite fundamental group and trivial $\pi_2(M)$. Every endomorphism $\phi\colon \pi_1(M) \to \pi_1(M)$ is induced by a (basepoint-preserving) continuous map $f\colon M \to M$.*

Furthermore, if g is also a continuous self-map of M such that f_* is conjugate to g_*, then $\deg f \equiv \deg g \bmod |\pi_1(M)|$; furthermore, f and g are homotopic to each other if and only if f_* is conjugate to g_* and $\deg(f) = \deg(g)$.

According to this proposition, homotopic information of self-maps can be completely determined by degree and induced homomorphism on π_1.

We also need a little elementary number theory:

Definition 2.5 Let $U_p = \{\text{all units in the ring } \mathbb{Z}_p\}$, $U_p^2 = \{a^2 \mid a \in U_p\}$, which is a subgroup of U_p. Denote $|U_p/U_p^2|$ by $\Psi(p)$.

The following theorem in number theory can be found in Ireland and Rosen [9, page 44]:

Lemma 2.6 *Let $p = 2^a p_1^{e_1} \cdots p_l^{e_l}$ be the prime decomposition of p. Then $U_p \cong U_{2^a} \times U_{p_1^{e_1}} \times \cdots \times U_{p_l^{e_l}}$, where $U_{p_i^{e_i}}$ is the cyclic group of order $p_i^{e_i-1}(p_i - 1)$. The group U_{2^a} is the cyclic group of order 1 and 2 for $a = 1$ and 2, respectively, and if $a > 2$, then it is the product of one cyclic group of order 2 and another of order 2^{a-2}.*

By Lemma 2.6 and elementary computation, we get:

Lemma 2.7 *Let $p = 2^a p_1^{a_1} \cdots p_l^{a_l}$ be the prime decomposition of p. Then*

$$\Psi(p) = \begin{cases} (\mathbb{Z}_2)^l & \text{if } a = 0, 1, \\ (\mathbb{Z}_2)^{l+1} & \text{if } a = 2, \\ (\mathbb{Z}_2)^{l+2} & \text{if } a > 2. \end{cases}$$

This Lemma is useful in the computation process that determines which S^3–manifolds have property H.

If $|\pi_1(M)| = p$, denote $U^2(|\pi_1(M)|)$ by U_p^2. Then we define group homomorphism $\mathcal{H}\colon \mathrm{Out}(\pi_1(M)) \to U^2(|\pi_1(M)|)$: for all $\phi \in \mathrm{Out}(\pi_1(M))$, take a self-map f of M, such that $f_* \in \phi$, and define $\mathcal{H}(\phi) = \deg(f) \in U^2(|\pi_1(M)|)$.

By Proposition 2.3, $\deg(f) \in U^2(|\pi_1(M)|)$ (after mod $|\pi_1(M)|$). By Proposition 2.4, $\mathcal{H}$ is well defined. By Proposition 2.3 again, $\mathcal{H}$ is surjective.

Let $K(M)=\{\phi\in \mathrm{Out}(\pi_1(M))\mid \exists f\colon M\to M,\ f_*\in\phi,\ \deg(f)=1\}$. We can see that $K(M)=\ker(\mathcal{H})$, $|K(M)|=|\mathrm{Out}(\pi_1(M))|/|U^2(|\pi_1(M)|)|$. By Proposition 2.4, $K(M)$ corresponds bijectively with

$$\{\text{degree 1 self-maps } f \text{ on } M\}\,/\,\text{homotopy}.$$

Let $K'(M)=\{\phi\in \mathrm{Out}(\pi_1(M))\mid \exists f\colon M\to M$ an orientation-preserving homeomorphism, $f_*\in\phi\}$, which is a subgroup of $K(M)$. $K'(M)$ corresponds bijectively with the orientation-preserving subgroup of mapping class group of M:

$$\mathcal{MCG}^+(M)=\{\text{orientation-preserving homeomorphism } f \text{ on } M\}\,/\,\text{homotopy}$$

For an S^3–manifold $M\neq L(p,q)$, M does not admit a degree -1 self-map, so $\mathcal{MCG}^+(M)=\mathcal{MCG}(M)$.

Remark For the standard definition of $\mathcal{MCG}(M)$, we should use isotopy, not homotopy. However, [1] shows that, for self-homeomorphisms on S^3–manifolds, homotopy implies isotopy.

To determine whether M has property 1H, we need only determine whether $K(M)=K'(M)$, or whether $|K(M)|=|\mathcal{MCG}^+(M)|$. For this, define the realization coefficient of M:

$$\mathrm{RC}(M)=\frac{|K(M)|}{|K'(M)|}=\frac{|\mathrm{Out}(\pi_1(M))|}{|U^2(|\pi_1(M)|)|\cdot|\mathcal{MCG}^+(M)|}.$$

So M has property 1H if and only if $\mathrm{RC}(M)=1$. We need only compute $|\mathrm{Out}(\pi_1(M))|$ and $|\mathcal{MCG}^+(M)|$, the computations are completed in Section 4 and Section 5. Section 4 only contains algebraic computations; we will give geometric generators of $\mathcal{MCG}^+(M)$ in Section 5, and determine the relations by results in Section 4.

Since $L(p,q)$ may also admit degree -1 self-maps, and it admits different Seifert structures, we will use a different way to determine $\mathcal{MCG}(M)$ in this case. Section 3 will deal with the lens space case first.

3 Property H of lens spaces

Suppose $L(p,q)$ is decomposed as $L(p,q)=N_1\cup_T N_2$, where each N_i is a solid torus and $T=\partial N_1=\partial N_2$ is the Heegaard torus. Let l be the core circle of N_1.

The following result can be found in [2, Theorem 2.5]:

Lemma 3.1 *For any homeomorphism $f\colon L(p,q)\to L(p,q)$, $f(T)$ is isotopic to T.*

Lemma 3.2 *Suppose f is a degree 1 self-map on $L(p,q)$, f is homotopic to an orientation-preserving homeomorphism if and only if*

$$f_*(l) = \begin{cases} \pm l & \text{if } p \nmid (q^2-1), \\ \pm l, \pm ql & \text{if } p \mid (q^2-1). \end{cases}$$

Proof By Proposition 2.4, we need only determine all the possible $n \in \mathbb{Z}_p$, such that there is an orientation-preserving homeomorphism f of $L(p,q)$, such that $f_*(l) = nl$.

Suppose f is an orientation-preserving homeomorphism of $L(p,q)$. By Lemma 3.1, $f(T)$ is isotopic to T. So we can isotope f so that $f(T) = T$. In this case, f sends N_i to N_i $(i = 1,2)$ or f exchanges N_i.

