The Lusternik–Schnirelmann category 
and the fundamental group

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We prove that
\[ \text{cat}_{LS} X \leq \text{cd}(\pi_1(X)) + \left\lceil \frac{\dim X - 1}{2} \right\rceil \]
for every CW–complex \( X \) where \( \text{cd}(\pi_1(X)) \) denotes the cohomological dimension of the fundamental group of \( X \).

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1 Introduction

The Lusternik–Schnirelmann category \( \text{cat}_{LS} X \) of a topological space \( X \) is the minimal number \( n \) such that there is an open cover \( \{U_0, \ldots, U_n\} \) of \( X \) by \( n+1 \) contractible in \( X \) sets (we note that sets \( U_i \) are not necessarily contractible). The Lusternik–Schnirelmann category has proven useful in different areas of mathematics. In particular, the classical theorem of Lusternik and Schnirelmann (see Cornea et al [3]) proven in the 30s states that \( \text{cat}_{LS} M \) gives a lower bound for the number of critical points on \( M \) of any smooth not necessarily Morse function. For nice spaces, such as CW–complexes, it is an easy observation that \( \text{cat}_{LS} X \leq \dim X \). In the 40s Grossman [8] (and independently in the 50s G W Whitehead [16; 3]) proved that for simply connected CW–complexes \( \text{cat}_{LS} X \leq \dim X/2 \). In the presence of a fundamental group as small as \( \mathbb{Z}_2 \) the Lusternik–Schnirelmann category can be equal to the dimension. An example is \( \mathbb{R} P^n \).

Nevertheless, Yu Rudyak conjectured that in the case of free fundamental group there should be a Grossman–Whitehead-type inequality at least for closed manifolds. There were partial results towards Rudyak’s conjecture (see Dranishnikov, Katz and Rudyak [6] and Strom [13]) until it was settled by the author [5]. Also it was shown in [5] that a Grossman–Whitehead-type estimate holds for complexes with fundamental group of cohomological dimension \( \leq 2 \). We recall that free groups (and only them by Stallings [12] and Swan [15]) have cohomological dimension one. In this paper we prove an inequality for complexes with fundamental groups having finite cohomological dimension. Complexes of type \( \mathbb{C}P^n \times B\pi \) show that our inequality is sharp when \( \pi \) is free.

Published: 17 April 2010

DOI: 10.2140/agt.2010.10.917
We conclude the introductory part with definitions and statements from [5] which are used in this paper. Let \( \mathcal{U} = \{ U_a \}_{a \in A} \) be a family of sets in a topological space \( X \). Formally, it is a function \( U: A \rightarrow 2^X \setminus \{ \emptyset \} \) from the index set to the set of nonempty subsets of \( X \). The sets \( U_a \) in the family \( \mathcal{U} \) will be called elements of \( \mathcal{U} \). The multiplicity of \( \mathcal{U} \) (or the order) at a point \( x \in X \), denoted \( \text{Ord}_x \mathcal{U} \), is the number of elements of \( \mathcal{U} \) that contain \( x \). The multiplicity of \( \mathcal{U} \) is defined as \( \text{Ord} \mathcal{U} = \sup_{x \in X} \text{Ord}_x \mathcal{U} \). A family \( \mathcal{U} \) is a cover of \( X \) if \( \text{Ord}_x \mathcal{U} \neq 0 \) for all \( x \). A cover \( \mathcal{U} \) is a refinement of another cover \( \mathcal{C} \) (\( \mathcal{U} \) refines \( \mathcal{C} \)) if for every \( U \in \mathcal{U} \) there exists \( C \in \mathcal{C} \) such that \( U \subset C \). We recall that the covering dimension of a topological space \( X \) does not exceed \( n \), \( \dim X \leq n \), if for every open cover \( \mathcal{C} \) of \( X \) there is an open refinement \( \mathcal{U} \) with \( \text{Ord} \mathcal{U} \leq n + 1 \).

**Definition 1.1** A family \( \mathcal{U} \) of subsets of \( X \) is called a \( k \)-cover, \( k \in \mathbb{N} \), if every subfamily of \( k \) elements forms a cover of \( X \).

The following is obvious (see Dranishnikov [5]).

**Proposition 1.2** A family \( \mathcal{U} \) that consists of \( m \) subsets of \( X \) is an \( (n + 1) \)-cover of \( X \) if and only if \( \text{Ord}_x \mathcal{U} \geq m - n \) for all \( x \in X \).

The following theorem can be found in Ostrand [10].

