The characterization of a line arrangement whose fundamental group of the complement is a direct sum of free groups

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Kwai Man Fan proved that if the intersection lattice of a line arrangement does not contain a cycle, then the fundamental group of its complement is a direct sum of infinite and cyclic free groups. He also conjectured that the converse is true as well. The main purpose of this paper is to prove this conjecture.

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1 Introduction

An arrangement of lines is a finite collection of $\mathbb{C}$–affine subspaces of dimension 1. For such an arrangement $\Sigma \subseteq \mathbb{C}^2$, there is a natural projective arrangement Σ^* of lines in $\mathbb{CP}^2$ associated to it, obtained by adding to each line its corresponding point at infinity. The problem of finding connections between the topology of $\mathbb{C}^2 - \Sigma$ and the combinatorial theory of Σ is one of the main problems in the theory of line arrangements (see for example Cohen and Suciu [3]). The main motivations for studying the topology of $\mathbb{C}^2 - \Sigma$ are derived from the areas of hypergeometric functions, singularity theory and algebraic geometry.

Given an arrangement Σ, we define the graph $G(\Sigma)$ in the way it is defined by Fan [7]. We first connect all higher multiple points of Σ which lie on a line L_i by an arbitrary simple arc $\alpha_i \subset L_i$ which does not go through any double point of Σ. Taking the union of the set of all higher multiple points and all of this simple arcs, we obtain $G(\Sigma)$. Note that if 3 points are on the same line we do not consider it as a cycle.

In 1994, Jiang and Yau [11] defined the concept of a "nice" arrangement. For Σ, they define a graph $G(V, E)$: The vertices are the multiple points of Σ, and vertices u, v are connected by an edge if there exists $l \in \Sigma$ such that $u, v \in l$. For $v \in V$ define a

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subgraph $G_{\Sigma}(v)$: The vertex set is v and all his neighbors from Σ (note this definition differs from the definition of Fan).

An arrangement Σ is *nice* if there is $V' \subset V$ such that $E_{\Sigma}(v) \cap E_{\Sigma}(u) = \varnothing$ for all $u, v \in V'$, and if we delete the vertex v and the edges of its subgraph $G_{\Sigma}(v)$ from G, for all $v \in V'$ we get a forest (ie a graph without cycles).

Jiang and Yau have proven several properties of nice arrangements:

(a) If A_1, A_2 are nice arrangements and their lattices are isomorphic, then their complements $\mathbb{C}^2 - A_1$ and $\mathbb{C}^2 - A_2$ are diffeomorphic. This property naturally implies that $\pi_1(\mathbb{C}^2 - A_1) \cong \pi_1(\mathbb{C}^2 - A_2)$.

(b) As a consequence of (a), they showed that the presentation of the fundamental group of the complement can be written explicitly and depends only on the lattice of the line arrangement.

In 2005, Wang and Yau [20] continued in this direction and proved that the results of Jiang and Yau hold for a much larger family of arrangements, which they call *simple arrangements*.

Falk [4] finds several examples of line arrangements with the same homotopy types but with different lattices. Fan [6] proved that for up to 6 lines, the fundamental group of a real line arrangement is determined by the lattice. Garber, Teicher and Vishne [9] proved this for a real line arrangement with up to 8 lines.

Let G, H be groups whose abelianizations are free abelian groups of finite rank. Choudary, Dimca and Papadima [2] defined a set Φ of natural group isomorphisms $\phi\colon G \to H$ (which they call a 1*–marking*). In the case of $G = \pi_1(\mathbb{C}^2 - \Sigma)$ (where Σ is an affine arrangement), ϕ takes the topological structure of $\mathbb{C}^2 - \Sigma$ into consideration. They prove that if A, B are line arrangements and B is nice in the sense of Jiang and Yau, then the lattices are isomorphic if and only if there is an isomorphism $\phi \in \Phi$ where $\phi\colon \pi_1(\mathbb{C}^2 - A) \xrightarrow{\sim} \pi_1(\mathbb{C}^2 - B)$.

Fan [7] showed that if the graph $G(\Sigma)$ is a forest (ie a graph without cycles), then the fundamental group is a direct sum of free groups. He also conjectured that the converse of his theorem is true. In [5] Fan proved that if the fundamental group of the complement is a direct sum of free groups, then the arrangement is composed of parallel lines.

In this paper, we prove his conjecture. Our theorem will state that if the fundamental group is isomorphic to a direct sum of free groups, then the graph has no cycles. We would like to emphasize that we make no restrictions on our isomorphisms.

The structure of the paper is as follows. In Section 2, we give basic definitions related to groups. In Section 3 we give basic definitions related to line arrangements and the fundamental group of the complement of a line arrangement. In Section 4, we define a function and a special set induced by it. Section 5 deals with some special properties of fundamental groups of the complements of line arrangements which are direct sum of free groups. In Section 6, we prove the main result of the paper.

2 Definitions and notation

This section presents the needed definitions for the paper.

2.1 Lower central series

We start by defining the *lower central series of a group G* which will be used throughout the paper.

Definition 2.1 (Commutator group and lower central series) Let G be a group. The *commutator group* of G is

$$G' = G_2 = [G, G] = \langle aba^{-1}b^{-1} \mid a, b \in G\rangle$$

The subgroup G' is normal in G with an abelian quotient. We can define the *lower central series of G* recursively:

$$\begin{aligned} G_1 &= G \\ G_2 &= [G, G] \\ G_3 &= [G, G_2] \\ &\vdots \\ G_n &= [G, G_{n-1}]. \end{aligned}$$

Since G_{n+1} contains the commutators of G_n, we have $G_{n+1} \triangleleft G_n$ and the quotient G_n/G_{n+1} is abelian for all $n \in \mathbb{N}$.

To understand these groups, the following identities are needed.

Proposition 2.2 (Witt–Hall identities [15])

(1) $[a, b][b, a] = e$.

(2) $[a, bc] = [a, b][a, c][[c, a], b]$.

(3) $[ab, c] = [a, [b, c]][b, c][a, c]$.

From the second and third identities we get:

Lemma 2.3 *Let G be a group and let $\{x_1, \dots, x_k\}$ be the generators of G. Then*

$$G_2/G_3 = \langle [x_i, x_j] \mid i \neq j,\ 1 \leq i, j \leq k \rangle.$$

Proof If $[xy, z] \in G_2$, then

$$[xy, z] = xyz(xy)^{-1}z^{-1} = xyzy^{-1}x^{-1}z^{-1} = x[y, z]x^{-1}[x, z].$$

Over G_3, we have $[x, [y, z]] = x[y, z]x^{-1}[y, z]^{-1} = e \Rightarrow [y, z] = x[y, z]x^{-1}$. We get

$$[xy, z] = [y, z][x, z] \quad \text{over } G_3.$$

This result means that G_2/G_3 is finitely generated by the set

$$\{[x_i, x_j] \mid i \neq j, 1 \leq i, j \leq k\},$$

where $x_1, \dots, x_k$ are the generators of G. □

Proposition 2.4 *Let $G = \langle x_1, \dots, x_k \mid R \rangle$, where the relations R are commutator type relations (ie every relation $r \in R$ can be written as $r = [w_1, w_2], w_1, w_2 \in G$). Then*

$$G_2/G_3 = \left\langle [x_i, x_j] \;\middle|\; \begin{array}{l} [x_k, x_l] = [x_l, x_k]^{-1}, \\ \text{all generators commute,} \\ R^* \end{array} \right\rangle$$

where R^* is set of the relations R written by means of the generators of G_2, taken modulo G_3.

The next Definition and theorem will give us a better understanding of G_2/G_3 and help us in the future.

Definition 2.5 [10, page 165] Let G be a group generated by $x_1, \dots, x_r$. We consider formal words or strings $b_1 \cdot b_2 \cdots b_n$ where each b is one of the generators. We also introduce formal commutators c_j and weights $\omega(c_j)$ by the rules:

(1) $c_i = x_i, i = 1, \dots, r$ are the commutators of weight 1; ie $\omega(x_i) = 1$.