If f exchanges N_i, suppose $T_i = \partial N_i$, and T_1 is pasted to T_2 by a linear homeomorphism A. Then there is the commutative diagram

$$\begin{array}{ccc} T_1 & \xrightarrow{f|_{T_1}} & T_2 \\ A\big\downarrow & & \big\downarrow A^{-1} \\ T_2 & \xrightarrow{f|_{T_2}} & T_1. \end{array}$$

Since A pastes the two solid tori to $L(p,q)$, A can be written as

$$\begin{pmatrix} r & p \\ s & q \end{pmatrix},$$

where $rq - sp = 1$. Also $f|T_i$ can be extended to a homeomorphism from N_i to N_j $(i \neq j)$, so $f|T_i$ sends meridian to meridian. Since f preserves the orientation, $f|T_i$ has the form

$$\pm \begin{pmatrix} 1 & 0 \\ m & -1 \end{pmatrix}.$$

From $A \circ f|_{T_2} \circ A = f|_{T_1}$, we have

$$\begin{pmatrix} r^2+mrp-sp & rp+mp^2-pq \\ sr+mrp-sq & sp+mrp-q^2 \end{pmatrix} = \begin{pmatrix} r & p \\ s & q \end{pmatrix} \begin{pmatrix} 1 & 0 \\ m & -1 \end{pmatrix} \begin{pmatrix} r & p \\ s & q \end{pmatrix} = \pm \begin{pmatrix} 1 & 0 \\ n & -1 \end{pmatrix}.$$

So $rp + mp^2 - pq = 0$, and then $q - r = mp$, $r \equiv q \bmod p$. Since $rq - sp = 1$, we have $q^2 \equiv 1 \bmod p$. In this case, $f_*(l) = \pm rl = \pm ql$.

On the other hand, when $q^2 = np + 1$, taking $r = q, s = n$,

$$f|T_1 = f|T_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then we can obtain an orientation-preserving homeomorphism f on $L(p,q)$ with $f_*(l) = \pm ql$.

If f sends N_i to N_i as a homeomorphism, then f must send l to a longitude of N_1 and so does f_* in $\pi_1(L(p,q))$: $f_*(l) = \pm l$. The homeomorphisms can be realized as in the last case. □

Thus we can compute $\mathrm{RC}(M)$ directly:

Proposition 3.3 *For the lens space $L(p,q)$,* $\mathrm{Out}(\pi_1(L(p,q))) \cong \mathrm{Out}(\mathbb{Z}_p) \cong U_p$,

$$\mathcal{MCG}^+(L(p,q)) = \begin{cases} \{e\} & \text{if } p = 2, \\ \mathbb{Z}_2 & \text{if } p \nmid (q^2-1) \text{ or } q = \pm 1, \\ \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if } p \mid (q^2-1) \text{ and } q \neq \pm 1, \end{cases}$$

$$\mathrm{RC}(L(p,q)) = \begin{cases} 1 & \text{if } p = 2, \\ \Psi(p)/2 & \text{if } p \nmid (q^2-1) \text{ or } q = \pm 1, \\ \Psi(p)/4 & \text{if } p \mid (q^2-1) \text{ and } q \neq \pm 1. \end{cases}$$

The $L(p,q)$ part of Theorem 1.1 follows from this Proposition and Lemma 2.7.

Lemma 3.4 *$L(p,q)$ admits a degree -1 self-map if and only if $4 \nmid p$ and all the odd prime factors of p are of the form $4k+1$.*

Proof By Proposition 2.3, we need only determine for which p, there is an integer q, such that $q^2 \equiv -1 \bmod p$.

Suppose $4 \nmid p$ and all odd prime factors of p are of the form $4k+1$. By Lemma 2.6, U_p is direct sum of some order $4k$ cyclic groups and the order of -1 in U_p is 2, so q exists.

On the other hand, if $4 \mid p$, there is no q such that $q^2 \equiv -1 \bmod p$, since odd squares are congruent to $1 \bmod 4$. If some prime factor p_1 of p is of the form $4k+3$, then $q^{4k+2} \equiv 1 \bmod p_1$, by Fermat's Little Theorem, and so again there is no q such that $q^2 \equiv -1 \bmod p$. □

By the same computation as Lemma 3.2, we get:

Lemma 3.5 *$L(p,q)$ admits an orientation-reversing homeomorphism if and only if $q^2 \equiv -1 \bmod p$. In this case, a degree -1 self-map f on $L(p,q)$ is homotopic to an orientation-reversing homeomorphism if and only if $f_*(l) = \pm ql$.*

If $L(p,q)$ admits an orientation-reversing homeomorphism, then $L(p,q)$ has property 1H if and only if $L(p,q)$ has property -1H. Synthesizing Lemma 3.4, Lemma 3.5 and Proposition 3.3, we get Proposition 1.2.

4 Out($\pi_1(M)$) of S^3–manifolds

We are only interested in the order of $\mathrm{Out}(\pi_1(M))$, so we only compute the order here. Moreover, we also give a presentation of $\mathrm{Out}(\pi_1(M))$, since it will help us in Section 5. All the arguments in this section are combinatorial.

If $(m, |G|) = 1$, we have $\mathrm{Out}(\mathbb{Z}_m \times G) \cong \mathrm{Out}(\mathbb{Z}_m) \times \mathrm{Out}(G) \cong U_m \times \mathrm{Out}(G)$. So the main aim of this section is to compute $\mathrm{Out}(G)$ for G in Theorem 2.2 without cyclic summands.

We know that $\mathrm{SU}(2) \subset O(4) \cong \mathrm{Iso}_+(S^3)$. Let $p\colon \mathrm{SU}(2) \to O(3)$ be the canonical two-to-one Lie group homomorphism. T^*, O^*, I^* and $D^*_{4\alpha_3}$ are the preimage of T, O, I and $D_{2\alpha_3}$ respectively. T, O, I are the symmetry groups of regular tetrahedron, octagon and icosahedron (isomorphic to A_4, S_4, A_5 respectively), and $D_{2\alpha_3}$ is the dihedral group.

Case 1 $G \cong T^*$ or O^* or I^*.

By [7, VIII-2], $\mathrm{Out}(T^*) \cong \mathrm{Out}(O^*) \cong \mathrm{Out}(I^*) \cong \mathbb{Z}_2$. The elements in $\mathrm{Out}(G^*)$ not equal to identity can be presented as follows (we can lift an element of $\mathrm{Out}(G)$ to $\mathrm{Out}(G^*)$ to obtain the presentation ($G \cong T, O, I$), and we will talk more about this method in the next case):

$$\begin{aligned}
T^*:&\quad \phi(x) = x^3, && \phi(y) = y^5,\\
O^*:&\quad \phi(x) = x^3, && \phi(y) = y^5,\\
I^*:&\quad \phi(x) = xyx^{-1}y^{-1}x^{-1}, && \phi(y) = x^2y^2.
\end{aligned}$$

Case 2 $G \cong D^*_{4\alpha_3} \cong \{x, y \mid x^2 = (xy)^2 = y^{\alpha_3}\}$.

We determine $\mathrm{Out}(D_{2\alpha_3})$ first. $D_{2\alpha_3} \cong \{x, y \mid x^2 = (xy)^2 = y^{\alpha_3} = 1\}$. Every element in $D_{2\alpha_3}$ can be presented by y^n or xy^n and order of xy^n is 2.

If $\alpha_3 = 2$, $D^*_{4\alpha_3} \cong Q_8 \cong \{\pm 1, \pm i, \pm j, \pm k\}$, $\mathrm{Out}(D^*_8) \cong S_3$. So we assume $\alpha_3 > 2$ in the following.

By elementary combinatorial arguments, we can get the following consequence (the condition $\alpha_3 > 2$ is used here):

(1) When α_3 is odd, $\mathrm{Out}(D_{2\alpha_3})$ is presented by $\phi(x) = x, \phi(y) = y^k$, where $1 \le k \le \alpha_3/2$, $(k, \alpha_3) = 1$.