**Theorem 1.3** (Kolmogorov–Ostrand) A metric space \( X \) is of dimension \( \leq n \) if and only if for each open cover \( \mathcal{C} \) of \( X \) and each integer \( m \geq n \), there exist \( m \) disjoint families of open sets \( \mathcal{V}_0, \ldots, \mathcal{V}_m \) such that their unions \( \bigcup \mathcal{V}_i \) is an \( (n + 1) \)-cover of \( X \) and it refines \( \mathcal{C} \).

Let \( f: X \rightarrow Y \) be a map and let \( X' \subset X \). A set \( U \subset X \) is fiberwise contractible to \( X' \) if there is a homotopy \( H: U \times [0, 1] \rightarrow X \) such that \( H(x, 0) = x \), \( H(U \times \{ 1 \}) \subset X' \), and \( f(H(x, t)) = f(x) \) for all \( x \in U \).

We refer to [5] for the proof of the following:

**Theorem 1.4** Let \( \mathcal{U} = \{ U_0, \ldots, U_k \} \) be an open cover of a normal topological space \( X \). Then for any \( m = 0, 1, 2, \ldots, \infty \) there is an open \((k + 1)\)-cover \( \mathcal{U}_m = \{ U_0, \ldots, U_{k+m} \} \) of \( X \) extending \( \mathcal{U} \) such that for \( n > k \), \( U_n = \bigcup_{i=0}^{k} V_i \) is a disjoint union with \( V_i \subset U_i \).

**Corollary 1.5** Let \( f: X \rightarrow Y \) be a continuous map of a normal topological space and let \( \mathcal{U} = \{ U_0, \ldots, U_k \} \) be an open cover of \( X \) by sets fiberwise contractible to \( X' \subset X \). Then for any \( m = 0, 1, 2, \ldots, \infty \) there is an open \((k + 1)\)-cover \( \mathcal{U}_m = \{ U_0, \ldots, U_{k+m} \} \) of \( X \) by sets fiberwise contractible to \( X' \).
2 Generalization of Ganea’s fibrations

Let $A \subset Z$ be a closed subset of a path-connected space and let $F$ denote the homotopy fiber of the inclusion. By $A_Z$ we denote the space of paths in $Z$ issued from $A$, i.e., the space of continuous maps $\phi : [0,1] \to Z$ with $\phi(0) \in A$ and we define a map $p_A : A_Z \to Z$ by the formula $p(\phi) = \phi(1)$. Note that $A_Z$ deforms to $A$ and $p_A$ is a Hurewicz fibration. Then by the definition $F$ is the fiber of $p_A$.

**Proposition 2.1** There is a Hurewicz fibration $\pi : F \to A$ with fiber $\Omega Z$, the loop space on $Z$.

**Proof** The map $q' : A_Z \to A \times Z$ that sends a path to the end points is a Hurewicz fibration as a pullback of the Hurewicz fibration $q : Z^{[0,1]} \to Z \times Z$ [11]. The fiber of $q$ is the loop space $\Omega Z$. Since $p_A = pr_2 \circ q'$, the fiber $F = p_A^{-1}(x) = (q')^{-1} pr_2^{-1}(x) = q^{-1}(A)$ is the total space of a Hurewicz fibration $q$ over $A$ with the fiber $\Omega Z$. □

We define the $k$–th *generalized Ganea’s fibration* $p_k : E_k(Z, A) \to Z$ over a path connected space $Z$ with a fixed closed subset $A$ as the fiberwise join product of $k + 1$ copies of the fibrations $p_A : A_Z \to Z$. Since $p_A$ is a Hurewicz fibration and the fiberwise join of Hurewicz fibrations is a Hurewicz fibration, so are all $p_k$ by Švarc [14]. Note that the fiber of $p_k$ is the join product $s^{k+1}F$ of $k + 1$ copies of $F$ (see Cornea et al [3] for more details). Also we note that for $A = \{z_0\}$ the fibration $p_k$ is the standard Ganea fibration. The following is a generalization of the Ganea–Švarc theorem.

**Theorem 2.2** Let $A \subset X$ be a subcomplex contractible in $X$. Then $\operatorname{cat}_{L,S}(X) \leq k$ if and only if the generalized Ganea fibration

$$p_k : E_k(Z, A) \to Z$$

admits a section.

**Proof** When $A$ is a point this statements turns into the classical Ganea–Švarc theorem [3; 14]. Since for $z_0 \in A$, the above fibration $p_k : E_k(Z, z_0) \to Z$ is contained in $p_k : E_k(Z, A) \to Z$, the classical Ganea–Švarc theorem implies the only if direction.