(2) If c_i and c_j are commutators, then $c_k = [c_i, c_j]$ and $\omega(c_k) = \omega(c_i) + \omega(c_j)$.

Theorem 2.6 (Basis theorem [10]) *If F is the free group with free generators $y_1, \dots, y_r$ and if in a sequence of basic commutators $c_1, \dots, c_t$ are those of weights $1, 2, \dots, n$ then an arbitrary element $f \in F$ has a unique representation $f = c_1^{e_1} c_2^{e_2} \cdots c_t^{e_t}$* mod F_{n+1}.

The basic commutators of weight n form a basis for the free abelian group F_n/F_{n+1}.

2.2 Line arrangements

Definition 2.7 (Line arrangement) A line arrangement $\Sigma = \{L_1, \dots, L_s\} \subseteq \mathbb{C}^2$ is a union of copies of $\mathbb{C}^1$.

Remark 2.8 Each time we are mentioning a line arrangement $\Sigma \subseteq \mathbb{C}^2$, we assume that there are no parallel lines in Σ. Pay attention we can project every line arrangement in $\mathbb{CP}^2$ to $\mathbb{C}^2$ such that the new arrangement does not contain parallel lines.

Definition 2.9 Let Σ be a line arrangement. An intersection point in Σ is called *simple* if there are precisely two lines which meet at that point. Otherwise, we call it *multiple*.

Definition 2.10 (Cycle) A cycle is a nonempty ordered set of multiple intersection points $\{p_1, \dots, p_k\}$, such that any pair of adjacent points p_j, p_{j+1} and the points p_1, p_k are connected by lines of the arrangement. Moreover, if $1 \le j \le k-2$, then the line connecting p_j to p_{j+1} and the line connecting p_{j+1} to p_{j+2} are different. Also, the line connecting p_{k-1} to p_k is different from the line connecting p_k to p_1.

Let $\Sigma \subseteq \mathbb{CP}^2$ be a line arrangement. The invariant $\beta(\Sigma)$ which is defined in [7], counts the number of independent cycles in the graph.

Fan [7] showed that $\beta(\Sigma) = 0$ if and only if $G(\Sigma)$ has no cycles.

In an analogous way for $\Sigma \subseteq \mathbb{C}^2$, we define $\beta(\Sigma) := \beta(\Sigma \cup L_\infty)$ where L_∞ is the projective line at infinity of $\mathbb{C}^2$ and $G(\Sigma) := G(\Sigma \cup L_\infty)$. Note that the new definitions are well defined.

One of the most important invariants of a line arrangement is the fundamental group of its complement, denoted by $\pi_1(\mathbb{C}^2 - \Sigma)$.

The next lemma presents its computation.

Lemma 2.11 (Constructing the fundamental group [1; 18; 19; 3, page 304]) *Let $\Sigma = \{L_1, \dots, L_n\} \subseteq \mathbb{C}^2$ be a line arrangement that is enumerated as in [3].*

We associate a generator Γ_i to each line L_i with base point as in [3] such that

$$G = \pi_1(\mathbb{C}^2 - \Sigma) = \langle \Gamma_1, \dots, \Gamma_n \mid R \rangle,$$

where R is a set of relations generated as follows.

Every intersection point of lines $L_{i_1}, \dots, L_{i_m}$ creates a set of relations

$$\Gamma_{i_1}^{x_1} \Gamma_{i_2}^{x_2} \cdots \Gamma_{i_m}^{x_m} = \Gamma_{i_m}^{x_m} \Gamma_{i_1}^{x_1} \cdots \Gamma_{i_{m-1}}^{x_{m-1}} = \Gamma_{i_2}^{x_2} \cdots \Gamma_{i_m}^{x_m} \Gamma_{i_1}^{x_1}$$

where $x_i \in G$ and $\Gamma_i^{x_i} = {x_i}^{-1} \Gamma_i x_i$.

It is easy to see that this set is equivalent to the following set:

$$[\Gamma_{i_j}^{x_j}, \Gamma_{i_1}^{x_1} \cdots \Gamma_{i_m}^{x_m}] = e, \quad 1 \leq j \leq m.$$

We get a similar presentation for a real arrangement by using the Moishezon–Teicher algorithm [16] and the van Kampen Theorem [13].

Notation 2.12 We denote by $\mathcal{P}$ the set of intersection points. For $p \in \mathcal{P}$, we denote by $\Gamma(p)$ the set of generators attached to the lines passing through the point p and by $\Gamma(p)^c$ the set of generators attached to the lines not passing through the point p.

3 Decomposition of G_2/G_3

Let G be a group. The abelianization of G, denoted by $\overline{G}$, is $\overline{G} = G/G_2$. If $g \in G$, we denote $\overline{g} = g \cdot G_2$.

Remark 3.1 Let $\Sigma \in \mathbb{C}^2$ be a line arrangement, and let $p \in \mathcal{P}$. Then, $G = \pi_1(\mathbb{C}^2 - \Sigma) = \langle \Gamma(p), \Gamma(p)^c \mid R \rangle$,

$$G/G_2 = \overline{G} = \mathrm{Ab}(G) = \langle \overline{\Gamma(p)}, \overline{\Gamma(p)^c} \mid R, [x,y] = e, x, y \in \Gamma(p) \cup \Gamma(p)^c \rangle.$$

This Lemma is an immediate implementation of the last section.

Lemma 3.2 (An implementation for line arrangements) *Let Σ be a line arrangement, then the abelian group G_2/G_3 can be written as*

$$G_2/G_3 = \left\langle [\Gamma_i, \Gamma_j] \;\middle|\; \begin{array}{l} [\Gamma_i, \Gamma_j] = [\Gamma_j, \Gamma_i]^{-1}, \\ [\Gamma_i], \Gamma_j][\Gamma_k, \Gamma_l] = [\Gamma_k, \Gamma_l][\Gamma_i, \Gamma_j], \\ \prod_{\Gamma_x \in \Gamma(p)} [\Gamma_x, \Gamma_y], \;\; p \in \mathcal{P}, \;\; \Gamma_y \in \Gamma(p) \end{array} \right\rangle.$$

Proof A simple implementation of Proposition 2.4 on the presentation of G from Lemma 2.11. □

Remark 3.3 We can see that if Γ_1 and Γ_2 are associated with lines meeting in one point and Γ_3 and Γ_4 are associated with lines meeting in a different point, there is no relation combining $[\Gamma_1, \Gamma_2]$ and $[\Gamma_3, \Gamma_4]$. Therefore,

$$G_2/G_3 = \bigoplus_{p \in \mathcal{P}} C_p$$

where

$$C_p=\left\langle [\Gamma_i,\Gamma_j],\Gamma_i,\Gamma_j\in\Gamma(p) \;\middle|\; \begin{array}{l} [\Gamma_i,\Gamma_j]=[\Gamma_j,\Gamma_i]^{-1}, \\ [\Gamma_i,\Gamma_j][\Gamma_k,\Gamma_l]=[\Gamma_k,\Gamma_l][\Gamma_i,\Gamma_j],\Gamma_i,\Gamma_j,\Gamma_k,\Gamma_l\in\Gamma(p), \\ \prod_{\Gamma_x\in\Gamma(p)}[\Gamma_x,\Gamma_y],\Gamma_y\in\Gamma(p) \end{array}\right\rangle.$$

We can see that the generators of the different groups C_p in the direct sum are disjoint. Consequently, let $x\in G_2/G_3$, then $x=\bigoplus_{p\in\mathcal{P}} c_p$, where $c_p\in C_p$.

For each $r\in\mathcal{P}$, consider the projection

$$\xi_r\colon G_2/G_3\to C_r$$

given by

$$\xi_r(x)=\xi_r\left(\bigoplus_{p\in\mathcal{P}} c_p\right)=c_r.$$

If $\Gamma_i\in\Gamma(r)^c$, then for all $\Gamma_j,\xi_r([\Gamma_i,\Gamma_j])=\xi_r([\Gamma_j,\Gamma_i])=e$. If $\Gamma_i,\Gamma_j\in\Gamma(r)$, then $\xi_r([\Gamma_i,\Gamma_j])=[\Gamma_i,\Gamma_j]$.