(2) When α_3 is even, $\mathrm{Out}(D_{2\alpha_3})$ is presented by $\phi(x) = x, \phi(y) = y^k$; $\phi(x) = xy, \phi(y) = y^k$, where $1 \le k \le \alpha_3/2$, $(k, \alpha_3) = 1$.

For p: $D^*_{4\alpha_3} \to D_{2\alpha_3}$, $\ker(p)$ is the center of $D^*_{4\alpha_3}$. Every automorphism ϕ' on $D^*_{4\alpha_3}$ sends the center to the center, so induces an automorphism ϕ on $D_{2\alpha_3}$. If two induced automorphism ϕ_1, ϕ_2 are conjugate in $D_{2\alpha_3}$, then two automorphisms ϕ'_1, ϕ'_2 on $D^*_{4\alpha_3}$ are conjugate. So we can work in this process: given a presentation of $\mathrm{Out}(D_{2\alpha_3})$, $\phi_1, \dots, \phi_k$, list all the possible liftings of every ϕ_i (there are at most four), and check whether there are any pair of liftings of the same ϕ_i are conjugate with each other. Then we get a presentation of $\mathrm{Out}(D^*_{4\alpha_3})$.

Lemma 4.1 *A presentation of* $\mathrm{Out}(D^*_{4\alpha_3})$ *is given by the following:*

(1) $\alpha_3 = 2$, $|\mathrm{Out}(D^*_8)| = 6$:
id; $\phi(x) = x,\ \phi(y) = xy$; $\phi(x) = y,\ \phi(y) = x$; $\phi(x) = y,\ \phi(y) = xy$; $\phi(x) = xy,\ \phi(y) = x$; $\phi(x) = xy,\ \phi(y) = y$.

(2) α_3 *odd*, $|\mathrm{Out}(D^*_{4\alpha_3})| = |U_{4\alpha_3}|/2$:
$\phi(x) = x,\ \phi(y) = y^k$; $\phi(x) = x^3,\ \phi(y) = y^k$, *here* $1 \le k \le \alpha_3, (k, \alpha_3) = 1, k$ *odd*.

(3) $\alpha_3 > 2$ *even*, $|\mathrm{Out}(D^*_{4\alpha_3})| = |U_{4\alpha_3}|/2$:
$\phi(x) = x,\ \phi(y) = y^k$; $\phi(x) = x^3 y,\ \phi(y) = y^k$, *here* $1 \le k \le \alpha_3, (k, \alpha_3) = 1$.

Case 3 $G \cong D'_{2^{k+2}\alpha_3} \cong \{x, y \mid x^{2^{k+2}} = y^{\alpha_3} = 1, xy = y^{-1}x\}$, here α_3 is odd.

In $D'_{2^{k+2}\alpha_3}$, every element can be written as $x^u y^v$. Since the subgroup generated by y is product of normal Sylow subgroups of $D'_{2^{k+2}\alpha_3}$, it is a characteristic subgroup. So for any automorphism ϕ of $D'_{2^{k+2}\alpha_3}$, there is $\phi(x) = x^u y^v,\ \phi(y) = y^w$, $(w, \alpha_3) = 1$.

To guarantee ϕ is a homomorphism, u should be odd, and it is enough for ϕ to be an automorphism. The inversion of ϕ is $\phi'(x) = x^{u'} y^{v'},\ \phi'(y) = y^{w'}$, $uu' \equiv 1 \bmod 2^{k+2}$, $ww' \equiv 1 \bmod \alpha_3$, $v + v'w \equiv 0 \bmod \alpha_3$. $\mathrm{Aut}(D'_{2^{k+2}\alpha_3})$ is given as

$$\phi(x) = x^u y^v,\ \phi(y) = y^w, (w, \alpha_3) = 1, u \text{ odd}.$$

So $|\mathrm{Aut}(D'_{2^{k+2}\alpha_3})| = 2^{k+1}\alpha_3 |U_{\alpha_3}|$.

For every automorphism $\phi(x) = x^u y^v, \phi(y) = y^w$, conjugate by $x^p y^q$, we get $\phi'(x) = x^u y^{(-1)^p(v-2q)}, \phi'(y) = y^{(-1)^p w}$. So the inner automorphism group of $D'_{2^{k+2}\alpha_3}$ has order $2\alpha_3$.

So we get $|\mathrm{Out}(D'_{2^{k+2}\alpha_3})| = 2^k |U(\alpha_3)|$. A presentation of $\mathrm{Out}(D'_{2^{k+2}\alpha_3})$ is

$$\phi(x) = x^u, \phi(y) = y^v,\ u \text{ odd}, 1 \le v \le \frac{\alpha_3}{2}, (v, \alpha_3) = 1.$$

Case 4 $G \cong T'_{8\cdot 3^{k+1}} \cong \{x, y, z \mid x^2 = y^2 = (xy)^2, z^{3^{k+1}} = 1, zxz^{-1} = y, zyz^{-1} = xy\}$.

Here we assume $k \geq 1$, since $T'_{24} \cong T^*_{24}$. We can observe that $N = \{x, y \mid x^2 = y^2 = (xy)^2\}$ is a normal Sylow subgroup of $T'_{8\cdot 3^{k+1}}$, so every automorphism ϕ must send N to itself. By conjugation, we can assume $\phi(x) = x, \phi(y) = y$ or xy.

There are eight possibilites for $\phi(z)$: $z^n, z^n x, z^n y, z^n xy, z^n x^2, z^n x^3, z^n yx, z^n y^3$, so ϕ may have sixteen forms. However, to guarantee ϕ to be an automorphism, ϕ can only be one of the following:

$$\phi(x) = x,\ \phi(y) = y,\quad \phi(z) = z^n,\quad n \equiv 1 \bmod 3,$$
$$\phi(x) = x,\ \phi(y) = xy,\ \phi(z) = z^n x,\ n \equiv 2 \bmod 3.$$

We can check that all these automorphisms are not conjugate to each other, so they give a presentation of $\mathrm{Out}(T'_{8\cdot 3^{k+1}})$, and $|\mathrm{Out}(T'_{8\cdot 3^{k+1}})| = 2\cdot 3^k$.

5 Mapping class group of S^3–manifolds

We determine the mapping class group of S^3–manifolds $M \neq L(p,q)$. In this section, all the manifolds have Seifert manifold structure $M = \{b; \beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3\}$. For these manifolds, $\mathcal{MCG}(M) = \mathcal{MCG}^+(M)$.

In [1; 12], the mapping class groups of S^3–manifolds have been determined, and some partial results are given in [8; 17; 18]. However, we would like to recompute the mapping class group based on the fact that all homeomorphisms on an S^3–manifold $M \neq L(p,q)$ can be isotoped to fiber-preserving homeomorphism [1; 10].

5.1 Geometric generators of mapping class group

At first, we construct two types of homeomorphisms of $M = \{b; \beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3\}$ (the second type may be available only for certain types of M). Then we prove these two types of homeomorphisms generate $\mathcal{MCG}(M)$.

