Chimneys, leopard spots and the identities of Basmajian and Bridgeman

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We give a simple geometric argument to derive in a common manner orthospectrum identities of Basmajian and Bridgeman. Our method also considerably simplifies the determination of the summands in these identities. For example, for every odd integer n, there is a rational function q_n of degree $2(n-2)$ so that if M is a compact hyperbolic manifold of dimension n with totally geodesic boundary S, there is an identity $\chi(S) = \sum_i q_n(e^{l_i})$ where the sum is taken over the orthospectrum of M. When $n=3$, this has the explicit form $\sum_i 1/(e^{2l_i}-1) = -\chi(S)/4$.

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1 Orthospectrum identities

Let M be a compact hyperbolic n–manifold with totally geodesic boundary S. An *orthogeodesic* is a properly immersed geodesic arc perpendicular to S at either end. The *orthospectrum* is the set of lengths of orthogeodesics, counted with multiplicity.

Basmajian [1] and Bridgeman–Kahn [2; 3] derived identities relating the orthospectrum of M to the area of S and the volume of M respectively. The following identity is implicit in [1]:

Basmajian's Identity [1] *There is a function a_n depending only on n, so that if M is a compact hyperbolic n–manifold with totally geodesic boundary S, and l_i denotes the (ordered) orthospectrum of M, with multiplicity, there is an identity:*

$$\operatorname{area}(S) = \sum_i a_n(l_i)$$

Basmajian's identity is not well known; in fact, Bridgeman and Kahn were apparently unaware of Basmajian's work when they derived the following by an entirely different method:

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Bridgeman's Identity [2; 3] *There is a function v_n depending only on n, so that if M is a compact hyperbolic n–manifold with totally geodesic boundary S, and l_i denotes the (ordered) orthospectrum of M, with multiplicity, there is an identity:*

$$\mathrm{volume}(M) = \sum_i v_n(l_i)$$

In this paper, we show that both theorems can be derived from a common geometric perspective. In fact, the derivation gives a very simple expression for the functions a_n and v_n, which we describe in Section 2. The derivation rests on a simple geometric decomposition.

Definition Let π and π' be totally geodesic $\mathbb{H}^{n-1}$'s in $\mathbb{H}^n$ with disjoint closure in $\mathbb{H}^n \cup S^{n-1}_\infty$. A *chimney* is the closure of the union of the geodesic arcs from π to π' that are perpendicular to π.

Thus, the boundary of the chimney consists of three pieces: the *base*, which is a round disk in π, the *side*, which is a cylinder foliated by geodesic rays, and the *top*, which is the plane π'. Note that the distance from the base to the top is realized by a unique orthogeodesic, called the *core*. The *height* of the chimney is the length of this orthogeodesic, and the *radius* is the radius of the base (these two quantities are related, and either one determines the chimney up to isometry).

Chimney Decomposition *Let M be a compact hyperbolic n–manifold with totally geodesic boundary S. Let M_S be the covering space of M associated to S. Then M_S has a canonical decomposition into a piece of zero measure, together with two chimneys of height l_i for each number l_i in the orthospectrum.*

Proof If S is disconnected, the cover M_S is also disconnected, and consists of a union of connected covering spaces of M, one for each component of S. The boundary of M_S consists of a copy of S, together with a union of totally geodesic planes. Each such plane is the top of a chimney, with base a round disk in S, and these chimneys are pairwise disjoint and embedded. Since M is geometrically finite, the limit set has measure zero, and therefore these chimneys exhaust all of M_S except for a subset of measure zero. Every oriented orthogeodesic in M lifts to a unique geodesic arc with initial point in M_S. Evidently this arc is the core of a unique chimney in the decomposition, and all chimneys arise this way. □

Basmajian's identity is immediate (in fact, though Basmajian does not express things in these terms, the argument we give is quite similar to his):

Proof S in M_S is decomposed into a set of measure zero together with the union of the bases of the chimneys. Thus

$$\text{area}(S) = 2\sum_i \text{ area of the base of a chimney of height } l_i. \qquad \square$$

Remark Thurston calls the chimney bases *leopard spots*; they arise in the definition of the skinning map (see eg Otal [7]).

Bridgeman's identity takes slightly more work, but is still elementary:

Proof If p is a point in M, and γ is an arc from p to S, there is a unique geodesic in the relative homotopy class of p which is perpendicular to S. Thus, the unit tangent sphere to p is decomposed into a set of measure zero, together with a union of round disks, one for each relative homotopy class of arc γ.