4 The stabilizer of an intersection point

Let $G=\pi_1(\mathbb{C}^2-\Sigma)$. Define

$$f\colon G/G_2\times G/G_2\longrightarrow G_2/G_3$$

by

$$f(\overline{a},\overline{b})=[a,b]/G_3.$$

This function is well-defined: If $\overline{a_1}=\overline{a_2}$ and $\overline{b_1}=\overline{b_2}$, then $a_2=a_1x$ where $x\in G_2$, $b_2=b_1y$ where $y\in G_2$. Then, by Proposition 2.2,

$$[a_2,b_2]=[a_1x,b_1y]=[a_1x,b_1][a_1x,y]=[a_1x,b_1]=[a_1,b_1][x,b_1]=[a_1,b_1],$$

so $f(\overline{a_1},\overline{b_1})=f(\overline{a_2},\overline{b_2})$.

The following lemma presents some properties of f:

Lemma 4.1 *Let $a,b,c\in G/G_2$. Then:*

(1) $f(a\cdot b,c)=f(a,c)\cdot f(b,c)$.

(2) $f(a,b\cdot c)=f(a,b)\cdot f(a,c)$.

(3) $f(a^n,b^m)=f(a,b)^{nm}$ for $m,n\in\mathbb{Z}$.

(4) $f(b,a)=(f(a,b))^{-1}$.

Proof (1) Let $A, B, C \in G$ such that $\overline{A} = a$, $\overline{B} = b$, and $\overline{C} = c$. This means that $\overline{AB} = ab$, so by definition, $f(ab, c) = [AB, C]/G_3 = ([A, C][B, C])/G_3$ which by definition is equal to $f(a, c) f(b, c)$.

(2) The proof is the same as (1).

(3) Simple induction on (1) and (2).

(4) Let $A, B \in G$ such that $\overline{A} = a$, $\overline{B} = b$. By definition,

$$f(b, a) = [B, A]/G_3 = ([A, B]/G_3)^{-1} = (f(a, b))^{-1}. \qquad \square$$

Now for any $\overline{x} \in G/G_2$ we define

$$S(\overline{x}) = \{y \in G/G_2 \mid f(\overline{y}, \overline{x}) = e_{G_3}\}.$$

By Lemma 4.1, $S(\overline{x})$ is a subgroup of G/G_2 .

From now on, we talk about a specific intersection point Q. The lines passing through this point are $\{L_{i_1}, \dots, L_{i_m}\}$. Define $M = \Gamma_{i_1}^{x_1} \Gamma_{i_2}^{x_2} \cdots \Gamma_{i_m}^{x_m}$. Then, as noted in Lemma 2.11, the relations induced from the point Q can be translated to $[\Gamma_{i_1}^{x_1}, M] = \cdots = [\Gamma_{i_m}^{x_m}, M] = e$.

Let $\overline{M} = \overline{\Gamma_{i_1}} \cdot \overline{\Gamma_{i_2}} \cdots \overline{\Gamma_{i_m}}$.

Since for each j, $\overline{\Gamma_{i_j}} = \overline{\Gamma_{i_j}^{x_j}}$ we get that

$$f(\overline{\Gamma_{i_j}}, \overline{M}) = [\Gamma_{i_j}^{x_j}, M]/G_3 = e_{G_3}. \tag{1}$$

Theorem 4.2 *Let $Q \in \mathcal{P}$ be an intersection point. Let $\Gamma(Q) = \{\Gamma_{i_1}, \dots, \Gamma_{i_m}\}$ and $M = \Gamma_{i_1}^{x_1} \cdots \Gamma_{i_m}^{x_m}$. Then*

$$S(\overline{M}) = \Big\langle \overline{\Gamma(Q)} \cup \Big(\bigcap_{\Gamma \in \Gamma(Q)} S(\overline{\Gamma}) \Big) \Big\rangle.$$

We call $S(\overline{M})$ the stabilizer *of the intersection point Q.*

Proof We start by proving that

$$S(\overline{M}) \supseteq \Big\langle \overline{\Gamma(Q)} \cup \Big(\bigcap_{\Gamma \in \Gamma(Q)} S(\overline{\Gamma}) \Big) \Big\rangle.$$

$\overline{\Gamma(Q)} \subseteq S(\overline{M})$: We have already shown that if $\Gamma \in \Gamma(Q)$, then $f(\overline{\Gamma}, \overline{M}) = \overline{e}$ and hence $f(\overline{M}, \overline{\Gamma}) = f((\overline{\Gamma}, \overline{M}))^{-1} = \overline{e}$.

$\bigcap_{\Gamma\in\Gamma(Q)} S(\overline{\Gamma}) \subseteq S(\overline{M})$: If $x \in \bigcap_{\Gamma\in\Gamma(Q)} S(\overline{\Gamma})$, then for any $\Gamma \in \Gamma(Q)$, we have $f(\overline{\Gamma}, x) = e$. This means that also the product $\prod_{\Gamma\in\Gamma(Q)} f(\overline{\Gamma}, x) = \overline{e}$, so by Lemma 4.1,

$$\overline{e} = \prod_{\Gamma\in\Gamma(Q)} f(\overline{\Gamma}, x) = f\Bigg(\bigg(\prod_{\Gamma\in\Gamma(Q)} \overline{\Gamma}\bigg), x\Bigg) = f(\overline{M}, x).$$

Therefore $x \in S(\overline{M})$.

Since $S(\overline{M})$ is a subgroup and we have shown that it contains the union of $\overline{\Gamma(Q)}$ and $\bigcap_{\Gamma\in\Gamma(Q)} S(\overline{\Gamma})$, it clearly contains the subgroup generated by the union.

To complete the proof, we prove the opposite inclusion:

$$S(\overline{M}) \subseteq \Bigg\langle \overline{\Gamma(Q)} \cup \bigg(\bigcap_{\Gamma\in\Gamma(Q)} S(\overline{\Gamma})\bigg)\Bigg\rangle.$$

Let $\overline{x} \in S(\overline{M}) \subseteq G/G_2 = \langle \overline{\Gamma(Q)}, \overline{\Gamma(Q)^c} \mid \overline{R}\rangle$. Then $\overline{x}$ can be written as $\overline{x} = \overline{z} \cdot \overline{y}$ where $\overline{z} \in \langle \overline{\Gamma(Q)}\rangle$, $\overline{y} \in \langle \Gamma(Q)^c\rangle$. We will prove $\overline{y} \in \bigcap_{\Gamma\in\Gamma(Q)} S(\overline{\Gamma})$.

Since $\overline{z^{-1}} \in \langle \overline{\Gamma(Q)}\rangle \subseteq S(\overline{M})$, so $\overline{y} = \overline{z^{-1}} \cdot \overline{z} \cdot \overline{y} = \overline{z^{-1}}\overline{x}$. Hence we get $\overline{y} \in S(\overline{M})$.

Let l_x be a line passing through Q and Γ_x be the generator associated with it. Recall that $\overline{y} \in S(\overline{\Gamma})$ if $f(\overline{\Gamma}, \overline{y}) = \overline{e}$. We need to prove that $f(\overline{\Gamma_x}, \overline{y}) = \overline{e}$ in G_2/G_3. From Remark 3.3 we know that G_2/G_3 is a direct sum of groups

$$G_2/G_3 = \bigoplus_{p\in\mathcal{P}} C_p,$$

so it remains to prove that $f([\overline{\Gamma_x}, \overline{y}]) = \overline{e}$ in C_p, for all $p \in \mathcal{P}$. Let $p \in \mathcal{P}$. We have to show that the projection of the coset $[\overline{\Gamma_x}, \overline{y}]/G_3$ on C_p is trivial.

We separate our proof into three cases.

Case 1 $p = Q$.

Since $\overline{y} \in \langle \overline{\Gamma(Q)^c}\rangle = \langle \overline{\Gamma(p)^c}\rangle$ then the projection is trivial by the definition of C_p.

Case 2 l_x does not pass through p.