The barycentric coordinates of a section to $p_k$ define an open cover $U_0, \ldots, U_k$ of $U_l$ with each $U_l$ contractible to $A$. Since $A$ is contractible in $Z$, all sets $U_l$ are contractible in $Z$. □
We call a map \( f: X \to Y \) a \textit{stratified locally trivial bundle} (with two strata) with fiber \((Z, A)\) if there \( X' \subset X \), such that \((f^{-1}(y), g^{-1}(y)) \cong (Z, A)\) for all \( y \in Y \), where \( g = f|_{X'} \), and there is an open cover \( \mathcal{U} = \{U\} \) of \( Y \) such that \((f^{-1}(U), g^{-1}(U)) \) is homeomorphic as a pair to \((Z \times U, A \times U)\) by means of a fiber preserving homeomorphism. Such a bundle is called a \textit{trivial stratified bundle} if one cannot take \( \mathcal{U} \) consisting of one element \( U = Y \).

Now let \( f: X \to Y \) be a stratified locally trivial bundle with a subbundle \( g: X' \to Y \) and a fiber \((Z, A)\). We define a space

\[
E_0 = \{ \phi \in C(I, X) \mid f \phi(I) = f \phi(0), \; \phi(0) \in g^{-1}(f \phi(0)) \}
\]

to be the space of all paths \( \phi \) in \( f^{-1}(y) \) for all \( y \in Y \) with the initial point in \( g^{-1}(y) \). The topology in \( E_0 \) is inherited from \( C(I, X) \). We define a map \( \xi_0: E_0 \to X \) by the formula \( \xi_0(\phi) = \phi(1) \). Then \( \xi_k: E_k \to X \) is defined as the fiberwise join of \( k + 1 \) copies of \( \xi_0 \). Formally, we define inductively \( E_k \) as a subspace of the join \( E_0 \ast E_{k-1} \):

\[
E_k = \bigcup \{ \phi * \psi \in E_0 * E_{k-1} \mid \xi_0(\phi) = \xi_{k-1}(\psi) \},
\]

which is the union of all intervals \([\phi, \psi] = \phi * \psi\) with the endpoints \( \phi \in E_0 \) and \( \psi \in E_{k-1} \) such that \( \xi_0(\phi) = \xi_{k-1}(\psi) \). There is a natural projection \( \xi_k: E_k \to X \) that takes all points of each interval \([\phi, \psi]\) to \( \phi(0) \).

Note that when \( f: X = Z \times Y \to Y \) is a trivial stratified bundle with the subbundle \( g: A \times Y \to Y, \; A \subset Z \), then \( E_k = E_k(Z, A) \times Y \) and \( \xi_k = p_k \times 1_Y \) where \( p_k: (E_k, A) \to Z \) is the generalized Ganea fibration.

**Lemma 2.3** Let \( f: X \to Y \) be a stratified locally trivial bundle between paracompact spaces with a fiber \((Z, A)\) in which \( A \) is contractible in \( Z \). Then:

(i) For each \( k \) the map \( \xi_k: E_k \to X \) is a Hurewicz fibration.

(ii) The fiber of \( \xi_k \) is the join of \( k + 1 \) copies of the fiber \( F \) of \( p_A: A \times Z \to Z \).

(iii) If the projection \( \xi_k \) has a section, then \( X \) has an open cover \( \mathcal{U} = \{U_0, \ldots, U_k\} \) by sets each of which admits a fiberwise deformation into \( X' \) where \( g: X' \to Y \) is the subbundle.

**Proof** (i) First, we note that this statement holds true for trivial stratified bundles. By the assumption there is a cover \( \mathcal{U} \) of \( Y \) such that \( f|_{f^{-1}(U)}: f^{-1}(U) \to U \) is a trivial stratified bundle and hence \( \xi_k \) is a Hurewicz fibration over \( f^{-1}(U) \) for all \( U \in \mathcal{U} \). Then by Hurewicz [9] (see also Dold [4]) we conclude that \( \xi_k \) is a Hurewicz fibration over \( X \).
(ii) We note that $k$ over $f^{-1}(y)$ coincides with the generalized Ganea fibration $p_k$ for $(Z, A)$. Therefore, the fiber of $\xi_k$ coincides with the fiber of $p_k$. Then we apply Proposition 2.1

(iii) Suppose $\xi_k$ has a section $\sigma: X \to E_k$. For each $x \in X$ the element $\sigma(x)$ of $\ast^{k+1} \Omega F$ can be presented as the $(k + 1)$–tuple

$$\sigma(x) = ((\phi_0, t_0), \ldots, (\phi_k, t_k))$$

where $\sum t_i = 1$ and $t_i \geq 0$.