Homeomorphism Type I As Definition 2.1, we first define the homeomorphism on $M' = F \times S^1$, and then extend it over three solid tori N_1, N_2, N_3.

Here F is the three punctured sphere, and we draw it as in Figure 1. Define ρ_1 to be the reflection with respect to the x–axis; σ_1 to be the homeomorphism on S^1, $\sigma_1(\theta) = -\theta$.

Let $f'_1 = \rho_1 \times \sigma_1$ on M'. This preserves the orientation of M', and reverses the orientation on F and S^1. The restriction of f'_1 to the boundary tori is $(\phi, \theta) \to (-\phi, -\theta)$, which sends $l_i = \alpha_i c_i + \beta_i h_i$ to $--l_i$. So we can extend f'_1 to a homeomorphism f_1 on M.

Homeomorphism Type II In this case we need $\beta_1/\alpha_1 = \beta_2/\alpha_2$. The two boundary components c_1, c_2 of F corresponding to $\beta_1/\alpha_1, \beta_2/\alpha_2$ are drawn in Figure 1.

Take the polar coordinate (r, θ) on D^2, assume c_1, c_2 are symmetric with respect to the π rotation on D^2. Define homeomorphism ρ_2 on F as follows: $\rho_2(r, \theta) = (r, \theta + \pi)$. Then ρ_2 exchanges c_1 and c_2.

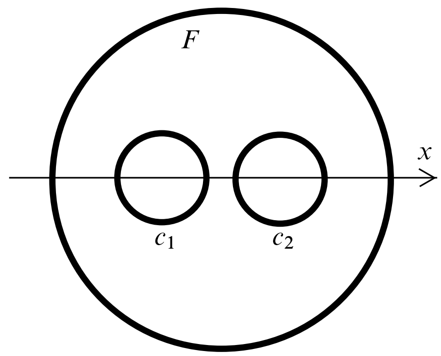

Figure 1

Let $f_2' = \rho_2 \times \mathrm{id}_{S^1}$ on M'. This preserves the orientation of M', F and S^1. Since $\beta_1/\alpha_1 = \beta_2/\alpha_2$, so f_2' exchanges l_1, l_2, and sends l_3 to itself. So we can extend f_2' to a homeomorphism f_2 on M.

We can see that these two types of homeomorphisms are involutions of M and they commute with each other.

We will prove that these two types of homeomorphisms generate $\mathcal{MCG}(M)$. First, we need this proposition [1, Proposition 3.1; 10, Lemmas 3.5, 3.6]:

Proposition 5.1 *Suppose M is an S^3–manifold which has a Seifert structure with orbifold S^2 with three singular points. Then any homeomorphism $f\colon M \to M$ is isotopic to a fiber-preserving homeomorphism with respect to the fibration.*

Lemma 5.2 *Suppose F is a three punctured sphere, $g\colon F \to F$ is a homeomorphism and $g|_{\partial F} = \mathrm{id}_{\partial F}$. Then g is isotopic to identity.*

Proof We denote the three boundary components of F by c_1, c_2, c_3. Take a simple arc α connecting c_1 and c_2.

A basic fact due to Dehn is that we can isotope g so that $g|_\alpha = \mathrm{id}_\alpha$, and we can still require g to be identity on ∂F. Cutting along α, we get an annulus F_1 and g induces a homeomorphism g_1 on F_1 such that $g_1|_{\partial F_1} = \mathrm{id}_{\partial F_1}$. The boundary component of F_1 consists of arcs c_1, c_2 and α is denoted by α'. Then we can isotopy g_1 to id_{F_1} and the isotopy process fix all points on α'.

Then we can paste the isotopy on F_1 to an isotopy on F, since the isotopy process fixes α' pointwise. Thus we can isotope g to id_F. □

Lemma 5.3 *Suppose that* $M = \{b; r_1, r_2, r_3\}$, $f \colon M \to M$ *is a fiber-preserving, orientation-preserving homeomorphism, the induced map* $\bar{f} \colon \mathcal{O}(M) \to \mathcal{O}(M)$ *preserves the orientation of orbifold, and* $\bar{f}(x_i) = x_i$ *for the singular points* $x_i, i = 1, 2, 3$. *Then* f *is homotopic to the identity.*

Proof Decompose M as the union of $M' = F \times S^1$ and solid tori N_1, N_2, N_3 as in Definition 2.1; the boundary torus of N_i is denoted by T_i. F can be identified with a subsurface of $\mathcal{O}(M)$: $\mathcal{O}(M)$ minus neighborhood of singular points. ∂F consists of three boundary components c_1, c_2, c_3, which correspond to singular points x_1, x_2, x_3 respectively (see Figure 2). Suppose $r_i = \beta_i / \alpha_i$.

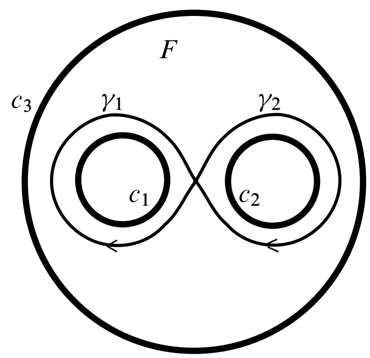

Figure 2

Since $\bar{f}(x_i) = x_i$, we can assume $f(M') = M'$ and $f(N_i) = N_i$.

Since N_i is a solid torus, the homeomorphism must send the meridian to meridian, so we have $(f|_{T_i})_*(\alpha_i c_i + \beta_i h_i) = \pm(\alpha_i c_i + \beta_i h_i)$ on the boundary torus T_i. Since $\bar{f}$ preserves the orientation of $\mathcal{O}(M)$, f preserves the orientation of M, and so preserves the orientation of regular fiber, we have $f_*(h) = h$, thus $(f|_{T_i})_*(h_i) = h_i$. Then we get $(f|_{T_i})_*(c_i) = c_i$.

Take two loops γ_1, γ_2 to be generators of $\pi_1(F)$ as shown in Figure 2. Since $(f|_{T_i})_*(c_i) = c_i$, and c_i is isotopic to γ_i in F, for the subgroup $\pi_1(F) < \pi_1(M')$, we have $(f|_{M'})_*(\pi_1(F)) = \pi_1(F)$ and also $(f|_{M'})_*(h) = h$.

For $g = \overline{f}|_F$, we have $g|_{\partial F} = \mathrm{id}$. By Lemma 5.2, we get a homotopy $H\colon (F, \partial F)\times I \to (F, \partial F)$, such that $H_0 = g$, $H_1 = \mathrm{id}$. So g_* is conjugate to $\mathrm{id}_{\pi_1(F)}$.

Conjugate by the same element in $\pi_1(F) < \pi_1(M')$, we get $(f|_{M'})_*$ is conjugate to $\mathrm{id}|_{\pi_1(M')}$. Since $i_*\colon \pi_1(M') \to \pi_1(M)$ is surjective, f_* conjugates to the identity. By Proposition 2.4, f is homotopic to the identity. □

Lemma 5.4 *Suppose that $f\colon M \to M$ is a fiber-preserving, orientation-preserving homeomorphism and $\overline{f}$ preserves the orientation of $\mathcal{O}(M)$. If f sends singular fiber with index (α_1, β_1) to singular fiber with index (α_2, β_2), then $\alpha_1 = \alpha_2$ and $\alpha_1 | (\beta_2 - \beta_1)$.*

Proof The notation is as in the last lemma.