In this case, $\overline{\Gamma_x} \in \langle \overline{\Gamma(p)^c}\rangle$ and therefore the projection is trivial, by the definition of C_p.

Case 3 $p \neq Q$ and l_x does pass through p.

By definition of S, $f(\prod_{\Gamma\in\Gamma(Q)} \overline{\Gamma}, \overline{y}) = \overline{e}$. This means by the properties of f (Lemma 4.1) that $\prod_{\Gamma\in\Gamma(Q)} f(\overline{\Gamma}, \overline{y}) = \overline{e}$. Since l_x passes through p, all the other lines passing through Q do not cross p. So if we project any $f(\overline{\Gamma_z}, \overline{y})$ where Γ_z is any other generator from $\Gamma(Q)$, we get the identity. So in C_p,

$$\overline{e} = \prod_{\Gamma \in \Gamma(Q)} f(\overline{\Gamma}, \overline{y}) = f(\overline{\Gamma_x}, \overline{y}),$$

and thus $f(\overline{\Gamma_x}, \overline{y}) = \overline{e}$. □

5 Fundamental groups which are semidirect product

Theorem 5.1 *Let $G = \left(\bigoplus_{i=1}^n A_i\right) \oplus \mathbb{Z}^l$ where A_i are free groups and let $\psi\colon G \longrightarrow G$ be a projection where $\operatorname{im}(\psi) = \langle y_1, \dots, y_k \rangle \cong \mathbb{F}_k$. Then, there exists r, $1 \le r \le n$, such that for all $g \in G, \operatorname{pr}_r(g) = e$ implies that $\psi(g) = e$ and $\psi = \psi \circ \operatorname{pr}_r$, where $\operatorname{pr}_r\colon G \to G$ is the projection onto the subgroup of G naturally isomorphic to A_r.*

Proof Let us denote the $\mathbb{Z}^l$ component as A_0. Since ψ is a projection, $\psi(y_1) = y_1$ and $\psi(y_2) = y_2$. Then $\psi([y_1, y_2]) = [y_1, y_2] \neq e$. Note

$$[y_1, y_2] = \bigoplus_{i=1}^n \operatorname{pr}_i([y_1, y_2]) + \operatorname{pr}_0([y_1, y_2]).$$

Since A_0 is abelian, $\operatorname{pr}_0([y_1, y_2]) = e$, and therefore

$$[y_1, y_2] = \bigoplus_{i=1}^n \operatorname{pr}_i([y_1, y_2]).$$

Since ψ is a homomorphism,

$$\begin{aligned}\psi([y_1, y_2]) &= \psi(\operatorname{pr}_1([y_1, y_2]) \oplus \cdots \oplus \operatorname{pr}_n([y_1, y_2])) \\ &= \psi(\operatorname{pr}_1([y_1, y_2])) \oplus \cdots \oplus \psi(\operatorname{pr}_n([y_1, y_2])) \neq e.\end{aligned}$$

This means that there is $r, 1 \le r \le n$, such that $\psi(\operatorname{pr}_r[y_1, y_2]) \neq e$, so if we denote $a := \operatorname{pr}_r(y_1)$ and $b := \operatorname{pr}_r(y_2)$ we get $[\psi(a), \psi(b)] \neq e$.

Since $a, b \in A_r$, if $x \in A_j$ where $j \neq r$, then $[x, a] = [x, b] = e$. We get that

$$[\psi(x), \psi(a)] = [\psi(x), \psi(b)] = e.$$

So $\psi(x)$ commutes with two noncommutative elements in a free group and hence $\psi(x) = e$, for all $x \in A_j$, $j \neq r$. This means that if $g \in G$ and $\operatorname{pr}_r(g) = e$, then $\psi(g) = e$.

For each $g \in G$, $g = v_1 \oplus \cdots \oplus v_n \oplus w, v_j \in A_j, w \in \mathbb{Z}^l$, we have that $\psi(g) = \psi(v_1 \oplus \cdots \oplus v_n \oplus w) = \psi(v_1) \oplus \cdots \oplus \psi(v_n) \oplus \psi(w) = \psi(v_r) = \psi(\mathrm{pr}_r(g))$. Therefore $\psi = \psi \circ \mathrm{pr}_r$. □

For proving the next result, we use the following theorem from Lyndon and Schupp [14]:

Theorem 5.2 *Let ϕ be an epimorphism from a finitely generated free group F to a free group G. Then F has a basis $Z = Z_1 \cup Z_2$ such that ϕ maps $\langle Z_1 \rangle$ isomorphically onto G and maps $\langle Z_2 \rangle$ to e.*

Corollary 5.3 *Let $G = \left(\bigoplus_{i=1}^n A_i\right) \oplus \mathbb{Z}^l$ where A_i are free groups and let $\psi\colon G \longrightarrow G$ be a projection where $\mathrm{im}(\psi) = \langle y_1, \dots, y_k \rangle \cong \mathbb{F}_k$. For the same r as in Theorem 5.1, we can find elements $\{z_1, \dots, z_m \mid m \geq 0\} \subseteq A_r$ such that*

$$\{\mathrm{pr}_r(y_1), \dots, \mathrm{pr}_r(y_k), z_1, \dots, z_m \mid m \geq 0\}$$

are generators of A_r and $\psi(z_i) = e$.

Proof From Theorem 5.1 and the fact that ψ is a projection ($\psi \circ \psi = \psi$), we get that $\mathrm{pr}_r \circ \psi \circ \mathrm{pr}_r \circ \psi = \mathrm{pr}_r \circ \psi$. Therefore, $\mathrm{pr}_r \circ \psi$ is a projection.

Now since $(\mathrm{pr}_r \circ \psi)(A_r) \subseteq A_r$,

$$(\mathrm{pr}_r \circ \psi)|_{A_r} \circ (\mathrm{pr}_r \circ \psi)|_{A_r} = ((\mathrm{pr}_r \circ \psi) \circ (\mathrm{pr}_r \circ \psi))|_{A_r} = (\mathrm{pr}_r \circ \psi)|_{A_r}.$$

By assumption,

$$(*) \qquad \mathrm{im}(\psi) = \langle y_1, \dots, y_k \rangle \quad \text{is a free group.}$$

Now we claim $\{\mathrm{pr}_r(y_1), \dots, \mathrm{pr}_r(y_k)\}$ are the generators of free group $\mathbb{F}_k$. If not, there is a nontrivial word $w = W(y_1, \dots, y_k)$ so that $\mathrm{pr_r}(w) = W(\mathrm{pr}_r(y_1), \dots, \mathrm{pr}_r(y_k)) = e$. Therefore

$$(\psi \circ \mathrm{pr}_r)(w) = \psi(\mathrm{pr}_r(w)) = \psi(e) = e.$$

By Theorem 5.1, $\psi = \psi \circ \mathrm{pr}_r$, therefore $\psi(w) = (\psi \circ \mathrm{pr}_r)(w) = e$ which means that $w = e$, a contradiction to $(*)$.

In conclusion, we get that $(\mathrm{pr}_r \circ \psi)|_{A_r}\colon A_r \to A_r$ is a projection such that $\mathrm{im}(\mathrm{pr}_r \circ \psi) = \langle \mathrm{pr}_r(y_1), \dots, \mathrm{pr}_r(y_k) \rangle \cong \mathbb{F}_k$ ($\mathrm{pr}_r(y_1), \dots, \mathrm{pr}_r(y_k)$ are the generators of $\mathbb{F}_k$).

By Theorem 5.2, A_i has a basis $B = B_1 \cup B_2$ such that $\mathrm{im}(\mathrm{pr}_r \circ \psi) = \langle B_1 \rangle$ and $(\mathrm{pr}_r \circ \psi)(B_2) = e$. Since $\mathrm{pr}_r \circ \psi$ is a projection, we can assume that $B_1 = \langle \mathrm{pr}_r(y_1), \dots, \mathrm{pr}_r(y_k) \rangle$. Let $B_2 = \{z_1, \dots, z_m \mid \mathrm{pr}_r \circ \psi(z_i) = e\}$, which means that $(\psi \circ \mathrm{pr}_r \circ \psi)(z_i) = e$. As we proved earlier, $\psi \circ \mathrm{pr}_r = \psi$, so $(\psi \circ \psi)(z_i) = e$. But $\psi \circ \psi = \psi$, therefore $\psi(z_i) = e$.

