

On Davis–Januszkiewicz homotopy types II: Completion and globalisation

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For any finite simplicial complex K , Davis and Januszkiewicz defined a family of homotopy equivalent CW-complexes whose integral cohomology rings are isomorphic to the Stanley–Reisner algebra of K . Subsequently, Buchstaber and Panov gave an alternative construction, which they showed to be homotopy equivalent to the original examples. It is therefore natural to investigate the extent to which the homotopy type of a space X is determined by such a cohomology ring. Having analysed this problem rationally in Part I, we here consider it prime by prime, and utilise Lannes’ T -functor and Bousfield–Kan type obstruction theory to study the p -completion of X . We find the situation to be more subtle than for rationalisation, and confirm the uniqueness of the completion whenever K is a join of skeleta of simplices. We apply our results to the global problem by appealing to Sullivan’s arithmetic square, and deduce integral uniqueness whenever the Stanley–Reisner algebra is a complete intersection.

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1 Introduction

For any finite simplicial complex K , the Stanley–Reisner algebra $\mathbb{Z}[K]$ is an important combinatorial invariant (see Stanley [14]), and may be graded by assigning dimension 2 to each of its generators. The corresponding R -algebra $R[K]$ is defined over any commutative ring R as $R \otimes \mathbb{Z}[K]$, and a topological space X realises $R[K]$ whenever $H^*(X; R)$ is isomorphic to $R[K]$ as graded R -algebras. In their pioneering work on toric topology [5], Davis and Januszkiewicz construct a family of realisations of $\mathbb{Z}[K]$ for every K , and show each of them to be homotopy equivalent to a certain universal example. We refer to a generic representative of this homotopy type as a *Davis–Januszkiewicz space* $\text{DJ}(K)$.

Questions of uniqueness then arise, and suggest that we investigate the relationship between the geometric properties of K and the number of homotopy types which realise $\mathbb{Z}[K]$. As usual, the problem is best approached by dealing separately with

its rational and p -adic versions, and applying Sullivan's arithmetic square to recover global information. We consider the rational situation in our paper [11], and study the case in which $\mathbb{Q}[K]$ is a complete intersection. The rational homotopy type of any realisation of $\mathbb{Q}[K]$ is then unique, up to weak equivalence of nilpotent spaces.

Our purpose here is to address the p -adic version of the problem. We obtain uniqueness statements for a somewhat larger class of complexes K , from which we deduce global uniqueness for complete intersections by appealing to the arithmetic square. So far, we have little evidence to suggest that these results may be extended to more general K , either p -adically or globally; nor, however, do we have any example of an exotic Davis–Januszkiewicz space that realises $\mathbb{Z}[K]$ yet fails to be homotopy equivalent to $\text{DJ}(K)$. On the other hand, a simple cohomological argument proves that the suspension of any such space *is* equivalent to $\Sigma \text{DJ}(K)$, for every K .

We refer readers to [11] for detailed background and prerequisites, and summarise the crucial points in Section 2 below, where we also state our main results explicitly. Unlike the rational version, we have found no helpful formulation of the p -adic problem in terms of model category theory. Nevertheless, we work in several algebraic and geometric categories which admit model structures, and appeal regularly to the language of homotopy colimits. We insist, for example, that our spaces lie in the model category TOP of k -spaces and continuous maps (see Vogt [16]). Sometimes we deal with based CW-complexes $(X, *)$, which lie in the pointed category TOP^+ ; then $*$ is a distinguished 0-cell and its inclusion into X is a cofibration.

So far as notation is concerned, we let p denote a fixed but arbitrary prime throughout our work. We usually abbreviate the cyclic group \mathbb{Z}/p to C in the interests of notational simplicity, and write \mathbb{Z}_p^\wedge for the p -adic integers. We denote the field of p elements by \mathbb{F}_p and the topological group of unimodular complex numbers by T , in order to distinguish them from the underlying group \mathbb{Z}/p and the underlying circle S^1 respectively. Over any commutative ring R with identity, we interpret the polynomial algebra generated by a set V of graded independent variables as the symmetric algebra $S_R(V)$ on V . Finally, for every pair of objects x_0 and x_1 of an arbitrary category \mathcal{C} , we write $\mathcal{C}(x_0, x_1)$ for the set of morphisms $x_0 \rightarrow x_1$.

For any space X we consider the p -adic completion X_p^\wedge described by Bousfield and Kan [2], which agrees with Sullivan's original p -completion in all our cases. Given a p -complete space X_K realising $\mathbb{Z}_p^\wedge[K]$, our underlying strategy is to develop methods of comparing X_K with a canonical representative $\text{hc}(K)$ for $\text{DJ}(K)$, and each of Sections 3 to 8 takes steps towards this goal. We organise the programme as follows.

In Section 3, we begin by assuming that X_K realises $\mathbb{F}_p[K]$, and apply Lannes' T -functor to compute the mod p cohomology ring of certain components of the mapping

space $\text{map}(BC^\sigma, X_K)$ for appropriate elementary abelian p -groups C^σ . We extend our computations to p -adic cohomology in Section 4, by taking advantage of the fact that the mod p ring is zero in odd dimensions. In Section 5 we develop the corresponding results for certain components of $\text{map}(BT^\sigma, X_K)$, by interpreting T_p^\wedge as a limit of cyclic subgroups \mathbb{Z}/p^r .

In order to bring these calculations to bear on the central problem, we provide a brief survey of higher limits of algebraic $\text{CAT}(K)$ -diagrams in Section 6. These feature prominently in the obstruction theory associated to Bousfield and Kan's spectral sequence for homotopy limits, which we investigate in Section 7; in particular, we show that there are no obstructions to defining a homotopy equivalence $\text{hc}(K)_p^\wedge \rightarrow X_K$ when K is the skeleton of a simplex, and deduce p -adic uniqueness in such cases. We extend the results to iterated joins in Section 8, where we also confirm that homotopy classes of self-equivalences of X_K are classified by their action on cohomology. Finally, in Section 9, we employ the arithmetic square to combine our p -adic conclusions with the rational calculations of [11], and prove global uniqueness for X_K when K is a complete intersection. The latter is equivalent to identifying K as the iterated join of a simplex with boundaries of simplices.

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2 Preliminaries and main results

We shall work with a canonical representative for $\text{DJ}(K)$, constructed as the homotopy colimit of a diagram of topological spaces. Our methods depend upon the cohomological and homotopy theoretic properties of such diagrams, so we recall the crucial definitions and notation in this section. Readers may consult [11] for further motivation and details.

We consider an ordered set V of vertices v_1, \dots, v_m . A *simplicial complex* K on $V = V(K)$ is given by a finite set of faces $\sigma \subseteq V$, which is closed with respect to the formation of subsets. The dimension $\dim(\sigma)$ of any face is given in terms of its cardinality by $|\sigma| - 1$, and $\dim K$ is the maximum dimension attained by its faces. We assume that the empty face \emptyset belongs to K unless otherwise stated, and write K^\times when we wish to emphasise that it has been omitted. The most basic

example is the simplex $\Delta(V)$; it consists of all possible subsets of V , and therefore contains any K on V as a subcomplex. Its r -skeleton $\Delta^{(r)}(V)$ includes those faces $\sigma \subset V$ of dimension $\leq r$, for any $0 \leq r \leq m - 1$. In particular, $\Delta^{(m-2)}(V)$ is the boundary $\partial(V) \subset \Delta(V)$, constructed by deleting the maximal face V . For any two disjoint simplicial complexes K and L , their *join* $K * L$ on $V(K) \cup V(L)$ has faces $\{\sigma \cup \tau : \sigma \in K, \tau \in L\}$; thus K and L are full subcomplexes.

Every face $\sigma \in K$ determines several subcomplexes. In particular, we need the simplex $\Delta(\sigma)$ on σ , and the *link* and *star* of σ , defined by

$$\ell_K(\sigma) := \{\tau \setminus \sigma : \sigma \subseteq \tau \in K\} \quad \text{and} \quad \text{st}_K(\sigma) := \{\tau : \sigma \cup \tau \in K\};$$

so $\text{st}_K(\sigma)$ coincides with $\Delta(\sigma) * \ell_K(\sigma)$ as subcomplexes of K . Given any two complexes K and L , we note that

$$\ell_{K*L}(\sigma \cup \tau) = \ell_K(\sigma) * \ell_L(\tau) \quad \text{and} \quad \text{st}_{K*L}(\sigma \cup \tau) = \text{st}_K(\sigma) * \text{st}_L(\tau)$$

as subcomplexes of $K * L$, for every face $\sigma \in K$ and $\tau \in L$.

The vertices masquerade as algebraically independent variables of degree 2, and generate a graded polynomial algebra $S_R(V)$. For any subset $\omega \subseteq V$, we abbreviate the square-free monomial $\prod_{\omega} v_i$ to v_ω , which has degree $2|\omega|$. We then define the *Stanley–Reisner algebra* $R[K]$ as $S_R(V)/(v_\omega : \omega \notin K)$; so the inclusions of the full subcomplexes $K, L \subset K * L$ induce an isomorphism

$$(1) \quad R[K] \otimes R[L] \xrightarrow{\cong} R[K * L].$$

Since $R[\Delta(\sigma)] = S_R(\sigma)$ for every face σ , an example is provided by

$$(2) \quad S_R(\sigma) \otimes R[\ell_K(\sigma)] \xrightarrow{\cong} R[\text{st}_K(\sigma)].$$

Any subcomplex $J \subseteq K$ induces a surjection $R[K] \rightarrow R[J]$, by annihilating the faces in $K \setminus J$; in particular, $K \subseteq \Delta(V)$ induces the canonical projection $S_R(V) \rightarrow R[K]$.

Every K determines a finite category $\text{CAT}(K)$, whose objects are the faces σ and morphisms the inclusions $i_{\sigma,\tau} : \sigma \subseteq \tau$. The empty face is initial, so the classifying space $B\text{CAT}(K)$ is contractible, whereas $B\text{CAT}(K^\times)$ is homeomorphic to the geometric realisation $|K|$. The maximal faces μ are characterised by the fact that they admit only identity morphisms. By construction, there is an isomorphism $\text{CAT}(K) \times \text{CAT}(L) \cong \text{CAT}(K * L)$ for any K and L , which we use to identify $\text{CAT}(\Delta(\sigma)) \times \text{CAT}(\ell_K(\sigma))$ with $\text{CAT}(\text{st}_K(\sigma))$ as necessary.

An *A*-*diagram* in an arbitrary category \mathbf{R} consists of a covariant functor $D : \mathbf{A} \rightarrow \mathbf{R}$ for some small category \mathbf{A} .

Definitions 2.1 For any based CW–complex $(X, *)$, the $\text{CAT}(K)$ –diagram

$$X^K: \text{CAT}(K) \longrightarrow \text{TOP}$$

assigns the cartesian product X^σ to each face σ , where $X^K(\emptyset) = *$; its value on $i_{\sigma,\tau}$ is the cofibration $X^\sigma \rightarrow X^\tau$, where the extra coordinates are set to $*$. The constant functor cst_{X^V} assigns $X^V \cong X^m$ to each face, and the identity map id_{X^V} to each inclusion. The natural transformation $i_K: X^K \rightarrow \text{cst}_{X^V}$ is induced by the inclusions $i_\sigma: X^\sigma \rightarrow X^V$, for every face σ .

We are interested in two particular values of X , which stem from the inclusion of the cyclic subgroup C in the circle T . To any subset $\sigma \subseteq V$, there corresponds the inclusion of the elementary abelian p –subgroup C^σ in the torus T^σ , whose classifying map represents the projection of a product of infinite dimensional lens spaces onto the corresponding product of complex projective spaces. We shall denote this map by $t_\sigma: BC^\sigma \rightarrow BT^\sigma$, and assume that it is a homomorphism of abelian topological groups, written multiplicatively. It may be interpreted as a natural transformation

$$(3) \quad t: BC^K \longrightarrow BT^K.$$

The colimit $c(K) := \text{colim}_{\text{CAT}(K)} BT^K$ lies in the category TOP^+ of pointed topological spaces, and is a subcomplex of BT^V via i_K .

Following Buchstaber and Panov [4], we view $c(K)$ as a distinguished representative for $\text{DJ}(K)$. In [11, Section 3], this property is expressed in terms of isomorphisms

$$(4) \quad H^*(c(K); R) \xrightarrow{\cong} \lim S_R(K) \xrightarrow{\cong} R[K],$$

where $S_R(K)$ is the $\text{CAT}^{\text{op}}(K)$ –diagram of graded commutative R –algebras whose value on σ is $S_R(\sigma)$, and on $\tau \supseteq \sigma$ is the canonical projection $p_{\tau,\sigma}$.

In order to investigate the homotopy theoretical properties of $c(K)$, it is natural to consider the *homotopy colimit* $\text{hc}(K) := \text{hocolim}_{\text{CAT}(K)} BT^K$. Following Hollender and Vogt’s elaboration [8] of the original definition of Bousfield and Kan [2], we describe $\text{hc}(K)$ as the bar construction $B(*, \text{CAT}(K), BT^K)$. As such, it lies in TOP^+ . The fact that BT^K is cofibrant in the category of $\text{CAT}(K)$ –diagrams [11, Lemma 2.7] ensures that the natural projection $\text{hc}(K) \rightarrow c(K)$ is a homotopy equivalence, and that either space may be used as a model for $\text{DJ}(K)$.