We can assume that $f(N_1) = N_2$. Since $f|_{N_1}\colon N_1 \to N_2$ is a homeomorphism, $f|_{N_1}$ sends the meridian to meridian, thus $(f|_{T_1})_*(\alpha_1 c_1 + \beta_1 h_1) = \pm(\alpha_2 c_2 + \beta_2 h_2) \in \pi_1(T_2)$. Since $\overline{f}$ preserves the orientation of $\mathcal{O}(M)$, we have $(\overline{f}|_F)_*(c_1) = c_2$, so $(f|_{T_1})_*(c_1) = c_2 + l h_2$. Since $\overline{f}$ preserves the orientation of $\mathcal{O}(M)$, f preserves the orientation of M and the Seifert structure of M, we have $(f|_{T_1})_*(h_1) = h_2$. Then we have

$$\alpha_2 c_2 + \beta_2 h_2 = (f|_{T_1})_*(\alpha_1 c_1 + \beta_1 h_1) = \alpha_1 c_2 + (l\alpha_1 + \beta_1) h_2 \in \pi_1(T_2).$$

Since c_2, h_2 is a basis of $\pi_1(T_2)$, we get $\alpha_1 = \alpha_2$ and $\alpha_1 | (\beta_2 - \beta_1)$. □

Proposition 5.5 *For an S^3–manifold $M \neq L(p,q)$, the mapping class group of M is generated by the homeomorphisms of type I and type II defined at the beginning of Section 5.1.*

Proof Suppose f is an orientation-preserving homeomorphism of M. Then by Proposition 5.1, we can isotope f to a fiber-preserving homeomorphism.

If necessary, compose f with homeomorphism of type I. For the new homeomorphism f_1, we can assume $\overline{f_1}$ preserves the orientation on $\mathcal{O}(M)$. If $\overline{f_1}$ sends a singular point x_1 to singular point x_2, by Lemma 5.4, we have $\alpha_1 = \alpha_2$ and $\alpha_1 | (\beta_2 - \beta_1)$. If necessary, rechoose the section F, we can assume $\beta_1 = \beta_2$. Composing with homeomorphism of type II, we get a new homeomorphism f_2 such that $\overline{f_2}(x_1) = x_1$, and $\overline{f_2}$ still preserves the orientation on $\mathcal{O}(M)$. By induction, we obtain a map f_3 that sends every singular fiber to itself.

Now f_3 satisfies the condition of Lemma 5.3, so f_3 is homotopic to identity. Since we compose f with homeomorphisms of type I and II to get $f_3 \sim \mathrm{id}$, we obtain that f is homotopic to composition of homeomorphisms of type I and II. □

5.2 Equivalence of two presentations

The presentations of $\pi_1(M)$ in Theorem 2.2 do not reflect the Seifert structure of S^3–manifolds. However, for a Seifert manifold $M = \{b; \beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3\}$, there is a natural presentation of $\pi_1(M)$ from the Seifert structure [16]:

$$\pi_1(M) \cong \{h, c_1, c_2 \mid [c_1, h] = [c_2, h] = 1, c_1^{\alpha_1} h^{\beta_1} = c_2^{\alpha_2} h^{\beta_2} = (c_1 c_2)^{-\alpha_3} h^{b\alpha_3 + \beta_3} = 1\}.$$

For simplicity, we call the presentation given in Theorem 2.2 the classical presentation, and denote it by G; we call the presentation given by the Seifert structure Seifert presentation, and denote it by G'.

The induced maps on π_1 of the homeomorphisms of type I and II are more easily obtained for the Seifert presentation:

- For a type I homeomorphism f_1, we have $(f_1)_*(c_1) = c_1^{-1}$, $(f_1)_*(c_2) = c_2^{-1}$, $(f_1)_*(h) = h^{-1}$.
- For a type II homeomorphism f_2, we have $(f_2)_*(c_1) = c_2$, $(f_2)_*(c_2) = c_1$, $(f_2)_*(h) = h$.

However, we have given a presentation of $\mathrm{Out}(\pi_1(M))$ by the classical presentation, so we shall show how the presentations correspond to each other. Then we can present the induced map on fundamental group of type I and II homeomorphisms by the known presentation of $\mathrm{Out}(\pi_1(M))$.

Denote by $i\colon G \to G'$, $j\colon G' \to G$ the isomorphism between the two presentations of $\mathrm{Out}(\pi_1(M))$ such that $ji = \mathrm{id}_G$, $ji = \mathrm{id}_{G'}$. We will give i, j explicitly in the following.

Case 1 $M = \{b; 1/2, 1/2, \beta_3/\alpha_3\}$, $m = (b+1)\alpha_3 + \beta_3$, $(m, 2\alpha_3) = 1$.

(i) If $\alpha_3 > 2$, classical presentation: $G = \{a, x, y | a^m = 1, [x, a] = [y, a] = 1, x^2 = (xy)^2 = y^{\alpha_3}\}$; Seifert presentation: $G' = \{h, c_1, c_2 \mid [c_1, h] = [c_2, h] = 1, c_1^2 h = c_2^2 h = (c_1c_2)^{-\alpha_3} h^{b\alpha_3 + \beta_3} = 1\}$.

$$\begin{aligned} &i(a) = h^{1-m}, \quad i(x) = c_1^{-m^2}, \quad i(y) = c_1 c_2^{-1}, \\ &j(h) = ax^2, \quad j(c_1) = a^{(m-1)/2} x^{-1}, \quad j(c_2) = a^{(m-1)/2} y^{-1} x^{-1}. \end{aligned}$$

(ii) If $\alpha_3 = 2$, we take the same classical presentation but another Seifert presentation, since this presentation can reflect the symmetry of the orbifold better. $G' = \{h, c_1, c_2, c_3 \mid [c_1, h] = [c_2, h] = [c_3, h] = 1, c_1^2 h = c_2^2 h = c_3^2 h = c_1 c_2 c_3 h^{-b} = 1\}$.

$$\begin{aligned} &i(a) = h^{2b+4}, \quad i(x) = h^{2b^2+4b+1} c_1^{-1}, \quad i(y) = h^{2b^2-4} c_2, \\ &j(h) = ax^2, \quad j(c_1) = a^{b+1} x, \quad j(c_2) = a^{b+1} y, \quad j(c_3) = a^{b+1} (xy)^{2b-1}. \end{aligned}$$

Case 2 $M=\{b;1/2,1/2,\beta_3/\alpha_3\}$, $m=(b+1)\alpha_3+\beta_3$, $m=2^k m'$.

Classical presentation: $G=\{a,x,y \mid a^{m'}=1,[x,a]=[y,a]=1,x^{2^{k+2}}=y^{\alpha_3}=1,xy=y^{-1}x\}$; Seifert presentation: $G'=\{h,c_1,c_2 \mid [c_1,h]=[c_2,h]=1,c_1^2h=c_2^2h=(c_1c_2)^{-\alpha_3}h^{b\alpha_3+\beta_3}=1\}$. Suppose the integer w satisfies $wm'\equiv 1 \bmod 2^{k+2}$.

$$i(a)=h^{1-m'w},\quad i(x)=(h^{(m'-1)/2}c_1^{-1})^w,\quad i(y)=c_1^{-1-2m}c_2,$$
$$j(h)=ax^2,\qquad j(c_1)=a^{(m'-1)/2}x^{-1},\qquad j(c_2)=a^{(m'-1)/2}x^{-1-2m}y.$$

Case 3 $M=\{b;1/2,\beta_2/3,\beta_3/3\}$, $m=6b+3+2(\beta_2+\beta_3),(m,12)=1$. Then we can assume $\beta_2=\beta_3=1$, so $m=6b+7$.

