On the Rozansky–Witten weight systems

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Ideas of Rozansky and Witten, as developed by Kapranov, show that a complex symplectic manifold X gives rise to Vassiliev weight systems. In this paper we study these weight systems by using $D(X)$, the derived category of coherent sheaves on X. The main idea (stated here a little imprecisely) is that $D(X)$ is the category of modules over the shifted tangent sheaf, which is a Lie algebra object in $D(X)$; the weight systems then arise from this Lie algebra in a standard way. The other main results are a description of the symmetric algebra, universal enveloping algebra and Duflo isomorphism in this context, and the fact that a slight modification of $D(X)$ has the structure of a braided ribbon category, which gives another way to look at the associated invariants of links. Our original motivation for this work was to try to gain insight into the Jacobi diagram algebras used in Vassiliev theory by looking at them in a new light, but there are other potential applications, in particular to the rigorous construction of the $(1+1+1)$–dimensional Rozansky–Witten TQFT, and to hyperkähler geometry.

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Introduction

Motivation

The *Kontsevich integral* is a beautiful and very powerful invariant of framed knots in S^3. It takes values in a certain graded algebra $\mathcal{A}$ of *Jacobi diagrams*, and is universal for the class of *Vassiliev (finite-type) invariants*, as well as determining all the *quantum invariants* (the Jones polynomial, etc.) associated to quantum groups. The definitive exposition is by Bar-Natan [3].

Over the last few years the theory of the Kontsevich integral has been considerably extended (see Le, Murakami and Ohtsuki [27] and Murakami and Ohtsuki [29]), resulting in a system of Kontsevich-like invariants for links, graphs and 3–manifolds possessing much of the functoriality of a topological quantum field theory. Despite these successes, basic questions about the topological interpretations of the Kontsevich invariant and of the algebra $\mathcal{A}$ itself remain largely unanswered.

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The standard way to study diagram spaces such as $\mathcal{A}$ is by means of *weight systems*, ie functions on it, which are most easily obtained from Lie algebras. A finite-dimensional Lie algebra $\mathfrak{g}$ with an invariant nondegenerate metric defines weight system homomorphisms from diagram spaces to spaces of *invariant tensors* on $\mathfrak{g}$; recognisable formulae often emerge from looking in this way at diagrammatic identities "at the level of Lie algebras", as for example in Bar-Natan, Garoufalidis, Rozansky and Thurston [4]. From this has emerged the idea, pursued in particular by Vogel [41], that the diagrams themselves form some kind of universal Lie-algebra-like structure.

In this paper we propose to study diagram algebras from an alternative point of view using *Rozansky–Witten weight systems* [36]. These arise from *complex symplectic manifolds*, according to Kapranov [20] and Kontsevich [24], and map diagram algebras to *Dolbeault cohomology groups* of such manifolds. Our original motivation for this study was to try to understand the extent to which diagrams behave like elements of cohomology; we were seeking to interpret $\mathcal{A}$ as some kind of ring of universal characteristic classes, and had already been studying certain diagrammatic formulae as if they were cohomological identities.

This is a reasonable point of view: after all, Kontsevich's formulation of *graph cohomology* [23] shows that indeed, diagrams may be thought of as representing elements of homology and cohomology, though this combinatorial framework affords little topological insight. Although Kontsevich has also given interpretations of graph cohomology via Gelfand–Fuchs cohomology and Lie algebras of formal Hamiltonian vector fields, we still hope that there is a more direct explanation for much of the theory. We would like to be able to view graph cohomology as the cohomology of some kind of interesting and meaningful geometrical classifying space (by analogy with *fatgraph* cohomology, which is the cohomology of the moduli space of Riemann surfaces), and then use the geometry of this space to give new explanations of the existence and properties of the knot and 3–manifold invariants. There is an obvious candidate, *outer space* [23], but it still seems rather too abstract for our purposes, and these goals remain unfulfilled. Fortunately, Rozansky–Witten theory is a fruitful subject to study in its own right.

In this paper we deal only with the nature of the Rozansky–Witten weight systems. That is, we are looking at diagrams "at the level of complex symplectic manifolds", and studying the analogies between Lie algebra and Rozansky–Witten weight systems, in a sense parallelling the paper [4]. An alternative focus would be to use the theory to derive results about hyperkähler geometry, in the manner of Hitchin and Sawon [19] and Nieper-Wißkirchen [30], but we will avoid this here. Likewise, though we touch here on the Rozansky–Witten link invariants, we will for the most part postpone the study of the associated topological invariants for a mythical sequel in which we would apply our techniques to set up the full $(1+1+1)$–dimensional Rozansky–Witten TQFT.

It is on the face of it very surprising that objects as disparate as Lie algebras and complex symplectic manifolds give rise to weight systems. The main point of our paper is to unify these two worlds, showing how to define and handle Rozansky–Witten weight systems in a way completely analogous to the Lie algebra ones. We show in fact that a complex symplectic manifold gives rise to something akin to a metric Lie algebra, and then investigate the ramifications of this analogy. The catch here is that this something is an object in a category other than the usual category of vector spaces; in fact, the category must be taken to be the *bounded derived category of coherent sheaves* on the manifold.

Now phrases like this used to strike terror into the hearts of the authors, and we presume some readers will also recoil slightly! But we are really convinced that the use of derived categories gives the most natural and elegant formulation of the Rozansky–Witten theory, and hope by our exposition to convince the reader likewise. An additional justification for their use is that in the construction of the $(1+1+1)$–dimensional Rozansky–Witten TQFT, the derived category should be the category associated to a circle.

It is also worth mentioning here a disadvantage of our approach, which is that the beautiful L_∞ structure described by Kapranov is thrown away. We would need this if we were interested in weight systems defined on higher graph cohomology, but these do not figure in the computation of the usual knot or TQFT invariants, so this is an acceptable loss.

A brief sketch of the results in this paper appears in the paper by the first author [34], and further questions appear in the paper by the first author and Sawon [35].

Overview of results

The derived category as a Lie algebra representation category Kapranov showed that one could use a certain L_∞–algebra structure to obtain the Rozansky–Witten weight systems: our approach is to work in the derived category and use Lie algebras type objects as in Chern–Simons theory. The first step is to identify the derived category as the representation category of a certain Lie algebra inside it.

Theorem 1 *Let X be a complex manifold. The shifted tangent bundle $T[-1]$ is a Lie algebra object in the bounded derived category $D(X)$, and $D(X)$ is the category of modules over $T[-1]$.*

To explain this, first note that a Lie algebra object in a additive symmetric tensor category means an object L in the category with a bracket morphism $L \otimes L \to L$ which satisfies suitable versions of the Jacobi and antisymmetry relations, so that a Lie algebra object in the category of vector spaces is a usual Lie algebra and a Lie algebra object in the category of graded vector spaces is a graded Lie algebra. A module M over L is then an object of the category with an action morphism $M \otimes L \to M$ satisfying an appropriate condition. By the statement that $D(X)$ is the category of modules over $T[-1]$ we mean that it acts canonically on every object in the category and every morphism is a module morphism: this is analogous to L being a Lie algebra object in its category of modules as the adjoint module.

Next we need to know a little about the derived category. This has as objects bounded chains complexes of coherent sheaves on X, so in particular for each coherent sheaf $\mathcal{E}$ on X there is the $\mathcal{E}[-i]$, consisting of $\mathcal{E}$ in degree i and zero elsewhere (we still write $\mathcal{E}$ for $\mathcal{E}[0]$). One fundamental fact about the derived category is that hom-sets can be identified with Ext groups, or equivalently, cohomology groups, so that

$$\mathrm{Hom}_{D(X)}(\mathcal{E}, \mathcal{F}[i]) \cong \mathrm{Ext}^i(\mathcal{E}, \mathcal{F}).$$

The next key ingredient is the Atiyah class $\alpha_{\mathcal{E}}$ for a coherent sheaf $\mathcal{E}$; this is a characteristic class which lives in $\mathrm{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \Omega)$, which we can identify as the hom-set $\mathrm{Hom}_{D(X)}(\mathcal{E} \otimes T[-1], \mathcal{E})$.

Thus the Atiyah class can be thought of as a morphism $\alpha_{\mathcal{E}}\colon \mathcal{E} \otimes T[-1] \to \mathcal{E}$. In particular the Atiyah class of the tangent bundle gives the Lie bracket $T[-1] \otimes T[-1] \to T[-1]$, and the other Atiyah classes $\alpha_{\mathcal{E}}$ give module maps.

Unfortunately these do not give the action of $T[-1]$ on every object of the derived category. So in fact we take the more elegant approach of realising the action of $T[-1]$ as a natural transformation α from $\mathrm{id} \otimes T[-1]$ to the identity functor of $D(X)$. This gives for every object A in the derived category a morphism $\alpha_A\colon A \otimes T[-1] \to A$, and naturality ensures that every morphism $A \to B$ intertwines the action on A and B.

The natural transformation $\alpha\colon \mathrm{id} \otimes T[-1] \to \mathrm{id}$ is obtained using an *integral transform*. It is a standard principle of "correspondences" that objects of $D(X \times X)$ define functors $D(X) \to D(X)$, and that morphisms of $D(X \times X)$ define natural transformations between them: in fact we get a functor from $D(X \times X)$ to the functor category $\mathrm{Fun}(D(X), D(X))$. Our natural transformation α is obtained from a morphism $\mathcal{O}_\Delta \otimes \pi^* T[-1] \to \mathcal{O}_\Delta$ in $D(X \times X)$ which is essentially one half of the Atiyah class of the structure sheaf of the diagonal.

Complex symplectic manifolds and invariant metrics A complex symplectic form is a nondegenerate holomorphic two-form, that is, an element of $H^0(X, \bigwedge^2 T^*)$ and we can identify this with a symmetric element of $\mathrm{Hom}_{D(X)}(T[-1] \otimes T[-1], \mathcal{O}_X[-2])$. This isn't quite an invariant metric on the Lie algebra object $L = T[-1]$: such a thing would be a symmetric L–module map $L \otimes L \to \mathbf{1}$, but we have an extra shift $[-2]$. To handle this we can work in the "extended derived category" $\widetilde{D}(X)$ whose hom-set $\mathrm{Hom}_{\widetilde{D}(X)}(A, B)$ is the graded group $\mathrm{Ext}^*(A, B)$, where the shift problem disappears. Thus:

Theorem 2 *Let X be a complex symplectic manifold. The shifted tangent bundle $T[-1]$ is a metric Lie algebra object in the extended bounded derived category $\widetilde{D}(X)$, and $D(X)$ is the category of modules over $T[-1]$.*

Symmetric and universal enveloping algebras The reader familiar with ideas of Vassiliev invariants will know that other Lie algebraic concepts such as the universal enveloping algebra, symmetric algebra, Poincaré–Birkoff–Witt isomorphism and Duflo isomorphism play important roles. We show that analogues of these makes sense for the Lie algebra object $T[-1]$ on any complex manifold. If we were working in an abelian category then the symmetric and universal enveloping algebras could be constructed as quotients or subobjects of the tensor algebra of the Lie algebra object, but as the derived category is not abelian, we have to work a bit harder. The symmetric algebra of $T[-1]$ is easily identifiable as $S = \bigoplus (\bigwedge^i T)[-i]$, the shifted exterior algebra on T. But the universal enveloping algebra is less obvious: we define $U = \pi_* \mathcal{H}om(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$, and prove that $D(X)$ can be thought of as the representation category of U also:

Theorem 3 *The object $U \in D(X)$ is an associative algebra object. There is a canonical map $L = T[-1] \to U$ with respect to which U is the universal enveloping algebra of L. The algebra U acts on all objects of $D(X)$ in a manner compatible with the action of $T[-1]$.*

The Poincaré–Birkhoff–Witt and Duflo isomorphisms have their analogues in this world. For standard Lie algebras, the natural symmetrisation map PBW: $S(\mathfrak{g}) \to U(\mathfrak{g})$ is an isomorphism of $\mathfrak{g}$–modules and hence induces an isomorphism on their invariant parts. The latter can be corrected by a strange automorphism of $S(\mathfrak{g})^{\mathfrak{g}}$ to give Duflo's algebra isomorphism between $S(\mathfrak{g})^{\mathfrak{g}}$ and $Z(\mathfrak{g}) = U(\mathfrak{g})^{\mathfrak{g}}$.

In our context, there is again an isomorphism of objects PBW: $S \cong U$ in $D(X)$. Our proof is an elaboration of ideas of Markarian [28]. The correct categorical way to think of invariants is as homomorphisms from the trivial object, which amounts in the derived category to cohomology. Thus the induced map on invariant parts gives an isomorphism between the polyvector field cohomology

$$\mathrm{HT}^*(X) = H^*(S) = H^*(X, \textstyle\bigwedge^* T)$$

and the Hochschild cohomology

$$\mathrm{HH}^*(X) = H^*(U) = \mathrm{Ext}^*(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$$

first demonstrated by Gerstenhaber and Schack. The analogue of Duflo's isomorphism between these algebras is Kontsevich's "theorem on a complex manifold". Although this isomorphism exists for all complex manifolds, in the case of complex symplectic manifolds it follows from the wheeling theorem of Bar-Natan, Le and Thurston [5].

A corollary of these theorems is the existence, given a complex symplectic manifold X, of sheaf-cohomology-valued Vassiliev weight systems defined on all the usual algebras of Jacobi diagrams, naturally compatible with operations such as gluing of legs, etc.

Ribbon categories and link invariants The theory of the Knizhnik–Zamolodchikov equation gives a way to produce an interesting ribbon category structure on the category of representations of $U\mathfrak{g} \otimes \mathbb{C}[\![\hbar]\!]$, which by Drinfel′d's work is equivalent to the category of representations of a quantum group. This result has an analogue in our context.

Theorem 4 *The category $\widetilde{D}(X)$ has a natural nonsymmetric ribbon tensor category structure when X is a complex symplectic manifold.*

Ribbon categories automatically define framed link invariants. The ones arising from $\widetilde{D}(X)$ agree with the invariants obtained by taking the Kontsevich integral and composing with the weight systems; they may be thought of as the "knot polynomial" type quantum invariants arising from complex symplectic manifolds. We do not however know of any analogue of Drinfel′d's theorem in this context.

Outline of the paper

Part of the intention of this paper is to make this material accessible to knot theorists, so there is much exposition of material that might be considered "well-known" to algebraic geometers.

The first two sections of the paper are an exposition of the "standard" approach to Rozansky–Witten weight systems.

In Section 1 we give a brief description of what weight systems are, and how they are obtained from finite-dimensional metric Lie algebras.

In Section 2 we describe, by analogy with Chern–Weil theory, the differential-geometric formulation of Rozansky–Witten invariants as integrals of suitable curvature forms. The original treatment of Rozansky and Witten used physics (path integrals) as a motivation and Riemannian geometry for the actual construction of weight systems for hyperkähler manifolds. We follow instead Kapranov's reworking in terms of hermitian differential geometry, which has the advantage of demonstrating that the construction does not actually depend on the hyperkähler metric, and will work for any complex symplectic manifold.

The next sections are essentially reformulations of the first two, introducing the language in which our theorems are going to be stated.

In Section 3 we reformulate the construction of weight systems from metric Lie algebras so that it generalises to metric Lie algebras in categories other than the category of vector spaces. This is all based on work of Vogel [41] and Vaintrob [40].

In Section 4 we explain the language of derived categories (first in a general way and then with specific reference to sheaf theory), which will be necessary in Section 5 when we reformulate the relevant differential geometry in terms of sheaf theory, following Kapranov. The key concept is the *Atiyah class*, the cohomological version of the curvature of a holomorphic bundle. In Section 6 we show how it gives a Lie bracket.

In Section 7 we explain various generalisations of the idea of a weight system to other graph algebras, and how these relate to Lie-theoretic concepts such as symmetric and universal enveloping algebras. In Section 8 we show how these concepts manifest themselves in the context of complex symplectic manifolds and how they give more interesting kinds of weight systems.

In Section 9 we show how to turn $D(X)$ into a ribbon category, thereby giving another way to explain the associated invariants of links. The paper concludes in Section 10 with a summary of the analogy between the world of Lie algebras and complex symplectic manifolds, which should extend to an analogy between Chern–Simons TQFT and Rozansky–Witten TQFT.

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1 Lie algebra weight systems

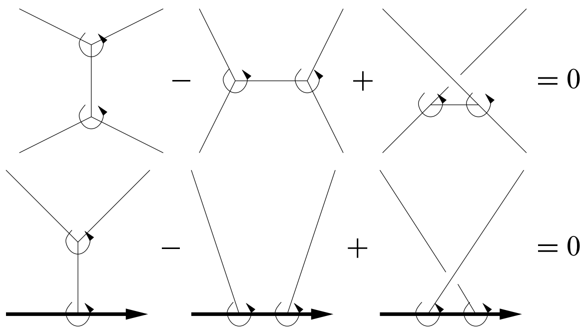
We begin with a brief description of the algebra $\mathcal{A}$ of Jacobi diagrams used in Vassiliev theory, and of how it is studied using weight systems arising from finite-dimensional metric Lie algebras. The more involved parts of the theory are deferred until Section 3. Apart from our grading convention, this is all standard; see Bar-Natan [3].

1.1 Jacobi diagrams

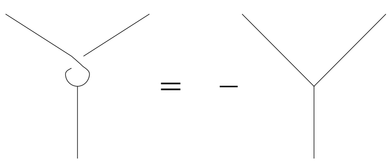
The Kontsevich integral is an invariant of framed oriented knots in S^3. It takes values in the complex, graded, algebra $\mathcal{A}$ of *Jacobi diagrams* defined as follows. Consider all isomorphism classes of connected trivalent graphs containing a preferred oriented circle and with a choice of cyclic orientation at each vertex not on the preferred circle. (The ones on the circle are canonically oriented because the circle is oriented.) Define $\mathcal{A}$ to be the complex span of such classes, quotiented by the vertex-antisymmetry, and IHX relations, pictured below. When the IHX relation involves an edge in the preferred circle it is called the STU relation. In this paper we will grade $\mathcal{A}$ by the total number of vertices of the graph, which is even. It is important to note that this is *twice* the conventional grading.

IHX:

STU:



Such graphs are usually described using planar pictures, in which the preferred circle is drawn as an external loop, and the rest of the graph is inscribed. One thinks of it as a graph "with legs" which is attached to the external circle. One useful function of such a planar projection is that every vertex may be given the canonical "anticlockwise" orientation, so the orientations need not be drawn explicitly. Any oriented abstract graph may be drawn in such a way. The antisymmetry relation then can be denoted by the following picture:



The space $\mathcal{A}$ is a commutative algebra, whose product # is given by connect-summing diagrams arbitrarily along their preferred oriented circles.

1.2 Lie algebra weight systems

To obtain numerical knot invariants from the Kontsevich integral, or simply to study the infinite-dimensional space $\mathcal{A}$, it is necessary to construct linear maps from $\mathcal{A}$ to some better understood rational vector space such as $\mathbb{Q}$. (In fact we will typically work with complex vector spaces in this paper.) Such a map is called a *weight system*.

The simplest way to obtain a weight system taking values in $\mathbb{C}$ is to pick a finite-dimensional Lie algebra $\mathfrak{g}$ with a metric b (a nondegenerate invariant symmetric bilinear form), and a finite-dimensional representation V of $\mathfrak{g}$. This information is completely encoded by the following three $\mathfrak{g}$–module maps:

$$a=[-,-]\colon \mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g},\qquad b\colon \mathfrak{g}\otimes\mathfrak{g}\to\mathbb{C},\qquad a_V\colon V\otimes\mathfrak{g}\to V.$$

Since b induces an isomorphism $\mathfrak{g}\cong\mathfrak{g}^*$, we may rewrite the Lie bracket as a skew trilinear form and think of it as a tensor $f\in\bigwedge^3\mathfrak{g}^*$. Additionally, we may "invert" the metric to define a Casimir element $c\in S^2\mathfrak{g}$. The action a_V is usually thought of as a tensor in $V^*\otimes V\otimes\mathfrak{g}^*$.

Now, a graph in $\mathcal{A}$ defines a way of contracting together these tensors to obtain a scalar in $\mathbb{C}$. Simply insert f at the internal vertices, a_V at the external vertices, and c on the internal edges, and contract the $\mathfrak{g}$–$\mathfrak{g}^*$ pairs and the V–V^* pairs as indicated by the graph. The vertex-orientation corresponds precisely to the information needed to insert f at a vertex; without it there would be a sign ambiguity. The symmetry of c means that no orientation on the edge is required. It is easy now to check that the relations in $\mathcal{A}$ are satisfied by this assignment, and that the weight system $w_{\mathfrak{g},V}\colon\mathcal{A}\to\mathbb{C}$ is well-defined.