The area of the disk in UT_p associated to γ can be computed as follows. Let $\widetilde{\gamma}$ be the unique lift of γ to M_S with one endpoint on S, and let $\widetilde{p}$, a lift of p, be the other endpoint of $\widetilde{\gamma}$. If N is the complete hyperbolic manifold with M as compact core and N_S denotes the cover of N associated to S (so that M_S is a convex subset of N_S), let h_S be the harmonic function on N_S whose value at every point q is the probability that Brownian motion starting at q exits the end associated to S. Note that $h_S = 1/2$ on S, and at every point q depends only on the distance from q to S. Then the area of the disk in UT_p associated to γ is $\Omega_{n-1} \cdot h_S(\widetilde{p})$, where $\Omega_{n-1} := 2\pi^{n/2}/\Gamma(n/2)$ denotes the area of a Euclidean sphere of dimension $n-1$ and radius 1.

Since the volume of the unit tangent bundle of M is $\Omega_{n-1} \cdot \text{volume}(M)$, it follows that the volume of M is equal to the integral of h_S over M_S. In each chimney, h_S restricts to a harmonic function h, equal to $1/2$ on the base, and whose value at each point depends only on the distance to the base. Hence

$$\text{volume}(M) = 2\sum_i \text{ integral of } h \text{ over a chimney of height } l_i. \qquad \square$$

Remark In fact, precisely because our derivation is utterly unlike that of [3], we do *not* know whether Bridgeman's function v_n is equal to the integral of h over an n–dimensional chimney of given height, only that there *is* such a function v_n with the desired properties. If $n = 2$, our v_2 and Bridgeman's v_2 agree, but the proof is not easy; one short derivation follows from [4], together with a geometric dissection argument.

2 Explicit formulae

In this section we show that the summands in the area and volume identities have a very nice explicit form. The expressions we obtain depend on the following elementary ingredients:

Quadrilateral A chimney is a solid of revolution, obtained by revolving a hyperbolic quadrilateral Q with three right angles and one ideal vertex about the S^{n-2} of directions perpendicular to one of the finite sides (which becomes the core of the chimney, the other finite side becoming the radius of the base). In a quadrilateral with three right angles and one ideal vertex, the length of one finite edge determines the other. If one finite edge has length l, let $\iota(l)$ denote the length of the other finite edge, so that ι is an involution on $(0,\infty)$. Then ι is defined implicitly by the fact that it is positive, and the identity

$$1/\cosh^2(l) + 1/\cosh^2(\iota(l)) = 1$$

or equivalently,

$$\sinh(\iota(l)) = 1/\sinh(l).$$

If we write $\alpha = e^l$ and $\beta = e^{\iota(l)}$, then α and β are related by

$$\beta + \beta^{-1} = 2\left(\frac{\alpha+\alpha^{-1}}{\alpha-\alpha^{-1}}\right).$$

Hyperbolic volume If B is a ball of radius r in n–dimensional hyperbolic space, let $V_n^H(r)$ denote the volume of B. One has the following integral formula for V_n^H:

$$V_n^H(r) = \Omega_{n-1}\int_0^r \sinh^{n-1}(t)\,dt$$

The base of an n–dimensional chimney of height l is just the volume of an $(n-1)$–dimensional ball in hyperbolic space of radius $\iota(l)$. When n is even, the integral $\int_0^{\iota(l)} \sinh^{n-1}(t)\,dt$ is a *polynomial* in $\beta+\beta^{-1}$, and therefore a *rational function* in α of degree $2(n-1)$. If the dimension of M is at least 3, the set of numbers e^l where l runs over the orthospectrum are algebraic (by Mostow rigidity), and contained in a quadratic extension of the trace field of M.

If S has even dimension, then the area of S is proportional to the Euler characteristic, by the Chern–Gauss–Bonnet theorem; in fact, for a hyperbolic manifold of dimension n where n is even, one has

$$\text{area}(S) = (2\pi)^{n/2}\chi(S)r_n$$

where r_n is a rational number depending on n.