This implies that A_r has generators

$$\{\mathrm{pr}_r(y_1), \dots, \mathrm{pr}_r(y_k), z_1, \dots, z_m\}$$

which satisfy the needed properties. □

Without loss of generality, we temporarily change to a local numeration.

Let $G = \pi_1(\mathbb{C}^2 - \Sigma) = \langle \Gamma_1, \dots, \Gamma_n, \Gamma_{n+1}, \dots, \Gamma_l \mid R \rangle$, where $\Gamma(A) = \{\Gamma_1, \dots, \Gamma_n\}$ are the generators of the lines which participate in a specific multiple intersection point called A.

Definition 5.4 [12] Let G be a group with a given presentation $G = \langle X \mid R \rangle$. A *Tietze transformation* T_i $(1 \le i \le 4)$ is a transformation of $\langle X \mid R \rangle$ to a presentation $G = \langle X' \mid R' \rangle$ of one of the following types:

(T_1) If r is a word in $\langle X \rangle$ and $r = e$ is a relation that holds in G, then let $X' = X, R' = R \cup \{r\}$.

(T_2) If $r \in R$ is such that the relation $r = e$ holds in the group $\langle X \mid R - r \rangle$, then let $X' = X, R' = R \setminus \{r\}$.

(T_3) If w is a word in $\langle X \rangle$ and $z \notin X$, put $X' = X \cup \{z\}, R' = R \cup \{wz^{-1}\}$.

(T_4) If $z \in X$ and w is a word in the other elements of X such that $wz^{-1} \in R$, then substitute w for z in every other element of R to get $\widetilde{R}$ and take $X' = X - \{z\}, R' = \widetilde{R}$.

Before introducing the notion of a *braid monodromy* [16], we have to make some constructions. From now, we will work in $\mathbb{C}^2$. Let E (resp. D) be a closed disk on x–axis (resp. y–axis), and let C be a part of an algebraic curve in $\mathbb{C}^2$ located in $E \times D$. Let $\pi_1\colon E \times D \to E$ and $\pi_2\colon E \times D \to D$ be the canonical projections, and let $\pi = \pi_1|_C\colon C \to E$. Assume π is a proper map, and $\deg \pi = n$. Let $N = \{x \in E \mid \#\pi^{-1}(x) < n\}$, and assume $N \cap \partial E = \varnothing$. Now choose $M \in \partial E$ and let $K = K(M) = \pi^{-1}(M)$. By the assumption that $\deg \pi = n$ $(\Rightarrow \#K = n)$, we can write $K = \{a_1, a_2, \dots, a_n\}$. Under these constructions, from each loop in $E - N$, we can define a braid in $B_n[M \times D, K]$ in the following way:

(1) Since $\deg \pi = n$, we can lift any loop in $E - N$ with a base point M to a system of n paths in $(E - N) \times D$ which start and finish at $\{a_1, a_2, \dots, a_n\}$.

(2) Project this system into D (by π_2), to get n paths in D which start and end at the image of K in D (under π_2). These paths actually form a motion.

(3) Induce a braid from this motion; see Garber [8].

To conclude, we can match a braid to each loop. So we get a map $\alpha\colon \pi_1(E-N, M) \to B_n[M \times D, K]$, which is also a group homomorphism. This map is called the *braid monodromy of C with respect to $E \times D, \pi_1, M$* [16; 8].

The following remark demonstrates the necessity of the condition that there are no parallel lines in the affine plane:

Remark 5.5 Let Σ be a line arrangement with no parallel line in the affine plane. Let $\Gamma_1, \dots, \Gamma_l$ be the generators of $\pi_1(\mathbb{C}^2 - \Sigma)$. Then

$$\Gamma_1 \cdots \Gamma_l \in Z(\pi_1(\mathbb{C}^2 - \Sigma)).$$

Proof Let $\alpha\colon \pi_1(\mathbb{C} - \Sigma) \to B_n$ be the braid monodromy of Σ. By Remark 4.7 in [3], the closed braid determined by the product $\alpha(\Gamma_1)\cdots\alpha(\Gamma_l)$ is actually a link of the curve at infinity. In a generic curve, we have $\alpha(\Gamma_1)\cdots\alpha(\Gamma_l) = \Delta^2$, where Δ is the braid which rotates by $180°$ counterclockwise all the strands together. One of the presentations of the fundamental group is $\langle \Gamma_1, \dots, \Gamma_l \mid \alpha(s_i)(\Gamma_j) = \Gamma_j,\ \forall i,\ \forall j \rangle$, where s_i is the loop created in the x–axis as a result of the projection of the line arrangement on the plane [13]. Therefore, $\alpha(s_i)$ is a braid acting on the generators of the fundamental group. As a result $\alpha(s_1 \cdots s_n)(\Gamma_j) = \Gamma_j$, for all j, which means $\Delta^2(\Gamma_j) = \Gamma_j$, for all j. It is known that $\Delta^2(x) = x^{\Gamma_1 \cdots \Gamma_l}$ and thus ${\Gamma_j}^{\Gamma_1 \cdots \Gamma_l} = \Gamma_j$. Hence, $\Gamma_1 \cdots \Gamma_l \in Z(\pi_1(\mathbb{C}^2 - \Sigma))$. □

Let $G = \pi_1(\mathbb{C}^2 - \Sigma) = \langle \Gamma_1, \dots \Gamma_n, \Gamma_{n+1}, \dots, \Gamma_l \mid R \rangle$, where $\Gamma(A) = \{\Gamma_1, \dots, \Gamma_n\}$ are the generators of the lines which pass through in a specific multiple intersection point called A.

Let us recall that the lines passing through Q are $\{L_{i_1}, \dots, L_{i_m}\}$. Recall also $M = \Gamma_{i_1}^{x_1} \Gamma_{i_2}^{x_2} \cdots \Gamma_{i_m}^{x_m}$.

Theorem 5.6 *With the same assumptions of Theorem 5.1 and Remark 5.5, let $H := \langle \Gamma_{i_1}, \dots, \Gamma_{i_{m-1}} \rangle$. Let N be the normal closure of $\{\Gamma(Q)^c, M\}$. Let $f_1\colon H \to G$ be the natural embedding and $f_2\colon G \to G/N$ the natural homomorphism. Then:*

1. *$G/N \cong \mathbb{F}_{m-1}$ which is generated by $f_2(\Gamma_{i_j})$, $1 \le j \le m-1$.*
2. *$H \cong \mathbb{F}_{m-1}$.*
3. *$G = N \rtimes \mathbb{F}_{m-1}$.*
4. *There exists a projection $h\colon G \to G$ $(h^2 = h)$, such that $\operatorname{im}(h) = H$ and $\ker(h) = N$.*

Proof (1) We can present $G/N = \langle \Gamma(Q), \Gamma(Q)^c \mid R, M, \Gamma(Q)^c \rangle$. Denote $\Gamma(Q)^c = \{z_1, \dots, z_k\}$. By applying iteratively Tietze's transformation (T_4) for every $z_i \in \Gamma(Q)^c$, we obtain $G/N = \langle \Gamma(Q) \mid \widehat{R}, \widehat{M} \rangle$, where $\widehat{R}, \widehat{M}$ are obtained from R, M by substituting e for z_i, $1 \le i \le k$, in every other element of R and M, respectively.

Let $v \in \mathcal{P}$ with $v \neq Q$ be an intersection point, and let r be the relations induced by v.

Now we can get the following:

(a) If there is a line L that goes through both points v and Q and the generator attached to L is Γ_{i_j} for some j, $1 \le j \le m$, then the relations r become $[\Gamma_{i_j}^x, 1, \dots, 1]$ for some x. These relations are trivial.

(b) If the points v and Q do not share any line, then the relations r become $[1, \dots, 1]$, which are also trivial.