A deeper analysis of $\text{hc}(K)$ involves certain secondary structures associated to $\text{CAT}(K)$, such as the *undercategory* $\sigma \downarrow \text{CAT}(K)$. For any face σ of K , this is obtained by restricting attention to those objects τ for which $\sigma \subseteq \tau$. The *overcategory* $\text{CAT}(K) \downarrow \sigma$

is defined by analogy, and $(\sigma \downarrow \text{CAT}(K))^{\text{op}}$ is isomorphic to $\text{CAT}^{\text{op}}(K) \downarrow \sigma$. Setting $P(\rho) := \rho \setminus \sigma$ defines a functor

$$(5) \quad P: \sigma \downarrow \text{CAT}(K) \longrightarrow \text{CAT}(\ell_K(\sigma)),$$

which is also an isomorphism. Writing the restriction of X^K to $\sigma \downarrow \text{CAT}(K)$ as $X^{\sigma \downarrow K}$ and taking appropriate colimits then yields homeomorphisms

$$(6) \quad \text{colim } X^{\sigma \downarrow K} \cong X^\sigma \times \text{colim } X^{\ell_K(\sigma)} \cong \text{colim } X^{\text{st}_K(\sigma)},$$

where the former is induced by P . In case $X = BT$, we apply $H^*(-; R)$ and appeal to (4) to obtain isomorphisms

$$(7) \quad \lim S_R(\sigma \downarrow \text{CAT}(K)) \cong S_R(\sigma) \otimes R[\ell_K(\sigma)] \cong R[\text{st}_K(\sigma)]$$

of graded commutative algebras.

To organise our proofs, it is convenient to formalise the following, in which $X = X_K$ denotes a space that realises $\mathbb{Z}[K]$ for any choice of K .

Definitions 2.2 If an isomorphism $\theta_X: H^*(X; \mathbb{Z}) \rightarrow \mathbb{Z}[K]$ is given, then the space X reflects K , and θ_X is the reflector. If θ_X is given with coefficients R , then X reflects K over R .

For any subcomplex $J \subseteq K$, a map $e: W_J \rightarrow X_K$ reflects the pair (K, J) whenever the square

$$(8) \quad \begin{array}{ccc} H^*(X_K; \mathbb{Z}) & \xrightarrow{e^*} & H^*(W_J; \mathbb{Z}) \\ \theta_X \downarrow & & \downarrow \theta_W \\ \mathbb{Z}[K] & \xrightarrow{p_{K,J}} & \mathbb{Z}[J] \end{array}$$

commutes, where $p_{K,J}$ denotes the canonical surjection.

For example, the classifying space BT^V reflects $\Delta(V)$ by means of the standard isomorphism $H^*(BT^V; \mathbb{Z}) \rightarrow S_{\mathbb{Z}}(V)$. More generally, the colimit $c(K)$ reflects K for any complex K , and the inclusion $c(K) \subset BT^V$ reflects the pair $(\Delta(V); K)$. This may be extended to any other X_K by interpreting BT^V as an Eilenberg–Mac Lane space $H(\mathbb{Z}^V, 2)$. Then θ_X determines a unique homotopy class of maps $q_X: X_K \rightarrow BT^V$, which represents the m -tuple of generators in $H^2(X_K, \mathbb{Z})$ and reflects $(\Delta(V), K)$ as before. Composing q_X with projection onto BT^σ creates a map $q_\sigma: X_K \rightarrow BT^\sigma$ for every face σ ; its image in integral cohomology realises the polynomial subalgebra $\mathbb{Z}[\sigma] \subset \mathbb{Z}[K]$, via θ_X .

Canonical reflectors $\theta_{\text{hc}(K)}: H^*(\text{hc}(K)) \rightarrow \mathbb{Z}[K]$ arise by taking limits over $\text{CAT}^{\text{op}}(K)$, and have good functorial properties. So we may often interchange $H^*(c(K))$ and $\mathbb{Z}[K]$ without further comment.

Given any e reflecting (K, J) , there is a homotopy commutative square

$$(9) \quad \begin{array}{ccc} W_J & \xrightarrow{e} & X_K \\ q_W \downarrow & & \downarrow q_X \\ BT^{V(J)} & \xrightarrow{j_{J,K}} & BT^{V(K)} \end{array}$$

where $j_{J,K}$ denotes coordinatewise inclusion. This reflects the subpair

$$(K, J) \subseteq (\Delta(V(K)), \Delta(V(J))),$$

in the sense that it induces a commutative cube in cohomology when combined with the four reflectors.

For any noetherian local graded commutative algebra A over \mathbb{Q} , a choice of homogeneous generators establishes a presentation

$$0 \longrightarrow I_A \longrightarrow S \longrightarrow A \longrightarrow 0,$$

where S is a finitely generated graded polynomial algebra and $I_A \subset S$ is a graded ideal. Then A is a *complete intersection* whenever I_A is generated by a regular sequence of homogeneous elements; this definition is independent of the choice of generators by [3, Theorem 2.3.3]. We note in [11, Section 5] that the Stanley–Reisner algebra $\mathbb{Q}[K]$ is a complete intersection precisely when K is an iterated join of the form $\Delta(U_0) * \partial(U_1) * \dots * \partial(U_t)$, for any partition $\{U_0, U_1, \dots, U_t\}$ of V . In this case, an isomorphism

$$(10) \quad S_{\mathbb{Q}}(U_0) \otimes \mathbb{Q}[\partial(U_1)] \otimes \dots \otimes \mathbb{Q}[\partial(U_t)] \cong \mathbb{Q}[K]$$

is given by (1) above.

Theorem 9.4 *Let X be a nilpotent CW–complex, and suppose that $\mathbb{Q}[K]$ is a complete intersection; then there is an isomorphism $\theta: H^*(X; \mathbb{Z}) \rightarrow \mathbb{Z}[K]$ if and only if there is a homotopy equivalence $f: \text{hc}(K) \rightarrow X$ such that $f^* = \theta$.*

We prove Theorem 9.4 by using the arithmetic square to combine the rational results of [11] with the following p –adic statement, which refers to a larger class of complexes.

Theorem 8.6 *Let X be a p –complete CW–complex, and suppose that K is an iterated join $\Delta^{(r_1)}(U_1) * \dots * \Delta^{(r_t)}(U_t)$ of skeleta of simplices; then there is an isomorphism $\theta: H^*(X; \mathbb{Z}_p^\wedge) \rightarrow \mathbb{Z}_p^\wedge[K]$ if and only if there is a homotopy equivalence $f: \text{hc}(K)_p^\wedge \rightarrow X$ such that $f^* = \theta$.*

After suspension, however, the situation becomes much simpler.

Theorem 9.5 *Let X be a CW-complex that realises $\mathbb{Z}[K]$ for any K ; then there is a homotopy equivalence $e: \Sigma X \rightarrow \Sigma \text{hc}(K)$.*

Our final result on completions is a homotopy classification of self-equivalences of $\text{hc}(K)$. We do not expect a global version to hold, even under the assumptions of Theorem 9.4.

Theorem 8.7 *For complexes K as in Theorem 8.6, any pair of self-equivalences $f, g: \text{hc}(K)_p^\wedge \rightarrow \text{hc}(K)_p^\wedge$ are homotopic if and only if $H^*(f; \mathbb{Z}_p^\wedge) = H^*(g; \mathbb{Z}_p^\wedge)$.*

For these K , it is also of interest to apply [7, Theorem 10.2(3)], and deduce that the t -fold suspension $\Sigma^t \mathcal{Z}_K$ of the *moment-angle complex* [5; 4] is homotopy equivalent to a wedge of spheres.

3 Stanley–Reisner algebras and the T -functor

We begin by focusing our attention on the mod p Stanley–Reisner algebra $\mathbb{F}_p[K]$, where K is a specific simplicial complex with generic face σ . It is convenient to denote $H^*(-; \mathbb{F}_p)$ by $H^*(-)$ throughout this section, and to abbreviate $H^*(BE)$ to HE for an arbitrary elementary abelian p -group E . Often, E will be of the form C^σ for some face σ of K .

For any space X reflecting K over \mathbb{F}_p , our aim is to construct maps $BE \rightarrow X$ with prescribed cohomological properties. We proceed by studying the mapping space $\text{map}(BE, X)$, and take advantage of the fact that its mod p cohomology ring may be computed by Lannes' functor T_E . We refer to Lannes [10] and Schwartz [13] for the basic properties of T_E , which we summarise below. We follow their notation by writing \mathcal{U} for the category of unstable modules over the mod p Steenrod algebra \mathcal{A}_p , and \mathcal{K} for the subcategory of unstable algebras over \mathcal{A}_p . In order to avoid technical difficulties, we shall only consider algebras that are finitely generated; thus $\mathcal{K}(A, A')$ is a finite set for any algebras A and A' .

By definition, T_E is left adjoint to the functor $HE \otimes -: \mathcal{U} \rightarrow \mathcal{U}$. The latter is compatible with the forgetful functor $\mathcal{K} \rightarrow \mathcal{U}$, and $T_E: \mathcal{K} \rightarrow \mathcal{K}$ is left adjoint to its restriction. By construction, T_C^n coincides with the n -fold iterate $T_C \cdots T_C$ for any n , and T_E preserves tensor products in \mathcal{U} and \mathcal{K} .

For any space Y , evaluation defines a map $BE \times \text{map}(BE, Y) \rightarrow Y$. Passing to cohomology and left adjointing yields a \mathcal{K} –morphism

$$(11) \quad a_Y: T_E H^*(Y) \longrightarrow H^*(\text{map}(BE, Y)),$$

which is an isomorphism whenever Y is p –complete and $T_E H^*(Y)$ has finite type and vanishes in degree 1. So T_E computes the mod p cohomology of the mapping space under these conditions [10, Theorem 3.2.1].

The splitting of $\text{map}(BE, Y)$ into the disjoint union $\coprod \text{map}(BE, Y)_f$ of its connected components is mirrored by the algebraic properties of T_E , as follows. For each unstable algebra A and \mathcal{K} –morphism $\phi: A \rightarrow HE$, its adjoint restricts to a map $T_E^0 A \rightarrow \mathbb{F}_p$ on the zero-dimensional part. We write $\mathbb{F}_p(\phi)$ when it is necessary to emphasise the induced $T_E^0 A$ –module structure on \mathbb{F}_p . The component of $T_E A$ associated to ϕ is defined by

$$(12) \quad T_E(A, \phi) := T_E A \otimes_{T_E^0 A} \mathbb{F}_p(\phi),$$

and Lannes’ linearisation principle [13, Theorem 3.8.6] leads to splittings

$$T_E A \cong \bigoplus_{\phi \in \mathcal{K}(A, HE)} T_E(A, \phi) \quad \text{and} \quad T_E^0 A \cong \bigoplus_{\phi \in \mathcal{K}(A, HE)} \mathbb{F}_p(\phi).$$

In particular, we may identify $T_E^0 A$ with the p –Boolean algebra $\mathbb{F}_p^{\mathcal{K}(A, HE)}$.

In the case that $A = H^*(Y)$ and $\phi = f^*$ for some $f: BE \rightarrow Y$, the morphism a_Y of (11) restricts to a \mathcal{K} –morphism

$$(13) \quad a_f: T_E(H^*(Y), f^*) \longrightarrow H^*(\text{map}(BE, Y)_f).$$

Since the algebraic components of $T_E H^*(Y)$ correspond bijectively to the connected components of $\text{map}(BE, Y)$, it follows that a_Y splits accordingly.

The following key example is based on the proof of [13, Proposition 9.8], in which $\text{map}(BE, BT^\omega)$ is identified with $E(\omega) \times BT^\omega$, where $E(\omega)$ denotes the discrete group of homomorphisms $E \rightarrow E^\omega$ for any subset $\omega \subseteq V$.

Example 3.1 As \mathcal{A}_p –algebras, $HT^\omega := H^*(BT^\omega)$ is canonically isomorphic to $S_{\mathbb{F}_p}(\omega)$, and multiplication on BT^ω induces a coproduct d . The \mathcal{K} –morphisms $\phi: HT^\omega \rightarrow HE$ correspond bijectively to group homomorphisms $E \rightarrow T^\omega$, and hence to $E(\omega)$. The coaugmentation $c_\phi: HT^\omega \rightarrow T_E(HT^\omega, \phi)$ is an isomorphism in \mathcal{K} , whose inverse r_ϕ is the adjoint of $(\phi \otimes 1) \circ d$.

For each $f: BE \rightarrow BT^\omega$, the composition

$$(14) \quad HT^\omega \xrightarrow{c_f} T_E(HT^\omega, f^*) \xrightarrow{a_f} H^*(\text{map}(BE, BT^\omega)_f)$$

computes the mod p cohomology of the mapping space, and is induced by evaluation map $(BE, BT^\omega)_f \rightarrow BT^\omega$ at the base point. Its inverse is induced by the adjoint of the action $s_f: BE \times BT^\omega \rightarrow BT^\omega$, defined by $(b, u) \mapsto uf(b)$.