Here we use the notation $t_i = t_i(x)$ and $\phi_i = \phi_i^x$. Clearly, $t_i(x)$ and $\phi_i^x$ are continuous functions of $x$.

A section $\sigma: X \to E_k$ defines a cover $\mathcal{U} = \{U_0, \ldots, U_k\}$ of $X$ as follows:

$$U_i = \{x \in X | t_i(x) > 0\}.$$  

By the construction of $U_i$ for $i \leq n$ for every $x \in U_i$ there is a canonical path connecting $x$ with $g^{-1} f(x)$. These paths define a fiberwise deformation $H: U_i \times [0, 1] \to X'$ of $U_i$ into $g^{-1} f(U_i) \subset X'$ by the formula $H(x, t) = \phi_i^x(1-t)$.  

\[\Box\]

3 The main result

We recall that the homotopical dimension of a space $X$, $\text{hd}(X)$, is the minimal dimension of a CW–complex homotopy equivalent to $X$ [3].

**Proposition 3.1** Let $p: E \to X$ be a fibration with $(n - 1)$–connected fiber where $n = \text{hd}(X)$. Then $p$ admits a section.

**Proof** Let $h: Y \to X$ be a homotopy equivalence with the homotopy inverse $g: X \to Y$ where $Y$ is a CW–complex of dimension $n$. Since the fiber of $p$ is $(n - 1)$–connected, the map $h$ admits a lift $h': Y \to E$. Let $H$ be a homotopy connecting $h \circ g$ with $1_Y$. By the homotopy lifting property there is a lift $H': X \times I \to E$ of $H$ with $H|_{X \times \{0\}} = h' \circ g$. Then the restriction $H|_{X \times \{1\}}$ is a section.  

\[\Box\]

We recall that $\lceil x \rceil$ denotes the smallest integer $n$ such that $x \leq n$.

**Lemma 3.2** Suppose that a stratified locally trivial bundle $f: X \to Y$ with a fiber $(Z, A)$ is such that $Z$ is $r$–connected, $A$ is $(r - 1)$–connected, $A$ is contractible in $Z$, and $Y$ is locally contractible. Then

$$\text{cat}_{LS} X \leq \dim Y + \left\lceil \frac{\text{hd}(X) - r}{r + 1} \right\rceil.$$
We show that for all $i$ $X$ iswise contractible to be an extension of $U$. Proof

Let $\dim Y = m$ and $\text{hd}(X) = n$. By Lemma 2.3 the fiber $K$ of the fibration $\xi_k: E_k \to X$ is the join product $*^{k+1}F$ of $k + 1$ copies of the fiber $F$ of the map $p_A: AZ \to Z$. By Proposition 2.1, $F$ admits a fibration $\phi: F \to A$ with fibers homotopy equivalent to the loop space $\Omega Z$. Since the base $A$ and the fibers are $(r - 1)$–connected, $F$ is $(r - 1)$–connected. Thus, $K$ is $(k + (k + 1)r - 1)$–connected. By Proposition 3.1 there is a section $\sigma: X \to E_k$ to the fibration $\xi_k: E_k \to X$, whenever $k(r + 1) + r \geq n$. Let $k$ be the smallest integer satisfying this condition. Thus, $k = \lceil (n - r)/(r + 1) \rceil$.

By Lemma 2.3 a section $\sigma: X \to E_k$ defines a cover $\mathcal{U} = \{U_0, \ldots, U_k\}$ by sets fiberwise contractible to $X'$ where $X' \subset X$ is the first stratum. Let $\mathcal{U}_m = \{U_0, \ldots, U_{k+m}\}$ be an extension of $\mathcal{U}$ to a $(k + 1)$–cover of $X$ from Corollary 1.5.