Classical presentation: $G=\{a,x,y \mid a^m=1,[x,a]=[y,a]=1,x^2=(xy)^3=y^3,x^4=1\}$; Seifert presentation: $G'=\{h,c_1,c_2 \mid [c_1,h]=[c_2,h]=1,c_1^3h=c_2^3h=(c_1c_2)^2h^{-1-2b}=1\}$.

$$i(a)=h^{6b+8},\quad i(x)=c_1c_2h^{-4b-4},\quad i(y)=c_2^{-1}h^{2b+2},$$
$$j(h)=ax^2,\quad j(c_1)=a^{2b+2}xy,\quad j(c_2)=a^{2b+2}y^{-1}.$$

Case 4 $M=\{b;1/2,\beta_2/3,\beta_3/3\}$, $m=6b+3+2(\beta_2+\beta_3),(m,12)\neq 1$. We assume $\beta_2=1,\beta_3=2$. Then $m=6b+9=3^k m'$, so we can also assume $m'=3n+1$.

Classical presentation: $G=\{a,x,y,z \mid a^{m'}=1,[x,a]=[y,a]=[z,a]=1,x^2=(xy)^2=y^2,zxz^{-1}=y,zyz^{-1}=xy,z^{3^{k+1}}=1\}$; Seifert presentation: $G'=\{h,c_1,c_2 \mid [c_1,h]=[c_2,h]=1,c_1^3h=c_2^3h^2=(c_1c_2)^2h^{-1-2b}=1\}$.

$$i(a)=h^{(1-m')^k},\qquad i(x)=c_1c_2h^{-4b-5},$$
$$i(y)=c_2c_1h^{2b+4},\quad i(z)=c_2^{-1}c_1^{-2}h^{-(1-m')^{k+1}/3+4b+5},$$
$$j(h)=ax^2z^3,\ j(c_1)=a^{-(m'-1)^2/3}z^{-1}x^{-1},\ j(c_2)=a^{4b+5+(m'-1)^2/3}xy^3z^{12b+16}.$$

Actually, in Case 5 and Case 6, we do not need the isomorphism to determine $\mathcal{MCG}(M)$. However, for completion, we list the isomorphisms here.

Case 5 $M=\{b;1/2,\beta_2/3,\beta_3/4\}$, $m=12b+6+4\beta_2+3\beta_3$. We can assume $\beta_2=1$.

Classical presentation: $G=\{a,x,y \mid a^m=1,[x,a]=[y,a]=1,x^2=(xy)^3=y^4,x^4=1\}$; Seifert presentation: $G'=\{h,c_1,c_2 \mid [c_1,h]=[c_2,h]=1,c_1^3h=c_2^4h^{\beta_3}=(c_1c_2)^2h^{-1-2b}=1\}$.

(i) $\beta_3=1$, so $m=12b+13$:

$$i(a)=h^{12b+14},\quad i(x)=c_1c_2h^{12b^2-6b-20},\quad i(y)=c_2^{-1}h^{12b^2+4b-10},$$
$$j(h)=ax^2,\qquad j(c_1)=a^{4b+4}xy,\qquad j(c_2)=a^{3b+3}y^{-1}.$$

(ii) $\beta_3 = 3$, so $m = 12b + 19$:

$$i(a) = h^{12b+20}, \quad i(x) = c_1 c_2 h^{12b^2+12b-20}, \quad i(y) = c_2^{-1} h^{12b^2+22b+4},$$
$$j(h) = ax^2, \qquad j(c_1) = a^{4b+6}xy, \qquad j(c_2) = a^{3b+4}y^{-1}.$$

Case 6 $M = \{b; 1/2, \beta_2/3, \beta_3/5\}$, $m = 30b + 15 + 10\beta_2 + 6\beta_3$. We can assume $\beta_2 = 1$.

Classical presentation: $G = \{a, x, y \mid a^m = 1, [x, a] = [y, a] = 1, x^2 = (xy)^3 = y^5, x^4 = 1\}$; Seifert presentation: $G' = \{h, c_1, c_2 \mid [c_1, h] = [c_2, h] = 1, c_1^3 h = c_2^5 h^{\beta_3} = (c_1c_2)^2 h^{-1-2b} = 1\}$.

(i) $\beta_3 = 1$ or 3:

$$i(a) = h^{30b+26+6\beta_3}, \quad i(x) = c_1 c_2 h^{-16b-13-3\beta_3}, \quad i(y) = c_2^{-1} h^{6b+5+\beta_3},$$
$$j(h) = ax^2, \qquad j(c_1) = a^{10b+8+2\beta_3}xy, \qquad j(c_2) = a^{6b+5+\beta_3}y^{-1}.$$

(ii) $\beta_3 = 2$ or 4:

$$i(a) = h^{30b+26+6\beta_3}, \quad i(x) = c_1^{-1} c_2^{-1} h^{16b+13+3\beta_3}, \quad i(y) = c_2 h^{-6b-5-\beta_3},$$
$$j(h) = ax^2, \qquad j(c_1) = a^{10b+8+2\beta_3}y^{-1}x^{-1}, \quad j(c_2) = a^{6b+5+\beta_3}x^2y.$$

5.3 Determination of mapping class group

Given the equivalence connecting the classical and Seifert presentations of $\pi_1(M)$, we can compute the $\mathcal{MCG}(M)$ now (for S^3–manifolds $M \neq L(p,q)$, $\mathcal{MCG}(M) = \mathcal{MCG}^+(M)$).

Case 1 $M = \{b; 1/2, 1/2, \beta_3/\alpha_3\}$, $m = (b+1)\alpha_3 + \beta_3$, $(m, 2\alpha_3) = 1$, $\pi_1(M) \cong \mathbb{Z}_m \times D^*_{4\alpha_3}$.

(i) We first assume $\alpha_3 > 2$. Since only one pair of singular fibers of M satisfies $\alpha_1 = \alpha_2$, and $\alpha_1 | (\beta_2 - \beta_1)$, M only admit one homeomorphism of type II.

Suppose f is the homeomorphism of type I: $f_*(c_1) = c_1^{-1}$, $f(c_2) = c_2^{-1}$, $f_*(h) = h^{-1}$. By the equivalence given in the last part, in the classical presentation, we have $f_*(a) = a^{-1}$, $f_*(x) = x^3$, $f_*(y) = y$.

Suppose g is the unique homeomorphism of type II: $g_*(c_1) = c_2$, $g_*(c_2) = c_1$, $g_*(h) = h$. In the classical presentation, we have $g_*(a) = a$, $g_*(x) = (xy)^{-1}$, $g_*(y) = y^{-1}$.