For any \mathcal{K} -morphism $\pi: A_1 \rightarrow A_2$, the induced homomorphism

$$T_E^0(\pi): \bigoplus_{\phi \in \mathcal{K}(A_1, HE)} \mathbb{F}_p(\phi) \cong T_E^0 A_1 \longrightarrow T_E^0 A_2 \cong \bigoplus_{\psi \in \mathcal{K}(A_2, HE)} \mathbb{F}_p(\psi)$$

is determined by composition $\circ \pi: \mathcal{K}(A_2, HE) \rightarrow \mathcal{K}(A_1, HE)$ with π . For any $\phi \in \mathcal{K}(A_1, HE)$, we denote the subset $(\circ \pi)^{-1}(\phi) \subset \mathcal{K}(A_2, HE)$ by $\text{ext}(\phi, \pi)$; its elements ψ satisfy $\psi \circ \pi = \phi$. We may then identify $T_E^0(\pi)$ with the direct sum of diagonal maps of the form $\mathbb{F}_p(\phi) \rightarrow \bigoplus_{\psi \in \text{ext}(\phi, \pi)} \mathbb{F}_p(\psi)$, and obtain an isomorphism

$$(15) \quad T_E^0 A_2 \otimes_{T_E^0 A_1} \mathbb{F}_p(\phi) \cong \bigoplus_{\psi \in \text{ext}(\phi, \pi)} \mathbb{F}_p(\psi).$$

For any unstable algebra A in \mathcal{K} we write $A\text{-}\mathcal{U}$ for the category whose objects M are simultaneously A -modules and unstable \mathcal{A}_p -modules, and whose structure maps $A \otimes M \rightarrow M$ are \mathcal{A}_p -linear; here \mathcal{A}_p acts on $A \otimes M$ via the Cartan formula. The morphisms in $A\text{-}\mathcal{U}$ are both A - and \mathcal{A}_p -linear. Since the T -functor commutes with tensor products, T_E extends to a functor $A\text{-}\mathcal{U} \rightarrow (T_E A)\text{-}\mathcal{U}$, which we also label T_E . For every morphism $\phi: A \rightarrow HE$, we may then define the relative component

$$T_E(M, \phi) := T_E M \otimes_{T_E^0 A} \mathbb{F}_p(\phi)$$

by analogy with (12), and obtain a splitting $T_E M \cong \bigoplus_{\phi \in \mathcal{K}(A, HE)} T_E(M, \phi)$. Furthermore, the coaugmentation $A \rightarrow T_E(A, \phi)$ makes $T_E(M, \phi)$ into an A -module, and $T_E(-, \phi)$ becomes an endofunctor on $A\text{-}\mathcal{U}$.

In particular, the morphism π of (15) turns A_2 into an object of $A_1\text{-}\mathcal{U}$, where there are isomorphisms

$$\begin{aligned} T_E(A_2, \phi) &\cong T_E A_2 \otimes_{T_E^0 A_1} \mathbb{F}_p(\phi) \\ &\cong T_E A_2 \otimes_{T_E^0 A_2} T_E^0 A_2 \otimes_{T_E^0 A_1} \mathbb{F}_p(\phi) \\ &\cong T_E A_2 \otimes_{T_E^0 A_2} \left(\bigoplus_{\psi \in \text{ext}(\phi, \pi)} \mathbb{F}_p(\psi) \right) \\ &\cong \bigoplus_{\psi \in \text{ext}(\phi, \pi)} T_E(A_2, \psi). \end{aligned}$$

One additional property of the T -functor is that T_E commutes with finite limits; in other words, for any finite category C and diagram $F: C \rightarrow \mathcal{K}$, there exists a \mathcal{K} -isomorphism $T_E(\lim F) \cong \lim T_E F$. In this situation, we are interested in computing the components of $T_E(\lim F)$. To avoid technical difficulties we assume that $F(c)$ is finitely generated as an algebra for every object c of C , and similarly for $A := \lim F$.

The limit is defined by an exact sequence of the form

$$0 \longrightarrow A \longrightarrow \prod_c F(c) \xrightarrow{\delta} \prod_{c_0 \rightarrow c_1} F(c_1),$$

and the natural projections $\pi_c: A \rightarrow F(c)$ turn δ into an A – \mathcal{U} –morphism. For every $\phi: A \rightarrow HE$ the functor $T_E(-, \phi)$ is exact on A – \mathcal{U} , because it is a retract of T_E . We therefore obtain an exact sequence

$$0 \longrightarrow T_E(A, \phi) \longrightarrow \prod_c T_E(F(c); \phi) \longrightarrow \prod_{c_0 \rightarrow c_1} T_E(F(c_1), \phi),$$

which identifies $T_E(\lim_c F, \phi)$ with $\lim_c T_E(F, \phi)$.

If $\text{ext}(\phi, \pi_c)$ is empty, then $T_E(F(c), \phi) = 0$. So we consider the full subcategory $C\langle F, \phi \rangle$ of C , whose objects are defined by insisting that $\text{ext}(\phi, \pi_c)$ be nonempty. For any morphism $c' \rightarrow c$ with c in $C\langle F, \phi \rangle$, it follows that c' also lies in $C\langle F, \phi \rangle$; so there exists an isomorphism

$$(16) \quad T_E(\lim_c F, \phi) \cong \lim_{C\langle F, \phi \rangle} T_E(F, \phi).$$

We apply these considerations to the diagram $S_{\mathbb{F}_p}(K): \text{CAT}^{\text{op}}(K) \rightarrow \mathcal{K}$ of (4), whose objects are interpreted as \mathcal{A}_p –algebras via Example 3.1. In particular, $\lim_{\text{CAT}^{\text{op}}(K)} S_{\mathbb{F}_p}(K)$ is isomorphic to $\mathbb{F}_p[K]$, and the \mathcal{K} –morphism $\pi_\sigma: \mathbb{F}_p[K] \rightarrow S_{\mathbb{F}_p}(\sigma)$ is the standard projection, for every $\sigma \in K$. We focus on the \mathcal{K} –morphisms $\phi_\sigma: \mathbb{F}_p[K] \rightarrow HC^\sigma$, defined in terms of (3) by $t_\sigma^* \circ \pi_\sigma$.

Proposition 3.1 *For any face σ of K , there is an isomorphism*

$$r_\sigma: T_{C\sigma}(\mathbb{F}_p[K], \phi_\sigma) \longrightarrow \mathbb{F}_p[\text{st}_K(\sigma)]$$

as objects of \mathcal{K} .

Proof The set $\text{ext}(\phi_\sigma, \pi_\tau)$ contains the single element $\phi_{\tau, \sigma}: S_{\mathbb{F}_p}(\tau) \rightarrow HC^\sigma$, defined by $t_\sigma^* \circ p_{\tau, \sigma}$, if and only if $\sigma \subseteq \tau$. Otherwise, $\text{ext}(\phi_\sigma, \pi_\tau)$ is empty; so for any particular σ , we may identify $\text{CAT}(K)\langle S_{\mathbb{F}_p}(K), \phi_\sigma \rangle$ with $\sigma \downarrow \text{CAT}(K)$.

For $\sigma \subseteq \tau$, Example 3.1 provides isomorphisms

$$r_\sigma(\tau): T_{C\sigma}(S_{\mathbb{F}_p}(\tau), \phi_{\tau, \sigma}) \longrightarrow S_{\mathbb{F}_p}(\tau)$$

that are functorial in τ , and so define an isomorphism

$$r_\sigma(K): T_{C\sigma}(S_{\mathbb{F}_p}(K), \phi_\sigma) \longrightarrow S_{\mathbb{F}_p}(\sigma \downarrow \text{CAT}(K))$$

of $\text{CAT}^{\text{op}}(K) \downarrow \sigma$ –diagrams in \mathcal{K} . Then take limits, using (7) and (16). □

Corollary 3.2 *For every maximal face μ of K , there exists a \mathcal{K} -isomorphism $r_\mu: T_{C^\mu}(\mathbb{F}_p[K], \phi_\mu) \rightarrow S_{\mathbb{F}_p}(\mu)$.*

Proof Since μ is maximal, its star is the simplex $\Delta(\mu)$. □

We now introduce the crucial map $f_\sigma: BC^\sigma \rightarrow X$, and prove the main results of this section. It is convenient to write the space $\text{map}(BC^\sigma, X)_{f_\sigma}$ as $m^\sigma(X)$, and evaluation at the identity of BC^σ as $\text{ev}: m^\sigma(X) \rightarrow X$.

Theorem 3.3 *If X is p -complete and reflects K by means of an isomorphism $\theta: H^*(X) \rightarrow \mathbb{F}_p[K]$, then*

- (1) *there exists a map $f_\sigma: BC^\sigma \rightarrow X$ such that $f_\sigma^* = \phi_\sigma \circ \theta$, and it is uniquely determined up to homotopy;*
- (2) *the space $m^\sigma(X)$ is p -complete, and reflects $\text{st}_K(\sigma)$ over \mathbb{F}_p ;*
- (3) *the map $\text{ev}: m^\sigma(X) \rightarrow X$ reflects $(K, \text{st}_K(\sigma))$ over \mathbb{F}_p ;*
- (4) *for maximal μ , there is a homotopy equivalence $m^\mu(X) \simeq (BT^\mu)_p^\wedge$.*

Proof (1) Every \mathcal{K} -morphism $H^*(X) \rightarrow HC^\sigma$ may be realised uniquely up to homotopy by [10, Théorème 3.1.1], because X is p -complete and $H^*(X)$ has finite type.

(2) By [10, Proposition 3.4.4], $m^\sigma(X)$ inherits the property of completeness from X . Also, $T_{C^\sigma}(H^*(X), \phi_\sigma)$ is of finite type and vanishes in degree 1, by Proposition 3.1. So $a_\sigma := a_{f_\sigma}: T_{C^\sigma}(H^*(X), \phi_\sigma) \rightarrow H^*(m^\sigma(X))$ of (13) is an isomorphism, and the required reflector is $\theta_{m^\sigma(X)} := r_\sigma \circ T_{C^\sigma}(\theta) \circ a_\sigma^{-1}$.

(3) By construction, $\text{ev}^*: H^*(X) \rightarrow H^*(m^\sigma(X))$ is the composition of f_σ^* with the coaugmentation c_σ for T_{C^σ} . So we may combine (2) with the functorial properties of T to obtain a commutative ladder

$$(17) \quad \begin{array}{ccccc} H^*(X) & \xrightarrow{c_\sigma} & T_{C^\sigma}(H^*(X), \phi_\sigma) & \xrightarrow{a_\sigma} & H^*(m^\sigma(X)) \\ \theta \downarrow & & \downarrow T_{C^\sigma}(\theta) & & \downarrow \theta_{m^\sigma(X)} \\ \mathbb{F}_p[K] & \xrightarrow{c_\sigma} & T_{C^\sigma}(\mathbb{F}_p[K], \phi_\sigma) & \xrightarrow{r_\sigma} & \mathbb{F}_p[\text{st}_K(\sigma)] \end{array}$$

whose outer rectangle is a commutative square of the form (8); hence ev is the required reflector.

(4) It suffices to combine (2), Corollary 3.2, and the fact that the homotopy type of $(BT^\mu)_p^\wedge$ is determined by its mod p cohomology. □

Remarks 3.2 (1) The mod p cohomology of X of Theorem 3.3 is concentrated in even dimensions, so $H^{\text{ev}}(X; \mathbb{Z}_p^\wedge)$ is torsion free and $H^{\text{od}}(X; \mathbb{Z}_p^\wedge)$ is zero. Since Theorem 3.3(3) implies that evaluation induces an epimorphism in mod p cohomology, the same must hold over \mathbb{Z}_p^\wedge .

(2) We may replace ϕ_σ in Proposition 3.1 with the homomorphism induced by the constant map $\text{cst}: BC^\sigma \rightarrow X$, which we write as $\kappa_\sigma: \mathbb{F}_p[K] \rightarrow HC^\sigma$. Then $C\langle F, \kappa_\sigma \rangle = C$, and we obtain an isomorphism $T_{C^\sigma}(\mathbb{F}_p[K], \kappa_\sigma) \rightarrow \mathbb{F}_p[K]$. It follows that $\text{ev}: \text{map}(BC^\sigma, X)_{\text{cst}} \rightarrow X$ induces an isomorphism in mod p cohomology, by adapting the proofs of Theorem 3.3(2) and (3); it is therefore an equivalence, because the source and target are p -complete.

4 Mapping spaces and p -adic cohomology: I

In this section we extend our calculations to p -adic cohomology for the mapping space $m^\sigma(X)$ of Theorem 3.3(2). We insist that X is p -complete and 1-connected, and reflects K over \mathbb{Z}_p^\wedge by means of an isomorphism $\theta: H^*(X; \mathbb{Z}_p^\wedge) \rightarrow \mathbb{Z}_p^\wedge[K]$. The reduction of θ mod p is therefore a reflector over \mathbb{F}_p , and Theorem 3.3 applies.

For our first result, we consider the evaluation map of Theorem 3.3(3), and abbreviate $V(\text{st}_K(\sigma))$ to $V_K(\sigma)$ throughout.

Lemma 4.1 *For any face $\sigma \in K$, the map $q_{m^\sigma(X)}$ may be chosen so as to make the diagram*

$$(18) \quad \begin{array}{ccc} m^\sigma(X) & \xrightarrow{\text{ev}} & X \\ q_{m^\sigma(X)} \downarrow & & \downarrow q_X \\ (BT^{V_K(\sigma)})_p^\wedge & \xrightarrow{j_\sigma} & (BT^V)_p^\wedge \end{array}$$

homotopy commutative, where j_σ denotes coordinatewise inclusion.

Proof Define $q_{m^\sigma(X)}$ by composing $q_X \circ \text{ev}$ with projection onto $(BT^{V_K(\sigma)})_p^\wedge$. The diagram induced by (18) in mod p cohomology is then commutative, by Theorem 3.3(3), and it remains to prove that $\text{ev}^* q_X^*(v_i)$ is also zero in $H^2(m^\sigma(X); \mathbb{Z}_p^\wedge)$ for any vertex $v_i \notin V_K(\sigma)$.