2 Rozansky–Witten weight systems

In this section we explain, following Kapranov [20], a construction via hermitian differential geometry of weight systems from complex symplectic manifolds. We are not actually going to use this approach in the rest of the paper, but it's likely that at first sight it will be more illuminating than the later sheaf cohomology approach; in any case, Kapranov's paper is a little terse, and we feel it is worthwhile to expand on his construction. Actually, his demonstration of the Lie structure only works for Kähler manifolds, so by extending this to all complex manifolds we are tidying up a little too.

2.1 Chern–Weil theory

In this context, it is natural to consider Rozansky–Witten theory as an variant of Chern–Weil theory. Instead of using the curvature of a smooth connection on a smooth complex vector bundle to give invariants in the de Rham cohomology of the base manifold, we will use a the curvature of a hermitian connection on a holomorphic vector bundle to give invariants in the Dolbeault cohomology of the base complex symplectic manifold.

Recall that if E is a smooth complex vector bundle over the smooth manifold X and $\Omega^p(X;E)$ is the space of smooth p–forms with values in E, then there is no canonical choice of differential on $\Omega^*(X;E)$. But if we pick a smooth connection on E, that is a covariant derivative $\nabla\colon \Omega^0(X;E)\to\Omega^1(X;E)$, then we induce operators $\nabla\colon \Omega^p(X;E)\to\Omega^{p+1}(X;E)$. It is a standard fact that the composite ∇^2 is given by wedging with the *curvature two-form* $F\in\Omega^2(X;\mathrm{End}(E))$. One can then use $\mathrm{GL}(E)$–invariant polynomials in F to define cohomology classes which are independent of the choice of connection. When the bundle E has rank r these polynomials are spanned by the functions $F\mapsto \mathrm{tr}(F^d)$, for $0\le d\le r$. The resulting cohomological invariants of E, suitably normalised, define its Chern classes modulo torsion; more precisely, the class

$$\mathrm{ch}_d(E)=\left[\frac{1}{d!}\,\mathrm{tr}\left(\frac{-F}{2\pi i}\right)^d\right]\in H^{2d}(X;\mathbb{Q})$$

is the d–th part of the Chern character of E.

For a holomorphic bundle E on a complex manifold X there is a preferred class of connections coming from smooth hermitian metrics on the bundle. These define curvature forms of type $(1,1)$. Now an $\mathrm{End}(E)$–valued $(1,1)$–form can also be thought of as a $(T^*\otimes\mathrm{End}(E))$–valued $(0,1)$–form, where T^* is the holomorphic cotangent bundle of X. After this identification we are free to use *more complicated operations* to combine the curvature with itself (as well as with the curvature of the holomorphic tangent bundle and a holomorphic symplectic form, if available), because the curvature now has three tensorial "indices" rather than the original two. The different possible combinations, which replace the invariant polynomials used above, are in fact parametrised by Jacobi diagrams such as those defining $\mathcal{A}$.

2.2 Curvature of a holomorphic bundle

In order to fix the notation, let us recall the basics of complex differential geometry. If X is a complex manifold then one may decompose the complexified tangent bundle into holomorphic and antiholomorphic parts: $T_{\mathbb{R}}X\otimes\mathbb{C}\cong T\oplus\overline{T}$. The exterior differential likewise splits as $d = \partial + \overline{\partial}$ and the complexified de Rham complex $(\Omega^*(X;\mathbb{C}), d)$ may be refined to obtain the Dolbeault complex $(\Omega^{*,*}(X;\mathbb{C}), \overline{\partial})$, with cohomology $H^{*,*}_{\overline{\partial}}(X;\mathbb{C})$. (If X has a Kähler metric then these Dolbeault cohomology groups may be identified with subspaces of the complexified de Rham cohomology of X, via the Hodge decomposition, but we will not usually assume X is Kähler in what follows.)

Unlike in the smooth case, if E is a holomorphic vector bundle on X, then there is a canonical operator $\overline{\partial}$ on the spaces of smooth E–valued forms $\Omega^{p,q}(X;E)$. To define it, write a form locally in terms of a basis of holomorphic sections of E, and apply the usual Dolbeault $\overline{\partial}$ operator to the smooth-form coordinates; one obtains a complex with cohomology $H^{*,*}_{\overline{\partial}}(X;E)$.

A smooth connection on a holomorphic bundle E, thought of as a covariant derivative $\nabla\colon \Omega^0(X;E) \to \Omega^1(X;E)$, splits into pieces of type $(1,0)$ and type $(0,1)$. It is said to be *compatible* with the holomorphic structure on E if its $(0,1)$ part equals the canonical $\overline{\partial}$ operator of E. One may then write $\nabla = \overline{\partial} + \nabla^{1,0}$, the last term being a *connection of type* $(1,0)$, which satisfies a version of the usual Leibniz rule in which ∂ replaces d. The resulting curvature two-form $F \in \Omega^2(X;\mathrm{End}(E))$ has no part of type $(0,2)$, because $F = (\overline{\partial} + \nabla^{1,0})^2$ and $\overline{\partial}^2 = 0$. In local coordinates, if one writes the covariant derivative operator ∇ as $d + A$ for some 1–form $A \in \Omega^1(X;\mathrm{End}(E))$, this compatibility amounts to saying that A is of type $(1,0)$.

If E has a smooth hermitian metric h then we may further require that ∇ is compatible with h by imposing that for all sections $s,t \in \Omega^0(X;E)$,

$$dh(s,t) = h(\nabla s, t) + h(s, \nabla t).$$

Computing d of this formula using a basis of local covariant-constant sections shows that the curvature $F = \nabla^2$ is of type $(1,1)$ (and in fact purely imaginary). Therefore $(\nabla^{1,0})^2 = 0$ and we can write the operator F as $\nabla^{1,0}\overline{\partial} + \overline{\partial}\nabla^{1,0}$ or even as $\nabla\overline{\partial} + \overline{\partial}\nabla$. This second form will be used below. Varying the hermitian form alters the form F by a $\overline{\partial}$–coboundary.

If two bundles E_1, E_2 have connections, then there is an induced connection on $E_1 \otimes E_2$ given by the Leibniz rule, and the resulting curvature is

$$F_{E_1\otimes E_2} = F_{E_1} \otimes \mathrm{id} + \mathrm{id} \otimes F_{E_2}.$$

Similarly, a connection on a bundle E induces one on its dual E^* by the formula

$$\langle \nabla\phi, s\rangle + \langle \phi, \nabla s\rangle = d\langle \phi, s\rangle,$$

where s is a section of E, ϕ is a section of E^* and the brackets indicate the contractions to complex valued forms on X. It is useful to think in terms of operators on the space of sections of E and write $F_{E^*}\phi = -\phi \circ F_E$.

The *Bianchi identity* is often written $\nabla F = 0$. The operator $\nabla = \nabla_{\mathrm{End}(E)}$ is the covariant derivative on sections of the bundle $\mathrm{End}(E)$ induced by the original connection ∇ on E. As an operator on $\Omega^0(X;E)$, $\nabla_{\mathrm{End}(E)} F = \nabla_E \circ F - F \circ \nabla_E$, so its vanishing amounts to nothing more than the fact that the operators $F = \nabla_E^2$ and ∇_E commute. In the holomorphic context, the $(1,0)$ part of the identity becomes the equation $\overline{\partial} F = 0$.

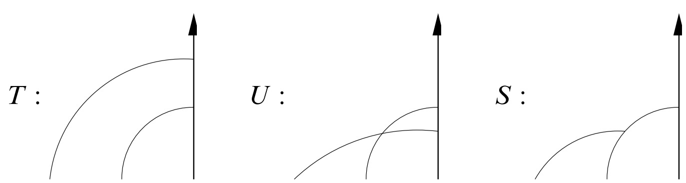
2.3 Complex manifolds and the Jacobi identity

Kapranov discovered that the curvature of a holomorphic bundle on a complex manifold satisfies a kind of Jacobi identity. This fact (which has nothing to do with hyperkähler or complex symplectic geometry) is absolutely basic to Rozansky–Witten theory.

Suppose E is a holomorphic bundle on X, with associated Dolbeault operator $\overline{\partial}_E$. Pick a smooth hermitian metric on E with associated connection ∇_E and curvature form $F_E \in \Omega^{1,1}(X, \mathrm{End}(E))$. Do the same for the holomorphic tangent bundle T. We will from now on drop the redundant "X" from notation such as $\Omega^*(X;E)$.

We want to think of the curvature as living in a slightly different space. Let Θ denote any identification of the form $\Omega^{p,q}(-) \cong \Omega^{0,q}(\bigwedge^p(T^*) \otimes -)$. Here we think of the right hand side as a subspace of $\Omega^{0,q}((T^*)^{\otimes p} \otimes -)$, and explicitly (this will affect signs in an inevitably messy way) set $\Theta(d\overline{z}^I \wedge dz^J \otimes s) = d\overline{z}^I \otimes dz^J \otimes s$. Define $R_E = \Theta F_E \in \Omega^{0,1}(T^* \otimes \mathrm{End}(E))$; this form will also be referred to as the curvature. Since F_E is $\overline{\partial}$–closed, so is R_E, as the appropriate $\overline{\partial}$ operators commute with Θ. Define R_T similarly.

Kapranov's result is that a certain three-term quadratic relation in the tensors R_E, R_T is a $\overline{\partial}$–coboundary. At the level of cohomology it will become the *STU relation* of Vassiliev theory, and in the special case $E = T$ the *IHX relation*. Define three elements of $\Omega^{0,2}(T^* \otimes T^* \otimes \mathrm{End}(E))$ called $R_E \circ_S R_T$, $R_E \circ_T R_E$, $R_E \circ_U R_E$ by taking the appropriate wedge products of 1–forms and contracting indices according to the three graphs shown below.



Explicitly, applying these elements to sections t_1, t_2 and e gives elements of $\Omega^{0,2}(E)$ which may be written $R_E(R_T(t_1,t_2),e)$, $R_E(t_1, R_E(t_2,e))$, $R_E(t_2, R_E(t_1,e))$.

Lemma 2.1 (STU relation) *If E is a holomorphic bundle over the complex manifold X, then in $\Omega^{0,2}(T^* \otimes T^* \otimes \mathrm{End}(E))$ we have the coboundary formula*

$$R_E \circ_T R_E + R_E \circ_U R_E + R_E \circ_S R_T = -\bar\partial(\Theta \nabla R_E).$$

Proof Via the Leibniz formula we obtain the operator identity

$$F_{T^* \otimes \mathrm{End}(E)} = F_{T^* \otimes E^* \otimes E} = F_{T^*} \otimes \mathrm{id} \otimes \mathrm{id} + \mathrm{id} \otimes F_{E^*} \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{id} \otimes F_E,$$

so that composing with R_E and evaluating on sections t, e of T, E we have in $\Omega^{1,2}(E)$ the identity

$$(F_{T^* \otimes \mathrm{End}(E)} R_E)(t, e) = -R_E(F_T t, e) - R_E(t, F_E e) + F_E(R_E(t, e)).$$

(The signs come from the curvature of the dual bundle; switching the order of 2–form and 1–form does not give signs.) Now applying Θ (carefully) to obtain an identity in $\Omega^{0,2}(T^* \otimes T^* \otimes \mathrm{End}(E))$ gives

$$-\Theta(F_{T^* \otimes \mathrm{End}(E)} R_E) = R_E \circ_S R_T + R_E \circ_T R_E + R_E \circ_U R_E.$$

The result now follows on rewriting the left-hand side using $FR_E = (\bar\partial\nabla + \nabla\bar\partial) R_E = \bar\partial(\nabla R_E)$ (because R_E is $\bar\partial$–closed) and the fact that Θ commutes with $\bar\partial$. □

Just as important from the point of view of constructing weight systems is the *symmetry* of the curvature form R_T of the tangent bundle. In fact there are two separate symmetries: the first comes from considering the torsion of the connection on T, while the second appears in the presence of a holomorphic symplectic form, and will be studied in the next section. Kapranov assumes in his paper that the hermitian metric on X is Kähler, so that the torsion of ∇_T vanishes (this is one definition of a Kähler metric, in fact). But the next proposition shows that vanishing of the torsion is unnecessary; one no longer has an exact symmetry, but symmetry modulo coboundaries, which is still perfectly acceptable to us.

If ∇ is a smooth connection on the real tangent bundle $T_{\mathbb{R}}$ of a smooth manifold, then the *torsion* is a 2–form with values in $T_{\mathbb{R}}$ given by the formula

$$\tau(t_1, t_2) = \nabla_{t_1} t_2 - \nabla_{t_2} t_1 - [t_1, t_2].$$

For a complex manifold with a smooth hermitian connection ∇ on its holomorphic tangent bundle T, we can tensor over $\mathbb{R}$ with $\mathbb{C}$ to obtain a connection all of $T[-1]_{\mathbb{C}} = T \oplus \bar{T}$, and use the same formula to define the torsion $\tau \in \Omega^2(T_{\mathbb{C}})$. The part $\tau^{1,0}$ with values in T turns out to be of type $(2, 0)$.

Proposition 2.2 (Partial symmetry) *The curvature form R_T is symmetric in its two inputs, up to a $\overline{\partial}$–coboundary. Specifically,*

$$R_T - \sigma \circ R_T = \overline{\partial}(\Theta\tau^{1,0})$$

where $\Theta\tau^{1,0} \in \Omega^{0,0}(T^* \otimes T^* \otimes T)$ is a version of the torsion and σ is the permutation of the two T^* factors.

Proof For arbitrary smooth sections t_1, t_2, t_3 of $T_{\mathbb{C}}$ we have the straightforward identity

$$\sum F(t_1, t_2)t_3 = \sum d_{t_1}\tau(t_2, t_3) + \sum \tau(t_1, [t_2, t_3]) = (\nabla\tau)(t_1, t_2, t_3),$$

all sums being over cyclic permutations of the three vector fields. (In the Levi-Civita case, the vanishing of the right-hand side implies one of the symmetries of the Riemann curvature.) Now assume t_1, t_3 are of type $(1,0)$ while t_2 is of type $(0,1)$, and look at the type $(1,0)$ part of this equation:

$$F(t_1, t_2)t_3 + F(t_2, t_3)t_1 = (\nabla\tau^{1,0})(t_1, t_2, t_3) = (\overline{\partial}\tau^{1,0})(t_1, t_2, t_3).$$

Applying Θ we can have an identity in $\Omega^{0,1}(T^* \otimes T^*)$ which when evaluated on t_2, t_1, t_3 says that

$$R_T(t_2)(t_1, t_3) - R_T(t_2)(t_3, t_1) = \Theta(\overline{\partial}\tau^{1,0})(t_2)(t_1, t_3). \qquad \square$$

Remark 2.3 The exterior product of forms followed by contraction with R_T defines a degree-one bilinear product on the Dolbeault complex $\Omega^{0,*}(T)$. This operation satisfies the graded Jacobi identity up to a coboundary, and in the Kähler case it is exactly symmetric, making it an "odd Lie bracket up to homotopy". Kapranov shows that together with higher-order derivatives of the curvature, it makes the Dolbeault complex $\Omega^{0,*}(T)$ into an L_∞*–algebra*. In the non-Kähler case, the above lemma suggests that there is an even weaker kind of infinity-structure in which there are also higher homotopies (controlled by derivatives of the torsion) arising from noncommutativity of the bracket. Such structures are beautiful and interesting, but we will not need them in this paper.

2.4 Complex symplectic manifolds

As we have seen, the curvature of a holomorphic vector bundle has a kind of intrinsic Jacobi identity property. To construct weight systems we also need a *metric* of some kind, and in keeping with the "switch of statistics" that has replaced a skew Lie bracket by a symmetric curvature tensor, we seek a *skew* rather than symmetric nondegenerate form.

A *complex symplectic manifold* is an (even-dimensional) complex manifold X^{2n} together with a nondegenerate holomorphic two-form $\omega \in \Omega^0(\bigwedge^2 T^*)$. The nondegeneracy implies that ω defines an isomorphism of holomorphic bundles $T \cong T^*$. (An obvious topological obstruction to existence is therefore the vanishing of the odd rational Chern classes of X.)

Using this isomorphism, we can convert the curvature $R_T \in \Omega^{0,1}(T^* \otimes \mathrm{End}(T))$ into a form $C_T \in \Omega^{0,1}(T^* \otimes T^* \otimes T^*)$:

$$C_T(t_1, t_2, t_3) = \omega(R_T(t_1, t_2), t_3).$$

Since ω is holomorphic, C_T too is $\overline{\partial}$–closed.

Lemma 2.4 (Full symmetry of curvature) *The curvature form*

$$C_T \in \Omega^{0,1}(T^* \otimes T^* \otimes T^*)$$

of a complex symplectic manifold is symmetric in its three factors, up to $\overline{\partial}$–coboundaries.

Proof We already have such symmetry in the first two factors. To show symmetry in the second and third, consider $F_{T^*\otimes T^*}\omega$. By the Leibniz rule and the rule for curvature of dual bundles we can write

$$F_{T^*\otimes T^*}\omega = -\omega \circ (F_T \otimes \mathrm{id}_T) - \omega \circ (\mathrm{id}_T \otimes F_T).$$

Applying Θ and rewriting this identity in terms of elements of $\Omega^{0,1}(T^* \otimes T^* \otimes T^*)$ gives

$$\Theta(\overline{\partial}(\nabla\omega)) = -C_T + C_T \circ \sigma_{23},$$

where σ_{23} is the permutation of the last two inputs. The left-hand side is the coboundary $\overline{\partial}(\Theta(\nabla\omega))$ and so the symmetry is proved. (Note that ∇ came from an arbitrary choice of hermitian metric on T; there is no reason why $\nabla\omega$ should be zero.) □

2.5 Rozansky–Witten weight systems

With the above preliminaries completed, we can now describe briefly the construction of weight systems on the space $\mathcal{A}$.

Theorem 2.5 *If X is a complex symplectic manifold and E is a holomorphic vector bundle on X, then there is a weight system*

$$RW_{X,E}\colon \mathcal{A} \to H^{0,*}_{\overline{\partial}}(X)$$

taking values in the Dolbeault cohomology of X.

Proof If Γ is a $2v$–vertex closed trivalent graph with *ordered vertices* and *oriented edges*, then one can obtain a form in $\Omega^{0,2v}(X)$ by a procedure like that of Section 1: wedge/tensor one copy of C_T for each vertex of Γ, and contract tensorially with one copy of ω^{-1} (that is, ω converted into a holomorphic section of $T \otimes T$) for each edge of Γ. Since all the elements in the construction are $\overline{\partial}$–closed, so is the result, by the Leibniz formula. There is clearly a choice of how one attaches C_T at a vertex – a choice of correspondence between the three legs and the three copies of T^* – but differences alter the resulting form by a coboundary, because of the symmetry property and the Leibniz formula. The choice of hermitian metric used to define C_T similarly only affects the result by a coboundary, so that the result is a well-defined element of $H^{0,2v}_{\overline{\partial}}(X)$. This basic construction clearly generalises to the case where the graph has an oriented Wilson loop: the form R_E is inserted at the vertices on the loop, which are canonically oriented.

Reversing the orientation of any edge or swapping the order of two vertices negates this element, because ω^{-1} and the cup product of 1–forms are skew. (The following example may help: If α, β are 1–forms with values in vector space V, W, then $\alpha \wedge \beta = -\sigma(\beta \wedge \alpha)$, where σ is the usual permutation. In particular $\alpha \wedge \alpha = -\sigma(\alpha \wedge \alpha)$; "$\alpha$ anticommutes with itself".) Therefore the map is really well-defined on *oriented* graphs, where an orientation is an ordering of the vertices and an orientation of the edges, considered up to an even number of transpositions and reversals. The remarkable fact is that this notion of orientation is canonically isomorphic to the standard convention (from Section 1) on Jacobi diagrams, in which each vertex has a cyclic ordering of its legs.

To see this, let V and E be, respectively, the sets of vertices and edges of Γ. Let F be the set of all flags (half-edges) of Γ, and for each vertex v and for each edge e let F_v and F_e be the obvious two- and three-element sets of incident flags. For any set S, use the notation $\mathrm{Det}(S)$ for the top exterior power $\mathrm{Det}(\mathbb{R}^S)$, so that orienting a vertex or an edge in the usual sense amounts to orienting the appropriate 1–dimensional vector space $\mathrm{Det}(F_v)$ or $\mathrm{Det}(F_e)$. Orientations of graphs under the two different conventions are measured by the spaces $\mathrm{Det}(V) \otimes \bigotimes_e \mathrm{Det}(F_e)$ and $\bigotimes_v \mathrm{Det}(F_v)$.