Let $O$ be an open cover of $Y$ such that $f$ is trivial stratified bundle over each $O \in O$. Let $C$ be an open cover of $Y$ such that for every $C \in C$ there is $O \in O$ such that $C \subset O$ and $C$ is contractible in $O$. Such a cover exists since $Y$ is locally contractible. By Theorem 1.3 there are $m + k + 1$ families of open sets $V_0, \ldots, V_{m+k}$ such that their union forms an $(m + 1)$–cover of $Y$ refining $C$. We define $V_i = \bigcup_\alpha V_i^\alpha$ to be the unions of all sets from $V_i = \{V_i^\alpha\}$. Then $V = \{V_0, \ldots, V_{m+k}\}$ is an open $(m + 1)$–cover of $Y$ such that for every $i$, $V_i = \bigcup_\alpha V_i^\alpha$ is a disjoint union of open sets $V_i^\alpha$ contractible to a point in $O_i^\alpha \subset O$.

We show that for all $i \in \{0, 1, \ldots, m + k\}$, the sets $W_i = f^{-1}(V_i) \cap U_i$ are contractible in $X$. Since

$$W_i = \bigcup_\alpha f^{-1}(V_i^\alpha) \cap U_i$$

is a disjoint union, it suffices to show that the sets $f^{-1}(V_i^\alpha) \cap U_i$ are contractible in $X$ for all $\alpha$. By Corollary 1.5 the set $U_i$ is fiberwise contractible into $X'$ for $i \leq m + k$. Hence we can contract $f^{-1}(V_i^\alpha) \cap U_i$ to $f^{-1}(V_i^\alpha) \cap X' \cong V_i^\alpha \times A$ in $X$. Then we apply a contraction to a point of $V_i^\alpha$ in $O_i^\alpha$ and $A$ in $F$ to obtain a contraction to a point of $f^{-1}(V_i^\alpha) \cap X' \cong V_i^\alpha \times A$ in $f^{-1}(O_i^\alpha) \cong O_i^\alpha \times F$.

Next we show that $\{W_i\}_{i=0}^{m+k}$ is a cover of $X$. Since $V$ is an $(m + 1)$–cover, by Proposition 1.2 every $y \in Y$ is covered by at least $k + 1$ elements $V_{i_0}, \ldots, V_{i_k}$ of $V$. Since $\mathcal{U}_m$ is a $(k + 1)$–cover, $U_{i_0}, \ldots, U_{i_k}$ is a cover of $X$. Hence $W_{i_0}, \ldots, W_{i_k}$ covers $f^{-1}(y)$.

**Theorem 3.3** For every CW–complex $X$ with the following inequality holds true:

$$\text{cat}_{LS} X \leq \text{cd}(\pi_1(X)) + \left\lceil \frac{\text{hd}(X) - 1}{2} \right\rceil$$

*Algebraic & Geometric Topology, Volume 10 (2010)*
Proof Let $\pi = \pi_1(X)$ and let $\tilde{X}$ denote the universal cover of $X$. We consider Borel’s construction:

$$
\begin{array}{ccc}
\tilde{X} & \leftarrow & \tilde{X} \times E\pi \\
\downarrow & & \downarrow \\
X & \leftarrow & \tilde{X} \times_\pi E\pi \overset{f}{\rightarrow} B\pi.
\end{array}
$$

We refer for the properties of Borel’s construction also known as the twisted product to [1]. Note that the 1–skeleton $X^{(1)}$ of $X$ defines a $\pi$–equivariant stratification $\tilde{X}^{(1)} \subset \tilde{X}$ of the universal cover. This stratification allows us to treat $f$ as a stratified locally trivial bundle with the fiber $(\tilde{X}, \tilde{X}^{(1)})$. We note that all conditions of Lemma 3.2 are satisfied for $r = 1$. Therefore,

$$
cat_{LS}(\tilde{X} \times_\pi E\pi) \leq \dim B\pi + \left\lfloor \frac{hd(\tilde{X} \times_\pi E\pi) - 1}{2} \right\rfloor.
$$

Since $g$ is a fibration with homotopy trivial fiber, the space $\tilde{X} \times_\pi E\pi$ is homotopy equivalent to $X$. Thus, $cat_{LS}(\tilde{X} \times_\pi E\pi) = cat_{LS} X$ and $hd(\tilde{X} \times_\pi E\pi) = hd(X)$. In view of the results of Eilenberg and Ganea [7] (see also Brown [2]) we may assume that $\dim B\pi = \cd(\pi)$ if $\cd(\pi) > 2$. The case when $\cd(\pi) \leq 2$ is treated in [5].

Acknowledgements This work was partially supported by NSF grant DMS-0904278. I would like to thank the anonymous referee for helpful comments on an earlier draft.

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Received: 23 September 2009 Revised: 16 February 2010

*Algebraic & Geometric Topology, Volume 10 (2010)*