When α_3 is odd, conjugating by $xy^{-(\alpha_3-1)/2}$, we get g_* is conjugated to $\phi(a) = a$, $\phi(x) = x^3$, $\phi(y) = y$. Comparing with the presentation of $\mathrm{Out}(D^*_{4\alpha_3})$ in Section 4, we have: when $m=1$, $f_*\sim g_*\not\sim \mathrm{id}$ (here $\sim$ means conjugate), $\mathcal{MCG}(M)\cong\mathbb{Z}_2$; when $m>1$, $\mathrm{id}\not\sim f_*\not\sim g_*\not\sim \mathrm{id}$, $\mathcal{MCG}(M)\cong\mathbb{Z}_2\times\mathbb{Z}_2$.

When α_3 is even, conjugating by $y^{\alpha_3/2}$, f_* is conjugated to $\phi(a)=a^{-1}$, $\phi(x)=x, \phi(y)=y$; conjugate by $xy^{\alpha_3/2+1}$, g_* is conjugated to $\phi(a)=a$, $\phi(x)=x^3y$, $\phi(y)=y$. Comparing with Section 4, we have: when $m=1$, $\mathrm{id}\sim f_*\not\sim g_*$, $\mathcal{MCG}(M)\cong\mathbb{Z}_2$; when $m>1$, $\mathrm{id}\not\sim f_*\not\sim g_*\not\sim \mathrm{id}$, $\mathcal{MCG}(M)\cong\mathbb{Z}_2\times\mathbb{Z}_2$.

(ii) When $\alpha_3=2$, the three singular fibers are symmetric with each other, so there are more homeomorphisms of type II.

We take the section F' of $M''=F'\times S^1$ as in Figure 3; here F' is a four-punctured sphere, while one puncture corresponds to a regular fiber, c_i corresponds to singular fiber l_i, $i=1,2,3$ respectively.

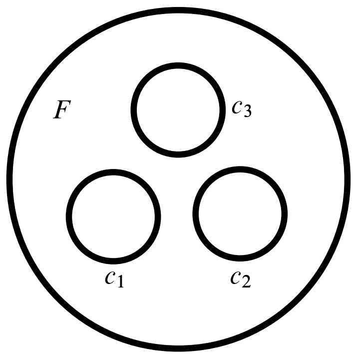

Figure 3

Suppose f is the homeomorphism of type I, $f_*(c_1)=c_1^{-1}, f(c_2)=c_2^{-1}, f(c_2)=c_2^{-1}, f_*(h)=h^{-1}$. By the isomorphism given in the last part, for the Seifert presentation, $f_*(a)=a^{-1}, f_*(x)=x^3, f_*(y)=y^3$; conjugating by xy, we have f_* is conjugated to $\phi(a)=a^{-1}, \phi(x)=x, \phi(y)=y$.

Suppose g, g' are two homeomorphisms of type II where g exchanges l_1, l_2, fixes l_3, while g' exchanges l_2, l_3, fixes l_1, and the type II homeomorphism that exchanges l_1, l_3 and fixes l_2 is equal to $gg'g$. The group generated by the g, g' actions on l_1, l_2, l_3 acts as the permutation group S_3, so the corresponding subgroup of $\mathcal{MCG}(G)$ is a quotient group of S_3.

Under the Seifert presentation, g, g' are: $g_*(c_1)=c_2, g_*(c_2)=c_1$, $g_*(c_3)=c_1^{-1}c_3c_1$, $g_*(h)=h$; $g'_*(c_1)=c_2^{-1}c_1c_2, g'_*(c_2)=c_3, g'_*(c_3)=c_2, g'_*(h)=h$. On the classical presentation, we have $g_*(a)=a, g_*(x)=y, g_*(y)=x$; $g'_*(a)=a, g'_*(x)=x^3, g'_*(y)=(xy)^{2b-1}$. Conjugating by y or xy, we have g'_* conjugates to $\psi(a)=a, \psi(x)=x, \psi(y)=xy$. Comparing with Section 4, we have: the action of g_* and g'_* on D_8^* generate the whole $\mathrm{Out}(D_8^*)\cong S_3$.

Considering f_*, we have: when $m=1$, $\mathcal{MCG}(M)\cong S_3$; when $m>1$, $\mathcal{MCG}(M)\cong \mathbb{Z}_2\times S_3$.

Case 2 $M=\{b;1/2,1/2,\beta_3/\alpha_3\}$, $m=(b+1)\alpha_3+\beta_3$, $m=2^k m'$, $\pi_1(M)\cong \mathbb{Z}'_m\times D'_{2^{k+2}\alpha_3}$.

Suppose f is the homeomorphism of type I. In the classical presentation, we have $f_*(a)=a^{-1}, f_*(x)=x^{-1}, f_*(y)=y$.

Suppose g is the unique homeomorphism of type II. In the classical presentation, we have $g_*(a)=a, g_*(x)=x^{2^{k+1}+1}y, g_*(y)=y^{-1}$. Conjugating by $xy^{(1+\alpha_3)/2}$, g_* conjugates to $\phi(a)=a, \phi(x)=x^{2^{k+1}+1}, \phi(y)=y$.

Comparing with the presentation of $\mathrm{Out}(D'_{2^{k+2}\alpha_3})$ in Section 4, we have: $\mathrm{id}\not\sim f_*\not\sim g_*\not\sim \mathrm{id}$, $\mathcal{MCG}(M)\cong\mathbb{Z}_2\times\mathbb{Z}_2$.

Case 3 $M=\{b;1/2,\beta_2/3,\beta_3/3\}$, $m=6b+3+2(\beta_2+\beta_3)$, $(m,12)=1$. We can assume $\beta_2=\beta_3=1$, $\pi_1(M)\cong\mathbb{Z}_m\times T_{24}^*$.

Suppose f is the homeomorphism of type I. In the classical presentation, we have $f_*(a)=a^{-1}, f_*(x)=y^{-1}x^{-1}y, f_*(y)=y^{-1}$. Conjugating by y, f_* conjugates to $\phi(a)=a^{-1}, \phi(x)=x^{-1}, \phi(y)=y^{-1}$.

Suppose g is the unique homeomorphism of type II. In the classical presentation, we have $g_*(a)=a, g_*(x)=y^{-1}xy, g_*(y)=y^{-1}x^{-1}$. Conjugating by $y^{-1}xy^2$, g_* conjugates to $\psi(a)=a, \psi(x)=x^{-1}, \psi(y)=y^{-1}$.

Comparing with the presentation of $\mathrm{Out}(T_{24}^*)$ in Section 4, we have: when $m=1$, $f_*\sim g_*\not\sim\mathrm{id}$, $\mathcal{MCG}(M)\cong\mathbb{Z}_2$; when $m>1$, $\mathrm{id}\not\sim f_*\not\sim g_*\not\sim\mathrm{id}$, $\mathcal{MCG}(M)\cong\mathbb{Z}_2\times\mathbb{Z}_2$.

Case 4 $M=\{b;1/2,\beta_2/3,\beta_3/3\}$, $m=6b+3+2(\beta_2+\beta_3), (m,12)\neq 1$. We assume $\beta_2=1, \beta_3=2$, so $m=6b+9=3^k m'$, and we can still assume $m'=3n+1$, $\pi_1(M)\cong\mathbb{Z}_{m'}\times T'_{8\cdot 3^{k+1}}$.