The isomorphism now follows by combining three simple natural (equivariant) isomorphisms: (i) $\mathrm{Det}(F) \cong \mathrm{Det}(V) \otimes \bigotimes_v \mathrm{Det}(F_v)$; (ii) $\mathrm{Det}(F) \cong \bigotimes_e \mathrm{Det}(F_e)$; and (iii) Det^2 is canonically trivial. The first two isomorphisms come from concatenating triples of flags according to the vertex order, or pairs of flags according to an arbitrary (irrelevant) edge order.

The fact that the construction respects the IHX and STU relations now follows from the earlier proposition about the Jacobi identity for the curvature. □

Remark 2.6 This last check is actually quite nasty, because the two different orientation conventions we are considering do not agree *locally*, and the equivalence between them is not so straightforward even *globally*. The categorical approach we adopt in the second half of the paper has a technical advantage in that it matches the correct orientation conventions *locally*, bypassing this annoying problem.

2.6 Examples

In many ways the best examples of complex symplectic manifolds are the *hyperkähler manifolds*, which were the subject of Rozansky and Witten's original work. A hyperkähler manifold is a real $4n$–dimensional manifold with a Riemannian metric of holonomy $\mathrm{Sp}(n)$. Because this group is contained in $\mathrm{GL}(n,\mathbb{H})$, one can introduce three parallel (which implies integrable) almost complex structures I, J, and K satisfying the usual quaternionic relation $IJK=-1$. Any imaginary unit quaternion q now defines a complex structure (for which the metric is Kähler, with Kähler form ω_q) and which possesses a holomorphic symplectic form: one only needs to check for example that the complex two-form $\omega=\omega_J+i\omega_K$ is I–holomorphic.

There is a partial converse: a *compact* complex symplectic manifold which is Kähler has a hyperkähler metric, by Yau's solution of the Calabi conjecture; see Beauville [6]. (This is a hard analytical existence theorem, and there is no known simple formula for the metric.) Kapranov's approach is therefore only really more general than Rozansky and Witten's if we are prepared to consider complex symplectic manifolds which are noncompact, non-Kähler, or both. There are a few compact non-Kähler examples due to Beauville and Guan [15], but there are plenty of noncompact hyperkähler manifolds coming from complex Lie group coadjoint orbits, geometric moduli spaces, etc. (See Hitchin [18].)

From the point of view of Vassiliev invariants, the *compact* case is (at least initially) the most interesting, because for a compact complex symplectic manifold X of real dimension $4n$, one can obtain *scalar*-valued weight systems of degree $2n$. To do this, integrate the invariants lying in $H^{0,2n}_{\bar\partial}(X)$ against the holomorphic volume form $\omega^n\in H^{2n,0}_{\bar\partial}(X)$. Further, in the *hyperkähler* case, Sawon [37] used the interplay between the Riemannian and hermitian constructions to show that these numbers are invariant under deformations of the hyperkähler metric and of the complex structure on X. He also performed some explicit calculations.

The current list of known compact hyperkähler manifolds is not very long. In dimension four, the K3 surface and 4–torus are the only examples. Each of these generates, via its Hilbert schemes of points (desingularised versions of its symmetric products), an infinite family of further examples. These are all *irreducible*, having holonomy not contained in a proper subgroup of $\mathrm{Sp}(n)$, and in particular not being products of lower dimensional hyperkähler manifolds. The only other known irreducible examples were both constructed by O'Grady [31; 32].

The relative paucity of examples – two countable families and some exceptions – might therefore seem to undermine the scope of the Rozansky–Witten weight systems. But in fact if one looks to Lie algebras one finds exactly the same situation – the two series of types A and BCD, and a few exceptions! In this sense there are "at least as many" Rozansky–Witten weight systems as Lie algebra ones. An obviously important issue is whether the Rozansky–Witten weight systems are really *new*, lying outside the span of the Lie algebra ones or not. Because of the difficulties in explicit calculation, we don't yet know the answer to this.

3 Lie algebra weight systems revisited

In this section we describe an alternative category-theoretic approach to the construction of weight systems from metric Lie algebras. It was introduced by Vogel [41] and Vaintrob [40], whose original motivation was to handle the weight systems arising from metric Lie superalgebras.

For such an algebra, the tensors f and b used in Section 1.2 have both skew and symmetric parts, leading to incompatibility with the standard orientation convention for Jacobi diagrams. The problem can be fixed by picking a direction in the plane and representing Jacobi diagrams always as Morsified planar graphs, rather than as abstract graphs. The approach leads inevitably to the idea of constructing weight systems from metric Lie algebras in any category for which the notion makes sense, and not just in the category of (super)vector spaces. We will justify all this abstract nonsense later in the paper by constructing interesting *examples* of such categories and Lie algebras.

3.1 Symmetric tensor categories

Here we will recall the standard definitions of symmetric tensor categories. For more detail see Bakalov and Kirillov [2], Chari and Pressley [9] or Kassel [22].

A category $\mathcal{C}$ is a *tensor* (or *monoidal*) category if it comes with a functor $\otimes\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ whose associativity is implemented by a natural isomorphism $\Phi\colon \otimes\circ(\otimes\times\mathrm{id})\to\otimes\circ(\mathrm{id}\times\otimes)$ satisfying the pentagon identity, and has a unit object 1 for tensor product, again with appropriate natural isomorphisms. We will for the moment ignore all these isomorphisms notationally, pretending that $\mathcal{C}$ is strictly associative, ie that these isomorphisms are equalities.

A symmetric tensor category is defined as follows. Let σ be the standard flip functor $\mathcal{C}\times\mathcal{C}\to\mathcal{C}\times\mathcal{C}$; $A\otimes B\mapsto B\otimes A$. The tensor category $\mathcal{C}$ is *symmetric* if there is a natural isomorphism $\tau\colon \otimes\to\otimes\circ\sigma$ – giving an isomorphism $\tau_{A,B}\colon A\otimes B\to B\otimes A$ for A, B objects of $\mathcal{C}$ – satisfying $\tau_{B,A}\circ\tau_{A,B}=\mathrm{id}$ and satisfying the hexagon relation

$$\tau_{A,B\otimes C}=(\mathrm{id}_B\otimes\tau_{A,C})\circ(\tau_{A,B}\otimes\mathrm{id}_C).$$

The hexagon would be more visible if we hadn't dropped the associators from the notation. The natural isomorphism τ is sometimes called the *symmetry*. The standard example to keep in mind here is the category of *supervector spaces*, which is symmetric but in a nontrivial way; the isomorphism τ will handle all the signs for us.

The notion of *duality* between objects in a tensor category $\mathcal{C}$ is a little tricky. The basic definitions are abstracted from properties of finite-dimensional vector spaces, but a little more is required in order to control double duals properly. An object A^* is a *right dual* of an object A if there is a *right evaluation* $\epsilon_A\colon A^*\otimes A\to 1$ and a *right co-evaluation* $\iota_A\colon 1\to A\otimes A^*$ which satisfy

$$(\mathrm{id}_A\otimes\epsilon_A)\circ(\iota_A\otimes\mathrm{id}_A)=\mathrm{id}_A$$
$$(\epsilon_A\otimes\mathrm{id}_{A^*})\circ(\mathrm{id}_{A^*}\otimes\iota_A)=\mathrm{id}_{A^*}\,.$$

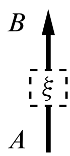
Such an object is unique up to a canonical isomorphism. We can similarly define a left dual ${}^*\!A$ with structural maps $\iota'_A\colon \mathbf{1}\to {}^*\!A\otimes A$ and $\epsilon'_A\colon A\otimes{}^*\!A\to\mathbf{1}$. A *rigid* tensor category is one in which all objects have left and right duals. This is enough to permit the construction of traces $\mathrm{Hom}(A,A)\to\mathrm{Hom}(\mathbf{1},\mathbf{1})$ on endomorphisms of any object, the construction of adjoints of morphisms, and identifications such as $\mathrm{Hom}(A\otimes B,C)\cong\mathrm{Hom}(A,C\otimes B^*)$.

Most of the categories we will use in this paper will be at least *additive* (and probably $\mathbb{C}$*–linear*), having abelian groups (or complex vector spaces) for morphism sets, bilinear composition, a direct sum operation $\oplus\colon\mathcal{C}\times\mathcal{C}\to\mathcal{C}$ and a zero object 0.

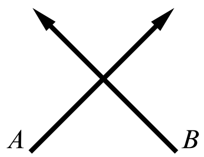
3.2 Penrose calculus

An important tool is Penrose's diagrammatic representation of the structure of a tensor category by planar pictures. A tensor product of objects is represented by a collection of labelled dots on a horizontal level; a morphism from one such to another is represented by drawing, inside a horizontal strip whose top and bottom edges are labelled appropriately, a box, labelled with the name of the morphism, and connected by strings from its top and bottom edges to the object dots. Composition of morphisms is represented by concatenation of diagrams moving up the page; tensor product of morphisms by horizontal juxtaposition. (If we were not assuming strict associativity then bracketings of objects and explicit associator morphisms would also be required.)

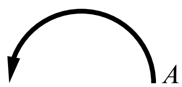
Special structural morphisms in the category are represented using special pictures as a short-hand for labelled boxes. The identity morphism on an object is always represented by a vertical arc labelled with that object, and the other possible structural morphisms are pictured below. The point of using these particular pictures is of course that the rather complicated algebraic *relations* satisfied by the structural morphisms now correspond to natural topological identities.



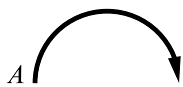
$\xi\colon A \to B$
(a morphism)



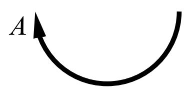
$\tau\colon A \otimes B \to B \otimes A$
(the symmetry)



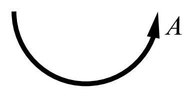
$\epsilon_A\colon A^* \otimes A \to \mathbf{1}$



$\epsilon'_A\colon A \otimes A^* \to \mathbf{1}$



$\iota_A\colon \mathbf{1} \to A \otimes A^*$



$\iota'_A\colon \mathbf{1} \to A^* \otimes A$

3.3 Lie algebras and modules

Here we take the usual definitions of a Lie algebra and a Lie algebra module and abstract them from the category of vector spaces to an arbitrary additive symmetric tensor category.

Let $\mathcal{C}$ be an additive symmetric tensor category. A *Lie algebra* in $\mathcal{C}$ is an object L equipped with a *bracket* morphism $\alpha\colon L \otimes L \to L$ which is skew-symmetric and satisfies the Jacobi identity:

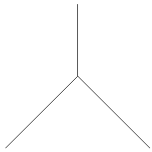
$$\alpha + \alpha \circ \tau = 0,$$
$$\alpha \circ (\alpha \otimes \mathrm{id}) + \alpha \circ (\alpha \otimes \mathrm{id}) \circ \tau_{123} + \alpha \circ (\alpha \otimes \mathrm{id}) \circ \tau_{321} = 0,$$

where τ_{123} and τ_{321} denote the actions on $L^{\otimes 3}$ of the three-cycles in the symmetric group S_3. Note that addition of morphisms makes sense because $\mathcal{C}$ is additive.

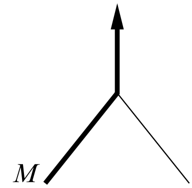
A (right) *module* over such a Lie algebra is an object M together with an *action* morphism $\alpha_M\colon M \otimes L \to M$ satisfying the identity

$$\alpha_M \circ (\mathrm{id} \otimes \alpha) = \alpha_M \circ (\alpha_M \otimes \mathrm{id}) - \alpha_M \circ (\alpha_M \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau).$$

Pictorially, the bracket and action are represented by the following diagrams, in a way that turns the above identities into the antisymmetry, IHX and STU relations.



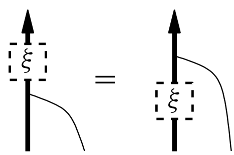
$\alpha\colon L \otimes L \to L$



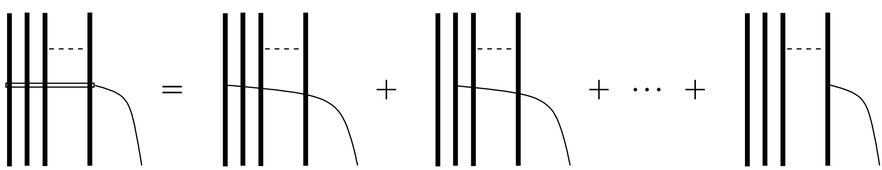
$\alpha_M\colon M \otimes L \to M$

Note any right module M can be given a natural left module structure $\check{\alpha}_M\colon L \otimes M \to M$ by $\check{\alpha}_M = -\alpha_M \circ \tau_{L,M}$.

An *L–module morphism* is a morphism $\xi\colon M \to N$ between L–modules such that $\xi \circ \alpha_M = \alpha_N \circ (\xi \otimes \mathrm{id})$. Pictorially this is shown below. The collection of L–modules and L–morphisms form a category mod–L.



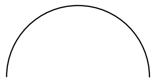
The *tensor product* of two L–modules is an L–module under a Leibniz rule such as $\alpha_{M\otimes N} = (\alpha_M \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau) + \mathrm{id} \otimes \alpha_N$, and therefore mod–$L$ is a tensor-category. The action on a tensor product is defined and notated as shown below. The crossings have been drawn in a slightly non-Morse way here, but we hope that the meaning is clear: they are τ morphisms forming an essential part of the correct definition of the action on tensor products.



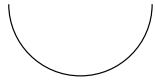
A *metric Lie algebra* is a Lie algebra equipped with an abstracted version of a nondegenerate symmetric invariant bilinear form. Thus, it comes with a *metric* morphism $\beta\colon L \otimes L \to \mathbf{1}$ and a *Casimir* $\gamma\colon \mathbf{1} \to L \otimes L$, each an L–module morphism satisfying nondegeneracy and symmetry axioms:

$$(\mathrm{id} \otimes \beta) \circ (\gamma \otimes \mathrm{id}) = \mathrm{id} = (\beta \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \gamma),$$
$$\beta = \beta \circ \tau, \qquad \gamma = \gamma \circ \tau.$$

In pictures, cup and cap denote these morphisms:

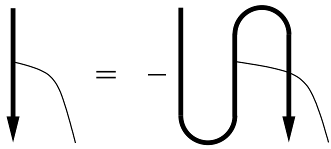


$\beta\colon L \otimes L \to \mathbf{1}$



$\gamma\colon \mathbf{1} \to L \otimes L$

When $\mathcal{C}$ is a rigid category, the dual of a module may be made a module by forcing the evaluation and coevaluation maps to be module maps. This is better defined by a picture than by a formula:



3.4 Weight systems

With this framework set up, we can state the theorem which will underlie our later explicit construction of the Rozansky–Witten weight systems:

Theorem 3.1 [40; 41] *Let $\mathcal{C}$ be a rigid, additive, symmetric tensor category, L a metric Lie algebra in $\mathcal{C}$, and M a dualizable module over L. Then there is a weight system*

$$w_{L,M}\colon \mathcal{A} \to \mathrm{Hom}(\mathbf{1},\mathbf{1}).$$

Proof Given any Jacobi diagram in $\mathcal{A}$, first draw it in the plane in a way compatible with its orientation. Morsify it so that the critical points and trivalent vertices lie at different levels, and so that the whole diagram is built from the generating morphisms we gave earlier, together with the Lie bracket and module action. Now compose the corresponding morphisms in $\mathcal{C}$. The proof of independence of the Morse and planar structures is the usual Reidemeister-move type argument, for which we refer to Vaintrob [40]. □

4 Sheaves and derived categories

Our main goal in this paper is to reinterpret the Rozansky–Witten weight systems in the context of the category-theoretic framework described above. The basic construction will be to associate to any complex manifold X a symmetric tensor category $D(X)$ and a Lie algebra object L in $D(X)$.

We begin in this section with a quick general explanation of the salient points about derived categories and about sheaves. Useful references for derived categories are Gelfand and Manin [13] and Richard Thomas [38]. For sheaves see Hartshorne [16] and Kashiwara and Schapira [21].

4.1 Derived categories

Let $\mathcal{C}$ be an *abelian category*; recall that this is an additive category in which every morphism has a kernel and a cokernel, and the two possible definitions of "image" (cokernel of kernel, or kernel of cokernel) agree. The standard example is the category of all (right, say) modules over a ring R, and in practice one may treat any abelian category as being of this form. From $\mathcal{C}$ we can form the category $\mathrm{Ch}(\mathcal{C})$ of *chain complexes* of objects of $\mathcal{C}$.

In homological algebra, one works primarily at the level of chain complexes, because taking homology groups prematurely can destroy some of the information they contain. For example, the homology-cohomology universal coefficient theorem shows that the operation of replacing a complex by its homology does not commute with the operation of taking the dual. When working in $\mathrm{Ch}(\mathcal{C})$, it is clearly reasonable to identify chain-homotopic maps and thereby to pass to a quotient *homotopy category* $\mathrm{Ho}(\mathcal{C})$, whose morphisms are the homotopy classes of maps between complexes.

But it is more sensible to regard in addition any *quasi-isomorphism* – a map between complexes which induces isomorphisms on homology – as an isomorphism. Although chain homotopy equivalences are certainly quasi-isomorphisms, the converse is not true; there may remain in $\mathrm{Ho}(\mathcal{C})$ quasi-isomorphisms without inverses. This can cause problems: for example, we often want to view a module as "equivalent" to any of its projective resolutions (quasi-isomorphic complexes of projective modules), but such resolutions need not actually be homotopy-equivalent to the original module.

The *derived category* $D(\mathcal{C})$ is defined by formally inverting these inside $\mathrm{Ho}(\mathcal{C})$: one introduces a calculus of fractions f/g (for f any morphism and g a quasi-isomorphism) essentially identical to the Ore localisation for noncommutative rings. Explicitly, any morphism in $D(\mathcal{C})$ between the complexes A^* and B^* may be represented by a diagram of each of the forms

$$A^* \xrightarrow{f} C^* \xleftarrow{g} B^* \qquad \text{and} \qquad A^* \xleftarrow{g} C^* \xrightarrow{f} B^*,$$

for some other complex C^*.

Any functor defined on $\mathrm{Ch}(\mathcal{C})$ which takes quasi-isomorphisms to isomorphisms – the abelian-group-valued homology functors $h^i\colon \mathrm{Ch}(\mathcal{C}) \to Ab$ being the obvious examples – therefore factors through $D(\mathcal{C})$, and in fact this universal property characterises $D(\mathcal{C})$.

Note that the objects of the derived category are the same as those of $\mathrm{Ch}(\mathcal{C})$, and that objects of the original category $\mathcal{C}$ may be identified with chain complexes whose only nonzero term lies in degree 0, so that there is an "inclusion" functor $\mathcal{C} \to D(\mathcal{C})$.

In $D(\mathcal{C})$ there are *shift functors* written $[n]\colon D(\mathcal{C}) \to D(\mathcal{C})$, for $n \in \mathbb{Z}$. The functor $[n]$ acts on a complex A^* by shifting it n places to the left, so that $A^*[n]^i = A^{i+n}$ and the differential is $d[n]^i = (-1)^n d^{i+n}$. It acts on chain maps by shifting the constituent maps compatibly. Any morphism $f\colon A^* \to B^*$ may be completed by a mapping cone construction into a 3–periodic sequence

$$\cdots \to A^* \to B^* \to C^*(f) \to A^*[1] \to \cdots$$

which becomes an exact sequence upon application of any functor $\mathrm{Hom}_{D(\mathcal{C})}(Z, -)$, and in particular upon taking cohomology. This shows that although $D(\mathcal{C})$ is not an abelian category, it is what is known as a *triangulated* category.

4.2 Derived functors

We are particularly interested here in *morphisms* in the derived category and in the way they compose. They turn out to be Ext–groups, with composition being the Yoneda product. In other words, the derived category is the place one should work if one wants to view and compose elements of a cohomology group like morphisms – which is exactly what we propose to do to reformulate Kapranov's construction of weight systems.

To explain this we need to consider derived functors. Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is an additive functor between abelian categories. Clearly it induces functors $\mathrm{Ch}(\mathcal{C}) \to \mathrm{Ch}(\mathcal{C})$ and $\mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$. But the obvious attempt to induce a functor $D(F)\colon D(\mathcal{C}) \to D(\mathcal{D})$ between the derived categories fails, because F does *not* necessarily take quasi-isomorphisms to quasi-isomorphisms. By considering mapping cones one can see that this property is equivalent to F taking all acyclic complexes (those quasi-isomorphic to zero) to acyclic complexes, which only holds for *exact* functors. To derive more general functors we need to restrict the kinds of complex under consideration.