Let us denote by $\widetilde{W}$ the word W after rewriting. Due to the above observations, using the presentation $G/N = \langle \Gamma(Q) \mid \widetilde{R}, \widetilde{M} \rangle$, then

$$G/N = \langle \Gamma_{i_1}, \dots, \Gamma_{i_m} \mid [\Gamma_{i_1}^{\widetilde{x_{i_1}}}, \dots, \Gamma_{i_m}^{\widetilde{x_{i_m}}}], \widetilde{M} \rangle.$$

This is equal to

$$\langle \Gamma_{i_1}, \dots, \Gamma_{i_m} \mid [\Gamma_{i_1}^{\widetilde{x_{i_1}}}, \widetilde{M}], \dots, [\Gamma_{i_m}^{\widetilde{x_{i_m}}}, \widetilde{M}], \widetilde{M} \rangle = \mathbb{F}_{m-1}.$$

(2) Assume otherwise. Then there exists a nontrivial word $w(\Gamma_{i_1}, \dots, \Gamma_{i_{m-1}}) = e$. So applying f_2, we get a nontrivial word $w(f_2(\Gamma_{i_1}), \dots, f_2(\Gamma_{i_{m-1}}))$ which is not possible by (1).

(3) From the first paragraph, we get that $f_2 \circ f_1$ is a surjective function from $\mathbb{F}_{m-1}$ to $\mathbb{F}_{m-1}$ and therefore it is an isomorphism.

(4) Derived directly from (2) and (3). □

By the last theorem: Let $Q \in \mathcal{P}$, $\Gamma(Q) = \{\Gamma_{i_1}, \dots, \Gamma_{i_m}\}$. Then there is a projection $h\colon G \to G$ such that $\operatorname{im}(h) = \langle \Gamma_{i_1}, \dots, \Gamma_{i_m} \rangle$ and $\ker(h)$ is the normal closure of $\langle \Gamma(Q)^c, M \rangle$.

By Theorem 5.1 and Corollary 5.3 we get that there is r, $1 \le r \le n$, such that $A_r = \{w_1, \dots, w_{n-1}, z_1, \dots, z_k \mid k \ge 0\}$ and $h = h \circ \mathrm{pr}_r$, $\mathrm{pr}_r \circ h$ is a projection, $\Gamma_i = h(w_i), h(z_i) = e, b_i = \Gamma_i {w_i}^{-1}, h(b_i) = e$ (note that h has the role of ψ).

Claim 5.7 $\overline{A_r} = \mathrm{pr}_{\overline{r}}(S(\overline{M}))$.

Proof We start by proving that $\mathrm{pr}_r(M) = e$: for all x, y and $j_1 \neq j_2$. Since $\Gamma_{i_{j_1}}$ and $\Gamma_{i_{j_2}}$ are different generators of the free group $\mathrm{im}(h)$, $h([\Gamma_{i_{j_1}}^x \, \Gamma_{i_{j_2}}^y]) \neq e$. We know that $h = h \circ \mathrm{pr}_r$, so we get

$$(h \circ \mathrm{pr}_r)([\Gamma_{i_{j_1}}^x \, \Gamma_{i_{j_2}}^y]) = h([\Gamma_{i_{j_1}}^x \, \Gamma_{i_{j_2}}^y]) \neq e,$$

so $\mathrm{pr}_r([\Gamma_{i_{j_1}}^x \, \Gamma_{i_{j_2}}^y]) \neq e$. Hence, for $x = x_{j_1}, y = x_{j_2}$, $\mathrm{pr}_r([\Gamma_{i_{j_1}}^{x_{j_1}}, \Gamma_{i_{j_2}}^{x_{j_2}}]) \neq e$. We know that $[\Gamma_{i_{j_1}}^{x_{j_1}}, M] = e$, so the elements $\mathrm{pr}_r(\Gamma_{i_j}^{x_j})$, $j = 1, 2$, commute with M but do not commute with each other in the free group A_r, therefore $\mathrm{pr}_r(M) = e$.

The next step is to prove that $\overline{A_r} \subseteq S(\overline{M})$. Let $a \in \overline{A_r}$. Then there exists $a^* \in A_r$ such that $\overline{a^*} = a$. Since $\mathrm{pr}_r(M) = e$ and $a^* \in A_r$, by the properties of direct sum $[a^*, M] = e$. Therefore $[a^*, M]/G_3 = e_{G_3}$. By definition $f(a, \overline{M}) = e_{G_3}$ which implies that $a \in S(\overline{M})$.

Hence, we have $\overline{A_r} \subseteq \mathrm{pr}_{\overline{r}}(S(\overline{M}))$. The opposite inclusion is trivial by definition, and thus $\overline{A_r} = \mathrm{pr}_{\overline{r}}(S(\overline{M}))$. $\square$

By Theorem 4.2, $S(\overline{M}) = \left\langle \{\overline{\Gamma_{i_1}}, \dots, \overline{\Gamma_{i_m}}\} \cup \left\{ \bigcap_{k=1}^m S(\overline{\Gamma_{i_k}}) \right\} \right\rangle$, therefore

$$\begin{aligned} \overline{A_r} &= \mathrm{pr}_{\overline{r}}\left(\left\langle \{\Gamma_{i_1}, \dots, \Gamma_{i_m}\} \cup \bigcap_{k=1}^m S(\Gamma_{i_k}) \right\rangle \right) \\ &= \left\langle \{\mathrm{pr}_{\overline{r}}(\Gamma_{i_1}), \dots, \mathrm{pr}_{\overline{r}}(\Gamma_{i_m})\} \cup \mathrm{pr}_{\overline{r}}\left(\bigcap_{k=1}^m S(\overline{\Gamma_{i_k}}) \right) \right\rangle. \end{aligned}$$

We claim that the right part of this generating set is trivial:

Claim 5.8 $\mathrm{pr}_{\overline{r}}\left(\bigcap_{k=1}^m S(\Gamma_{i_k}) \right) = \{e\}$.

Proof It is known that for any two sets A, B and a function F that $F(A \cap B) \subseteq F(A) \cap F(B)$. Therefore $\mathrm{pr}_{\overline{r}}\left(\bigcap_{k=1}^m S(\Gamma_{i_k}) \right) \subseteq \bigcap_{k=1}^m \left(\mathrm{pr}_{\overline{r}}(S(\Gamma_{i_k})) \right)$.

As we mentioned $A_r = \langle \mathrm{pr}_r(\Gamma_1), \dots, \mathrm{pr}_r(\Gamma_m), z_1, \dots, z_q \mid q \geq 0 \rangle$.

In other words, $A_r = \langle \beta_1, \dots, \beta_{m+q} \rangle$ where

$$\beta_k = \begin{cases} \mathrm{pr}_r(\Gamma_{i_k}) & k \leqslant m, \\ z_{k-m} & m+1 \leqslant k \leqslant m+q. \end{cases}$$

Let $\overline{g} \in \mathrm{pr}_{\overline{r}}(S(\Gamma_{i_k})) \subseteq \overline{A_r}$, therefore $\overline{g} = \overline{\beta_1}^{t_1} \cdots \overline{\beta_{q+m}}^{t_{m+q}}$. Without loss of generality, we can choose that $g = \beta_1^{t_1} \cdots \beta_{m+q}^{t_{m+q}}$. By Theorem 5.1 we know that

$$G = A_1 \oplus \cdots \oplus A_r \oplus \cdots \oplus A_n \oplus \mathbb{Z}^\ell.$$

We define $D = A_1 \oplus \cdots \oplus \widehat{A_r} \oplus \cdots \oplus A_n \oplus \mathbb{Z}^\ell$, then $G = A_r \bigoplus D$.

Hence, there exists $g_2 \in D$ such that $\overline{g} \oplus \overline{g_2} \in S(\overline{\Gamma_{i_k}})$.