The following corollary appears to be new:

Rational Identity *For every odd integer n, there is a rational function q_n of degree $2(n-2)$, with integral coefficients, so that if M is a compact hyperbolic manifold of (odd) dimension n with totally geodesic boundary S, there is an identity*

$$\chi(S) = \sum_i q_n(e^{l_i})$$

where χ denotes Euler characteristic (which takes values in $\mathbb{Z}$) and l_i denotes the orthospectrum of M (with multiplicity). Note that for $n \geq 3$, the numbers e^{l_i} are all contained in a fixed number field K (depending on M).

Example It is elementary to compute q_n for small n. For example:

$$q_3(x) = \frac{4}{1-x^2}$$
$$q_5(x) = \frac{5x^6 - 33x^4 + 63x^2 - 27}{8(x^2-1)^3}$$

The denominator is easily seen to be an integer multiple of $(x^2-1)^{n-2}$.

Remark In the case of 3 dimensions, the identity has the following form. Let M be a hyperbolic 3–manifold with totally geodesic boundary S. Then

$$\sum_i \frac{1}{e^{2l_i} - 1} = -\chi(S)/4.$$

This is vaguely reminiscent of McShane's identity [5], which says that for S a hyperbolic once-punctured torus, there is an identity

$$\sum_i \frac{1}{1+e^{l_i}} = 1/2$$

where the sum is taken over lengths l_i of *simple* closed geodesics *in the surface* S.

If there is a simple relation between our identities and McShane's identity, it is not obvious. However, Mirzakhani [6] showed how to derive and generalize McShane's identity as a sum over *embedded* orthogeodesics on a surface with boundary. The appearance of orthogeodesics in yet another identity is quite suggestive of a more substantial connection, though we do not know what it might be.

To determine the summands in the volume identity, one needs the following additional ingredients:

ϕ–Quadrilateral If Q is a hyperbolic quadrilateral with three right angles and one vertex with angle ϕ, then one of the lengths l of the edges ending at right angles determines the other $\iota_\phi(l)$, defined implicitly by the identity

$$\sinh(\iota_\phi(l)) = \sinh(\iota(l))\cos(\phi) = \cos(\phi)/\sinh(l).$$

Spherical volume If B is a ball of radius r in n–dimensional spherical space, let $V_n^S(r)$ denote the volume of B. One has the following integral formula for V_n^S:

$$V_n^S(r) = \Omega_{n-1}\int_0^r \sin^{n-1}(t)\,dt$$

Harmonic Let h be the harmonic function on $\mathbb{H}^n$ equal to the indicator function of a round disk D in S_∞^{n-1}, so that $h = 1/2$ on the plane π bounded by ∂D. For q bounded away from D by π, if t is the distance from q to π, then $h(q)$ is Ω_{n-1}^{-1} times the volume of a ball in S^{n-1} of radius θ, where $\sin(\theta) = 1/\cosh(t)$.

Level sets Nearest point projection from an equidistant surface to a totally geodesic hyperplane multiplies distances by $1/\cosh(t)$. If C is a chimney of height l (and radius $\iota(l)$), let C_t be the level set at distance t from the base. Orthogonal projection of C_t to the base of the chimney is surjective if $t \le l$, and otherwise surjects onto an annulus with outer radius $\iota(l)$, and inner radius $\iota_\phi(l)$, where ϕ is defined implicitly by $\sin(\phi) = \cosh(l)/\cosh(t)$.

The area of C_t is therefore

$$\operatorname{area}(C_t) = \begin{cases} \cosh^{n-1}(t)V_{n-1}^H(\iota(l)) & \text{if } t \le l, \\ \cosh^{n-1}(t)\big(V_{n-1}^H(\iota(l)) - V_{n-1}^H(\iota_\phi(l))\big) & \text{if } t \ge l. \end{cases}$$

Putting this all together, we get an explicit integral formula for v_n:

$$\begin{aligned} v_n(l)/2 = &\int_0^l \cosh^{n-1}(t)V_{n-1}^H(\iota(l))V_{n-1}^S(\arcsin(1/\cosh(t)))\Omega_{n-1}^{-1}\,dt \\ &+ \int_l^\infty \cosh^{n-1}(t)\big(V_{n-1}^H(\iota(l)) - V_{n-1}^H(\iota_\phi(l))\big)V_{n-1}^S(\arcsin(1/\cosh(t)))\Omega_{n-1}^{-1}\,dt \end{aligned}$$

Notice when n is even this can be expressed in closed form in terms of elementary functions (compare with the formulae and the derivation in [3, pages 4-11]).

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