Recall that for any object $A \in \mathcal{C}$, the functor $\mathrm{Hom}_{\mathcal{C}}(-, A)\colon \mathcal{C}^{\mathrm{op}} \to Ab$ is *left-exact*, and that if it is also *right-exact* then A is called *injective*. Let $\mathrm{Inj}(\mathcal{C})$ denote the full subcategory of injective objects of $\mathcal{C}$. If every object $A \in \mathcal{C}$ has an *injective resolution* – a quasi-isomorphism $A \to I^*$ to a complex of injective objects $I^* \in \mathrm{Ch}(\mathrm{Inj}(\mathcal{C}))$ – then we say that $\mathcal{C}$ *has enough injectives*.

Now any quasi-isomorphism out of an injective complex is a homotopy equivalence; that is, we may construct its inverse in $\mathrm{Ho}(\mathcal{C})$. Consequently, any two injective resolutions of an object are homotopy-equivalent, and if $\mathcal{C}$ has enough injectives then there is an equivalence of categories between $\mathrm{Ho}(\mathrm{Inj}(\mathcal{C}))$ and $D(\mathcal{C})$. In this case one can define the *right-derived functor* RF of $F\colon \mathcal{C} \to \mathcal{D}$ by just replacing $D(\mathcal{C})$ by $\mathrm{Ho}(\mathrm{Inj}(\mathcal{C}))$, applying F to get to $\mathrm{Ho}(\mathcal{D})$, and then passing to the quotient $D(\mathcal{D})$. Explicitly, if B^* is an object of $D(\mathcal{C})$, one simply replaces it by an injective resolution (well-defined up to homotopy equivalence) and applies F to construct $RF(B^*)$.

The *classical derived functors* $R^i F$ associated to F are just the composites of the homology functors h^i with RF. For the functor $F = \mathrm{Hom}_{\mathcal{C}}(A, -)$ applied to an object $B \in \mathcal{C}$, we have $R^i F(B) = \mathrm{Ext}^i_{\mathcal{C}}(A, B)$, because the procedure above agrees with the traditional definition of the Ext–groups: namely, take an injective resolution of B, apply $\mathrm{Hom}_{\mathcal{C}}(A, -)$, and take cohomology.

4.3 Morphisms in the derived category

Now we can explain the structure on the morphism sets in the derived category which we need. The key fact is that for objects A, B in $\mathcal{C}$ we have

$$\mathrm{Hom}_{D(\mathcal{C})}(A, B[i]) = \mathrm{Ext}^i_{\mathcal{C}}(A, B).$$

Here is a sketch proof. First replace B by an injective resolution I^*, so that there is the isomorphism $\mathrm{Hom}_{D(\mathcal{C})}(A, B[i]) = \mathrm{Hom}_{D(\mathcal{C})}(A, I^*[i])$. Now elements of this latter group are represented a priori by diagrams $A \to C^* \leftarrow I^*$ whose second map is a quasi-isomorphism; but because quasi-isomorphisms out of an injective complex are invertible in $\mathrm{Ho}(\mathcal{C})$, we only need to look at actual homotopy classes of maps $A \to I^*$. Finally, chain homotopy classes of maps between chain complexes A^*, B^* are given by the zeroth cohomology of the chain complex $\mathrm{Hom}^*(A^*, B^*)$. Taking the shift into account gives the result.

One important consequence is that we see that $\mathcal{C}$ is embedded in $D(\mathcal{C})$ as a full subcategory, because $\mathrm{Hom}_{D(\mathcal{C})}(A, B) = \mathrm{Ext}^0_{\mathcal{C}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B)$ for objects $A, B \in \mathcal{C}$.

There is a generalisation of the principle: for general objects of $D(\mathcal{C})$, complexes A^*, B^*, we have

$$\mathrm{Hom}_{D(\mathcal{C})}(A^*, B^*[i]) = \mathrm{Ext}^i_{\mathcal{C}}(A^*, B^*)$$

where the right-hand side is a "hyperext" group, computed by taking an injective resolution of each of the terms of B^*, applying $\mathrm{Hom}(A^*, -)$ and taking the total cohomology of the resulting double complex.

It is also possible to show that the Yoneda product on Ext groups of objects of $\mathcal{C}$

$$\mathrm{Ext}^i_{\mathcal{C}}(A, B) \otimes \mathrm{Ext}^j_{\mathcal{C}}(B, C) \to \mathrm{Ext}^{i+j}_{\mathcal{C}}(A, C)$$

corresponds to the composition of morphisms $A \to B[i]$ and $B \to C[j]$ in $D(\mathcal{C})$ that one gets after applying the shift $[i]$ to the latter.

4.4 Derived categories of coherent sheaves

In this section we will look specifically at the case of the derived category of $\mathcal{O}_X$–modules on a complex manifold X.

Let X be a finite-dimensional complex manifold X and let $\mathcal{O}_X$ be the structure sheaf, that is its sheaf of germs of holomorphic functions. We are interested in the sheaves taking into account the complex structure on X, these are the *sheaves of* $\mathcal{O}_X$*–modules*, in other words sheaves $\mathcal{E}$ with a natural map of sheaves $\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{E} \to \mathcal{E}$. The obvious example is the sheaf of germs of holomorphic sections of a holomorphic vector bundle. This example is *locally-free* in the sense that any point has a neighbourhood U over which the sections are isomorphic to the sheaf $\mathcal{O}_U^{\oplus k}$, for some k. The converse also holds: any locally-free sheaf is the sheaf of sections of a holomorphic vector bundle.

We restrict the class of sheaves further by considering coherent sheaves. A *coherent sheaf* is a sheaf of $\mathcal{O}_X$–modules which is locally a quotient of a finite-rank locally-free sheaf. On a smooth projective variety it actually has a global finite resolution by locally-free sheaves. The coherent sheaves form an abelian category and it is the *bounded* derived category of this that we refer to as the derived category of X and denote simply by $D(X)$.

We will use letters such as E, F to denote locally-free sheaves, script letters such as $\mathcal{E}, \mathcal{F}$ for general coherent sheaves, and letters such as A, B for typical objects in $D(X)$. The tangent sheaf of X will be written T and its dual T^* or Ω. (We abuse the star to indicate either the dual of a sheaf or a complex of sheaves; it should be clear from the context which is intended.)

The *sheaf cohomology* groups $H^*(-)$ are the classical derived functors of the global section functor $\Gamma = \mathrm{Hom}(\mathcal{O}_X, -)$, which takes sheaves to abelian groups. Thus, one computes $H^*(\mathcal{E})$ by taking an injective resolution of $\mathcal{E}$, applying Γ to obtain a chain complex of abelian groups, and then taking the cohomology. It can be helpful to have other points of view: one can compute them using Čech cohomology, and for a holomorphic vector bundle one can also think differential-geometrically using the Dolbeault isomorphism $H^q(E) \cong H^{0,q}_{\bar\partial}(E)$. For a compact complex manifold, all cohomology groups are finite-dimensional.

In a similar vein, we can define the groups $\mathrm{Ext}^*(\mathcal{E}, \mathcal{F})$ by applying the classical derived functors of $\mathrm{Hom}(-,-)$ to the pair $\mathcal{E}, \mathcal{F}$. They can also be computed by taking an injective resolution of $\mathcal{F}$.

We can use the description of morphism sets from the previous section to state the following result which is a key point for the construction of the Rozansky–Witten weight systems: if X is a complex manifold and $\mathscr{E}$ is a coherent sheaf on X then the sheaf cohomology groups of $\mathscr{E}$ are expressible as morphism sets in the derived category as follows:

$$H^q(\mathscr{E}) = \mathrm{Ext}^q(\mathscr{O}_X, \mathscr{E}) = \mathrm{Hom}_{D(X)}(\mathscr{O}_X, \mathscr{E}[q]).$$

As another example of this logic, we can describe the cup product

$$H^i(\textstyle\bigwedge^j T^*) \otimes H^k(\bigwedge^l T^*) \to H^{i+k}(\bigwedge^{j+l} T^*)$$

as the Yoneda product operation which, given a pair of morphisms

$$\mathscr{O}_X \to \textstyle\bigwedge^j T^*[i], \qquad \mathscr{O}_X \to \bigwedge^l T^*[k],$$

applies the functor $- \otimes \bigwedge^j T^*[i]$ to the second, composes the two, and then performs exterior multiplication $\bigwedge^j T^* \otimes \bigwedge^l T^* \to \bigwedge^{j+l} T^*$.

Inside the category of coherent sheaves there is an internal hom-functor: we can define $\mathscr{H}om(\mathscr{E}, \mathscr{F})$ to be the *sheaf* of local homomorphisms $\mathscr{E} \to \mathscr{F}$. This has a right derived functor which could be written $R\,\mathscr{H}om(-,-)$ but which we will denote for simplicity by $\mathscr{E}xt(-,-)$. The complex of sheaves $\mathscr{E}xt(\mathcal{E}, \mathcal{F})$ can be computed by taking a locally-free resolution of $\mathscr{E}$ and applying $\mathscr{H}om(-, \mathscr{F})$.

The category of coherent sheaves is a tensor category under the product $\otimes_{\mathscr{O}_X}$, and the left derived functor of this product equips the derived category $D(X)$ with the structure of a symmetric tensor category. (We use an underline to distinguish the derived functor $\underline{\otimes}$ from the underived $\otimes$ when applying it to complexes of sheaves, for which such a distinction is necessary. But in a context where "everything is derived" we often revert to the notation $\otimes$.) The identity object of $D(X)$ is the structure sheaf and the symmetry τ is the usual graded symmetry for chain complexes. In fact there is a rigid structure: the dual of an object A is given by $A^* = \mathscr{E}xt(A, \mathscr{O}_X)$. With this definition, the double dual functor is canonically isomorphic to the identity – something which is not true for the naive (underived) dualising functor $\mathrm{Hom}(-, \mathscr{O}_X)$ defined on the category of coherent sheaves.

4.5 Standard operations with sheaves

For full details of these operations and their relations, see Kashiwara and Schapira [21].

If $f\colon X \to Y$ is a holomorphic map then there are induced pushforward f_* and pullback functors f^* defined going between the categories of coherent sheaves of $\mathscr{O}_X$–modules and $\mathscr{O}_Y$–modules. These functor f^* is left-adjoint to f_*, and this relationship is preserved on the level of the derived category: there are natural isomorphisms

$$\mathrm{Hom}_{D(X)}(Lf^*A, B) \cong \mathrm{Hom}_{D(Y)}(A, Rf_*B).$$

One of the fundamental properties of the derived category of coherent sheaves is that Rf_* also has a right-adjoint $f^!\colon D(Y) \to D(X)$, the Grothendieck–Verdier functor, so that

$$\mathrm{Hom}_{D(X)}(B, f^! A) \cong \mathrm{Hom}_{D(Y)}(Rf_* B, A).$$

This functor $f^!$ can be defined as $Lf^* \underline{\otimes}\, Lf^* \omega_Y \underline{\otimes}\, \omega_X[\dim X - \dim Y]$, where ω denotes the canonical line bundle, $\bigwedge^{\dim X} T^*$.

In fact these adjunctions hold "internally" in the derived category: there are natural isomorphisms

$$Rf_* \mathscr{E}xt(Lf^* A, B) \cong \mathscr{E}xt(A, Rf_* B)$$
$$\mathscr{E}xt(Rf^* B, A) \cong Rf_* \mathscr{E}xt(B, f^! A).$$

Other useful functorial identities are the tensoriality of the pullback

$$Lf^*(A \underline{\otimes} A') \cong Lf^* \underline{\otimes}\, Lf^* A'$$

and the projection formula

$$Rf_*(B \underline{\otimes}\, Lf^* A) \cong Rf_* B \underline{\otimes}\, A.$$

4.6 Integral transforms

Suppose we have two complex manifolds X and Y. Then there is a functor, *integral transform*, from $D(X \times Y)$ to the category $\mathrm{Fun}(D(X), D(Y))$ of functors $D(X) \to D(Y))$. Consider the diagram of projections:

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

We can view an object P of $D(X \times Y)$ as a "correspondence" and define a functor $\widehat{P}\colon D(X) \to D(Y)$, by pulling up to $D(X \times Y)$, tensoring with P and then pushing down to Y:

$$\widehat{P}(A) = R\pi_{Y*}(\pi_X^*(A) \underline{\otimes}\, P).$$

Here the pullback is exact and need not be derived. When $\widehat{P}$ is an equivalence of categories, this is called a *Fourier–Mukai transform*.

Moreover, a morphism $\Theta\colon P \to Q$ in $D(X \times Y)$ gives a natural transformation $\widehat{\Theta}$ between the functors $\widehat{P}$ and $\widehat{Q}$. Explicitly, we get for each object $A \in D(X)$ a morphism

$$\widehat{\Theta}_A\colon R\pi_{Y*}(\pi_X^*(A) \underline{\otimes}\, P) \to R\pi_{Y*}(\pi_X^*(A) \underline{\otimes}\, Q)$$

by applying the functors $\pi_X^*(A) \underline{\otimes} -$ and then $R\pi_{Y*}$ to the morphism Θ. So indeed we have a functor $\widehat{\ }\colon D(X\times Y) \to \mathrm{Fun}(D(X), D(Y))$. In what follows we will usually drop the hat notation, using for example the same notation for morphisms in $D(X\times Y)$ and their induced natural transformations.

Let us give some simple easily-checked examples of integral transforms in the most important case, when $X = Y$. In this case we denote the two projections by π_1, π_2, and we also consider the diagonal map $\Delta\colon X \to X \times X$.

The structure sheaf of the diagonal is an object $\mathcal{O}_\Delta \in D(X)$, given by the pushforward $\mathcal{O}_\Delta = \Delta_*\mathcal{O}_X$. This object gives the identity functor $D(X) \to D(X)$. If we look at the shifted version $\mathcal{O}_\Delta[n]$ it defines the shift functor $[n]\colon D(X) \to D(X)$.

We can define similarly define objects $T_\Delta = \Delta_* T$ and $\Omega_\Delta = \Delta_*\Omega$ of $D(X \times X)$, which are sheaves supported on the diagonal. It is easy to see that $\pi_1^* T \otimes \mathcal{O}_\Delta \cong T_\Delta$, and consequently (by means of the projection formula) that T_Δ defines the "tensor with T" functor

$$\mathrm{id} \otimes T\colon D(X) \to D(X).$$

A little more subtly, for any object $A \in D(X)$ we can define the *derived* pushforward $R\Delta_* A$ in $D(X \times X)$ and therefore get an integral transform $D(X) \to D(X)$, which turns out to be just the operation of derived tensor with A (in $D(X)$). First notice that there is an isomorphism of functors: $R\Delta_* \cong \pi_1^* \underline{\otimes}\, \mathcal{O}_\Delta$. This follows from a straight-forward use of the projection formula:

$$\begin{aligned}\pi_i^*(-) \underline{\otimes}\, \mathcal{O}_\Delta \cong \pi_i^*(-) \underline{\otimes}\, R\Delta_*\mathcal{O}_X &\cong R\Delta_*(\Delta^*\pi^*(-) \underline{\otimes}\, \mathcal{O}_X)\\ &\cong R\Delta_*(\mathrm{id}_X(-) \underline{\otimes}\, \mathcal{O}_X) \cong R\Delta_*(-).\end{aligned}$$

Now we can see that indeed there is an isomorphism of functors $\widehat{R\Delta_* A} \cong \mathrm{id} \underline{\otimes} A$: just apply again the projection formula:

$$\begin{aligned}\widehat{R\Delta_* A}(-) = R\pi_{2*}(\pi_1^*(-) \underline{\otimes}\, R\Delta_* A) &\cong R\pi_{2*}R\Delta_*(\Delta^*\pi_1^*(-) \underline{\otimes}\, A)\\ &\cong \mathrm{id}_X(\mathrm{id}_X(-) \underline{\otimes}\, A) = (-) \underline{\otimes}\, A.\end{aligned}$$

5 The Atiyah class

The construction we are interested in rests on the idea of the *Atiyah class*, the sheaf-theoretic (and ultimately derived-categorical) analogue of the curvature of a holomorphic bundle. It is an extremely attractive and useful concept, so we devote this section to a thorough explanation of its definition and properties.

5.1 The Atiyah class for vector bundles

If E is a holomorphic vector bundle on a complex manifold X then we can construct from a connection the curvature 1–form R_E used in Section 2. Under the isomorphisms

$$H^{0,1}_{\bar\partial}(E^* \otimes E \otimes T^*) \cong H^1(E^* \otimes E \otimes T^*) \cong \mathrm{Ext}^1(E \otimes T, E)$$

we can view it as a class $\alpha_E \in \mathrm{Ext}^1(E \otimes T, E)$. Atiyah [1] showed how to construct this characteristic class in a purely sheaf-theoretic manner, giving it a more canonical realisation.

One way to do this is as follows. If E is a vector bundle, the bundle of 1–jets of E is the sheaf $E \oplus E \otimes \Omega$ with the twisted action of $\mathcal{O}_X$ given by

$$f \cdot (s, t \otimes \theta) = (fs, ft \otimes \theta + s \otimes df)$$

which describes first-order Taylor expansions of sections of E. There is an exact sequence

$$0 \to E \otimes \Omega \to JE \to E \to 0,$$

and the Atiyah class $\alpha_E \in \mathrm{Ext}^1(E, E \otimes \Omega) = \mathrm{Ext}^1(E \otimes T, E)$ is defined to be the extension class. The extension class can be thought of as the obstruction to existence of a section of the sequence – in the case of a locally-free sheaf E, such a thing would be a holomorphic connection on E – and may be built in Čech cohomology using the differences between local holomorphic splittings (which always exist). Another way to construct it is to work purely homologically: tensoring the sequence with the dual bundle E^* gives another exact sequence whose associated long exact sequence contains the map

$$H^0(E^* \otimes E) \xrightarrow{\delta} H^1(E^* \otimes E \otimes \Omega),$$

and the Atiyah class is the image under δ of the identity section of $\mathrm{End}(E)$.

The jet sequence/extension class definition also works for general coherent sheaves $\mathcal{E}$, but the Čech cohomology representation is more complicated in this case, since computing the relevant Ext group requires a resolution. We give a general recipe later in the section.

5.2 Properties of the Atiyah class

The first important property of the Atiyah class we need is its *naturality*. Suppose $f\colon E \to F$ is a map of bundles, and regard each of α_E and α_F as a morphism in $D(X)$.

Then $f[1]\circ\alpha_E = \alpha_F\circ(f\otimes \mathrm{id}_T)$, in other words the diagram below commutes.

$$\begin{array}{ccc} E\otimes T & \xrightarrow{\alpha_E} & E[1] \\ f\otimes\mathrm{id}_T \downarrow & & f[1]\downarrow \\ F\otimes T & \xrightarrow{\alpha_F} & F[1] \end{array}$$

One way to prove this is to look at the long exact sequences arising from F^* tensor the jet sequence of E, from E^* tensor the jet sequence of E, and from F^* tensor the jet sequence of F. Since f induces maps of jet sequences, the latter two long exact sequences have maps to the first one, and both identity elements map to $f\in H^0(E^*\otimes F)$, proving a commutativity which when written using morphisms is the one above.

A second important property is the behaviour under *tensor product*. One can show that the jet sequence for $E\otimes F$ is the sum, in the sense of extensions, of the jet sequences $JE\otimes F + E\otimes JF$. Thus the Atiyah class satisfies a Leibniz rule as one might expect, which can be written sloppily as

$$\alpha_{E\otimes F} = \alpha_E\otimes\mathrm{id}_F + \mathrm{id}_E\otimes\alpha_F$$

if we view this as an identity among morphisms $T\otimes E\otimes F\to E\otimes F[1]$. Strictly speaking, some permutations should be inserted to make this make sense, but there are no sign problems until we deal with complexes of sheaves.

Finally, the Atiyah class of the tangent bundle α_T has a *symmetry* and lies in fact in $\mathrm{Ext}^1(S^2T,T)$. This corresponds in differential geometry to the vanishing of the torsion (see Section 2) and is explained elegantly by Kapranov as follows. If E is a sheaf on X we can consider sheaves of E*–torsors* over X, meaning sheaves whose local sections are affine spaces modelled on the abelian group of local sections of E. Thus, the sheaf *Conn* of local holomorphic connections on X is a $T^*\otimes T^*\otimes T$–torsor. The torsion defines a map from this sheaf to the sheaf of abelian groups $\bigwedge^2T^*\otimes T$, and hence an exact sequence

$$0\to \mathit{Conn}_{tf}\to \mathit{Conn}\to \bigwedge\nolimits^2T^*\otimes T\to 0,$$

where the first term is the sheaf of torsion-free connections, a torsor over $S^2T^*\otimes T$. Each torsor defines an obstruction element in H^1 of its appropriate model sheaf. These elements are related by the long exact sequence arising from

$$0\to S^2T^*\otimes T\to T^*\otimes T^*\otimes T\to \bigwedge\nolimits^2T^*\otimes T\to 0,$$

so the fact that $\bigwedge^2T^*\otimes T$ is a trivial torsor means it represents the trivial element, and so the Atiyah class comes from a symmetric element.