Let $\varphi\colon G \to D$ be the natural projection, and define $\eta_k := \varphi(\Gamma_{i_k})$. Hence, it implies that $\Gamma_{i_k} = \mathrm{pr}_r(\Gamma_{i_k}) \oplus \eta_k$.

$$\begin{aligned} e_{G_3} &= f(\overline{\Gamma_{i_k}}, \overline{g \oplus g_2}) = f(\overline{\mathrm{pr}_r(\Gamma_{i_k}) \oplus \eta_k}, \overline{g \oplus g_2}) \\ &= f(\overline{\mathrm{pr}_r(\Gamma_{i_k})}, \overline{g}) f(\overline{\mathrm{pr}_r(\Gamma_{i_k})}, \overline{g_2}) f(\overline{\eta_k}, \overline{g}) f(\overline{\eta_k}, \overline{g_2}). \end{aligned}$$

By the definition of f, this implies that

$$[\overline{\mathrm{pr}_r(\Gamma_{i_k})}, \overline{g}][\overline{\mathrm{pr}_r(\Gamma_{i_k})}, \overline{g_2}][\overline{\eta_k}, \overline{g}][\overline{\eta_k}, \overline{g_2}] \in G_3.$$

Since $\mathrm{pr}_r(\Gamma_{i_k}), g \in A_r$, and $g_2, \eta_k \in D$, then $[\mathrm{pr}_r(\Gamma_{i_k}), g_2] = [\eta_k, g] = e_{G_3}$. Therefore $[\mathrm{pr}_r(\Gamma_{i_k}), g][\eta_k, g_2] \in G_3 = (A_r)_3 \oplus D_3$, which means $[\mathrm{pr}_r(\Gamma_{i_k}), g] \in (A_r)_3$ and $[\eta_k, g_2] \in D_3$.

The fact that $(A_r)_2/(A_r)_3 = \langle [\beta_i, \beta_j] \mid 1 \leq i < j \leq m+q \rangle$ is derived directly from the definition of A_r. Hence, $(A_r)_3 \ni [\mathrm{pr}_r(\Gamma_{i_k}), g] = [\beta_k, g] = [\beta_k, \beta_1^{t_1} \cdots \beta_{m+q}^{t_{m+q}}] = [\beta_k, \beta_1]^{t_1} \cdots [\beta_k, \beta_k]^{t_k} \cdots [\beta_k, \beta_{m+q}]^{t_{m+q}}$.

Note that $[\beta_k, \beta_k] = e$. By [10], $\{[\beta_k, \beta_j] \mid j \neq k\}$ are generators (or inverses) of $(A_r)_2/(A_r)_3$ which is a free abelian group. Since every generator appears at most once, we get $t_j = 0$ for all $j \neq k$.

In conclusion, $\overline{g} = \overline{\beta_k}^{t_k} \in \langle \overline{\beta_k} \rangle = \langle \overline{\mathrm{pr}_r(\Gamma_{i_k})} \rangle$. So $\overline{g} \in \langle \overline{\mathrm{pr}_{\overline{r}}(\Gamma_{i_k})} \rangle$.

To summarize, we have shown that if $\overline{g} \in \langle \mathrm{pr}_{\overline{r}} \left(S(\Gamma_{i_k}) \right) \rangle$ then $\overline{g} \in \langle \mathrm{pr}_{\overline{r}} \left(\Gamma_{i_k} \right) \rangle$. Therefore, $\langle \mathrm{pr}_{\overline{r}} \left(S(\Gamma_{i_k}) \right) \rangle \subseteq \langle \mathrm{pr}_{\overline{r}}(\Gamma_{i_k}) \rangle$.

In conclusion, $\bigcap_{k=1}^m \langle \mathrm{pr}_{\overline{r}}(S(\Gamma_{i_k})) \rangle \subseteq \bigcap_{k=1}^m \langle \mathrm{pr}_{\overline{r}}(\Gamma_{i_k}) \rangle = e$, so $\mathrm{pr}_{\overline{r}}\big(\bigcap_{k=1}^m S(\Gamma_{i_k})\big) = \{e\}$ as claimed. □

Since $\Gamma_{i_1} \cdots \Gamma_{i_m} = e$, $\overline{A_r} = \langle \overline{\Gamma_{i_1}}, \ldots, \overline{\Gamma_{i_{m-1}}} \rangle$. As a result, $\overline{A_r} = \mathrm{pr}_{\overline{r}} \langle \Gamma_{i_1}, \ldots, \Gamma_{i_{m-1}} \rangle$. Consequently, $\mathrm{rank}(A_r) \leq m-1$.

We know that $A_r = \langle w_1, \ldots, w_{m-1}, z_1, \ldots, z_t \mid t \geq 0 \rangle$. Combining these two facts together and by Theorem 5.2, we get $A_r = \langle w_1, \ldots, w_{m-1} \rangle$, where $y_i = (\mathrm{pr}_r \circ h)(\Gamma_{i_j}^{x_j})$.

From Theorem 5.1, $h \circ \mathrm{pr}_r = h$. Since h is a projection $h^2 = h$. Hence, we get

$$\mathrm{pr}_r \circ h \circ \mathrm{pr}_r \circ h = \mathrm{pr}_r \circ h \circ h = \mathrm{pr}_r \circ h,$$

which means that $\mathrm{pr}_r \circ h$ is a projection too. Recall that

$$A_r = \langle (\mathrm{pr}_r \circ h)(\Gamma_{i_1}), \ldots, (\mathrm{pr}_r \circ h)(\Gamma_{i_{m-1}}) \rangle.$$

Therefore $(\mathrm{pr}_r \circ h)(w_j) = (\mathrm{pr}_r \circ h) \circ (\mathrm{pr}_r \circ h)(\Gamma_{i_j}) = (\mathrm{pr}_r \circ h)(\Gamma_{i_j}) = w_j$. Hence we get that $(\mathrm{pr}_r \circ h)(A_r) = A_r$.

Now if l_i is a line that does not pass through the point Q and Γ_i is the generator of l_i, then $h(\Gamma_i) = e$ and therefore $(\mathrm{pr}_r \circ h)(\Gamma_i) = \mathrm{pr}_r(e) = e$.

From this investigation, we get the following theorem.

Theorem 5.9 *Let Σ be a line arrangement and $\pi_1(\mathbb{C}^2 - \Sigma) \simeq (\bigoplus_{i=1}^n A_i) \oplus \mathbb{Z}^l$ where A_i is a free group. Then for any multiple point Q in the arrangement with k lines, namely, $\{l_1, \dots, l_k\}$, there exists r, $1 \le r \le n$,and a projection onto A_r, $\varphi_Q \colon G \to G$ such that $A_r = \langle \varphi_Q(\Gamma_1), \dots, \varphi_Q(\Gamma_k)\rangle \cong \mathbb{F}_{k-1}$. If l_j is a line which does not pass through the point, then $\varphi_Q(\Gamma_j) = e$.*

Moreover, if $p_1, \dots, p_m$ are the multiple points of Σ and n_i is the number of lines pass through the point p_i, then $G \cong \left(\bigoplus_{i=1}^m C_i\right) \oplus B$, where $C_i \cong \mathbb{F}_{n_i - 1}$. If l is a line which does not pass through p_i and let Γ be its corresponding generator, then $\mathrm{pr}_i(\Gamma) = e$ (where pr_i is the projection onto C_i).

6 Main theorems

Theorem 6.1 *Let $\Sigma \subseteq \mathbb{C}^2$ be a line arrangement which has no pair of parallel lines. Then if*

$$\pi_1(\mathbb{C}^2 - \Sigma) = \bigoplus_{i=1}^{r} A_i \oplus \mathbb{Z}^l,$$

where A_i are free groups. Then $\beta(\Sigma) = 0$.

Proof Assume by negation that $\beta(\Sigma) \neq 0$ which means that there is at least one cycle in the graph of Σ. Let us pick a minimal cycle in the following sense. The cycle contains r points, namely $\{p_1, \dots, p_r\}$ and p_i is connected in the cycle only to p_{i-1} and p_{i+1} (the indices are taken modulo r). In other words, a cycle with no subcycles.