Here M does not admit a homeomorphism of type II. Suppose f is the homeomorphism of type I. In the classical presentation, we have $f_*(a)=a^{-1}$, $f_*(x)=y$, $f_*(y)=x, f_*(z)=xz^{-1}$. Conjugating by z^{-1}, f_* is conjugate to $\phi(a)=a^{-1}, \phi(x)=x, \phi(y)=xy, \phi(z)=z^{-1}x$.

Comparing with the presentation of $\mathrm{Out}(T'_{8\cdot 3^{k+1}})$ in Section 4, we have $f_*\not\sim\mathrm{id}$, $\mathcal{MCG}(M)\cong\mathbb{Z}_2$.

Case 5 $M=\{b;1/2,\beta_2/3,\beta_3/4\}$, $m=12b+6+4\beta_2+3\beta_3$, $\pi_1(M)\cong\mathbb{Z}_m\times O_{48}^*$.

Case 6 $M=\{b;1/2,\beta_2/3,\beta_3/5\}$, $m=30b+15+10\beta_2+6\beta_3$, $\pi_1(M)\cong\mathbb{Z}_m\times I^*_{120}$.

In these two cases, M does not admit a homeomorphism of type II, so $\mathcal{MCG}(M)\cong\mathbb{Z}_2$ or is trivial. Suppose f is the homeomorphism of type I.

When $m>1$, then the fiber h corresponds with an element of type $(\bar{1},u)\in\pi_1(M)\cong\mathbb{Z}_m\times O^*_{48}$ or $\pi_1(M)\cong\mathbb{Z}_m\times I^*_{120}$. So we have $f_*(\bar{1},u)=(-\bar{1},g(u))$, and $f_*\not\simeq \mathrm{id}$, so $\mathcal{MCG}(M)\cong\mathbb{Z}_2$.

When $m=1$, by Section 4, we have $\mathrm{Out}(O^*_{48})\cong\mathrm{Out}(I^*_{120})\cong\mathbb{Z}_2$, and $U^2_{48}\cong U^2_{120}\cong\mathbb{Z}_2$. But $|K(M)|=|\mathrm{Out}(\pi_1(M))|/|U^2(|\pi_1(M)|)||$, so we have $K(M)=\{\mathrm{id}\}$. Since f is a degree one self-map on M, f is homotopic to identity, thus $\mathcal{MCG}(M)\cong\{e\}$.

Bringing together the above results, we get the following:

Theorem 5.6 *The mapping class groups of S^3–manifolds are shown as follows:*

- (i) $M=S^3$, $\mathcal{MCG}(M)\cong\{e\}$.
- (ii) $M=L(p,q)$:
  - (a) $q=\pm1$, or $p\nmid q^2-1$, $p\nmid q^2+1$, $\mathcal{MCG}(M)\cong\mathbb{Z}_2$.
  - (b) $p\mid q^2-1$, $q\neq\pm1$, $\mathcal{MCG}(M)\cong\mathbb{Z}_2\times\mathbb{Z}_2$.
  - (c) $p\mid q^2+1$, $q\neq\pm1$, $\mathcal{MCG}(M)\cong\mathbb{Z}_4$.
- (iii) $\pi_1(M)\cong\mathbb{Z}_m\times D^*_{4\alpha_3}$:
  - (a) $\alpha_3=2$: when $m=1$, $\mathcal{MCG}(M)\cong S_3$; when $m>1$, $\mathcal{MCG}(M)\cong\mathbb{Z}_2\times S_3$.
  - (b) $\alpha_3>2$: when $m=1$, $\mathcal{MCG}(M)\cong\mathbb{Z}_2$; when $m>1$, $\mathcal{MCG}(M)\cong\mathbb{Z}_2\times\mathbb{Z}_2$.
- (iv) $\pi_1(M)\cong\mathbb{Z}'_m\times D'_{2^{k+2}\alpha_3}$, $\alpha_3>1$ odd, $\mathcal{MCG}(M)\cong\mathbb{Z}_2\times\mathbb{Z}_2$.
- (v) $\pi_1(M)\cong\mathbb{Z}'_m\times T^*_{24}$: when $m=1$, $\mathcal{MCG}(M)\cong\mathbb{Z}_2$; when $m>1$, $\mathcal{MCG}(M)\cong\mathbb{Z}_2\times\mathbb{Z}_2$.
- (vi) $\pi_1(M)\cong\mathbb{Z}'_m\times T'_{8\cdot3^{k+1}}$, $k>0$, $\mathcal{MCG}(M)\cong\mathbb{Z}_2$.
- (vii) $\pi_1(M)\cong\mathbb{Z}'_m\times O^*_{48}$: when $m=1$, $\mathcal{MCG}(M)\cong\{e\}$; when $m>1$, $\mathcal{MCG}(M)\cong\mathbb{Z}_2$.
- (viii) $\pi_1(M)\cong\mathbb{Z}'_m\times I^*_{120}$: when $m=1$, $\mathcal{MCG}(M)\cong\{e\}$; when $m>1$, $\mathcal{MCG}(M)\cong\mathbb{Z}_2$.

6 Conclusions

The computational results of $\mathrm{Out}(\pi_1(M))$, $\mathcal{MCG}(M)$, $\mathrm{RC}(M)$ ($M\neq L(p,q)$) are shown in Table 1. By elementary computation, we can get the part of Theorem 1.1 for $M\neq L(p,q)$ easily.

$\pi_1(M)$	$\lvert\mathrm{Out}(\pi_1(M))\rvert$	$\lvert\mathcal{MCG}(M)\rvert$	$\mathrm{RC}(M)$
$\mathbb{Z}_m \times D_8^*$	$6\lvert U_m\rvert$	6 $m=1$ 12 $m>1$	1 $m=1$ $\Psi(m)/2$ $m>1$
$\mathbb{Z}_m \times D_{4\alpha_3}^*$, $\alpha_3>2$	$\lvert U_m\rvert\lvert U_{4\alpha_3}\rvert/2$	2 $m=1$ 4 $m>1$	$\Psi(4\alpha_3)/4$ $m=1$ $\Psi(m)\Psi(4\alpha_3)/8$ $m>1$
$\mathbb{Z}_m \times D'_{2^{k+2}\alpha_3}$, $\alpha_3>1$ odd	$2^k\lvert U_m\rvert\lvert U_{\alpha_3}\rvert$	4	$\Psi(m)\Psi(\alpha_3)/2$
$\mathbb{Z}_m \times T_{24}^*$	$2\lvert U_m\rvert$	2 $m=1$ 4 $m>1$	1 $m=1$ $\Psi(m)/2$ $m>1$
$\mathbb{Z}_m \times T'_{8\cdot 3^{k+1}}$, $k>0$	$2\cdot 3^k\lvert U_m\rvert$	2	$\Psi(m)$
$\mathbb{Z}_m \times O_{48}^*$	$2\lvert U_m\rvert$	1 $m=1$ 2 $m>1$	1 $m=1$ $\Psi(m)/2$ $m>1$
$\mathbb{Z}_m \times I_{120}^*$	$2\lvert U_m\rvert$	1 $m=1$ 2 $m>1$	1 $m=1$ $\Psi(m)/2$ $m>1$

Table 1

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