Let $q_i: X \rightarrow BT_p^\wedge$ represent v_i . The Serre spectral sequence of the induced spherical fibration

$$(19) \quad T_p^\wedge \longrightarrow F_i \longrightarrow X$$

shows that $H^*(F_i; \mathbb{F}_p)$ has finite type, and F_i is p -complete because q_i is a map of 1-connected p -complete spaces. Also, $q_i \circ f_\sigma: BC^\sigma \rightarrow BT_p^\wedge$ is null-homotopic for any $v_i \notin V_K(\sigma)$, so f_σ lifts to a map $f_i: BC^\sigma \rightarrow F_i$. By [15], the functor $\text{map}(BC^\sigma, -)$ converts $F_i \rightarrow X \rightarrow BT_p^\wedge$ into a fibration

$$\text{map}(BC^\sigma, F_i)_{f_i} \xrightarrow{g_i} m^\sigma(X) \xrightarrow{k_i} \text{map}(BC^\sigma, BT_p^\wedge)_{\text{cst}},$$

whose base is homotopy equivalent to BT_p^\wedge by Example 3.1. We abbreviate the fibre to $m^\sigma(F_i)$, and note that it is path connected and p -complete, because k_i is a map of 1-connected p -complete spaces; and $H^*(m^\sigma(F_i); \mathbb{F}_p)$ is of finite type, because the same holds for $H^*(m^\sigma(X); \mathbb{F}_p)$. We may then deduce from [10, Proposition 3.4.4] that T_{C^σ} computes $H^*(m^\sigma(F_i); \mathbb{F}_p)$. But the Gysin sequence of (19) confirms that $H^*(F_i; \mathbb{Z}_p^\wedge)$ is torsion-free, so the same is also true for $H^*(m^\sigma(F_i); \mathbb{Z}_p^\wedge)$ by [9, Theorem A].

Now $k_i^* v_i \in H^2(m^\sigma(X); R)$ is the Euler class $e(k_i)$ of the principal T_p^\wedge -fibration k_i for any coefficients R , and equals $\text{ev}^* q_X^*(v_i)$ by construction. Thus $e(k_i)$ vanishes mod p for every $v_i \notin V(\text{st}_K(\sigma))$, and we need only prove that the same is true p -adically. The mod p Gysin sequence shows that $g_i^*: H^*(m^\sigma(X); \mathbb{F}_p) \rightarrow H^*(m^\sigma(F_i); \mathbb{F}_p)$ is a monomorphism; and there are isomorphisms $H^*(Y; \mathbb{F}_p) \cong H^*(Y; \mathbb{Z}_p^\wedge) \otimes \mathbb{F}_p$ for $Y = m^\sigma(X)$ and $m^\sigma(F_i)$, because $H^*(Y; \mathbb{Z}_p^\wedge)$ is torsion-free in both cases. Then Nakayama’s lemma confirms that g_i^* is also a p -adic monomorphism, and $e(k_i) = 0$ as sought. □

Corollary 4.2 *For any maximal face $\mu \in K$, the map $q_{m^\mu(X)}$ is a homotopy equivalence and determines a homotopy commutative square*

$$\begin{array}{ccc} m^\mu(X) & \xrightarrow{\text{ev}} & X \\ q_{m^\mu(X)} \downarrow & & \downarrow q_X \\ (BT^\mu)_p^\wedge & \xrightarrow{j_\sigma} & (BT^V)_p^\wedge. \end{array}$$

Proof The statement follows from Lemma 4.1 by setting $\sigma = \mu$, and identifying $q_{m^\mu(X)}$ as a homotopy equivalence by Theorem 3.3(4). □

Definition 4.3 For any maximal face μ , the map $h_\mu: BT^\mu \rightarrow X$ is the composition of $\text{ev} \circ q_{m^\mu(X)}^{-1}$ with completion. For any other face σ , the map $h_\sigma: BT^\sigma \rightarrow X$ is $h_\mu \circ i_{\sigma, \mu}$ for some choice of $\mu \supseteq \sigma$.

Remarks 4.1 (1) In Section 5, we shall show that the homotopy class of h_σ is independent of the choice of μ , and therefore that the diagram

$$(20) \quad \begin{array}{ccc} BT_{h_\sigma}^\sigma & \xrightarrow{i_{\sigma,\tau}} & BT^\tau \\ & \searrow & \swarrow h_\tau \\ & X & \end{array}$$

is homotopy commutative for every $\sigma \subset \tau$. In fact h_μ reflects the pair $(K, \Delta(\mu))$ over \mathbb{Z}_p^\wedge by construction, so the diagram induced by (20) is certainly commutative in p -adic cohomology.

(2) In [6], Dehon and Lannes study maps $BT^r \rightarrow Y$ for spaces such that $H^*(Y; \mathbb{Z})$ is torsion free and concentrated in even degrees. They deduce that $H^*(-; \mathbb{Z})$ classifies such maps up to homotopy. Their proof establishes a similar result for the p -adic cohomology of p -complete spaces, and may also be adapted to show that the homotopy class of h_σ is independent of μ .

(3) For any face σ , Theorem 3.3(1) provides a homotopy

$$(21) \quad h_\sigma \circ t_\sigma \simeq f_\sigma: BC^\sigma \longrightarrow X.$$

Theorem 4.4 *If X is p -complete and reflects K over \mathbb{Z}_p^\wedge , then*

- (1) *the space $m^\sigma(X)$ reflects $\text{st}_K(\sigma)$ over \mathbb{Z}_p^\wedge ;*
- (2) *the map $\text{ev}: m^\sigma(X) \rightarrow X$ reflects the pair $(K, \text{st}_K(\sigma))$ over \mathbb{Z}_p^\wedge .*

Proof (1) For any pair of faces $\sigma \subseteq \tau$, write $i_{\sigma,\tau} \circ t_\sigma$ as $t_{\sigma,\tau}: BC^\sigma \rightarrow BT^\tau$. Then (21) confirms that $h_\tau \circ t_{\sigma,\tau}: BC^\sigma \rightarrow X$ is homotopic to f_σ . Passing to mapping spaces and employing Example 3.1 defines a composition

$$l_{\sigma,\tau}: BT^\tau \xrightarrow{e_{\sigma,\tau}} \text{map}(BC^\sigma, BT^\tau)_{t_{\sigma,\tau}} \xrightarrow{h_\tau \circ} m^\sigma(X),$$

where $e_{\sigma,\tau}(u)$ acts by $b \mapsto ut_{\sigma,\tau}(b)$, and $\text{ev} \circ l_{\sigma,\tau} \simeq h_\tau$. Since $H^*(\text{ev}; \mathbb{Z}_p^\wedge)$ is epic by Remarks 3.2(1), all choices of h_τ induce the same homomorphism $H^*(l_{\sigma,\tau}; \mathbb{Z}_p^\wedge)$. So for any $\sigma \subseteq \tau \subseteq \rho$ we obtain a commutative triangle

$$(22) \quad \begin{array}{ccc} & H^*(m^\sigma(X); \mathbb{Z}_p^\wedge) & \\ & \swarrow & \searrow \\ H^*(BT^\rho; \mathbb{Z}_p^\wedge) & \xrightarrow{\quad} & H^*(BT^\tau; \mathbb{Z}_p^\wedge), \end{array}$$

thus defining a $(\text{CAT}^{\text{op}}(K) \downarrow \sigma)$ -diagram $HT^{\sigma \downarrow K}$ of graded commutative \mathbb{Z}_p^\wedge -algebras, and an induced homomorphism

$$(23) \quad \theta': H^*(m^\sigma(X); \mathbb{Z}_p^\wedge) \longrightarrow \lim HT^{\sigma \downarrow K}.$$

By (7), the target is isomorphic to $\mathbb{Z}_p^\wedge[\text{st}_K(\sigma)]$.

Now consider the homomorphism $\theta'': H^*(m^\sigma(X); \mathbb{F}_p) \rightarrow \mathbb{F}_p[\text{st}_K(\sigma)]$, given by reducing θ' mod p ; it may be identified with the isomorphism $\theta_{m^\sigma(X)}$ of (17) by noting the commutativity of the square

$$\begin{array}{ccc} T_{C\sigma}(H^*(X; \mathbb{F}_p), \phi_\sigma) & \xrightarrow{a_\sigma} & H^*(m^\sigma(X); \mathbb{F}_p) \\ T_{C\sigma}(\pi_\tau \circ \theta) \downarrow & & \downarrow I_{\sigma, \tau}^* \\ T_{C\sigma}(S_{\mathbb{F}_p}(\tau), \phi_{\tau, \sigma}) & \xrightarrow{r_\sigma(\tau)} & S_{\mathbb{F}_p}(\tau) \end{array}$$

for any $\tau \supseteq \sigma$, and taking limits over $\text{CAT}^{\text{op}}(K) \downarrow \sigma$. Applying the exact sequence $\mathbb{Z}_p^\wedge \xrightarrow{p} \mathbb{Z}_p^\wedge \rightarrow \mathbb{F}_p$ yields a commutative ladder of short exact sequences

$$(24) \quad \begin{array}{ccccc} H^*(m^\sigma(X); \mathbb{Z}_p^\wedge) & \xrightarrow{p} & H^*(m^\sigma(X); \mathbb{Z}_p^\wedge) & \longrightarrow & H^*(m^\sigma(X); \mathbb{F}_p) \\ \theta' \downarrow & & \theta' \downarrow & & \downarrow \theta'' \\ \mathbb{Z}_p^\wedge[\text{st}_K(\sigma)] & \xrightarrow{p} & \mathbb{Z}_p^\wedge[\text{st}_K(\sigma)] & \longrightarrow & \mathbb{F}_p[\text{st}_K(\sigma)] \end{array}$$

because $H^*(m^\sigma(X); \mathbb{Z}_p^\wedge)$ is free. But θ'' is an isomorphism, so θ' is also an isomorphism by the snake lemma, and is therefore the required reflector.

(2) The homomorphism $\text{ev}^*: H^*(X; \mathbb{Z}_p^\wedge) \rightarrow H^*(m^\sigma(X); \mathbb{Z}_p^\wedge)$ induces morphisms of triangles (22) over the inclusion functor $\sigma \downarrow \text{CAT}(K) \rightarrow \text{CAT}(K)$. The required square commutes, by Lemma 4.1. □

Corollary 4.5 *If $\text{st}_K(\sigma)$ coincides with K , then $\text{ev}: m^\sigma(X) \rightarrow X$ is a homotopy equivalence and the reflectors satisfy $\theta' \circ \text{ev}^* = \theta_X$ over \mathbb{Z}_p^\wedge .*

Proof Setting $\sigma \downarrow \text{CAT}(K) = \text{CAT}(K)$ in the proof of Theorem 4.4(2) shows that ev^* is an isomorphism of p -adic cohomology, as required. □

Remark 4.2 In the special case $X = c(K)$ of Theorem 4.4, a more precise description of $m^\sigma(X)$ is possible by appeal to Example 3.1. The action $s_{t_{\sigma, \tau}}: BC^\sigma \times BT^\tau \rightarrow BT^\tau$ is given by $(b, u) \mapsto ut_{\sigma, \tau}(b)$ for every $\tau \supseteq \sigma$, and extends to a morphism $BC^\sigma \times BT^{\sigma \downarrow K} \rightarrow BT^K$ of diagrams over the inclusion $\sigma \downarrow \text{CAT}(K) \rightarrow \text{CAT}(K)$. Taking colimits and adjoining then creates a map

$$c(\text{st}_K(\sigma)) \longrightarrow m^\sigma(c(K)),$$

which induces the isomorphism θ'' of (24) in mod p cohomology. So its p -completion is a homotopy equivalence.

5 Mapping spaces and p -adic cohomology: II

In this section we prove that the homotopy class of the map h_σ of Definition 4.3 is uniquely determined, and compute the p -adic cohomology of $\text{map}(BT^\sigma, X)_{h_\sigma}$. We retain the prevailing notation, whereby X is p -complete, simply connected, and reflects K over \mathbb{Z}_p^\wedge by means of an isomorphism θ .

We focus on the action s_{t_σ} of BC^σ on BT^σ , given by $(b, u) \mapsto ut_\sigma(b)$ in Example 3.1 and denoted here by s_σ . There is a principal BC^σ -fibration

$$(25) \quad BC^\sigma \xrightarrow{j_\sigma} BC^\sigma \times BT^\sigma \xrightarrow{s_\sigma} BT^\sigma,$$

where j_σ includes the kernel of s_σ and is given by $b \mapsto (b^{-1}, t_\sigma(b))$. Any map $g: BT^\sigma \rightarrow X$ may be factorised through evaluation at the identity as

$$(26) \quad BT^\sigma \xrightarrow{g'} \text{map}(BC^\sigma, X)_{g \circ t_\sigma} \xrightarrow{\text{ev}} X,$$

where g' is the adjoint of $g \circ s_\sigma$, and satisfies $g'(u)(c) = g(ut_\sigma(c))$ for all $u \in BT^\sigma$ and $c \in BC^\sigma$.

Lemma 5.1 *The map*

$$\text{map}(BT^\sigma, X)_g \longrightarrow \text{map}(BT^\sigma, \text{map}(BC^\sigma, X)_{g \circ t_\sigma})_{g'}$$

induced by s_σ is a homotopy equivalence, whose inverse is induced by evaluation at the identity of BC^σ .