5.3 Functorial definition of the Atiyah class

We will need to extend the definition of the Atiyah class from bundles to general objects of the derived category. For each object A, we would like an element $\alpha_A \in \mathrm{Ext}^1(A \otimes T, A)$, or equivalently a morphism in $D(X)$

$$\alpha_A\colon A \otimes T \to A[1].$$

(Remark: Here and subsequently, if a tensor product here is *obviously* derived, as for example when we are dealing with $D(X)$ as a tensor category, we do not distinguish it by an underline.) The naturality square from the previous section suggests that such morphisms should form the components of a natural transformation

$$\alpha\colon \mathrm{id} \otimes T \to \mathrm{id}[1]$$

and this is exactly what we establish below. One way to do this is to build explicitly a representative for complexes of locally-free sheaves, starting from the above version for single sheaves. (We will shortly give a Čech description of the Atiyah class which could be used to do this.) But there is a far more elegant way to construct α directly.

Consider the product $X \times X$, with the two projections π_1, π_2 and the diagonal $\Delta \subseteq X \times X$. Associated to Δ is the ideal sheaf $\mathcal{I}_\Delta$ of holomorphic functions on $X \times X$ vanishing on Δ, and there is an exact sequence

$$0 \to \mathcal{I}_\Delta/\mathcal{I}_\Delta^2 \to \mathcal{O}_{X\times X}/\mathcal{I}_\Delta^2 \to \mathcal{O}_{X\times X}/\mathcal{I}_\Delta \to 0,$$

whose three terms are identifiable respectively as: the cotangent sheaf $\Omega_\Delta \cong \pi_1^*\Omega \otimes \mathcal{O}_\Delta$ of Δ; the structure sheaf of the first infinitesimal neighbourhood of Δ; and the structure sheaf $\mathcal{O}_\Delta$ of Δ. This sequence defines an extension class

$$\alpha \in \mathrm{Ext}^1_{X\times X}(\mathcal{O}_\Delta, \Omega_\Delta) = \mathrm{Hom}_{D(X\times X)}(\mathcal{O}_\Delta, \Omega_\Delta[1]) = \mathrm{Hom}_{D(X\times X)}(T_\Delta, \mathcal{O}_\Delta[1]).$$

Therefore it gives, by integral transform, a natural transformation

$$\alpha\colon \mathrm{id} \otimes T \to \mathrm{id}[1]$$

between the "tensor with T" and "shift by 1" functors, as required.

We think of the morphism $\alpha \in \mathrm{Hom}_{D(X\times X)}(T_\Delta, \mathcal{O}_\Delta[1])$ as the "universal Atiyah class" for X. We should check that from it we can indeed recapture the earlier definition of the Atiyah class, in the case when $A = E$ is a single locally-free sheaf. For this we only need to observe that if we apply the functor $R\pi_{2*}(\pi_1^* E \underline{\otimes} -)$ to the sequence

$$0 \to \Omega_\Delta \to \mathcal{O}_{X\times X}/\mathcal{I}_\Delta^2 \to \mathcal{O}_\Delta \to 0$$

we get the jet sequence for E. Consequently, the universal Atiyah class $\alpha\colon \mathcal{O}_\Delta \to \Omega_\Delta[1]$, which extends the first sequence into a distinguished triangle, is sent to the Atiyah class α_E, which extends the latter to a distinguished triangle. (Recall that derived functors preserve distinguished triangles.)

The properties of the Atiyah class that we observed for bundles still hold in this more general context. The naturality follows automatically from the construction via the universal class α. The Leibniz tensor product rule still holds for this generalised Atiyah class, with the symmetry τ taking care of the signs, and the symmetry property, which is special to the tangent sheaf T, is unchanged.

5.4 Explicit representation of the Atiyah class

Although we have tried to define the Atiyah class in the most elegant way possible, the abstract definition sometimes needs to be supplemented by a way of actually calculating it in examples. We give here a rather long exposition of how to do this, and most readers should probably ignore it, since in fact we only need this result at one point in Section 7.

Recall the construction of the connecting homomorphism in the long exact sequence of cohomology of sheaves. If

$$0 \to E \to F \to G \to 0$$

is an exact sequence then we take injective resolutions of these three sheaves, obtaining an exact sequence of complexes of sheaves

$$0 \to I^* \to J^* \to K^* \to 0,$$

apply the section functor Γ to get an exact sequence of complexes of abelian groups

$$0 \to \Gamma(I^*) \to \Gamma(J^*) \to \Gamma(K^*) \to 0.$$

Then the standard Snake lemma construction defines the coboundaries

$$H^i(G) \to H^{i+1}(E).$$

A much more tangible version is obtained by using Čech complexes instead. Fix some good cover of X and let $C^*(E) = C^0(E) \to C^1(E) \to \cdots$ be the associated Čech complex. Then we have an exact sequence of complexes of abelian groups

$$0 \to C^*(E) \to C^*(F) \to C^*(G) \to 0.$$

Via the usual double complex proof (look at $C^*(I^*)$ where I^* is an injective resolution of E) we know that Čech and sheaf cohomology are isomorphic, and since this isomorphism is functorial, it follows that the connecting homomorphisms coming from this exact sequence coincide with the ones from the first construction via injective resolutions.

If we now generalise to the case of hypercohomology, where we are computing the cohomology of a complex E^* of sheaves, the same arguments go through "with an additional index" as follows. By definition we compute the hypercohomology by forming a double complex of injective resolutions of the sheaves of E^*, and taking its total cohomology. But an alternative method of computation is to form a double complex $C^*(E^*)$ of Čech complexes of the sheaves making up E^*, and to take its total cohomology. (The proof that these two methods are functorially isomorphic uses a triple complex!) The connecting homomorphisms in the long exact sequence of hypercohomology come from the Snake lemma applied to a three-term sequence of double complexes (by using their total differentials), and we can construct them similarly in Čech cohomology.

These principles give us a way to write down representatives of the Atiyah class. We deal first with the case of a single locally-free sheaf E (this is very easy but it is a great help in explaining the more complicated case) and then with a complex of locally-free sheaves. As a general object in the derived category $D(X)$ is quasi-isomorphic to such a complex, this is all we ever need.

Recall that $\alpha_E \in \mathrm{Ext}^1(E, E \otimes \Omega)$ is an obstruction class: it is the image of the identity under the connecting homomorphism

$$\mathrm{Ext}^0(E, E) \to \mathrm{Ext}^1(E, E \otimes \Omega)$$

coming from the long exact sequence of classical derived funtcors $\mathrm{Ext}^i(E, -)$ applied to the jet exact sequence

$$0 \to E \otimes \Omega \to JE \to E \to 0.$$

Let's view Ext^i in this context as the composite of functors $h^i \circ R\,\mathrm{Hom} = h^i \circ R\Gamma \circ R\,\mathcal{H}om$ (h^i denotes, as usual, the i–th cohomology of a complex). Since E is locally-free, the functor $\mathcal{H}om(E, -)$ is exact, and hence $R\,\mathcal{H}om(E, E \otimes \Omega)$ is simply the sheaf $\mathcal{H}om(E, E \otimes \Omega)$. Therefore

$$\mathrm{Ext}^i(E, E \otimes \Omega) = H^i(\mathcal{H}om(E, E \otimes \Omega))$$

can be viewed as simply a sheaf cohomology group.

The relevant connecting homomorphism δ comes from the diagram of Čech cochain groups

$$\begin{array}{ccccccccc} 0 & \to & C^0(\mathcal{H}om(E, E \otimes \Omega)) & \to & C^0(\mathcal{H}om(E, JE)) & \to & C^0(\mathcal{H}om(E, E)) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & C^1(\mathcal{H}om(E, E \otimes \Omega)) & \to & C^1(\mathcal{H}om(E, JE)) & \to & C^1(\mathcal{H}om(E, E)) & \to & 0. \end{array}$$

Let us compute $\delta(1)$. Fix a good cover $\{U_i\}$ of X over which E is locally trivial. Begin with the 0–cochain $\{1_i\}_i$ in the top right (1_i represents the identity of $E|_{U_i}$). We lift this to a cochain in the top middle. Since $JE = E \oplus E \otimes \Omega$ and the top right map is just projection to E, the lift must be of the form $\{1_i \oplus \nabla_i\}_i$, where $\nabla_i\colon E|_{U_i} \to (E \otimes \Omega)|_{U_i}$ satisfies (for $f \in \Gamma(\mathcal{O}_X, U_i)$ and $s \in \Gamma(E, U_i)$)

$$\nabla_i(f \cdot s) = f \cdot \nabla_i s + f \cdot ds$$

and is therefore a *connection* on E over U_i. Since E is trivial on each U_i, such a thing exists.

Now applying the Čech coboundary and lifting to the bottom left corner, we end up with the cochain $\{\nabla_i - \nabla_j\}_{ij} \in C^1(\mathcal{H}om(E, E \otimes \Omega))$. Clearly we have recovered the fact that the Atiyah class is the obstruction to existence of a global holomorphic connection.

When (E^*, ∂) is a *complex* of locally-free sheaves we modify this construction as follows. Once more we view Ext^i as the composite of functors $h^i \circ R\,\mathrm{Hom} = h^i \circ R\Gamma \circ R\,\mathcal{H}om$. Since $E^* \otimes \Omega$ is locally-free, the functor $\mathcal{H}om(E^* \otimes \Omega, -)$ (taking complexes of sheaves to complexes of sheaves) is exact, and hence $R\,\mathcal{H}om(E^*, E^* \otimes \Omega)$ is simply the complex of sheaves $\mathcal{H}om(E^*, E^* \otimes \Omega)$. Therefore

$$\mathrm{Ext}^i(E^*, E^* \otimes \Omega) = H^i(\mathcal{H}om(E^*, E^* \otimes \Omega)).$$

is just a hypercohomology group. It can be computed from the total cohomology of the double complex $C^*(\mathcal{H}om(E^*, E^* \otimes \Omega))$ (the Čech complex of a complex of locally-free sheaves).

To compute the connecting homomorphism we use the analogue of the diagram above. This time the groups are the total cochain groups of double complexes and the vertical coboundary maps are the total differentials in these double complexes, namely $d + (-1)^p \partial$, where d is the Čech differential and ∂ the differential on the complex E^*. We begin with the collection of identity maps $\{1_j^i\} \in C^0(\mathcal{H}om^0(E^*, E^*))$. (Here the lower index denotes the set of the cover and the upper one the position in the complex, so that 1_j^i is the identity $E^i(U_j) \to E^i(U_j)$.) For each sheaf E^i and open set of the cover U_j we pick a local connection ∇_j^i so that the lift of the identity is

$\{1^i_j \oplus \nabla^i_j\} \in C^0(\mathcal{H}om^0(E^*, JE^*))$. Now apply the vertical coboundary and lift into the bottom left corner, namely

$$C^0(\mathcal{H}om^1(E^*, E^* \otimes \Omega)) \oplus C^1(\mathcal{H}om^0(E^*, E^* \otimes \Omega)).$$

The result is that the Atiyah class is represented by

$$\{\partial\nabla^i_j - \nabla^i_j\partial\}_j \oplus \{\nabla^i_j - \nabla^i_k\}_{jk} \in C^0(\mathcal{H}om^1(E^*, E^* \otimes \Omega)) \oplus C^1(\mathcal{H}om^0(E^*, E^* \otimes \Omega)).$$

Of course in the special extremal case that E^* is a single sheaf the first term drops out and we get back the representative we already computed. In the other extremal case where the E^* are globally trivial (for example on an affine space), the second term drops out and we just have $\partial\nabla - \nabla\partial$ as representative. We refer to this statement as *Markarian's lemma* 1, since it comes from his paper [28] (in which it is an exercise for the reader).

Finally we observe that to compute the Atiyah class for an arbitrary (not locally-free) sheaf or complex of sheaves $\mathcal{E}^*$ we can just resolve first by a (double) complex of locally-free ones and then use the above method to obtain a representative of the Atiyah class.

5.5 Final comments on the Atiyah class

There are a few further comments we will need soon.

The Atiyah class of the diagonal Recall that the universal Atiyah class

$$\alpha \in \mathrm{Ext}^1_{X\times X}(T_\Delta, \mathcal{O}_\Delta)$$

comes (after taking an adjoint) from the infinitesimal neighbourhood sequence

$$0 \to \Omega_\Delta \to \mathcal{O}_{X\times X}/\mathcal{I}^2_\Delta \to \mathcal{O}_\Delta \to 0.$$

This morphism is very closely related to the Atiyah class of $\mathcal{O}_\Delta$ itself, which lies in $\mathrm{Ext}^1_{X\times X}(\mathcal{O}_\Delta \otimes T_{X\times X}, \mathcal{O}_\Delta)$ and is (by definition) the extension class of the jet sequence

$$0 \to \mathcal{O}_\Delta \otimes \Omega_{X\times X} \to J(\mathcal{O}_\Delta) \to \mathcal{O}_\Delta \to 0.$$

We can decompose $\Omega_{X\times X} = \pi_1^*\Omega_X \oplus \pi_2^*\Omega_X$ and therefore identify

$$\begin{aligned}\mathrm{Ext}^1_{X\times X}(\mathcal{O}_\Delta \otimes T_{X\times X}, \mathcal{O}_\Delta) &= \mathrm{Ext}^1_{X\times X}(\mathcal{O}_\Delta \otimes \pi_1^*T, \mathcal{O}_\Delta) \oplus \mathrm{Ext}^1_{X\times X}(\mathcal{O}_\Delta \otimes \pi_2^*T, \mathcal{O}_\Delta)\\ &= \mathrm{Ext}^1_{X\times X}(T_\Delta, \mathcal{O}_\Delta) \oplus \mathrm{Ext}^1_{X\times X}(T_\Delta, \mathcal{O}_\Delta).\end{aligned}$$

It is easy to check that the jet sequence is a Baer sum of two copies of the infinitesimal neighbourhood sequence and hence that under this identification the Atiyah class $\alpha_{\mathcal{O}_\Delta}$

is equal to the sum $\alpha \oplus \alpha$ of two copies of the universal Atiyah class. This remark will be important in understanding the STU relation for the universal Atiyah class α.

Locality From abstract functoriality, or directly from the local representation of the Atiyah class, one can see the following locality property: if E is an object of $D(X)$ and $U \subseteq X$ is an open set, then the following diagram commutes:

$$\begin{array}{ccc} E & \stackrel{\alpha_X}{\to} & E \otimes \Omega_X \\ \downarrow & & \downarrow \\ i^* E & \stackrel{\alpha_U}{\to} & i^* E \otimes \Omega_U \end{array}$$

Functoriality under pullback A final property we need is about pullbacks of the Atiyah class. This is that the diagram

$$\begin{array}{ccc} f^* E \otimes T_Y & \stackrel{\alpha_{f^*E}}{\to} & f^* E[1] \\ \downarrow & & \downarrow \\ f^* E \otimes f^* T_X & \stackrel{f^*(\alpha_E)}{\to} & f^* E[1], \end{array}$$

where the left-hand downward map is $\mathrm{id} \otimes df$, commutes. As an example, consider the Atiyah class $\alpha_{T_{X\times X}} \in \mathrm{Ext}^1(T_{X\times X} \otimes T_{X\times X}, T_{X\times X})$. It can obviously be decomposed into two pieces via the usual splitting, with the first living in $\mathrm{Ext}^1(\pi_1^* T \otimes \pi_1^* T, \pi_1^* T)$. This piece equals $\pi_1^* \alpha_T$, by the above naturality.

6 Rozansky–Witten weight systems revisited

In this section we bring together the abstract nonsense of the previous three sections and show how it provides an elegant formulation of Rozansky–Witten weight systems.

6.1 The Lie algebra object of a complex manifold

The first main theorem of the paper is the following interpretation of a complex manifold as “being” in some sense a Lie algebra.

Theorem 6.1 *Suppose X is a complex manifold. Then the shifted tangent sheaf $T[-1]$ is a Lie algebra object in the derived category $D(X)$; furthermore, every object in $D(X)$ is canonically a module over $T[-1]$, and every morphism in $D(X)$ is a module map.*

Proof We just need to define the structure morphisms and check the identities for them. To obtain the bracket, start with the Atiyah class of T, viewed as a morphism $\alpha_T\colon T \otimes T \to T[1]$. Now apply an additional shift by $[-2]$ to each side and the result is the bracket

$$T[-1] \otimes T[-1] \to T[-1].$$

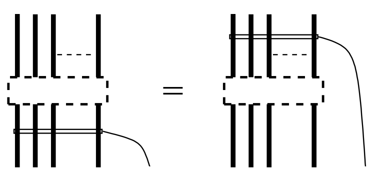
The module action for any object $A \in D(X)$, likewise, is just obtained by shifting the Atiyah class α_A by $[-1]$; it is a morphism

$$A \otimes T[-1] \to A.$$

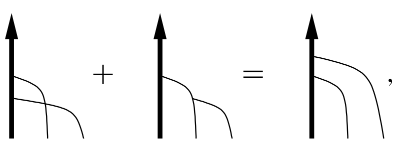
Skew-symmetry of the bracket comes because the *unshifted* Atiyah class is symmetric, and the shifts of $[-1]$ switch the parity. The Jacobi (IHX) identity and module (STU) identity are just the fact that the two morphisms above are *invariant* under the action of the Atiyah class, which is a consequence of its naturality. Explicitly, for the STU case: consider the morphism $\alpha_A\colon A \otimes T \to A[1]$. This commutes with taking Atiyah classes on each side, according to the diagram

$$\begin{tikzcd}
(A \otimes T) \otimes T \arrow[r, "\alpha_{A\otimes T}"] \arrow[d, "\alpha_A \otimes \mathrm{id}"'] & (A \otimes T)[1] \arrow[d, "\alpha_A[1]"'] \\
A \otimes T[1] \arrow[r, "\alpha_A[1]"] & A[2].
\end{tikzcd}$$

Using the Leibniz rule to evaluate the top line, and putting in the shifts (this affects the signs a little) gives the STU relation. It makes more sense with pictures; naturality and the Leibniz rule for the Atiyah class amount to the identity



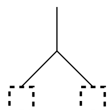
for *any* (boxed) morphism between two tensor products of objects in $D(X)$. Applying this naturality to $\alpha_A[-1]$ (the case where the box is actually a trivalent vertex) gives the familiar



and the IHX relation is the special case $A = T[-1]$. Note once more the way that the signs are *locally* compatible with the "correct" vertex-oriented orientations of graphs. □

There are two things to note here. Firstly, the Lie algebra structure on X is nilpotent: an n–fold composition defined using the Atiyah class (corresponding pictorially to n–vertex tree with $n+1$ inputs and 1 output) lies in $\mathrm{Ext}^n(T^{\otimes n+1}, T)$, which is zero for n larger than the complex dimension of X.

Secondly, the bracket $\alpha\colon T \otimes T \to T[1]$ induces by composition a bracket on the vector space $\bigoplus_n \mathrm{Hom}(\mathcal{O}_X, T[n]) = H^*(T)$. One might regard this as the simplest "computable"manifestation of the Atiyah class, but unfortunately it is zero. This is because the composite of any two elements may be pictured as below, and sliding the trivalent vertex down past one of the boxes creates the Atiyah class of the structure sheaf (the unit object) which is zero.



6.2 Metric Lie algebras from complex symplectic manifolds

The second main theorem is the similar interpretation of complex *symplectic* manifolds as "being" *metric* Lie algebras.