By Theorem 5.9, we can write $G = (\bigoplus_{i=1}^r C_i) \oplus B_1$ where B_1 is not necessarily abelian. If l is a line that does not pass through the points in the cycle and let Γ be its corresponding generator, then $\mathrm{pr}_i(\Gamma) = e, 1 \le i \le r$. Define $N = \langle \Gamma_{X_1}, \dots, \Gamma_{X_t}, \Gamma_1 \cdots \Gamma_n \rangle$, where $\Gamma_{X_1}, \dots, \Gamma_{X_t}$ are the generators of lines that do not participate in an intersection point p_i, $1 \le i \le r$. Denote $Z = \Gamma_1 \cdots \Gamma_n$. Let $H := G/N$.

Let n_i be the number of lines passing through p_i.

On one hand, since we have $b := \sum_{i=1}^{r} (n_i - 1)$ lines participating in the cycle, then if we denote by $\Gamma_1, \dots, \Gamma_b$ the generators associated with the lines participating in the cycle, then $H \cong \langle \Gamma_1, \dots, \Gamma_b \mid \widetilde{R}, \widetilde{Z} \rangle$ where $\widetilde{R}$ is the relations which are derived from the original relations. Therefore, it is easy to see that $\operatorname{rank}(\bar{H}) \leq b - 1$.

On the other hand, for $1 \leq i \leq r$ and $1 \leq j \leq t$, $\operatorname{pr}_i(\Gamma_{X_j}) = e$ and also $\Gamma_1 \cdots \Gamma_n \in Z(G)$ so $\operatorname{pr}_i(\Gamma_1 \cdots \Gamma_n) = e$. Thus $\operatorname{pr}_i(N) = e$ and hence $C_i/N \cong C_i$. This implies that $H = G/N \cong \left(\bigoplus_{i=1}^{r} C_i \oplus B_1\right)/N$. Since $C_i \cong \mathbb{F}_{n_i - 1}$ and $\overline{C_i} \cong \mathbb{Z}^{n_i - 1}$, we have $\bar{H} = (\bigoplus_{i=1}^{r} \overline{C_i}) \oplus \overline{B_1/N}$ which implies that $\bar{H} = (\bigoplus_{i=1}^{r} \mathbb{Z}^{n_i - 1}) \oplus \overline{B_1/N} \cong \mathbb{Z}^b \oplus \overline{B_1/N}$. Thus $\operatorname{rank}(\bar{H}) \geq b$, a contradiction. □

We are now ready to prove the converse of Fan's theorem.

Theorem 6.2 *Let $\Sigma \subseteq \mathbb{CP}^2$ be a line arrangement. If*

$$\pi_1(\mathbb{CP}^2 - \Sigma) = \bigoplus_{i=1}^{r} A_i \oplus \mathbb{Z}^l,$$

where A_i are free groups, then $\beta(\Sigma) = 0$.

Proof Suppose that $\Sigma \subset \mathbb{CP}^2$ is an arrangement of complex projective lines such that $\pi_1(\mathbb{CP}^2 \setminus \Sigma)$ is isomorphic to a direct product of free groups. Let $L_0 \subset \mathbb{CP}^2$ be a complex projective line which is in general position to Σ. Then L_0 and Σ intersect at double points. Choose an arbitrary complex projective line of Σ and denote this line by L_1. Note that $\mathbb{C}^2 \cong \mathbb{CP}^2 \setminus L_0 \cong \mathbb{CP}^2 \setminus L_1$. By applying the product theorem of Oka and Sakamoto [17], we have

$$\begin{aligned}
\pi_1(\mathbb{CP}^2 \setminus (\Sigma \cup L_0)) &= \pi_1((\mathbb{CP}^2 \setminus L_1) \setminus ((\Sigma \setminus L_1) \cup L_0)) \\
&\cong \pi_1((\mathbb{CP}^2 \setminus L_1) \setminus (\Sigma \setminus L_1)) \oplus \pi_1((\mathbb{CP}^2 \setminus L_1) \setminus L_0) \\
&\cong \pi_1(\mathbb{CP}^2 \setminus \Sigma) \oplus \mathbb{Z}.
\end{aligned}$$

This calculation shows that $\pi_1(\mathbb{CP}^2 \setminus (\Sigma \cup L_0))$ is a product of free group. Note that

$$\pi_1(\mathbb{CP}^2 \setminus (\Sigma \cup L_0)) = \pi_1((\mathbb{CP}^2 \setminus L_0) \setminus (\Sigma \setminus L_0)).$$

Note that $\Sigma \setminus L_0$ is a union of complex lines in the complex plane $\mathbb{CP}^2 \setminus L_0$ such that no two lines of this arrangement are parallel.

By Theorem 6.1 $\beta(\Sigma \setminus L_0) = 0$, therefore $\beta(\Sigma) = \beta(\Sigma \setminus L_0) = 0$. □

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References

[1] **W A Arvola**, *The fundamental group of the complement of an arrangement of complex hyperplanes*, Topology 31 (1992) 757–765 MR1191377

[2] **A D R Choudary**, **A Dimca**, **Ş Papadima**, *Some analogs of Zariski's theorem on nodal line arrangements*, Algebr. Geom. Topol. 5 (2005) 691–711 MR2153112

[3] **D C Cohen**, **A I Suciu**, *The braid monodromy of plane algebraic curves and hyperplane arrangements*, Comment. Math. Helv. 72 (1997) 285–315 MR1470093

[4] **M Falk**, *Homotopy types of line arrangements*, Invent. Math. 111 (1993) 139–150 MR1193601

[5] **K-M Fan**, *On parallel lines and free groups* arXiv:0905.1178

[6] **K M Fan**, *Position of singularities and the fundamental group of the complement of a union of lines*, Proc. Amer. Math. Soc. 124 (1996) 3299–3303 MR1343691

[7] **K-M Fan**, *Direct product of free groups as the fundamental group of the complement of a union of lines*, Michigan Math. J. 44 (1997) 283–291 MR1460414

[8] **D Garber**, *On the fundamental groups of complements of plane curves*, PhD thesis, Bar-Ilan University (2001)

[9] **D Garber**, **M Teicher**, **U Vishne**, *π_1–classification of real arrangements with up to eight lines*, Topology 42 (2003) 265–289 MR1928653

[10] **M Hall, Jr**, *The theory of groups*, second edition, Chelsea Publishing Co., New York (1976) MR0414669 Reprint of the 1968 edition

[11] **T Jiang**, **S S-T Yau**, *Diffeomorphic types of the complements of arrangements of hyperplanes*, Compositio Math. 92 (1994) 133–155 MR1283226

[12] **D L Johnson**, *Presentations of groups*, London Math. Soc. Lecture Notes Ser. 22, Cambridge Univ. Press (1976) MR0396763

[13] **E R V Kampen**, *On the fundamental group of an algebraic curve*, Amer. J. Math. 55 (1933) 255–267 MR1506962

[14] **R C Lyndon**, **P E Schupp**, *Combinatorial group theory*, Ergebnisse der Math. und ihrer Grenzgebiete 89, Springer, Berlin (1977) MR0577064

[15] **W Magnus**, **A Karrass**, **D Solitar**, *Combinatorial group theory.Presentations of groups in terms of generators and relations*, revised edition, Dover, New York (1976) MR0422434

[16] **B Moishezon**, **M Teicher**, *Braid group technique in complex geometry. I. Line arrangements in* $\mathbb{C}\mathrm{P}^2$, from: "Braids (Santa Cruz, CA, 1986)", (J S Birman, A Libgober, editors), Contemp. Math. 78, Amer. Math. Soc. (1988) 425–555 MR975093

[17] **M Oka**, **K Sakamoto**, *Product theorem of the fundamental group of a reducible curve*, J. Math. Soc. Japan 30 (1978) 599–602 MR513072

[18] **P Orlik**, **H Terao**, *Arrangements of hyperplanes*, Grund. der Math. Wissenschaften 300, Springer, Berlin (1992) MR1217488

[19] **R Randell**, *The fundamental group of the complement of a union of complex hyperplanes*, Invent. Math. 69 (1982) 103–108 MR671654

[20] **S Wang**, **S S-T Yau**, *Rigidity of differentiable structure for new class of line arrangements*, Comm. Anal. Geom. 13 (2005) 1057–1075 MR2216152

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