Proof In the context of (25), $g \circ s_\sigma \circ j_\sigma$ is null-homotopic; it follows from Remarks 3.2(2) that $\text{ev}: \text{map}(BC^\sigma, X)_{g \circ s_\sigma \circ j_\sigma} \rightarrow X$ is a homotopy equivalence. So Zabrodsky’s Lemma [18, 3.1] implies that $\circ s_\sigma: \text{map}(BT^\sigma, X) \rightarrow \text{map}(BC^\sigma \times BT^\sigma, X)_L$ is also an equivalence, where L denotes the set of homotopy classes whose composition with s_σ is trivial. Restricting to the component of g and adjoining yields the required equivalence; by construction, evaluation at the basepoint is a right inverse. \square

By analogy with (25), BC^σ also acts on $\text{map}(BC^\sigma, X)_{g \circ t_\sigma}$ according to the formula $(b, f) \mapsto fb$, where $fb(c) := f(bc)$.

Lemma 5.2 *The map $g': BT^\sigma \rightarrow \text{map}(BC^\sigma, X)_{g \circ t_\sigma}$ is BC^σ -equivariant.*

Proof For $(b, c) \in BC^\sigma \times BC^\sigma$, we have $g'(ut_\sigma(b))(c) = g(ut_\sigma(b)t_\sigma(c))$ by (26), and $g'b(c) = g(ut_\sigma(bc))$; these agree, as t_σ is a homomorphism. \square

We now insist that g induces the homomorphism $H^*(h_\sigma; \mathbb{Z}_p^\wedge)$, and write $g = g_\sigma$ to emphasise this choice. Thus $g_\sigma \circ t_\sigma \simeq f_\sigma$ by Theorem 3.3(1), and $\text{map}(BC^\sigma, X)_{g_\sigma \circ t_\sigma}$ becomes $m^\sigma(X)$. Applying p -adic cohomology to (26) shows that $H^*(g'_\sigma; \mathbb{Z}_p^\wedge) = H^*(h'_\sigma; \mathbb{Z}_p^\wedge)$ by Remarks 3.2(1), so we may study our uniqueness problem in terms of maps $BT^\sigma \rightarrow m^\sigma(X)$, or equivalently, by restricting attention to subcomplexes of the form $\text{st}_K(\sigma) \subseteq K$.

We may apply Lemma 4.1 to construct a map $q_\sigma: m^\sigma(X) \rightarrow (BT^\sigma)_p^\wedge$ such that $q_\sigma \circ g'_\sigma$ is completion, and consider the homotopy fibration

$$(27) \quad X_\ell \xrightarrow{i} m^\sigma(X) \xrightarrow{q_\sigma} (BT^\sigma)_p^\wedge.$$

Combining the Serre spectral sequence for i with Theorem 4.4 shows that i reflects $(\text{st}_K(\sigma), \ell_K(\sigma))$ over \mathbb{Z}_p^\wedge . We take θ' of (23) as the reflector for $m^\sigma(X)$, and its quotient by the appropriate ideals as the reflector for X_ℓ .

As a consequence of Lemma 5.2, we may apply the Borel construction to g'_σ , and obtain a homotopy pullback diagram of principal BC^σ -fibrations

$$(28) \quad \begin{array}{ccc} BT^\sigma & \xrightarrow{g'_\sigma} & m^\sigma(X) \\ p \downarrow & & \downarrow \bar{p} \\ BT^\sigma & \xrightarrow{\bar{g}'_\sigma} & \bar{m}^\sigma(X) \end{array}$$

where $\bar{m}^\sigma(X) := EBC^\sigma \times_{BC^\sigma} m^\sigma(X)$ and p denotes the p -th power homomorphism. So $\bar{m}^\sigma(X)$ is simply connected, and applying $\pi_2(-)$ to (28) yields the commutative square

$$(29) \quad \begin{array}{ccc} \mathbb{Z}^\sigma & \longrightarrow & (\mathbb{Z}^\sigma)_p^\wedge \oplus (\mathbb{Z}^{V(\ell_K(\sigma))})_p^\wedge \\ p_* \downarrow & & \downarrow p_* \oplus 1 \\ \mathbb{Z}^\sigma & \longrightarrow & (\mathbb{Z}^\sigma)_p^\wedge \oplus (\mathbb{Z}^{V(\ell_K(\sigma))})_p^\wedge \end{array}$$

where the horizontal homomorphisms include the completed direct summand $(\mathbb{Z}^\sigma)_p^\wedge$, and p_* is multiplication by p .

We now identify $\pi_2(\bar{m}^\sigma(X))$ with $H^2(\bar{m}^\sigma(X); \mathbb{Z}_p^\wedge)$ by the universal coefficient theorem, and deduce the existence of a map $\bar{q}_\sigma: \bar{m}^\sigma(X) \rightarrow (BT^\sigma)_p^\wedge$ such that $\bar{q}_\sigma \circ \bar{g}'_\sigma$ is completion. The homotopy fibre of \bar{q}_σ is X_ℓ because $\bar{q}_\sigma \circ \bar{p}$ and $p \circ q_\sigma$ are homotopic, and the Serre spectral sequence of the fibration $X_\ell \rightarrow \bar{m}^\sigma(X) \rightarrow (BT^\sigma)_p^\wedge$ shows

that $H^*(\bar{m}^\sigma(X); \mathbb{Z}_p^\wedge)$ is of finite type, torsion free, and generated by 2–dimensional classes.

Lemma 5.3 *The space $\bar{m}^\sigma(X)$ reflects $\text{st}_K(\sigma)$ over \mathbb{Z}_p^\wedge ; the reflector $\bar{\theta}'$ is determined by the commutative square*

$$\begin{CD} H^*(\bar{m}^\sigma(X); \mathbb{Z}_p^\wedge) @>\bar{\theta}'>> S_{\mathbb{Z}_p^\wedge}(\sigma) \otimes \mathbb{Z}_p^\wedge[\ell_K(\sigma)] \\ @V\bar{p}^*VV @VVp\otimes 1V \\ H^*(m^\sigma(X); \mathbb{Z}_p^\wedge) @>\theta'>> S_{\mathbb{Z}_p^\wedge}(\sigma) \otimes \mathbb{Z}_p^\wedge[\ell_K(\sigma)] \end{CD}$$

where p acts on the polynomial generators of $S_{\mathbb{Z}_p^\wedge}(\sigma)$ as multiplication by p .

Proof The Serre spectral sequence for the p –adic cohomology of the fibration \bar{p} of (28) collapses at the E_2 –page, and shows that \bar{p}^* is a monomorphism. It is given in dimension 2 by the dual of (29), and the result follows. \square

At this point it is helpful to work for a while with an arbitrary p –complete space Y that reflects $\text{st}_K(\sigma)$ over \mathbb{Z}_p^\wedge . We write the reflector as θ_Y , on the understanding that it agrees with θ' whenever Y is $m^\sigma(X)$; in these circumstances, we also identify $h_\sigma: BT^\sigma \rightarrow Y$ with $h'_\sigma: BT^\sigma \rightarrow m^\sigma(X)$. Of course h_σ induces the homomorphism $\pi_\sigma \circ \theta_Y$ in p –adic cohomology, where π_σ denotes the projection $S_{\mathbb{Z}_p^\wedge}(\sigma) \otimes \mathbb{Z}_p^\wedge[\ell_K(\sigma)] \rightarrow S_{\mathbb{Z}_p^\wedge}(\sigma)$. We write g_σ for any other map inducing the same homomorphism, and identify g_σ with g'_σ in case $Y = m^\sigma(X)$.

To analyse g_σ further, we consider its restriction $g_{\sigma/r}: BC_r^\sigma \rightarrow Y$, where $C_r < T$ denotes the cyclic subgroup of order p^r for any $r > 0$. We factorise t_σ into natural inclusions as

$$BC^\sigma \xrightarrow{t_{\sigma/r}} BC_r^\sigma \xrightarrow{t_{\sigma/r}} BT^\sigma,$$

so that $g_{\sigma/r} = g_\sigma \circ t_{\sigma/r}$. The induced homomorphism $H^*(h_{\sigma/r}; \mathbb{Z}_p^\wedge)$ is given by $t_{\sigma/r}^* \circ \pi_\sigma \circ \theta_Y$, and Theorem 3.3(1) implies that $g'_{\sigma/1} \simeq h'_{\sigma/1}$ because both are homotopic to f_σ . We abbreviate $\text{map}(BC_r^\sigma, Y)_{h_{\sigma/r}}$ to $m_r^\sigma(Y)$ henceforth.

Proposition 5.4 *If Y is p –complete and reflects $\text{st}_K(\sigma)$ by means of an isomorphism $\theta_Y: H^*(Y; \mathbb{Z}_p^\wedge) \rightarrow S_{\mathbb{Z}_p^\wedge}(\sigma) \otimes \mathbb{Z}_p^\wedge[\ell_K(\sigma)]$, then*

- (1) *there is a unique homotopy class of maps $h_{\sigma/r}: BC_r^\sigma \rightarrow Y$ such that $(h_{\sigma/r})^* = t_{\sigma/r}^* \circ \pi_\sigma \circ \theta_Y$;*
- (2) *the map $\text{map} \circ t_{\sigma/r}: m_r^\sigma(Y) \rightarrow m^\sigma(Y)$ is a homotopy equivalence.*

We prove Proposition 5.4 by induction on r , following application to our main result. It is convenient to denote $\text{map}(BT^\sigma, X)_{h_\sigma}$ by $M^\sigma(X)$, and to rewrite the induced map $\circ t_\sigma: M^\sigma(X) \rightarrow m^\sigma(X)$ as u_σ .

Theorem 5.5 *If X is p -complete and reflects K over \mathbb{Z}_p^\wedge , then*

- (1) *there is a unique homotopy class of maps $h_\sigma: BT^\sigma \rightarrow X$ that reflect $(K, \Delta(\sigma))$ over \mathbb{Z}_p^\wedge ;*
- (2) *the map u_σ is a homotopy equivalence, and the map $\text{ev}: M^\sigma(X) \rightarrow X$ reflects $(K, \text{st}_K(\sigma))$ over \mathbb{Z}_p^\wedge ;*
- (3) *there is a map $j: X_\ell \rightarrow M^\sigma(X)$ that reflects $(\text{st}_K(\sigma), \ell_K(\sigma))$ over \mathbb{Z}_p^\wedge ;*
- (4) *there is a homotopy equivalence $M^\sigma(X) \simeq (BT^\sigma)_p^\wedge \times X_\ell$.*

Proof (1) Write $Y = m^\sigma(X)$, and let $l(\sigma)$ be a set of representative maps $BT^\sigma \rightarrow Y$ for the homotopy classes that induce $\pi_\sigma \circ \theta'$ in p -adic cohomology. Since Y is p -complete and the natural map $\text{hocolim}_r BC_r^\sigma \rightarrow BT^\sigma$ is a mod- p equivalence, the maps

$$\text{map}(BT^\sigma, Y)_{l(\sigma)} \longrightarrow \text{map}(\text{hocolim}_r BC_r^\sigma, Y)_{l(\sigma)} \longrightarrow \text{holim}_r m_r^\sigma(Y)$$

are both equivalences. Furthermore, evaluation at the identity induces compatible equivalences $m_r^\sigma(Y) \rightarrow Y$ for all $r \geq 1$, by Proposition 5.4. Thus $l(\sigma)$ contains only the homotopy class of h'_σ , and (26) implies the result.

(2) The proof of (1) shows that $\text{ev} \circ u_\sigma: M^\sigma(Y) \rightarrow Y$ is also evaluation at the identity, and an equivalence. So Corollary 4.5 confirms that u_σ is an equivalence. Then apply Lemma 5.1 and Corollary 4.5 to replace Y by X in the source and target respectively; the required reflectors are θ_X for X and $\theta' \circ (u_\sigma^{-1})^*$ for $M^\sigma(X)$.

(3) By (2), the fibration (27) may be rewritten as

$$(30) \quad X_\ell \xrightarrow{j} M^\sigma(X) \xrightarrow{r_\sigma} BT^\sigma,$$

where $i \simeq u_\sigma \circ j$ and $r_\sigma = q_\sigma \circ u_\sigma$. The reflectors are those of (2) and (27).

(4) An action $BT^\sigma \times M^\sigma(X) \rightarrow M^\sigma(X)$ is given by $(u, f) \mapsto fu$, where $fu(v) = f(uv)$ for any $u, v \in BT^\sigma$ and $f \in M^\sigma(X)$. Restricting to $BT^\sigma \times X_\ell$ along $1 \times j$ induces an isomorphism of p -adic cohomology, and is therefore an equivalence. \square

We now return to the proof of Proposition 5.4; the base cases $r = 1$ are immediate, so we assume that $r \geq 2$ throughout.

Proof (1) Pulling the p -th power map on BT^σ back along $t_{\sigma/r-1}$ and combining the result with (28) and (29) yields a homotopy commutative ladder

$$(31) \quad \begin{array}{ccccccc} BC_r^\sigma & \xrightarrow{t_{\sigma/r}} & BT^\sigma & \xrightarrow{g'_\sigma} & m^\sigma(Y) & \xrightarrow{q_\sigma} & BT^\sigma \\ p \downarrow & & p \downarrow & & \downarrow \bar{p} & & \downarrow p \\ BC_{r-1}^\sigma & \xrightarrow{t_{\sigma/r-1}} & BT^\sigma & \xrightarrow{\bar{g}'_\sigma} & \bar{m}^\sigma(Y) & \xrightarrow{\bar{q}_\sigma} & BT^\sigma. \end{array}$$

Each vertical map has homotopy fibre BC^σ and induces a monomorphism in p -adic cohomology, so Lemma 5.3 confirms that $H^*(\bar{g}'_{\sigma/r-1}; \mathbb{Z}_p^\wedge)$ agrees with $H^*(\bar{h}'_{\sigma/r-1}; \mathbb{Z}_p^\wedge)$ as homomorphisms $H^*(\bar{m}^\sigma(Y); \mathbb{Z}_p^\wedge) \rightarrow H^*(BC_{r-1}^\sigma; \mathbb{Z}_p^\wedge)$. But $\bar{m}^\sigma(Y)$ also reflects $\text{st}_K(\sigma)$; so the inductive hypotheses show that $\bar{g}'_{\sigma/r-1}$ is homotopic to $\bar{h}'_{\sigma/r-1}$, and that evaluation

$$(32) \quad \text{map}(BC_{r-1}^\sigma, \bar{m}^\sigma(Y))_{\bar{h}'_{\sigma/r-1}} \xrightarrow{\circ t_{\sigma/r-1}} \text{map}(BC_{r-1}^\sigma, \bar{m}^\sigma(Y))_{\bar{h}'_{\sigma/r-1}} \xrightarrow{\text{ev}} \bar{m}^\sigma(Y)$$

at the identity is a homotopy equivalence.