A complex symplectic form $\omega \in H^0(T^* \otimes T^*)$ may be rewritten as a morphism $T \otimes T \to \mathcal{O}_X$ and then, by shifting by $[-2]$, as a *symmetric* morphism

$$T[-1] \otimes T[-1] \to \mathcal{O}_X[-2].$$

As a morphism in $D(X)$, this is invariant under the action of $T[-1]$ and so satisfies the identities stated in Section 3. However, it cannot quite be regarded as metric on $T[-1]$ because of the shifts $[-2]$ appearing on the right-hand side. A metric on L is meant to be a morphism $L \otimes L \to 1$, which in our case would be a morphism $T[-1] \otimes T[-1] \to \mathcal{O}_X$, without the shift. To handle this difficulty we alter $D(X)$ into a category $\widetilde{D}(X)$: we define it to have the same objects as $D(X)$ but redefine the space of morphisms $A \to B$ to be the graded vector space $\mathrm{Ext}^*(A, B)$ instead of just $\mathrm{Ext}^0(A, B)$. Composition of morphisms is defined in the obvious way and is graded bilinear. After this extension, the above shifts cease to cause problems. In summary:

Theorem 6.2 *If X is a complex symplectic manifold then $T[-1]$ is a metric Lie algebra in the extended derived category $\widetilde{D}(X)$, and $\widetilde{D}(X)$ is a module category over $T[-1]$.*

Consequently we can apply the general categorical construction of weight systems from Section 3:

Theorem 6.3 *If X is a complex symplectic manifold and A is an object of $D(X)$ then there is a weight system*

$$w_{X,A}\colon \mathcal{A} \to H^*(\mathcal{O}_X).$$

Remark 6.4 A different way to define $\widetilde{D}(X)$ is as follows. Embed $D(X)$ in the derived category $D^u(X)$ of unbounded complexes using the functor $i = \bigoplus_{n\in\mathbb{N}}[n]$: an object $A \in D(X)$ is sent to $i(A) = \bigoplus_{n\in\mathbb{N}} A[n]$. The set of morphisms $i(A) \to i(B)$ in $D^u(X)$ is the rather large $\bigoplus_{m,n\in\mathbb{N}} \mathrm{Hom}_{D(X)}(A[m], B[n])$, but the shift functor [1] acts on this space, and the set of morphisms which *commute* with this action is the very reasonable

$$\bigoplus_{n\in\mathbb{N}} \mathrm{Hom}_{D(X)}(A, B[n]) = \mathrm{Ext}^*(A, B).$$

By this procedure of essentially looking at the [1]–invariant subcategory of $D^u(X)$, we define $\widetilde{D}(X)$.

This odd-looking construction has the advantage of being exactly parallel to the procedure of replacing the category of finite-dimensional complex $\mathfrak{g}$–modules with modules over $\mathbb{C}[\![\hbar]\!]$. One replaces every space V with the graded space $\bigoplus_{n\in\mathbb{N}} V \cdot \hbar^n$ ($= V \otimes_{\mathbb{C}} \mathbb{C}[\![\hbar]\!]$), and uses only the maps of $\mathbb{C}[\![\hbar]\!]$ modules, that is the $\hbar$–equivariant linear maps between these.

This construction is necessary in the theory of Vassiliev invariants if we want to obtain from a Lie algebra $\mathfrak{g}$ a weight system defined on the graded completion of $\mathcal{A}$. In order to avoid convergence problems we have to multiply the Casimir element by an indeterminate $\hbar^2$ and the metric by $\hbar^{-2}$ to obtain weight systems $\widehat{\mathcal{A}} \to \mathbb{Q}[\![\hbar^2]\!]$. Salvaging some of the grading in this way is absolutely essential to the correspondence between the Kontsevich integral and invariants coming from quantum groups, and to the deformation of the category of representations of $\mathfrak{g}$ via the Knizhnik–Zamolodchikov equation. In Section 9 we will see the parallel deformation for $\widetilde{D}(X)$.

Remark 6.5 There are weaker geometrical structures we could consider. If X is a *holomorphic Casimir manifold*, possessing a holomorphic bivector $w \in H^0(\Lambda^2 T)$ (not required to be nondegenerate) then $T[-1]$ is a Casimir Lie algebra in $\widetilde{D}(X)$, in an analogous way. The Casimir is the symmetric morphism $\mathcal{O}_X \to (T[-1] \otimes T[-1])[2]$. We can also formulate the even weaker analogue of a vector space with a classical r–matrix too: this is a complex manifold X with (for example) a sheaf $\mathcal{E}$ and an element $r \in \mathrm{Ext}^*(\mathcal{E} \underline{\otimes} \mathcal{E}, \mathcal{E} \underline{\otimes} \mathcal{E})$ satisfying the 4T relation of Vassiliev theory. But this is probably not very useful.

6.3 The STU relation for the universal Atiyah class

We've seen that for any object $A \in D(X)$, its Atiyah class $\alpha_A\colon A \otimes T[-1] \to A$ together with that of the tangent sheaf $\alpha_T\colon T[-1] \otimes T[-1] \to T[-1]$ satisfy the STU relation, which can be written nonpictorially as

$$\alpha_A \circ \alpha_T = [\alpha_A, \alpha_A] \in \mathrm{Hom}_{D(X)}(A \otimes T[-1] \otimes T[-1], A).$$

This strongly suggests that the universal Atiyah class morphism $\alpha\colon T_\Delta[-1] \to \mathcal{O}_\Delta$ in $D(X \times X)$, together with the pullback $\pi^*(\alpha_T)\colon \pi^*T[-1] \otimes \pi^*T[-1] \to \pi^*T[-1]$ (the sources of the above morphisms), should satisfy the corresponding "universal" relation

$$\alpha \circ \pi^*(\alpha_T) = [\alpha, \alpha] \in \mathrm{Hom}_{D(X\times X)}(\mathcal{O}_\Delta \otimes \pi^*T[-1] \otimes \pi^*T[-1], \mathcal{O}_\Delta).$$

This is in fact the case because of the relation between α and the Atiyah class of $\mathcal{O}_\Delta$. Certainly we have the identity

$$[\alpha_{\mathcal{O}_\Delta}, \alpha_{\mathcal{O}_\Delta}] = \alpha_{\mathcal{O}_\Delta} \circ \alpha_{T_{X\times X}},$$

and if we extract the first component parts of these identities under the usual splitting of $T_{X\times X}$ we get the desired equality.

7 The symmetric and universal enveloping algebras of $T[-1]$

Let $\mathfrak{g}$ be a Lie algebra. The symmetric algebra $S(\mathfrak{g})$ and the universal enveloping algebra $U(\mathfrak{g})$ are defined as quotients of the tensor algebra $\mathcal{T}(\mathfrak{g}) = \bigoplus \mathfrak{g}^{\otimes n}$:

$$S(\mathfrak{g}) := \mathcal{T}(\mathfrak{g})/\langle x \otimes y - y \otimes x\rangle, \quad U(\mathfrak{g}) := \mathcal{T}(\mathfrak{g})/\langle [x, y] - x \otimes y + y \otimes x\rangle.$$

Each inherits an associative algebra structure and $\mathfrak{g}$–module structure from the tensor algebra; the symmetric algebra also inherits a grading. The universal enveloping algebra has a universal property for Lie algebra homomorphisms from $\mathfrak{g}$ into associative algebras, and the representation theory of $U(\mathfrak{g})$ coincides with that of $\mathfrak{g}$.

Via symmetrization there is a splitting $S(\mathfrak{g}) \hookrightarrow \mathcal{T}(\mathfrak{g})$ and composing this with the quotient map $\mathcal{T}(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g})$ gives a vector space isomorphism called the Poincaré–Birkhoff–Witt map:

$$\mathrm{PBW}\colon S(\mathfrak{g}) \to U(\mathfrak{g}).$$

This is a $\mathfrak{g}$–module map, so it induces a vector space isomorphism on the invariant parts:

$$\mathrm{PBW}\colon S(\mathfrak{g})^{\mathfrak{g}} \cong U(\mathfrak{g})^{\mathfrak{g}}.$$

We have seen that the object $L = T[-1]$ is a Lie algebra for any complex manifold and that $D(X)$ is a category of modules over L. We now pursue this analogy further: we construct objects S and U, the symmetric and universal enveloping algebras of L, and a PBW isomorphism between them. The third main theorem of the paper is:

Theorem 7.1 *The object $S = \bigoplus(\bigwedge^k T)[-k]$ is the symmetric algebra of L, while $U = \pi_* \mathcal{E}xt(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$ is its universal enveloping algebra. These objects satisfy the expected universal properties, and U acts on all objects of $D(X)$ compatibly with the action of L. There is a morphism* PBW: $S \to U$*, the PBW morphism, which is an isomorphism of objects (but not of algebras) in $D(X)$.*

The construction of S is straightforward, but verifying the properties of U is quite difficult, and relies on some ideas of Markarian [28]. Căldăraru [8] independently explored similar ideas, to a different purpose, and recently Ramadoss [33] studies a similar problem.

To simplify notation, in this section *all functors will be derived*, so $\otimes$ means $\underline{\otimes}$, f^* means Lf^*, and f_* means Rf_*. (With this convention we could write $\mathcal{H}om$ for $\mathcal{E}xt$, but we won't.) We will also write simply π for the projection $\pi_1\colon X \times X \to X$.

7.1 The symmetric algebra

The symmetric power $S^k(T[-1])$ is actually the object $(\bigwedge^k T)[-k]$, because the shift $[-1]$ changes the parity of the flip map τ in $D(X)$ and therefore changes symmetrisation to antisymmetrisation. Thus, the symmetric algebra of $T[-1]$ is the object

$$S = \bigoplus(\bigwedge^k T)[-k].$$

It is a finite sum and is equipped with the commutative algebra structure induced by exterior multiplication. It is easy to see that it is category-theoretically the symmetric algebra $S(L)$ of L. Firstly there is a canonical map $L \to S$. Secondly, given any map from L to a commutative algebra object A, we get a lift $S \to A$ by symmetrisation (view $\bigwedge^k T$ as a subsheaf of the tensor sheaf) followed by multiplication in the normal way. This gives a commutative algebra homomorphism, uniquely determined by the original $L \to A$.

7.2 The universal enveloping algebra: plan of attack

The usual construction in the category of vector spaces builds $U(\mathfrak{g})$ as a quotient of $T(\mathfrak{g})$. We cannot do the same construction in $D(X)$ because it is not an abelian category, merely triangulated. In any case, we want to have a reasonable description of the object U, not simply an abstract definition as a quotient. Our definition $U = \pi_* \mathcal{E}xt(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$ is quite explicit, but it is unfortunately relatively hard to show that it really is the universal enveloping algebra of L, in the sense of category theory. (While this isn't really essential to our study of Rozansky–Witten invariants, it is worth establishing in its own right and is conceptually important in studying the TQFT.)

Here are the steps we must take to prove the theorem.

(1) Show that U is an associative algebra object.

(2) Construct a natural map $L \to U$ which is a Lie algebra homomorphism (with respect to the commutator bracket on U).

(3) Show that the universal property holds: every Lie algebra morphism $L \to A$ for some other associative algebra A extends (under $L \to U$) to an associative algebra morphism $U \to A$.

(4) Construct a map $S \to U$ which is an isomorphism of objects in $D(X)$.

(5) Show that U acts on all objects in $D(X)$, compatibly (under $L \to U$) with the action of L.

It is relatively straightforward to perform Steps (1), (2) and (5) and this is handled in the next subsection.

Step (4), the construction of the PBW morphism, was done by Markarian [28] in lemma 1 ("proof: left to reader.") and definition-proposition 1 ("proof: local check is enough."). Not being experts, we didn't find these exercises at all trivial, so we worked out the details, the first in Section 5 (the local representation of the Atiyah class) and the second below. (Although these are a bit long-winded, we felt it would be useful to supply details as an aid to anyone else who has tried to understand Markarian's paper.)

Step (3) is the most frustrating step: we know of no direct way of constructing the requisite maps $U \to A$. So instead we fall back on a rather abstract method of proof which relies on Steps (1), (2) and (4) and a theorem of Hinich and Vaintrob. This is contained in the penultimate subsection, after which there are some further remarks on the structure of S and U.

7.3 The construction of U

We define $U = \pi_* \mathcal{E}xt(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$, an object of $D(X)$.

Step (1) This object U is an associative algebra in $D(X)$. To see this, first observe that $A = \mathcal{E}xt(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$ is an associative algebra in $D(X \times X)$. If we apply the pushforward to the multiplication map $A \otimes A \to A$ then we get a map

$$\pi_*(A \otimes A) \to \pi_* A$$

which is not quite what we want. However, there is a natural map (the adjunction unit)

$$\pi^*\pi_* A \to A$$

in $D(X \times X)$ corresponding to the identity under the adjunction isomorphism

$$\mathrm{Hom}_{D(X\times X)}(\pi^*\pi_* A, A) \cong \mathrm{Hom}_{D(X)}(\pi_* A, \pi_* A)$$

and if we tensor this with itself we get a map

$$\pi^*(\pi_* A \otimes \pi_* A) = \pi^*\pi_* A \otimes \pi^*\pi_* A \to A \otimes A$$

whereupon the adjunction isomorphism (reversed) gives us a map

$$\pi_* A \otimes \pi_* A \to \pi_*(A \otimes A).$$

Precomposing with this gives us the required multiplication $U \otimes U \to U$. It is straightforward to check that it is still associative and unital.

Step (2) Next, we define the canonical Lie algebra homomorphism $\gamma\colon L \to U$. Consider the universal Atiyah class morphism

$$\alpha \in \mathrm{Hom}_{D(X\times X)}(T_\Delta, \mathcal{O}_\Delta[1]) \cong \mathrm{Hom}_{D(X\times X)}(\pi^* T[-1] \otimes \mathcal{O}_\Delta, \mathcal{O}_\Delta)$$

in the adjoint form (moving $\mathcal{O}_\Delta$ to the RHS)

$$\alpha \in \mathrm{Hom}_{D(X\times X)}(\pi_1^* T[-1], \mathcal{E}xt(\mathcal{O}_\Delta, \mathcal{O}_\Delta))$$

and apply the adjunction

$$\mathrm{Hom}_{D(X\times X)}(\pi_1^* T[-1], \mathcal{E}xt(\mathcal{O}_\Delta, \mathcal{O}_\Delta)) \cong \mathrm{Hom}_{D(X)}(T[-1], \pi_*\,\mathcal{E}xt(\mathcal{O}_\Delta, \mathcal{O}_\Delta))$$

to get the required map $\gamma\colon T[-1] \to \pi_*\,\mathcal{E}xt(\mathcal{O}_\Delta, \mathcal{O}_\Delta))$.

We must show that this is a morphism of Lie algebras when U is given the commutator bracket, that is that the diagram

$$\begin{array}{ccc} T[-1] \otimes T[-1] & \overset{\gamma\otimes\gamma}{\to} & U \otimes U \\ \alpha_T \downarrow & & \downarrow [\,,\,] \\ T[-1] & \overset{\gamma}{\to} & U \end{array}$$

commutes. We recall $U = \pi_*\,\mathcal{E}xt(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$ and use again the adjunction

$$\mathrm{Hom}(-, U) \cong \mathrm{Hom}(\pi^*(-), \mathcal{E}xt(\mathcal{O}_\Delta, \mathcal{O}_\Delta))$$

to compute the two sides of this square. Write $\mathcal{E}$ for $\mathcal{E}xt(\mathcal{O}_\Delta, \mathcal{O}_\Delta))$ as a notational convenience.

The adjoint to the composition around the top is

$$\pi^*T[-1]\otimes\pi^*T[-1] \overset{\pi^*(\gamma\otimes\gamma)}{\to} \pi^*(\pi_*\mathscr{E}\otimes\pi_*\mathscr{E}) \overset{[\,,]}{\to} \mathscr{E}.$$

The right-hand map here, the commutator, is given in terms of the algebra structure on U which actually comes from a similar adjunction, so it can be factorised

$$\pi^*(\pi_*\mathscr{E}\otimes\pi_*\mathscr{E}) = \pi^*\pi_*\mathscr{E}\otimes\pi^*\pi_*\mathscr{E} \overset{p\otimes p}{\to} \mathscr{E}\otimes\mathscr{E}\to\mathscr{E}$$

where p is the adjunction unit $\pi^*\pi_*\mathscr{E}\to\mathscr{E}$. The composite $\pi^*T[-1]\otimes\pi^*T[-1]\to\mathscr{E}\otimes\mathscr{E}$ obtained is by definition $\alpha\otimes\alpha$ so the whole map can be thought of as the commutator $[\alpha,\alpha]$.

The lower side of the square is adjoint to

$$\pi^*T[-1]\otimes\pi^*T[-1] \overset{\pi^*(\alpha_T)}{\to} \pi^*T[-1] \overset{\alpha}{\to} \mathscr{E}.$$

Now the equality of these two compositions is just the STU identity for the universal Atiyah class, proved in the previous section.

Step (5) The object U acts on objects as follows. We define a morphism

$$\mathscr{O}_\Delta\otimes\pi^*U\to\mathscr{O}_\Delta$$

in $D(X\times X)$ by taking the composition

$$\mathscr{O}_\Delta\otimes\pi^*\pi_*\mathscr{E}\to\mathscr{O}_\Delta\otimes\mathscr{E}\to\mathscr{O}_\Delta$$

using the unit of the adjunction and the natural multiplication action of $\mathscr{E}$ on $\mathscr{O}_\Delta$. This morphism induces a natural transformation $-\otimes U\to -$ which makes the algebra U act on $D(X)$.

This is compatible with the action of $L=T[-1]$ on objects. To see this we need to show that the diagram

$$\begin{array}{ccc} \mathscr{O}_\Delta\otimes\pi^*T[-1] & \overset{\alpha}{\to} & \mathscr{O}_\Delta \\ {\scriptstyle \mathrm{id}\otimes\pi^*\gamma}\downarrow & & \downarrow \\ \mathscr{O}_\Delta\otimes U & \to & \mathscr{O}_\Delta \end{array}$$

commutes. But equivalently we can transfer the $\mathscr{O}_\Delta$s to the other side and look at

$$\begin{array}{ccc} \pi^*T[-1] & \overset{\alpha}{\to} & \mathscr{E} \\ {\scriptstyle \pi^*\gamma}\downarrow & & \downarrow \\ \pi^*\pi_*\mathscr{E} & \to & \mathscr{E} \end{array}$$

whose commutativity is in fact the definition of γ.

7.4 The PBW isomorphism

We can finally construct the PBW isomorphism. Start with the canonical map $\gamma\colon L \to U$ coming from the Atiyah class. By tensoring it up in the normal way it extends to an algebra homomorphism from the tensor algebra $T(L)$ to U, and by composing with the (non-algebra-morphism) symmetrisation map $S \to T(L)$ we get our PBW map.

To prove that this is an isomorphism of objects of $D(X)$, we can work locally: such objects are just complexes of sheaves, and a map is an isomorphism in $D(X)$ if it induces an isomorphism of cohomology *sheaves*. Isomorphisms of sheaves can of course be checked locally in an affine patch of X. Let $i\colon Y \hookrightarrow X$ be an affine chart: from the above remarks about locality of the Atiyah class (or by an abstract functorial diagram-chase), we see that restricting $\mathrm{PBW}_X\colon S_X \to U_X$ gives the corresponding morphism $\mathrm{PBW}_Y\colon S_Y \to U_Y$. So it is enough to show that the PBW morphism is an isomorphism when X is affine.

To do this it helps to transfer from the category of coherent sheaves on X to the equivalent category of (left) A–modules, where $A = \Gamma(\mathcal{O}_X)$. Of course $\mathcal{O}_X$ becomes the left regular module A, Ω_X becomes the module of Kähler differentials Ω^1_A and T_X becomes the module of derivations $\mathrm{Der}(A, A)$. The object S is therefore represented by the exterior algebra $\bigwedge_A \mathrm{Der}(A, A)$. (All tensor products in this section are over $\mathbb{C}$ unless otherwise noted.)

Extending this dictionary, sheaves on $X \times X$ become $A - A$–bimodules, that is A^e–modules, where A^e is the enveloping algebra $A \otimes A^{\mathrm{op}}$. In particular we have that $\mathcal{O}_{X\times X}$ corresponds to $A \otimes A$ (the free $A - A$–bimodule of rank 1) whereas $\mathcal{O}_\Delta$ corresponds to A as a bimodule. The cotangent sheaf $\Omega_{X\times X}$ corresponds to the bimodule

$$\Omega^1_{A^e} \cong \Omega^1_A \otimes A \oplus A \otimes \Omega^1_A,$$

where the two right-hand terms are of course the pullbacks $\pi_1^*\Omega_X$ and $\pi_2^*\Omega_X$. Taking pushforward $(\pi_1)_*$ simply corresponds to forgetting the right A–module action of a bimodule, making it just a left A–module.

Computing the object $\mathcal{E}xt(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$ is equivalent to computing the bimodule $\mathcal{E}xt(A, A)$, where the $\mathcal{E}xt$ here is the derived functor of internal hom in the category of $A - A$–bimodules" (since A is commutative, the set of bimodule homomorphisms is itself a bimodule). To compute it we can use a resolution of the first factor by free bimodules, such as the Hochschild (bar) complex:

$$B(A) = \qquad \to A^{\otimes n} \to A^{\otimes n-1} \to \cdots \to A \otimes A.$$

Here the $A^{\otimes n}$ term is taken to be in degree $2-n$, so we can in fact write

$$B^{-n}(A) = A \otimes A^{\otimes n} \otimes A, \qquad n \geq 0.$$

(Again, the tensor products are over $\mathbb{C}$ and the action of A^e is on the outer factors). The differentials are the usual Hochschild differentials:

$$\partial(a_0 \otimes (a_1 \otimes \cdots \otimes a_n) \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i \cdot a_{i+1} \otimes \cdots \otimes a_{n+1}.$$

The multiplication map $B^0(A) = A \otimes A \to A$ gives the resolving quasi-isomorphism $B(A) \to A$.