Let l_Y denote the set of homotopy classes of lifts $BC_r^\sigma \rightarrow m^\sigma(Y)$ of the map $\bar{p} \circ g'_{\sigma/r}: BC_r^\sigma \rightarrow \bar{m}^\sigma(Y)$. Applying the functor $\text{map}(BC_r^\sigma, -)$ to (31) creates a homotopy pullback square

$$(33) \quad \begin{array}{ccc} \text{map}(BC_r^\sigma, m^\sigma(Y))_{l_Y} & \xrightarrow{q_\sigma \circ} & \text{map}(BC_r^\sigma, BT^\sigma)_{l_B} \\ \bar{p} \circ \downarrow & & \downarrow p \circ \\ \text{map}(BC_r^\sigma, \bar{m}^\sigma(Y))_{\bar{p} \circ g'_{\sigma/r}} & \xrightarrow{\bar{q}_\sigma \circ} & \text{map}(BC_r^\sigma, BT^\sigma)_{p \circ t_{\sigma/r}} \end{array}$$

where l_B denotes a set of representatives for homotopy classes of lifts of $p \circ t_{\sigma/r}$. The vertical maps are principal $\text{map}(BC_r^\sigma, BC^\sigma)$ -fibrations by [15], and $\bar{q}_\sigma \circ$ may be identified with the corresponding map

$$\text{map}(BC_{r-1}^\sigma, \bar{m}^\sigma(Y))_{\bar{h}'_{\sigma/r-1}} \longrightarrow \text{map}(BC_{r-1}^\sigma, BT^\sigma)_{t_{\sigma/r-1} \circ p}$$

by Zabrodsky’s lemma. It follows from the inductive hypothesis that the lower fibration of (33) is equivalent to $\bar{q}_\sigma: \bar{m}^\sigma(Y) \rightarrow BT^\sigma$, and therefore that the common homotopy fibre of the horizontal maps is Y_ℓ , which is 1-connected. Thus $q_\sigma \circ$ induces a bijection $l_Y \leftrightarrow l_B$ of components. But we are given that $H^*(g'_{\sigma/r}; \mathbb{Z}_p^\wedge) = H^*(h'_{\sigma/r}; \mathbb{Z}_p^\wedge)$, from which we deduce that $q_\sigma \circ g'_{\sigma/r} \simeq q_\sigma \circ h'_{\sigma/r}$; so $g'_{\sigma/r}$ and $h'_{\sigma/r}$ define the same element of l_Y , and are therefore homotopic. Applying $\text{ev}: m^\sigma(Y) \rightarrow Y$ completes the proof.

(2) The pullback square (33) combines with the inductive hypotheses to show that $\text{ev}: \text{map}(BC_r^\sigma, m^\sigma(Y))_{h'_{\sigma/r}} \rightarrow m^\sigma(Y)$ is an equivalence, and Corollary 4.5 allows us

to replace $m^\sigma(Y)$ with Y , and $h'_{\sigma/r}$ with $h_{\sigma/r}$. Then $\circ \iota_{\sigma/r}$ is also an equivalence, by analogy with (32). □

6 Higher limits of diagrams

In this section we recall the basic properties of higher limits of diagrams $\Phi: C \rightarrow \text{AB}$ of abelian groups, where C is usually of the form $\text{CAT}^{\text{op}}(K)$. These are crucial for later sections, and also featured in [11, Section 3].

Following Oliver’s elaboration [12] of [2], we interpret $\lim^i_C \Phi$ as the i -th cohomology group of the cochain complex $(C^*(C, \Phi), \delta)$, where

$$(34) \quad C^n(C; \Phi) = \prod_{c_0 \rightarrow c_1 \rightarrow \dots \rightarrow c_n} \Phi(c_n)$$

for every $n \geq 0$, and the product is taken over all chains of morphisms in C . The differential $\delta: C^n(C; \Phi) \rightarrow C^{n+1}(C; \Phi)$ is the alternating sum $\sum_{k=0}^n (-1)^k \delta^k$, where δ^k is defined on $u \in C^n(C; \Phi)$ by

$$\delta^k(u)(c_0 \rightarrow \dots \rightarrow c_{n+1}) := \begin{cases} u(c_0 \rightarrow \dots \rightarrow \widehat{c}_k \rightarrow \dots \rightarrow c_{n+1}) & \text{for } k \leq n, \\ \Phi(c_n \rightarrow c_{n+1})u(c_0 \rightarrow \dots \rightarrow c_n) & \text{for } k = n + 1. \end{cases}$$

We may replace $C^*(C; \Phi)$ by its quotient $N^*(C; \Phi)$ of normalised cochains, obtained by restricting the products (34) to chains of nonidentities. So

$$\lim^i_C \Phi := H^i(C^*(C; \Phi), \delta) \cong H^i(N^*(C; \Phi), \delta)$$

for $i \geq 0$, and the limits are themselves abelian groups. The same construction works for small diagrams in an arbitrary abelian category A , and the limits inherit any additional algebraic structure from the objects of A .

Given a constant diagram $\text{cst}_M: C \rightarrow \text{AB}$, it follows that its higher limits are determined by the cohomology of the classifying space BC , via an isomorphism $\lim^i_C \text{cst}_M \cong H^i(BC; M)$. In particular, $\lim^i_{\text{CAT}^{\text{op}}(K)} \text{cst}_M$ vanishes for $i \geq 1$ because $B\text{CAT}^{\text{op}}(K)$ is contractible, and $\lim^i_{\text{CAT}^{\text{op}}(K^\times)} \text{cst}_M$ is isomorphic to $H^i(K; M)$ because $B\text{CAT}^{\text{op}}(K^\times)$ is homeomorphic to the realisation $|K|$. Both are isomorphic to M when $i = 0$.

For any face $\sigma \in K$, we refer to a diagram $\Phi: \text{CAT}^{\text{op}}(K) \rightarrow \text{AB}$ as σ -atomic when $\Phi(\tau) = 0$ for every $\tau \neq \sigma$. We consider the following generalisation of [11, Lemma 3.9], in which it is convenient to set $\tilde{H}^n(\emptyset; M) := 0$ for $n \geq 0$, and $\tilde{H}^{-1}(\emptyset; M) := M$, for any abelian group M .

Proposition 6.1 *If Φ is σ -atomic, then there exists an isomorphism*

$$\lim_{\text{CAT}^{\text{op}}(K)}^i \Phi \cong \tilde{H}^{i-1}(\ell_K(\sigma); \Phi(\sigma))$$

for any $i \geq 0$.

Proof Let $J = \ell_K(\sigma)$, and define $\Phi': \text{CAT}^{\text{op}}(J) \rightarrow \text{AB}$ by $\Phi'(\tau \setminus \sigma) := \Phi(\tau)$ for every $\tau \supseteq \sigma$. Thus Φ' is \emptyset -atomic, and $\Phi' \circ P^{\text{op}} = \Phi|_{\text{CAT}^{\text{op}}(K) \downarrow \sigma}$ in terms of the functor P of (5). The isomorphisms

$$N^*(\text{CAT}^{\text{op}}(K); \Phi) \cong N^*(\text{CAT}^{\text{op}}(K) \downarrow \sigma; \Phi) \cong N^*(\text{CAT}^{\text{op}}(J); \Phi')$$

of normalised chain complexes determine an isomorphism

$$\lim_{\text{CAT}^{\text{op}}(K)}^i \Phi \cong \lim_{\text{CAT}^{\text{op}}(J)}^i \Phi'$$

for every $i \geq 0$.

If $J = \{\emptyset\}$, the claim is clear. Otherwise, write $\Phi'(\emptyset)$ as M and consider the short exact sequence

$$(35) \quad 0 \longrightarrow \Phi' \longrightarrow \text{cst}_M \longrightarrow \Upsilon \longrightarrow 0$$

of $\text{CAT}^{\text{op}}(J)$ -diagrams, where Υ is trivial on \emptyset and constant on the subcategory $\text{CAT}^{\text{op}}(J^\times)$. The long exact sequence associated to (35) combines with the isomorphism $N^*(\text{CAT}^{\text{op}}(J); \Upsilon) \cong N^*(\text{CAT}^{\text{op}}(J^\times); \text{cst}_M)$ to ensure that

$$\lim_{\text{CAT}^{\text{op}}(J)}^i \Phi' \cong \lim_{\text{CAT}^{\text{op}}(J)}^{i-1} \Upsilon \cong \tilde{H}^{i-1}(J; M)$$

for $i \geq 2$, and reduces to the short exact sequence

$$0 \longrightarrow M \longrightarrow H^0(J; M) \longrightarrow \lim^1 \Phi' \longrightarrow 0$$

for $i = 1$. So the result holds for all i . □

We may associate a σ -atomic diagram Φ_σ to every Φ , by the formula

$$\Phi_\sigma(\tau) = \begin{cases} \Phi(\sigma) & \text{when } \tau = \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Such diagrams play a fundamental rôle in proving vanishing results for higher limits; in particular, they act as building blocks for the following notions of filtration, which depend on the poset structure of $\text{CAT}^{\text{op}}(K)$.

Given Φ as above, $\Phi_s, \Phi_{\geq s}: \text{CAT}^{\text{op}}(K) \rightarrow \text{AB}$ are defined by

$$\Phi_s(\sigma) = \begin{cases} \Phi(\sigma) & \text{if } |\sigma| = s, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \Phi_{\geq s}(\sigma) = \begin{cases} \Phi(\sigma) & \text{if } |\sigma| \geq s, \\ 0 & \text{otherwise.} \end{cases}$$

Thus Φ_s decomposes as the product diagram $\prod_{|\sigma|=s} \Phi_\sigma$. Also, $\Phi_{\geq 0} = \Phi$ and $\Phi_{\geq \dim K + 1} = \Phi_{\dim K + 1}$ hold by definition, and $\Phi_{\geq \dim K + 2}$ is trivial. Furthermore, there is a short exact sequence

$$(36) \quad 0 \longrightarrow \Phi_s \longrightarrow \Phi_{\geq s} \longrightarrow \Phi_{\geq s+1} \longrightarrow 0$$

for every $s > 0$, because there are no morphisms $\sigma \rightarrow \tau$ when $|\tau| < |\sigma|$.

Lemma 6.2 *If $\lim_{\text{CAT}^{\text{op}}(K)}^i \Phi_\sigma = 0$ for every face σ , then we have $\lim_{\text{CAT}^{\text{op}}(K)}^i \Phi = 0$.*

Proof The decomposition of Φ_s into atomic factors ensures that $\lim_{\text{CAT}^{\text{op}}(K)}^i \Phi_s$ is zero, and the long exact sequence associated to (36) confirms that

$$\lim_{\text{CAT}^{\text{op}}(K)}^i \Phi_{\geq s} \longrightarrow \lim_{\text{CAT}^{\text{op}}(K)}^i \Phi_{\geq s+1}$$

is a monomorphism for every $0 \leq s \leq \dim K + 1$. So the composition

$$\lim_{\text{CAT}^{\text{op}}(K)}^i \Phi \longrightarrow \lim_{\text{CAT}^{\text{op}}(K)}^i \Phi_{\geq \dim K + 2} = 0$$

is also a monomorphism, and the result follows. □

Lemma 6.3 *If $i \geq \dim K + 2$, then we have that $\lim_{\text{CAT}^{\text{op}}(K)}^i \Phi = 0$.*

Proof The cochain group $N^i(\text{CAT}^{\text{op}}(K); \Psi)$ vanishes for $i \geq \dim K + 2$. □

Given any two diagrams $\Phi: \text{CAT}^{\text{op}}(K) \rightarrow \text{AB}$ and $\Psi: \text{CAT}^{\text{op}}(L) \rightarrow \text{AB}$, we define their *external product* $\Phi \times \Psi: \text{CAT}^{\text{op}}(K * L) \rightarrow \text{AB}$ by $\Phi \times \Psi(\sigma \cup \tau) := \Phi(\sigma) \times \Psi(\tau)$.

Lemma 6.4 *For any $i \geq 1$, there is an isomorphism*

$$\lim_{\text{CAT}^{\text{op}}(K * L)}^i \Phi \times \Psi \cong \lim_{\text{CAT}^{\text{op}}(K)}^i \Phi \times \lim_{\text{CAT}^{\text{op}}(L)}^i \Psi.$$

Proof This follows directly from the isomorphism

$$C^*(\text{CAT}^{\text{op}}(K * L); \Phi \times \Psi) \cong C^*(\text{CAT}^{\text{op}}(K); \Phi) \times C^*(\text{CAT}^{\text{op}}(L); \Psi)$$

of cochain complexes. □

7 p -adic homotopy uniqueness for $\Delta^{(r)}(V)$

In this section we show that the homotopy type of the p -adic completion $\text{hc}(K)_p^\wedge$ is determined by its p -adic cohomology, for any r -skeleton $\Delta^{(r)}(V)$ of the $(m-1)$ -simplex. We restrict attention to 1-connected spaces, so that completion is well-behaved. We also maintain the notation of previous sections, but make the additional assumption that any space denoted by X or X_J is p -complete, for every simplicial complex J .