The object $\mathcal{E}xt(A, A)$ is thus represented by the complex of bimodules $\mathrm{Hom}(B(A), A)$, but in order to see the algebra structure most naturally we should resolve the second factor too, taking the quasi-isomorphic complex $\mathrm{Hom}(B(A), B(A))$. We are really interested in the object $U = \pi_* \mathcal{E}xt(A, A)$, which corresponds to the complex of *left* A–modules $\mathrm{Hom}(B(A), B(A))$ (we just forget the right module structure).

Now we can calculate explicitly the canonical map $\gamma\colon T[-1] \to \pi_* \mathcal{E}xt(A, A)$, which is the adjoint of the universal Atiyah class map α. For this, we use Markarian's lemma 1 (from Section 5).

Since the Hochschild complex corresponds to a complex of trivial sheaves on $X \times X$ (remember that X is still assumed affine), each of them has a global flat connection, namely the trivial connection. $\nabla = d$. In the world of $A-A$–bimodules these connections are given by the maps

$$\nabla\colon B^{-n}(A) \to B^{-n}(A) \otimes_{A^e} \Omega^1_{A^e}$$

which, using the decomposition of the module $\Omega^1_{A^e}$, becomes

$$\nabla\colon A \otimes A^{\otimes n} \otimes A \to (\Omega^1_A \otimes A^{\otimes n} \otimes A) \oplus (A \otimes A^{\otimes n} \otimes \Omega^1_A)$$

and, viewing the left-hand side as a free A^e–module with basis $A^{\otimes n}$, is given by

$$\begin{aligned}\nabla(a_0 \otimes (a_1 \otimes \cdots \otimes a_n) \otimes a_{n+1}) \\ = da_0 \otimes (a_1 \otimes \cdots \otimes a_n) \otimes a_{n+1} + a_0 \otimes (a_1 \otimes \cdots \otimes a_n) \otimes da_{n+1}.\end{aligned}$$

Now the Atiyah class of $\mathcal{O}_\Delta$ is represented by $\partial\nabla - \nabla\partial$, where ∂ represents the Hochschild differential, and we compute explicitly the difference

$$(\partial\nabla - \nabla\partial)(a_0 \otimes (a_1 \otimes \cdots \otimes a_n) \otimes a_{n+1}).$$

It's easy to see that the terms coming from the Hochschild differential when $1 \le i \le n-1$ cancel out, leaving only the outer ($i = 0, n$) terms, and we get the answer

$$a_0 \cdot da_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes a_{n+1} + (-1)^n a_0 \otimes a_1 \otimes \cdots \otimes da_n \cdot a_{n+1}.$$

Since the actual universal Atiyah class α is the part involving $\pi_1^* T$ we get a representation of

$$\alpha \in \mathrm{Hom}_{A^e}(B(A), B(A) \otimes_{A^e} (\Omega^1_A \otimes A[1]))$$

given by

$$\alpha(a_0 \otimes (a_1 \otimes \cdots \otimes a_n) \otimes a_{n+1}) = a_0 \cdot da_1 \otimes (a_2 \otimes \cdots \otimes a_n) \otimes a_{n+1}.$$

Applying the adjunction we see that the map $\gamma\colon T \to U[1]$ is represented by the left module morphism

$$\mathrm{Der}(A, A) \to \mathrm{Hom}^1(B(A), B(A))$$

given by

$$\gamma(\xi)(a_0 \otimes (a_1 \otimes \cdots \otimes a_n) \otimes a_{n+1}) = a_0 \cdot \xi(a_1) \otimes (a_2 \otimes \cdots \otimes a_n) \otimes a_{n+1}.$$

It follows that $\gamma^{\otimes n}\colon T^{\otimes n} \to \mathcal{E}xt(\mathcal{O}_\Delta, \mathcal{O}_\Delta)[n]$ is represented by a morphism

$$\bigotimes_A^n \mathrm{Der}(A, A) \to \mathrm{Hom}^n(B(A), B(A))$$

such that

$$\gamma^{\otimes n}(\xi_1 \otimes \cdots \otimes \xi_n)(a_0 \otimes (a_1 \otimes \cdots \otimes a_n) \otimes a_{n+1}) = a_0 \cdot \xi_1(a_1) \cdot \xi_2(a_2) \cdots \xi_n(a_n) \otimes a_{n+1}.$$

If we compose with the quasi-isomorphism $\mathrm{Hom}(B(A), B(A)) \cong \mathrm{Hom}(B(A), A)$ given by composition on $B^0(A)$ with the multiplication map, we see that $\gamma^{\otimes n}(\xi_1 \otimes \cdots \otimes \xi_n)$ lies in $\mathrm{Hom}(B^{-n}(A), A)$ and is given by

$$a_0 \otimes (a_1 \otimes \cdots \otimes a_n) \otimes a_{n+1} \mapsto a_0 \cdot \xi_1(a_1) \cdot \xi_2(a_2) \cdots \xi_n(a_n) \cdot a_{n+1}$$

Finally we symmetrise over the $X\xi_i$ to obtain the map representing the degree n part of PBW,

$$\textstyle\bigwedge^n T[-n] \to \pi_* \mathcal{E}xt(\mathcal{O}_\Delta, \mathcal{O}_\Delta).$$

But $\mathrm{Hom}_{A^e}(B(A), A)$ is the n–th Hochschild cochain group, and the symmetrised map we obtain is just the standard Hochschild–Kostant–Rosenberg map which defines an isomorphism on cohomology

$$\mathrm{HKR}\colon \textstyle\bigwedge_A^n \mathrm{Der}(A, A)) \to HH^n(A, A).$$

This ends the proof that the PBW map is an isomorphism.

7.5 The universal property of U

To complete the proof of Step (3), we need the following theorem of Hinich and Vaintrob [17].

Theorem 7.2 *Let $\mathcal{C}$ be a linear tensor category admitting infinite direct sums and symmetrisers. Let L be a Lie algebra in $\mathcal{C}$. Then there exists a universal enveloping algebra $L \to U(L)$ of L in $\mathcal{C}$. Furthermore there is a PBW isomorphism $S \cong U(L)$.*

Their proof of this theorem is essentially to start from the symmetric algebra $S(L)$, which exists given the conditions on $\mathcal{C}$, and then to redefine its product using a kind of universal algebraic construction (and the language of operads). See also Deligne and Morgan [10].

Assuming the properties already proved in steps 1, 2, 4 and this theorem, we can now complete the proof of the universal property of our object U. By the Hinich–Vaintrob theorem, we know that in $D(X)$ a universal enveloping algebra $U(L)$ (with the desired universal property) does exist. (We don't need to worry about infinite direct sums: the symmetric algebra in our case is a finite sum.) All we need to do is prove that our object U is isomorphic, as an algebra, to the Hinich–Vaintrob object $U(L)$. This is done by exploiting the universal property of $U(L)$ as follows.

Since U is an associative algebra and $L \to U$ is a Lie algebra homomorphism, this map extends to an algebra homomorphism $U(L) \to U$. If we can prove that this is an isomorphism of objects in $D(X)$ then we are done.

We have also the natural algebra homomorphism $T(L) \to U(L)$ obtained by extending $L \to U(L)$ to a map of algebras, and the symmetrisation morphism (viewing the symmetric algebra as a subspace of the tensor algebra) $S(L) \to T(L)$. Consider the composition of these with our map $U(L) \to U$:

$$S(L) \to T(L) \to U(L) \to U.$$

The composite of the first two maps is the universal PBW isomorphism $S \cong U(L)$ constructed by Hinich and Vaintrob. On the other hand, the composite of the latter two morphisms is the natural map $T(L) \to U$ extending $L \to U$ to a map of algebras, and thus the whole composition is by definition *our* PBW isomorphism $S \to U$. Therefore the final map $U(L) \to U$ is also an isomorphism.

7.6 Invariant parts

The PBW isomorphism between S and U restricts to their invariant parts. In standard Lie theory the invariant part of a module can be thought of as $V^{\mathfrak{g}} \cong \mathrm{Hom}_{\mathfrak{g}}(\mathbb{C}, V)$, and this gives the right way to generalise the notion to the categorical setting: the invariant part of an object $A \in D(X)$ is $\mathrm{Hom}_{D(X)}(\mathbb{O}_X, A)$, which is a cohomology space.

In our context we see that

$$\mathrm{Hom}(\mathbb{O}_X, S) = H^*(\textstyle\bigwedge^* T)$$

is the cohomology of polyvector fields on X, called $\mathrm{HT}^*(X)$ by Kontsevich. The degree k piece is

$$HT^k(X) = \bigoplus_{i+j=k} H^i(\textstyle\bigwedge^j T).$$

It is also worth identifying the invariant part of the symmetric algebra of the *dual* $\Omega[1]$ of the Lie algebra $T[-1]$, which is the usual Dolbeault cohomology of X but with a grading shift: in this context its natural part of degree k is

$$H^0\Big(\bigoplus_j \textstyle\bigwedge^j T^*[k+j]\Big) = \displaystyle\bigoplus_{i-j=k} H^i(\textstyle\bigwedge^j T^*).$$

There is an obvious "cap product" action of this cohomology ring on $\mathrm{HT}^*(X)$.

On the other hand, the invariant part of U is

$$\mathrm{HH}^*(X) = \mathrm{Ext}^*_{X\times X}(\mathbb{O}_\Delta, \mathbb{O}_\Delta),$$

the Ext–algebra of the structure sheaf of the diagonal in $X \times X$, with the Yoneda product as algebra structure. This should be thought of as the Hochschild cohomology of the manifold X: the usual definition of the Hochschild cohomology of an algebra A is as $\mathrm{Ext}^*_{A\otimes A^{\mathrm{op}}}(A, A)$ – that is, the Ext–algebra in the category of $A-A$–bimodules, and the above definition of $\mathrm{HH}^*(X)$ is clearly the sheaf-theoretic analogue.

The PBW isomorphism between S and U induces an isomorphism HKR: $\mathrm{HT}^*(X) \cong \mathrm{HH}^*(X)$. This version of the Hochschild–Kostant–Rosenberg theorem is originally due to Gerstenhaber and Schack [14]. Kontsevich showed how to alter it into an *algebra* isomorphism, and we discuss this in Section 8.

Remark 7.3 Using Hinich and Vaintrob's results [17], the Hochschild cohomology $\mathrm{HH}^*(X)$ can also be described "externally" as the quotient of $\bigoplus_{i+j=n} H^i(X, T^{\otimes j})$ by relations saying that the action of the Atiyah class equals the commutator (ie the relations for a universal enveloping algebra).

7.7 Alternative approach to U

There is a slightly different way to define U using the Grothendieck–Verdier functor. In some ways this is more natural, but it looks even more abstract.

Recall that $\Delta\colon X \to X \times X$ denotes the diagonal embedding and $\pi\colon X \times X \to X$ denotes projection onto the first factor. Then we can set $U = \Delta^! \mathcal{O}_\Delta$. (Recall from Section 4.5 that $\Delta^!$ is the right adjoint of Δ_*.) Then we can see that there are isomorphisms of functors:

$$\Delta_* \cong \mathcal{O}_\Delta \otimes \pi^*(-), \qquad \Delta^! \cong \pi_* \mathcal{E}xt(\mathcal{O}_\Delta, -).$$

This follows from the projection formula and the fact that $\pi \circ \Delta = \mathrm{id}$. We have

$$\mathcal{O}_\Delta \otimes \pi^*(-) \cong \Delta_*(\mathcal{O}_X \otimes \Delta^* \pi^*(-)) \cong \Delta_*(-).$$

So we can write Δ_* as the composition $(\mathcal{O}_\Delta \otimes -) \circ \pi^*$. By the uniqueness of adjoints we can write the right adjoint $\Delta^!$ as the composite of the right adjoints of the components, viz:

$$\Delta^! \cong \pi_* \circ (\mathcal{E}xt(\mathcal{O}_\Delta, -)) = \pi_* \mathcal{E}xt(\mathcal{O}_\Delta, -).$$

Note that the adjunctions, such as $\Delta_* \Delta^! \to \mathrm{id}$, translate into the composition of adjunctions, such as $\mathcal{O}_\Delta \otimes \pi^* \pi_* \mathcal{E}xt(\mathcal{O}_\Delta, -) \to \mathcal{O}_\Delta \otimes \mathcal{E}xt(\mathcal{O}_\Delta, -) \to \mathrm{id}$.

In this approach, to give the action of U via a natural transformation $- \otimes U \to \mathrm{id}$, it suffices to give a map $\Delta_* U \to \mathcal{O}_\Delta$. As $U = \Delta^! \mathcal{O}_\Delta$ we take the adjunction $\eta\colon \Delta_* \Delta^! \mathcal{O}_\Delta \to \mathcal{O}_\Delta$. If U is thought of as $\pi_* \mathcal{E}xt(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$ then the map is the composition $\mathcal{O}_\Delta \otimes \pi^* \pi_* \mathcal{E}xt(\mathcal{O}_\Delta, -) \to \mathcal{O}_\Delta \otimes \mathcal{E}xt(\mathcal{O}_\Delta, -) \to \mathrm{id}$ of the two basic adjunctions.

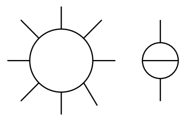
Similarly, the canonical Lie algebra homomorphism $T[-1] \to U$ can be defined as the right adjoint of the universal Atiyah class morphism $\alpha\colon \Delta_* T[-1] \to \mathcal{O}_\Delta$.

Remark 7.4 The associative algebra object U which acts on the objects of $D(X)$ has been constructed from functors on derived categories induced by the diagonal map $\Delta\colon X \to X \times X$ and the projection map $\pi\colon X \times X \to X$. Starting with a finite group G an analogous construction can be performed using functors on representation categories induced by the diagonal map $\Delta\colon G \to G \times G$ and the projection map $\pi\colon G \times G \to G$, in this case the resulting algebra object in the representation category of G which acts on everything in the category is nothing other than the group algebra of G, equipped with the adjoint action. Details of this will appear elsewhere.

8 Further weight systems

We now look at the roles played in Vassiliev theory by the symmetric and universal enveloping algebras of a Lie algebra, and construct weight systems from complex symplectic manifolds in this context. We begin by introducing a new space of diagrams resembling $\mathcal{A}$.

Define $\mathcal{B}$ to be the vector space spanned by not-necessarily connected unitrivalent graphs with the same vertex-orientation convention at their trivalent vertices, and subject to the antisymmetry and IHX relations as before.



Again we use the total number of trivalent and univalent vertices as grading, though we can also *bigrade* the algebra and write $\mathcal{B}^{v,l}$ for the part with v internal trivalent vertices, and l legs. The vector space $\mathcal{B}$ is naturally a commutative algebra via $\amalg$, the disjoint union of diagrams.

There is an isomorphism of graded, complex vector spaces

$$\chi\colon \mathcal{B} \to \mathcal{A}$$

given by taking an l–legged diagram in $\mathcal{B}$ to the average of the $l!$ diagrams obtained by attaching its legs in all possible orders to an oriented circle (see Bar-Natan [3]). The isomorphism χ is not an algebra isomorphism, so it is sometimes convenient to regard $\mathcal{B}$ and $\mathcal{A}$ as one space, using χ, which has two competing products. However there *is* an interesting algebra isomorphism between $\mathcal{A}$ and $\mathcal{B}$ which is described below.

8.1 Further weight systems from Lie algebras

We can now construct further weight systems, and will encapsulate them all in the following theorem. We give a proof in this familiar context, as this proof will go over pretty much exactly to the complex symplectic context in Section 8.3.

Theorem 8.1 (See Bar-Natan [3].) *Suppose that $\mathfrak{g}$ is a finite-dimensional metric Lie algebra. Let $S(\mathfrak{g})$ be its symmetric algebra, and $U(\mathfrak{g})$ be its universal enveloping algebra.*

(1) *There is a graded, multiplicative weight system $w_{\mathfrak{g}}\colon \mathcal{B} \to S(\mathfrak{g})^{\mathfrak{g}}$.*

(2) *There is a graded, multiplicative weight system $w^{\mathfrak{g}}\colon \mathcal{B} \to S(\mathfrak{g}^*)^{\mathfrak{g}}$.*

(3) *Given V a finite-dimensional representation of $\mathfrak{g}$, there is a multiplicative weight system $w_V\colon \mathcal{A} \to \mathrm{End}(V)$; composing with the trace we get a weight system $w_V\colon \mathcal{A} \to \mathbb{C}$.*

(4) *There is a multiplicative weight system $w_{\mathfrak{g}}\colon \mathcal{A} \to U(\mathfrak{g})^{\mathfrak{g}}$. If V is a finite-dimensional representation of $\mathfrak{g}$ then composing with the natural map $U(\mathfrak{g})^{\mathfrak{g}} \to \mathrm{End}(V)$ gives the weight system in* (3) *above.*

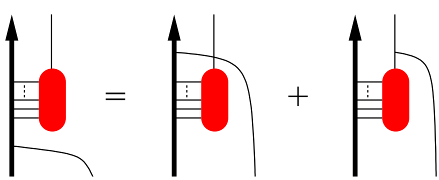
(5) *The maps χ and* PBW *correspond in the sense that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\chi} & \mathcal{A} \\ w_{\mathfrak{g}} \downarrow & & \downarrow w_{\mathfrak{g}} \\ S(\mathfrak{g})^{\mathfrak{g}} & \xrightarrow{\mathrm{PBW}} & U(\mathfrak{g})^{\mathfrak{g}} \end{array}$$

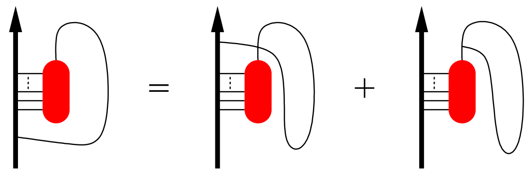
Proof These weight systems are all defined essentially by taking any Morse, planar projection of a representing graph and viewing it as a morphism in the category of $\mathfrak{g}$–modules. That it will be independent of the choice of projection and Morsification is due precisely to the axioms of a Lie algebra object in a category and of modules over it. This is the work of Vogel and Vaintrob. The work here is really in properly identifying the target.

Parts (1) and (2) are straightforward. Given a diagram in $\mathcal{B}$, represent it in the plane with all legs pointing upwards (in case (2), point them downwards and make obvious alterations). The legs will have to be ordered arbitrarily from left to right to do this. The picture defines an element of $\mathrm{Hom}_{\mathfrak{g}}(\mathbb{C}, \mathcal{T}(\mathfrak{g}))$; composing with the canonical map $\mathcal{T}(\mathfrak{g}) \to S(\mathfrak{g})$ gives a result independent of leg ordering. Multiplicativity in the first case follows by placing diagrams side-by-side. In the second case, the target space can be thought of as the algebra $\mathrm{Hom}_{\mathfrak{g}}(\mathbb{C}, S(\mathfrak{g}^*))$ and the map is multiplicative, but this is less important.

For part (3), cut the diagram in $\mathcal{A}$ at some point of its oriented circle, and open it out to an upward-oriented interval, with attached graph drawn to the right. This picture defines an element of $\mathrm{Hom}_{\mathfrak{g}}(V, V)$. The result is independent of the location of the cut by the standard argument from Bar-Natan, which we draw here.



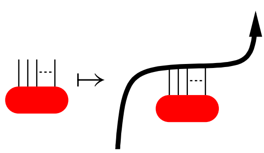
This expresses the fact that the "oval with legs" (representing any graph with legs) is an L–module map.



This follows by applying the Casimir and metric, and untangling the pictures suitably. Note that although $\mathrm{Hom}_{\mathfrak{g}}(V, V)$ need not be a commutative algebra, the image of the weight system is a commutative subalgebra.

Part (4) is only a little different. This time we cut the circle and draw the remaining interval *horizontally*, pointing to the right, with the rest of the graph below it. Removing the oriented interval gives a graph with legs ordered from left to right. This defines an element of $\mathcal{T}(\mathfrak{g})^{\mathfrak{g}}$ which projects to something in $U(\mathfrak{g})^{\mathfrak{g}}$. The IHX relations in $\mathcal{A}$ are clearly respected, and the STU relations also because of the universal property of the canonical morphism $\mathfrak{g} \to U(\mathfrak{g})$. Independence of the point of cutting follows from the same pictorial argument as above.