Our aim is to prove that $K = \Delta^{(r)}(V)$ satisfies the following.

Conditions 7.1

- (C1) There is an isomorphism $\theta: H^*(X; \mathbb{Z}_p^\wedge) \rightarrow \mathbb{Z}_p^\wedge[K]$ if and only if there is a homotopy equivalence $f: \text{hc}(K)_p^\wedge \rightarrow X$ such that $f^* = \theta$.
- (C2) Any two self-equivalences $f, g: \text{hc}(K)_p^\wedge \rightarrow \text{hc}(K)_p^\wedge$ are homotopic if and only if $H^*(f; \mathbb{Z}_p^\wedge) = H^*(g; \mathbb{Z}_p^\wedge)$.

To confirm (C1), it suffices to assume the existence of θ and construct a mod p equivalence $f: \text{hc}(K) \rightarrow X$ such that $f^* = \theta$. Our strategy is to apply the results of Section 4 and Section 5.

By Corollary 4.2, there are maps $h_\sigma: BT^\sigma \rightarrow X$ for all faces $\sigma \in K$, which induce the restrictions of θ in p -adic cohomology. By Theorem 5.5(1), the h_σ are compatible up to homotopy with the inclusions $BT^\sigma \rightarrow BT^\tau$ for all $\sigma \subseteq \tau$. They therefore define a map $(\text{hc}(K))^{(1)} \rightarrow X$ on the 1-skeleton of the homotopy colimit $\text{hc}(K)$. The Bousfield–Kan spectral sequence for homotopy inverse limits, together with work of Wojtkowiak [17] clarifying the situation for low dimensions, provides an obstruction theory for extending this map to $\text{hc}(K)$. The obstruction groups are given by the higher limits

$$\lim_{\text{CAT}^{\text{op}}(K)}^{i+1} \pi_i(M^\sigma(X)).$$

If these groups vanish for all $i \geq 1$, then the desired extension exists. So for any $i \geq 1$, it is convenient to define the diagram

$$\Pi_i: \text{CAT}^{\text{op}}(K) \longrightarrow \text{AB}$$

by assigning $\pi_i(M^\sigma(X))$ to each face σ , and assigning the homomorphism induced by $M^\tau(X) \rightarrow M^\sigma(X)$ to each morphism $\tau \supseteq \sigma$.

The obstruction theory also applies to the question of uniqueness. There is a restriction map

$$R: [\text{hc}(K), X] \longrightarrow \lim_{\text{CAT}^{\text{op}}(K)}^0 [BT^K, X]$$

of homotopy classes, whose target may be identified with $[\text{hc}(K)^{(1)}, X]$. If, for some element $h_1 := (h_\sigma)_{\sigma \in K}$ of the latter, the limits $\lim_{\text{CAT}^{\text{op}}(K)}^i \Pi_i$ vanish for all $i \geq 1$, then $R^{-1}(h_1)$ contains at most one element; in other words, there is at most one extension $\text{hc}(K) \rightarrow X$ of h_1 , up to homotopy.

So we are interested in simplicial complexes K that satisfy the following.

Condition 7.2 (VHL) For any p -complete space X equipped with an isomorphism $\theta: H^*(X; \mathbb{Z}_p^\wedge) \rightarrow \mathbb{Z}_p^\wedge[K]$, the groups

$$\lim_{\text{CAT}^{\text{op}}(K)}^i \Pi_i \quad \text{and} \quad \lim_{\text{CAT}^{\text{op}}(K)}^{i+1} \Pi_i$$

vanish for all $i \geq 1$.

Proposition 7.3 *If K satisfies (VHL), then it satisfies (C1) and (C2).*

Proof If K satisfies (VHL), the above considerations show that there exists a map $f: \text{hc}(K) \rightarrow X$ which induces θ . It therefore determines the homotopy equivalence $f: \text{hc}(K)_p^\wedge \rightarrow X$ required for (C1).

Let $f, g: \text{hc}(K)_p^\wedge \rightarrow \text{hc}(K)_p^\wedge$ be self-equivalences, and $H^*(f; \mathbb{Z}_p^\wedge) = H^*(g; \mathbb{Z}_p^\wedge)$. Without loss of generality, we may assume that f is the identity. Then for every face $\sigma \in K$, the restriction $g|_{BT^\sigma}$ reflects the pair $(K, \Delta(\sigma))$ over \mathbb{Z}_p^\wedge , and so by Theorem 5.5(1) is homotopic to h_σ ; in particular, g lies in $R^{-1}((h_\sigma)_{\sigma \in K})$. All obstruction groups vanish by assumption, so 1 and g are homotopic, as required for (C2). \square

We now turn to the distinguishing properties of the skeleta $\Delta^{(r)}(V)$. For any space X that reflects K over \mathbb{Z}_p^\wedge , we denote the homotopy fibre of the map $q_X: X \rightarrow BT^V$ of Lemma 4.1 by F_X .

Lemma 7.4 *For $K = \Delta^{(r-1)}(V)$ and any face σ , we have that*

- (1) *the space F_X is $2r$ -connected;*
- (2) *$\ell_K(\sigma)$ is empty if σ is maximal, and is $\Delta^{(r-|\sigma|-1)}(V \setminus \sigma)$ otherwise.*

Proof (1) The missing faces of K have cardinality at least $r + 1$, so q_X induces an isomorphism of p -adic cohomology in dimensions $\leq 2r + 1$. The same is therefore true in homology. Both X and BT^V are simply connected and p -complete, so $\pi_j(F_X) = 0$ for $1 \leq j \leq 2r$ by Whitehead’s Theorem.

(2) The faces ρ of $\ell_K(\sigma)$ are restricted only by the requirements that they are subsets of $V \setminus \sigma$, and that $|\sigma| + |\rho| \leq r$. \square

Lemma 7.4(2) combines with (9) and Theorem 5.5(3)–(4) to create a homotopy pullback square

$$(37) \quad \begin{array}{ccc} X_{\ell_K(\sigma)} & \longrightarrow & M^\sigma(X) \\ q_{\ell_K(\sigma)} \downarrow & & \downarrow q_{M^\sigma(X)} \\ (BT^{V \setminus \sigma})_p^\wedge & \longrightarrow & (BT^V)_p^\wedge \end{array}$$

for every nonmaximal face $\sigma \in K$; when σ is maximal, there is an equivalence $M^\sigma(X) \simeq (BT^\sigma)_p^\wedge$.

Theorem 7.5 *Suppose that $K = \Delta^{(r)}(V)$, and X reflects K over \mathbb{Z}_p^\wedge : then we have that $\lim_{\text{CAT}^{\text{op}}(K)}^j \Pi_i = 0$ for all $j \geq i$.*

Proof If $i = 1$, then $\Pi_1(\sigma) = 0$ for every face σ , by Theorem 5.5(2); therefore $\lim_{\text{CAT}^{\text{op}}(K)}^j \Pi_1 = 0$ for $j \geq 1$.

If $i = 2$, then $\Pi_2(\sigma) = (\mathbb{Z}^\sigma)_p^\wedge$ when σ is maximal, and $(\mathbb{Z}^V)_p^\wedge$ otherwise, by (37). So there exists a short exact sequence

$$0 \longrightarrow \Pi_2 \longrightarrow \text{cst}_{(\mathbb{Z}^V)_p^\wedge} \longrightarrow \Psi \longrightarrow 0,$$

where Ψ is nonzero only on maximal faces, and decomposes as a product of atomic diagrams. Hence $\lim_{\text{CAT}^{\text{op}}(K)}^j \Pi_2 \cong \lim_{\text{CAT}^{\text{op}}(K)}^{j-1} \Psi$, and Proposition 6.1 confirms that $\lim_{\text{CAT}^{\text{op}}(K)}^j \Pi_2 = 0$ for $j \geq 2$.

If $i \geq 3$, abbreviate the atomic diagrams $(\Pi_i)_\sigma$ to Π_σ , and the groups $\lim_{\text{CAT}^{\text{op}}(K)}^j \Pi_\sigma$ to l_σ^j . It follows from Lemma 7.4(1) and (37) that $\Pi_\sigma = 0$ for all σ satisfying $|\sigma| \leq r - i/2$, because the fibre of $q_{\ell_K(\sigma)}$ is $2(r - |\sigma|)$ -connected. So $l_\sigma^j = 0$ for $j \geq i$. The remaining faces satisfy $i > 2(r - |\sigma|)$, and therefore $i > (r - |\sigma|)$. But Proposition 6.1 and Lemma 7.4(2) identify l_σ^j with $\tilde{H}^{j-1}(\Delta^{(r-|\sigma|-1)}(V \setminus \sigma); \Phi(\sigma))$. Hence $l_\sigma^j = 0$ for $j \geq i$ in these cases also, and the result follows by applying Lemma 6.2. \square

Theorem 7.5 shows that K satisfies (VHL), so the following is an immediate consequence of Proposition 7.3.

Corollary 7.6 *Every complex $\Delta^{(r)}(V)$ satisfies conditions (C1) and (C2).*

For low dimensional K , our main results hold without restriction.

Theorem 7.7 (1) *For $\dim(K) \leq 1$, there is an isomorphism $\theta: H^*(X; \mathbb{Z}_p^\wedge) \rightarrow \mathbb{Z}_p^\wedge[K]$ if and only if there is a homotopy equivalence $f: \text{hc}(K)_p^\wedge \rightarrow X$ such that $f^* = \theta$.*

(2) *For $\dim(K) = 0$, two self equivalences $f, g: \text{hc}(K)_p^\wedge \rightarrow \text{hc}(K)_p^\wedge$ are homotopic if and only if $H^*(f; \mathbb{Z}_p^\wedge) = H^*(g; \mathbb{Z}_p^\wedge)$.*

Proof All relevant mapping spaces are simply connected, so (1) follows from the fact that $\lim^j \Pi_i(\sigma)$ vanishes for dimensional reasons when $j \geq 3$. If $\dim(K) = 0$ then $K = \Delta^{(0)}(V)$ for some V ; so (2) follows from Corollary 7.6. \square

8 Joins of complexes and higher limits

We have shown in Section 7 that skeleta of simplices satisfy condition (VHL), and the aim of Proposition 8.5 is to extend the class of complexes with this property. Preparing for the proof occupies most of this section, throughout which we insist that $X = X_{K*L}$ reflects $K * L$ over \mathbb{Z}_p^\wedge . We maintain our convention that any space denoted by X or X_J is p -complete, for every simplicial complex J .

Lemma 8.1 *For any i_K reflecting $(K * L, K)$ and i_L reflecting $(K * L, L)$ over \mathbb{Z}_p^\wedge , the homotopy pullback of the diagram*

$$X_K \xrightarrow{i_K} X \xleftarrow{i_L} X_L$$

is p -complete and contractible.

Proof The pullback is p -complete by [2, Section II.5], and the Eilenberg-Moore spectral sequence computes its mod p cohomology. The E_2 -term is given by

$$\mathrm{Tor}_{\mathbb{F}_p[K] \otimes \mathbb{F}_p[L]}^j(\mathbb{F}_p[K], \mathbb{F}_p[L]) = \begin{cases} 0 & \text{for } j \neq 0, \\ \mathbb{F}_p & \text{for } j = 0, \end{cases}$$

so the spectral sequence collapses and the pullback is contractible. \square

The homotopy fibre of i_L is therefore the loop space ΩX_K , and its inclusion into X_L is null-homotopic.

Lemma 8.2 *For any i_L reflecting $(K * L, L)$ over \mathbb{Z}_p^\wedge , and any face $\tau \in L$, the square*

$$\begin{array}{ccc} M^\tau(X_L) & \xrightarrow{\mathrm{ev}} & X_L \\ M^\tau(i_L) \downarrow & & \downarrow i_L \\ M^\tau(X) & \xrightarrow{\mathrm{ev}} & X \end{array}$$

is a homotopy pullback and functorial in τ .

Proof First apply the functor $\text{map}(BT^\tau, -)$ to i_L , and use evaluation at the base point to obtain a commutative square whose upper left-hand entry is $M^\tau(X_L)$. By Theorems 4.2 and 5.5(2), the upper evaluation reflects $(L, \text{st}_L(\tau))$ over \mathbb{Z}_p^\wedge ; and there is a canonical map $d: M^\tau(X_L) \rightarrow P$ to the pullback P .

The Eilenberg–Moore spectral sequence computes $H^*(P; \mathbb{F}_p)$. The isomorphisms

$$\text{Tor}_{\mathbb{F}_p[K] \otimes \mathbb{F}_p[L]}^j(\mathbb{F}_p[K] \otimes \mathbb{F}_p[\text{st}_L(\tau)], \mathbb{F}_p[L]) \cong \text{Tor}_{\mathbb{F}_p}^j(\mathbb{F}_p[\text{st}_L(\tau)], \mathbb{F}_p)$$

show that the higher Tor groups vanish, and that the spectral sequence collapses at the E_2 -page. It follows that d is a homotopy equivalence, as required. Functoriality of the construction is immediate. \square

Lemma 8.3 *Suppose that L satisfies condition (VHL); then there exists a map $j_L: \text{hc}(L) \rightarrow X$ reflecting $(K * L, L)$ over \mathbb{Z}_p^\wedge , and it is unique up to homotopy.*

Proof Consider a maximal face μ of the subcomplex $K \subseteq K * L$, so that $\ell_{K * L}(\mu) = L$. The fibre inclusion $X_L \rightarrow M^\mu(X)$ of Theorem 5.5(3) may be combined with the equivalence $\text{hc}(L) \rightarrow X_L$ that arises because L satisfies (VHL). Then composition with evaluation at the basepoint defines j_L as $\text{hc}(L) \rightarrow M^\mu(X) \rightarrow X$, which reflects $(K * L, K)$ over \mathbb{Z}_p^\wedge by Theorem 5.5.

In order to prove homotopy uniqueness, we utilise the obstruction theory of Section 7. In this case the obstruction groups are $\lim_{\text{CAT}^{\text{op}}(L)}^i \pi_i(M^\tau(X))$ for $i \geq 1$, and it remains to show that they vanish.

First apply Lemma 8.1 and Lemma 8.2 to the maps j_K and j_L . The homotopy exact sequences of the principal fibrations $\Omega \text{hc}(K) \rightarrow M^\tau(\text{hc}(L)) \rightarrow M^\tau(X)$ decompose into short exact sequences

$$(38) \quad 0 \longrightarrow \pi_i(M^\tau(\text{hc}(L))) \longrightarrow \pi_i(M^\tau(X)) \longrightarrow \text{cst}_{\pi_i(\text{hc}(K))} \longrightarrow 0$$

of $\text{CAT}^{\text{op}}(L)$ -diagrams. But L satisfies (VHL) by assumption, and the higher limits of constant diagrams are zero on $\text{CAT}^{\text{op}}(L)$. So (38) implies $\lim_{\text{CAT}^{\text{op}}(L)}^i \pi_i(M^\tau(X)) = 0$ for $i \geq 1$, as required. \square

Lemma 8.4 *Suppose that K and L satisfy (VHL); then for any $\sigma \in K$ and $\tau \in L$ there exist maps $l_K(\sigma, \tau)$ reflecting $(\text{st}_K(\sigma) * \text{st}_L(\tau), \text{st}_K(\sigma))$ and $l_L(\tau, \sigma)$ reflecting $(\text{st}_K(\sigma) * \text{st}_L(\tau), \text{st}_L(\tau))$ over \mathbb{Z}_p^\wedge , such that the diagram*

$$M^\sigma(\text{hc}(K)) \xrightarrow{l_K(\sigma, \tau)} M^{\sigma \cup \tau}(X) \xleftarrow{l_L(\tau, \sigma)} M^\tau(\text{hc}(L))$$

is homotopy functorial in σ and τ and has contractible homotopy pullback.

Proof In order to define $l_K(\sigma, \tau)$ and $l_L(\tau, \sigma)$, observe that the respective restrictions of $h_{\sigma \cup \tau}: BT^{\sigma \cup \tau} \rightarrow X$ to BT^σ and BT^τ are homotopic to h_σ and h_τ , by Theorem 5.5(1). So the adjoint properties of mapping spaces provide homeomorphisms

$$(39) \quad M^\tau(M^\sigma(X)) \cong M^{\sigma \cup \tau}(X) \cong M^\sigma(M^\tau(X)).$$

By Theorem 5.5(2) and Lemma 8.3, there exists $j_L(\sigma): \text{hc}(L) \rightarrow M^\sigma(X)$ reflecting $(\text{st}_K(\sigma) * L, L)$ over \mathbb{Z}_p^\wedge . Then $l_L(\tau, \sigma)$ is induced by applying $\text{map}(BT^\tau, -)$ to $j_L(\sigma)$, and reflects $(\text{st}_K(\sigma) * \text{st}_L(\tau), \text{st}_L(\tau))$ over \mathbb{Z}_p^\wedge by Theorem 5.5(2). The construction is functorial in τ , and homotopy functorial in σ because Lemma 8.3 defines $j_L(\sigma)$ uniquely up to homotopy. Interchanging $\sigma \in K$ and $\tau \in L$ defines $l_K(\sigma, \tau)$, which reflects $(\text{st}_K(\sigma) * \text{st}_L(\tau), \text{st}_K(\sigma))$ over \mathbb{Z}_p^\wedge ; the construction is functorial in σ and homotopy functorial in τ .

The homotopy pullback of the diagram is contractible by Lemma 8.1. □

Proposition 8.5 *The class of simplicial complexes satisfying (VHL) is closed under the formation of finite joins.*

Proof Let K and L satisfy (VHL). By Lemma 8.4, there are isomorphisms

$$\pi_i(M^\sigma(\text{hc}(K))) \times \pi_i(M^\tau(\text{hc}(L))) \cong \pi_i(M^{\tau \cup \sigma}(X))$$

that are functorial on $\text{CAT}^{\text{op}}(K) \times \text{CAT}^{\text{op}}(L)$, and so on $\text{CAT}^{\text{op}}(K * L)$. The result follows from Lemma 6.4, by induction on the number of factors. □

We believe that other combinatorial operations on simplicial complexes may also respect condition (VHL).

Theorem 8.6 *Let X be a p -complete CW-complex, and suppose that K is an iterated join $\Delta^{(r_1)}(U_1) * \dots * \Delta^{(r_t)}(U_t)$ of skeleta of simplices; then there is an isomorphism $\theta: H^*(X; \mathbb{Z}_p^\wedge) \rightarrow \mathbb{Z}_p^\wedge[K]$ if and only if there is a homotopy equivalence $f: \text{hc}(K)_p^\wedge \rightarrow X$ such that $f^* = \theta$.*

Proof Any such K satisfies condition (VHL) by Theorem 7.5 and Proposition 8.5, and therefore satisfies (C1) by Proposition 7.3. □

Theorem 8.7 *For complexes K as in Theorem 8.6, any pair of self-equivalences $f, g: \text{hc}(K)_p^\wedge \rightarrow \text{hc}(K)_p^\wedge$ are homotopic if and only if $H^*(f; \mathbb{Z}_p^\wedge) = H^*(g; \mathbb{Z}_p^\wedge)$.*

Proof As before, K satisfies (C2) by Proposition 7.3. □

9 The integral homotopy type

Our final aim is to prove Theorems 9.4 and 9.5. For any space Y , we denote the product of all its p -adic completions by $Y^\wedge := \prod_p Y_p^\wedge$, and its rationalisation by Y_0 ; its finite adèle type $Y_{\mathbb{A}_f}$ is both the rationalisation of Y^\wedge and the formal completion of Y_0 . When Y is nilpotent, these spaces fit into Sullivan’s arithmetic pullback square

$$(40) \quad \begin{array}{ccc} Y & \xrightarrow{\wedge} & Y^\wedge \\ \downarrow & & \downarrow \\ Y_0 & \xrightarrow{\wedge} & Y_{\mathbb{A}_f} . \end{array}$$

Given any partition $\mathcal{U} := \{U_0, \dots, U_t\}$ of V , we write $K(\mathcal{U})$ for the iterated join $\Delta(U_0) * \partial(U_1) * \dots * \partial(U_t)$. So there is an isomorphism

$$\mathbb{Q}[K(\mathcal{U})] \cong \mathbb{Q}[\Delta(U_0)] \otimes \mathbb{Q}[\partial(U_1)] \otimes \dots \otimes \mathbb{Q}[\partial(U_t)],$$

which implies the following observation of [11, Section 5].

Proposition 9.1 *The simplicial complexes $K(\mathcal{U})$ are precisely those for which $\mathbb{Q}[K]$ is a complete intersection.*

For the rest of this section we insist that X is a 1-connected space realising the complete intersection $\mathbb{Q}[K(\mathcal{U})]$, and maintain our convention of denoting the homotopy fibre of $q_X: X \rightarrow BT^V$ by F_X . In particular, the cofibre of any $q_{\text{hc}(\partial(U))}$ is the Thom complex $\text{Th}(U)$ of the universal product of line bundles over BT^U , so $q_{\text{hc}(\partial(U))}$ is equivalent to the projection of the corresponding sphere bundle, and has homotopy fibre the $(2|U| - 1)$ -dimensional sphere $S(U)$; hence $F_{\text{hc}(K(\mathcal{U}))}$ is given by $S(\mathcal{U}) := S(U_1) \times \dots \times S(U_t)$.

Lemma 9.2 *The p -adic completion, rationalisation, and finite adèle type of the fibration $F_X \rightarrow X \rightarrow BT^V$ are all themselves fibrations; moreover, F_X is 1-connected and rationally equivalent to $S(\mathcal{U})$.*

Proof The first statement follows from [2, Section II.5] since BT^V is 1-connected; the 1-connectedness of F_X arises directly from the definitions. Part I of [11, Proposition 5.11] confirms the existence of a homotopy equivalence $h: \text{hc}(K(\mathcal{U}))_0 \rightarrow X_0$, for which $q_X \circ h \simeq q_{\text{hc}(K(\mathcal{U}))}$ as rationalised maps. So h lifts to a rational equivalence $S(\mathcal{U}) \rightarrow F_X$ of homotopy fibres. □

Now let $q^\#: X^\# \rightarrow BT^V$ denote the homotopy pullback of the fibration q_X^\wedge along the completion map $BT^V \rightarrow (BT^V)^\wedge$; then the inclusion $F_X^\wedge \rightarrow X^\#$ of the fibre is homotopic to the principal T^V -fibration classified by $q^\#$.

Lemma 9.3 *The homotopy fibre $F^\#$ of the induced map $X \rightarrow X^\#$ is a product $\prod_{i=1}^t H(\mathbb{Z}^\wedge/\mathbb{Z}, 2|U_i| - 2)$ of Eilenberg–Mac Lane spaces.*

Proof By construction, there is a homotopy pullback diagram

$$\begin{array}{ccc} F_X & \xrightarrow{\wedge} & F_X^\wedge \\ \downarrow & & \downarrow \\ X & \longrightarrow & X^\# \end{array}$$

in which the homotopy fibre of the horizontal maps agrees with that of $(F_X)_0 \rightarrow (F_X)_{\mathbb{A}_f}$. Since the fibre of $S(\mathcal{U})_0 \rightarrow S(\mathcal{U})_{\mathbb{A}_f}$ has the required form, the result follows from (40) and Lemma 9.2. □

Theorem 9.4 *Let X be a nilpotent CW-complex, and suppose that $\mathbb{Q}[K]$ is a complete intersection; then there is an isomorphism $\theta: H^*(X; \mathbb{Z}) \rightarrow \mathbb{Z}[K]$ if and only if there is a homotopy equivalence $f: \text{hc}(K) \rightarrow X$ such that $f^* = \theta$.*

Proof Using Proposition 9.1, write K as $K(\mathcal{U})$ for some partition \mathcal{U} of V . Then θ extends to an isomorphism $H^*(X; \mathbb{Z}_p^\wedge) \rightarrow \mathbb{Z}_p^\wedge[K(\mathcal{U})]$ for each prime p , which is induced by an equivalence $\text{hc}(K(\mathcal{U}))_p^\wedge \rightarrow X_p^\wedge$ via Theorem 8.6. So there exists a map $f': \text{hc}(K(\mathcal{U})) \rightarrow X^\wedge$, such that $q_X^\wedge \circ f'$ factors through the completion $BT^V \rightarrow (BT^V)^\wedge$ using arguments of Sections 7 and 8. In other words, the square

$$\begin{array}{ccc} \text{hc}(K(\mathcal{U})) & \xrightarrow{q_{\text{hc}(K(\mathcal{U}))}} & BT^V \\ f' \downarrow & & \downarrow^\wedge \\ X^\wedge & \xrightarrow{q_X^\wedge} & (BT^V)^\wedge \end{array}$$

is homotopy commutative, and f' factors through $f'': \text{hc}(K(\mathcal{U})) \rightarrow X^\#$.

To lift f'' to X , note that the obstructions lie in the groups $H^{i+1}(\text{hc}(K(\mathcal{U})); \pi_i(F^\#))$. But $\pi_i(F^\#) = 0$ in odd dimensions by Lemma 9.3; so all obstructions vanish, and a lift $f: \text{hc}(K(\mathcal{U})) \rightarrow X$ exists. Then $f^* = \theta$ by construction, and f is an equivalence by Whitehead’s theorem, because the source and target are 1-connected. □

Theorem 9.5 *Let X be a CW-complex that realises $\mathbb{Z}[K]$ for any K ; then there is a homotopy equivalence $e: \Sigma X \rightarrow \Sigma \text{hc}(K)$.*

Proof The Stanley–Reisner algebra $\mathbb{Z}[K]$ decomposes additively as a sum $\bigoplus_{\sigma \in K} A_\sigma$ of graded subgroups, generated by monomials whose support is a particular σ . Every A_σ is realised in $H^*(X; \mathbb{Z})$ by a map $q'_\sigma: X \rightarrow \wedge_\sigma BT$, defined by composing q_σ with projection onto the iterated smash product. Using the cogroup structure of ΣX , we sum the suspensions $\Sigma q'_\sigma$ over all faces σ , to obtain a map

$$h_X: \Sigma X \longrightarrow \bigvee_{\sigma \in K} \Sigma(\wedge_\sigma BT).$$

This induces a cohomology isomorphism of 1-connected CW-complexes, and is therefore a homotopy equivalence. Identical reasoning applies to $\text{hc}(K)$, so it suffices to define h as $h_{c(K)}^{-1} \circ h_X$. \square

We may use the splitting of [1, Theorem 1.18] as an elegant alternative to $h_{c(K)}$; in either event, Theorem 9.5 identifies X with the wedge $\bigvee_{\sigma \in K} \text{Th}(\sigma)$ of Thom complexes, after one suspension.

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