The comparison with part (3) arises as follows.



Part (5) is now a straightforward check. □

The above construction works for any metric Lie algebra object in a category, so the case of complex symplectic manifolds will follow naturally. One point worth making is that it is clear from this construction that the notion of "invariant part" of a module M should be the hom-set $\mathrm{Hom}_L(\mathbf{1}, M)$.

8.2 The Duflo isomorphism and wheeling

We described earlier the PBW isomorphism between spaces of invariants $S(\mathfrak{g})^{\mathfrak{g}}$ and $U(\mathfrak{g})^{\mathfrak{g}}$. Each of these spaces is a commutative algebra, but the PBW map is not generally an algebra isomorphism. There *is* however an algebra isomorphism, the *Duflo isomorphism*, between $S(\mathfrak{g})^{\mathfrak{g}}$ and $U(\mathfrak{g})^{\mathfrak{g}}$ – for semisimple Lie algebras it is equivalent to Harish-Chandra's isomorphism, but in the more general form it is due to Duflo [12].

To define it, consider the invariant polynomial function $s_i(x) = \mathrm{tr}(\mathrm{ad}(x)^i)$ on $\mathfrak{g}$ as an element of the dual symmetric algebra $S^i(\mathfrak{g}^*)^{\mathfrak{g}}$. This space acts on $S(\mathfrak{g})^{\mathfrak{g}}$ by the symmetrised contraction map. We will think of it as a kind of cap product, and will write $f \cap -\colon S^*(\mathfrak{g})^{\mathfrak{g}} \to S^{*-i}(\mathfrak{g})^{\mathfrak{g}}$ for the action of $f \in S^i(\mathfrak{g}^*)^{\mathfrak{g}}$.

Define the *modified Bernoulli numbers* $\{b_{2i}\}_{i=1}^{\infty}$ by the power series

$$\sum_{i=1}^{\infty} b_{2i} x^{2i} = \frac{1}{2} \log \frac{\sinh(x/2)}{x/2},$$

and define the Duflo power series

$$j^{1/2} = \exp \sum b_{2i} s_{2i}$$

in the completion of $S(\mathfrak{g}^*)^{\mathfrak{g}}$.

Remark 8.2 This function plays a very important role in the Weyl character formula, amongst other things. For a semisimple Lie algebra we could identify the invariant polynomials $S^i(\mathfrak{g}^*)^{\mathfrak{g}}$ with $H^{2i}(BG)$, so that s_i would correspond to $i!$ times the i–th term of the Chern character of the vector bundle on BG corresponding to the adjoint representation. In $H^*(BG)$, it corresponds to the equivariant $\widehat{A}$–genus of the complex adjoint representation [7].

The Wheeling Theorem of Bar-Natan, Le and Thurston [5] is a strange and deep property of the algebras $\mathcal{A}$ and $\mathcal{B}$ which corresponds to the Duflo isomorphism.

The algebra $\mathcal{B}$ acts on itself by a leg-gluing operation which we will here denote by a "cap" notation. Thurston uses a "hat" or "differential operator" notation. This operation is defined on diagrams C and D by

$$C \cap D = \sum \text{all ways of joining all of the legs of } C \text{ to some of the legs of } D.$$

If C has more legs than D then, $C \cap D$ is zero. The capping operation $- \cap -\colon \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}$ is not a graded map, but if $\mathcal{B}$ is given an alternative "Euler characteristic" grading, namely $\mathcal{B}_n = \bigoplus_{l-v=n} \mathcal{B}^{v,l}$, then $- \cap -\colon \mathcal{B}_* \otimes \mathcal{B}^* \to \mathcal{B}^*$ is graded.

Let w_l denote the wheel with l legs and let $\Omega \in \mathcal{B}$ be the *wheeling element* given by the following formula.

$$\Omega := \exp_{\amalg} \sum_{i=1}^{\infty} b_{2i} w_{2i} \in \mathcal{B}.$$

It is in the subspace $\mathcal{B}_0$, so that the *wheeling map* $\Omega \cap -\colon \mathcal{B}^* \to \mathcal{B}^*$ is a graded map. Note that although Ω really lives in the completion of $\mathcal{B}$, there is no "convergence problem" when we define the wheeling map.

Theorem 8.3 (Wheeling theorem [5]) *The composition of the wheeling map $\Omega \cap -$ with the symmetrisation map is an algebra isomorphism $\mathcal{B} \to \mathcal{A}$.*

If $\mathfrak{g}$ is a finite-dimensional metric Lie algebra then we can combine the weight systems and the above isomorphisms into the following commutative diagram, whose top and bottom rows are both algebra isomorphisms. (Note in particular that Ω maps to $j^{1/2}$, a fact originally pointed out in [4].)

$$\begin{array}{ccccc} \mathcal{B} & \xrightarrow{\Omega\cap-} & \mathcal{B} & \xrightarrow{\chi} & \mathcal{A} \\ \downarrow & & \downarrow & & \downarrow \\ S(\mathfrak{g})^{\mathfrak{g}} & \xrightarrow{j^{1/2}\cap-} & S(\mathfrak{g})^{\mathfrak{g}} & \xrightarrow{\mathrm{PBW}} & U(\mathfrak{g})^{\mathfrak{g}} \end{array}$$

Note that this whole diagrammatic method could be used profitably for handling higher graph cohomology and L_∞–algebras – Kapranov uses the language of operads. However, for our present purposes, this extra structure is not important.

8.3 Further weight systems from complex symplectic manifolds

We can now translate Theorem 8.1 directly into the context of complex symplectic manifolds. Thus the following theorem has a parallel proof to that of Theorem 8.1 using the structures discussed in the above subsection.

Theorem 8.4 *Suppose (X, ω) is a complex symplectic manifold.*

(1) *There is a bigraded, multiplicative weight system $\mathrm{RW}_X\colon \mathcal{B}^{*,*} \to H^*(\bigwedge^* T)$.*

(2) *There is a bigraded, multiplicative weight system $\mathrm{RW}^X\colon \mathcal{B}^{*,*} \to H^*(\bigwedge^* \Omega)$.*

(3) *Given an object $A \in \widetilde{D}(X)$, there is a graded multiplicative weight system $\mathcal{A} \to \mathrm{Ext}^*(A, A)$, and composing with the trace we get a weight system $\mathcal{A} \to H^*(\mathcal{O}_X)$.*

(4) *There is a graded, multiplicative weight system $\mathcal{A} \to \mathrm{Ext}^*_{D(X\times X)}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) = \mathrm{HH}^*(X)$. If $A \in D(X)$ and we compose with the natural map $\mathrm{HH}^*(X) \to \mathrm{Hom}_{D(X)}(A, A)$, we recapture the weight system from* (3).

(5) *The HKR map $\mathrm{HT}^*(X) \to \mathrm{HH}^*(X)$ induces the following commutative diagram of vector spaces:*

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\chi} & \mathcal{A} \\ \downarrow & & \downarrow \\ \mathrm{HT}^*(X) & \xrightarrow{\mathrm{HKR}} & \mathrm{HH}^*(X) \end{array}$$

8.4 Wheels and wheeling for complex symplectic manifolds

The wheeling theorem for complex symplectic manifolds takes the following form:

Theorem 8.5 *Let X be a complex symplectic manifold. Then there is a commutative diagram*

$$\begin{array}{ccccc} \mathcal{B} & \xrightarrow{\Omega\cap -} & \mathcal{B} & \xrightarrow{\chi} & \mathcal{A} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{HT}^*(X) & \xrightarrow{\hat{A}^{1/2}\cap -} & \mathrm{HT}^*(X) & \xrightarrow{\mathrm{HKR}} & \mathrm{HH}^*(X) \end{array}$$

in which the two rows are algebra isomorphisms.

Proof The fact that the bottom line is an algebra isomorphism is due to Kontsevich [25]. Note that it holds for any complex manifold.

The intertwining weight system maps are only defined when the manifold is complex symplectic. That the square on the left commutes follows from the following lemma, independently computed by Hitchin and Sawon [19]. □

Lemma 8.6 $RW^X(\Omega) = \hat{A}^{1/2}(TX) \in \bigoplus H^{2k}(\bigwedge^{2k} T^*)$.

Proof All we need is that an l–leg wheel w_l, with its hub oriented and labelled by a locally-free sheaf E and legs pointed downwards maps under the weight system to

$$\mathrm{RW}^{X,E}(w_l) = \mathrm{tr}(\tilde{F}_E^l) \in H^l(\textstyle\bigwedge^l T^*).$$

This is a restated lemma of Atiyah [1]: we are using the Dolbeault point of view, in which $\tilde{F}_E \in \Omega^{1,1}(\mathrm{End}(E))$ is the renormalised curvature form $\tilde{F}_E = -1/2\pi i F_E$ of a smooth hermitian connection on E.

In particular, wheels in the honest algebra $\mathcal{B}$ correspond to the above case when $E = T[-1]$. The only effect of the degree shift here is to make the trace negative (it is really the supertrace of an odd object) and thus the l–wheel gives $-l!\,\mathrm{ch}(TX)$. (We did not specify an orientation on the hub of the wheel, but we can introduce one arbitrarily when comparing the definitions: odd wheels in $\mathcal{B}$ are zero, corresponding to the vanishing of the odd Chern classes of a complex symplectic manifold.)

Now recall that $\Omega \in \mathcal{B}$ is defined as

$$\Omega = \exp \sum b_{2n} w_{2n}$$

with
$$\sum b_{2n} x^{2n} = \frac{1}{2} \log \frac{\sinh(x/2)}{x/2}.$$

Under the weight system, each $2k$–wheel goes to a term $-\operatorname{tr}(\widetilde{F}^{2k})$, where we take trace in the fundamental representation. Because the weight system is multiplicative, disjoint union becomes cup product: we get

$$\exp \operatorname{tr} -\frac{1}{2} \log \frac{\sinh(\widetilde{F}/2)}{\widetilde{F}/2},$$

which is just the Chern–Weil definition of the $\widehat{A}^{1/2}$. □

It is worth pointing out that in general, a wheel whose hub is labelled by an object $A \in D(X)$ maps to the characteristic class $l! \operatorname{ch}_l(A)$, by which we mean the alternating sum of the terms $\operatorname{ch}_l(\mathscr{E})$ for the sheaves in the complex A, multiplied by $l!$. Evaluations of this nature will feature in our future work but are not needed for now.

Note that for a complex symplectic manifold, the $\widehat{A}$ and Todd classes are equal, because $T \cong T^*$ means that the line bundles appearing in the splitting principle occur in conjugate pairs, and the first Chern class is therefore zero. But the $\widehat{A}$ class is the correct one to use, as it appears in Kontsevich's theorem, which holds for *any* complex manifold (even if c_1 is not zero).

The appearance of the class $\widehat{A}^{1/2}$ requires further investigation. Does it have a meaning in index theory: for example, is there an interesting class of manifolds whose $\widehat{A}^{1/2}$–genus is integral? According to Sawon [37], it is not integral for compact hyperkähler manifolds.

9 Ribbon categories and link invariants

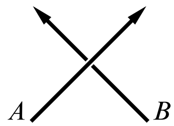
In this section we combine the complex symplectic manifold weight systems with the Kontsevich integral to obtain an interesting ribbon category, from which link invariants may be obtained by the standard methods of Turaev [39].

9.1 Ribbon categories

In Section 3.1 we introduced symmetric tensor categories; braided tensor categories are to the braid groups as symmetric tensor categories are to the symmetric groups. A tensor category is *braided* if there is a natural isomorphism τ, the braiding, between $\otimes$ and $\otimes \circ \sigma$ (where σ is the obvious flip functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$), satisfying the hexagon relation

$$\tau_{A,B\otimes C} = (\mathrm{id}_B \otimes \tau_{A,C}) \circ (\tau_{A,B} \otimes \mathrm{id}_C),$$

where the component $A \otimes B \to B \otimes A$ of the natural isomorphism is written $\tau_{A,B}$. The hexagon would be more visible if we hadn't dropped the associators from the notation. The braiding depicted as follows:



Combining the hexagon condition with naturality of τ yields the Yang–Baxter (or braid, or Reidemeister III) equation

$$(\tau_{B,C} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau_{A,C}) \circ (\tau_{A,B} \otimes \mathrm{id}) = (\mathrm{id} \otimes \tau_{A,B}) \circ (\tau_{A,C} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau_{B,C}).$$

The reader is strongly encouraged to draw the picture. A symmetric tensor category is a braided tensor category in which the square of τ is the identity.

In a braided tensor category, there is an action of the n string braid group on the n–th tensor power of any object. In a symmetric tensor category, this factors through an action of the symmetric group.

A *ribbon category* (or *balanced rigid braided tensor category*) is a braided tensor category with a *twist* θ which is a natural automorphism of the identity functor that commutes with duality and interacts with tensor product according to the formula

$$\theta_{A \otimes B} = \tau_{B,A} \tau_{A,B} (\theta_A \otimes \theta_B).$$

Using this it is possible to make natural isomorphisms $A \cong A^{**}$ and $A^* \cong {}^*A$ in ways compatible with tensor product, and which can be neglected notationally.

The idea behind ribbon categories is that morphisms are thought of as being two-sided ribbons, rather than strings. The twist θ represents a full-twist of the ribbon and is illustrated diagrammatically as below left or sometimes as on the right as it is easier to draw.

The reader is invited to discover the topological identity lurking in the tensor product interaction described above. Essentially by definition, a ribbon category gives rise to invariants of framed links in the following way. If the components of a framed link are coloured with objects in the ribbon category then any morse diagram of the link can be interpreted as a morphism from the unit object to itself. This element of $\mathrm{Hom}(\mathbf{1}, \mathbf{1})$ is an invariant of the coloured, framed, oriented link. Quantum invariants can be obtained in the this fashion using representation categories of quantum groups.

9.2 The ribbon structure of $\widetilde{D}(X)$

The construction of an interesting ribbon structure on $\widetilde{D}(X)$ is parallel to the construction of an interesting braided structure on the category of modules over a finite-dimensional metric Lie algebra using the Knizhnik–Zamolodchikov equation. One starts with the usual (symmetric) category $\mathfrak{g}$–mod, tensors with $\mathbb{C}[\![\hbar]\!]$, and uses the KZ equation to introduce a new, interesting braiding structure. The resulting category turns out to be equivalent to the category of representations of the quantum group $U_\hbar\mathfrak{g}$. See Drinfel′d [11] and Bakalov and Kirillov [2], for example.

In our case, we start with the derived category $D(X)$ of a complex symplectic manifold. We know this is a symmetric tensor category under derived tensor product and the usual flip map, with the structure sheaf as identity, and is a ribbon category when we bring in the derived duals of objects and the natural contraction maps. We "tensor with $\mathbb{C}[\![\hbar]\!]$", replacing $D(X)$ by the extended version $\widetilde{D}(X)$, in which the shift [2] plays the role of $\hbar$. Then we use the Kontsevich integral, in the tangle-functor version of Le and Murakami, to introduce the ribbon structures. The result is that the derived category of coherent sheaves on an complex symplectic manifold can be quantized in exactly the same way as the category of representations of a metric Lie algebra. This is our final main theorem.

Theorem 9.1 *For a complex symplectic manifold X, the extended derived category $\widetilde{D}(X)$ has a natural nonsymmetric ribbon tensor category structure.*

Proof All we have to do is define the various structure morphisms and check the identities. Explicitly, we need to compute the associator $\Phi_{A,B,C}$, the braiding $\tau_{A,B}$, the duality morphisms $\epsilon_A, \iota_A, \epsilon'_A, \iota'_A$ and the twist coefficients θ_A.

According to Le and Murakami [26], the Kontsevich integral defines a representation of the category of *nonassociative tangles* (also known as *quasi-* or *q–tangles*). This category is generated by morphisms corresponding to exactly the things we need above, and their explicit Kontsevich integrals can be found in Le and Murakami. Each is a formal power series of diagrams consisting of chords based on an underlying collection of oriented intervals: three for Φ, two for τ, and one for the other morphisms. For example, Φ has an expression as a power series in the two diagrams shown below, composed vertically and sometimes thought of as noncommuting indeterminates:

Having obtained these power series, we label the vertical strings of the diagram by objects of $\widetilde{D}(X)$ and evaluate using the weight systems. In particular, given two objects A, B of $\widetilde{D}(X)$, let $H_{A,B} \in \mathrm{Hom}_{\widetilde{D}(X)}(A \otimes B, A \otimes B)$ and $C_A \in \mathrm{Hom}_{\widetilde{D}(X)}(A, A)$ be the morphisms corresponding to the following graphs:

Each is really an element of Ext^2, that is $H_{A,B} \in \mathrm{Ext}^2(A \otimes B, A \otimes B)$ and $C_A \in \mathrm{Ext}^2(A, A)$. These "chord" and "Casimir" elements are really all we need to evaluate the Kontsevich integrals. For example, the new braiding morphism $\tau_{A,B}$ may be described as

$$\tau_{A,B} = \tau_{\mathrm{old}} \circ \exp(H_{A,B}/2) \in \mathrm{Ext}^*(A \otimes B, B \otimes A) = \mathrm{Hom}_{\widetilde{D}(X)}(A \otimes B, B \otimes A),$$

where τ_{old} is the original symmetric braiding. The associator $\Phi_{A,B,C}$ is written as a polynomial in the noncommuting variables $H_{A,B} \otimes \mathrm{id}_C$, $\mathrm{id}_A \otimes H_{B,C}$. (Note that the power series become truncated because of the boundedness of the Ext–groups.) The other morphisms depend directly on the Casimir. For example the framing twist is

$$\theta_A = \exp(C_A/2) \in \mathrm{Ext}^*(A, A) = \mathrm{Hom}_{\widetilde{D}(X)}(A, A).$$

The fact that all the relations of a ribbon category are satisfied is then automatic from the topological invariance of the Kontsevich integral of framed oriented tangles. □

A few remarks are now in order. Firstly, one can multiply the symplectic form by $\hbar$ and then "take the limit $\hbar \to 0$" to recapture the original symmetric structure on $\widetilde{D}(X)$.

Secondly, we don't know whether it is possible to make a "gauge transformation" (in the manner of Drinfel′d) to a form which is strictly associative but has a more complicated braiding (as in the case of quantum groups). This theorem (discussed in [2]) does not seem to have a purely geometrical formulation which can be carried over into our context.

Thirdly, the above construction actually goes through for a holomorphic Casimir manifold. In fact, only chord diagrams are used in the construction, so a complex manifold X with a suitable "r–matrix" would be sufficient.

Finally, we observe that the braided structure is completely different from the braid group actions on derived categories constructed by Seidel and Thomas or Rouquier.

10 Conclusion

In the table below we present a dictionary giving a translation between the worlds of Chern–Simons theory (derived from usual Lie algebras) and Rozansky–Witten theory (derived from complex symplectic manifolds).

	Chern–Simons	Rozansky–Witten
Category	Vector spaces	$D(X)$, derived category of X
Lie algebra object	$\mathfrak{g}$	$T[-1]$, shifted tangent bundle
Modules	$\rho\colon \mathfrak{g} \to \mathrm{End}(V)$	objects A of $D(X)$
Invariant part of enveloping algebra	$Z(\mathfrak{g}) \cong U(\mathfrak{g})^{\mathfrak{g}}$	$\mathrm{Ext}^*_{X\times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$
Invariant part of symmetric algebra	$S(\mathfrak{g})^{\mathfrak{g}}$	$H^*(X, \bigwedge^* T)$
Wheeling theorem	$S(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\cong} Z(\mathfrak{g})$ Duflo	$H^*(X, \bigwedge^* T) \xrightarrow{\cong} \mathrm{Ext}^*_{X\times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$ Kontsevich
Invariant metric	$\mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$	$\omega \in \Gamma(\bigwedge^2 T^*)$
Universal knot invariant	$\{\text{knots}\} \to Z(\mathfrak{g})[\![h]\!]$	$\{\text{knots}\} \to \mathrm{Ext}^*_{X\times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$
Knot invariant from module	$\{\text{knots}\} \to \mathbb{C}[\![h]\!]$	$\{\text{knots}\} \to H^*(X, \mathcal{O})$
Ribbon category	$U_\hbar(\mathfrak{g})$–modules	$\widetilde{D}(X)$

Table 1: Dictionary between Chern–Simons Theory and Rozansky–Witten Theory

This table only goes as far as the knot invariants arising in each theory, but it should extend into a correspondence between the full TQFTs (a sketch of this appears in [34]).

There are many interesting questions arising from the existence of the Rozansky–Witten invariants, their similarity with Chern–Simons constructions, and their potential applications in knot theory. In Roberts and Sawon [35] we mentioned many of these, so there is little point repeating them here.

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