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Braids inside the Birman–Wenzl–Murakami algebra

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We determine the Zariski closure of the representations of the braid groups that factor through the Birman–Wenzl–Murakami algebra, for generic values of the parameters α, s. For α, s of modulus 1 and close to 1, we prove that these representations are unitarizable, thus deducing the topological closure of the image when in addition α, s are algebraically independent.

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1 Introduction

Let B_n denote the braid group on n strands, defined by the presentation with generators $\sigma_1, \dots, \sigma_{n-1}$ and relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| \geq 2$ (which imply that the generators are all conjugates of each other). We consider here the linear representations of B_n afforded by the so-called Birman–Wenzl–Murakami algebras. For a field K of characteristic 0 and $\alpha, s \in K$ with $\alpha, s, s - s^{-1}$ nonzero, the algebra $\mathrm{BMW}_n(s, \alpha)$ can be defined as the quotient of the group algebra KB_n by the three relations

$$(\sigma_1 - s)(\sigma_1 + s^{-1})(\sigma_1 + \alpha^{-1}) = 0$$

and $$\left(1 - \frac{\sigma_2 - \sigma_2^{-1}}{s - s^{-1}}\right)\sigma_1^{\pm 1}\left(1 - \frac{\sigma_2 - \sigma_2^{-1}}{s - s^{-1}}\right) = \alpha^{\pm 1}\left(1 - \frac{\sigma_2 - \sigma_2^{-1}}{s - s^{-1}}\right).$$

For generic values of α, s, the algebra $\mathrm{BMW}_n(s, \alpha)$ is semisimple and its structure is known, thus providing many representations of the braid groups.

The algebra $\mathrm{BMW}_n(s, \alpha)$ is a deformation of Brauer's centralizer algebra (see below), which admits for quotient the so-called Iwahori–Hecke algebra of type A_{n-1}, namely the quotient $H_n(s)$ of KB_n by the relation

$$(\sigma_1 - s)(\sigma_1 + s^{-1}) = 0.$$

With this relation, rewritten as $\sigma_1 - \sigma_1^{-1} = s - s^{-1}$, the last two relations of $\mathrm{BMW}_n(s, \alpha)$ are void, making $H_n(s)$ appear as a quotient of $\mathrm{BMW}_n(s, \alpha)$.

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Besides the representations induced by this quotient, $\mathrm{BMW}_n(s,\alpha)$ admits another special representation, already singled out in Birman and Wenzl [3], which is known to induce a faithful representation of B_n by work of Krammer [8; 9] and Bigelow [2]. We call it the Krammer representation $R_K\colon B \to \mathrm{GL}_{n(n-1)/2}(K)$.

Let $R\colon B_n \to \mathrm{GL}_N(K)$ be a linear representation afforded by some linear representation of $\mathrm{BMW}_n(s,\alpha)$. We are interested here in the image $R(B_n) \subset \mathrm{GL}_N(K)$. Since R is defined only up to conjugacy, the first natural question is: what is the closure of $R(B_n)$ for the Zariski topology?

Letting $B_n' = (B_n, B_n)$ denote the commutator subgroup, we state some the results obtained in terms of $R(B_n')$, because the statements are simpler. Since $B_n' = \mathrm{Ker}(B_n \twoheadrightarrow \mathbb{Z})$ is defined by $\sigma_1 \mapsto 1$, the corresponding statements for $R(B_n)$ are easy to deduce from them.

For α, s generic (say, algebraically independent over $\mathbb{Q} \subset K$), this problem was solved by the author in [13] for the Hecke algebra representations and in [14] for the Krammer representation. The present paper is thus a sequel of these two previous works. In [13] we proved that the Zariski closure of B_n inside the whole Hecke algebra had for Lie algebra the Lie subalgebra of the group algebra of the symmetric group $\mathfrak{S}_n$ generated (for the bracket $[a,b] = ab - ba$) by the transpositions, and decomposed this reductive Lie algebra. We called this Lie algebra the infinitesimal Hecke algebra (of type A_{n-1}). Here we exhibit a reductive Lie subalgebra of the Brauer centralizer algebra that plays a similar role, and that we decompose accordingly. A consequence is the following, which generalizes [14, Theorem 2]. Recall from [3] that $\mathrm{BMW}_n(s,\alpha)$ is split semisimple over K (see Birman and Wenzl [3, Theorem 3.7], the proof given there being valid over $\mathbb{Q}(s,\alpha)$, not only $\mathbb{C}(s,\alpha)$).

Theorem 1.1 *Let $R\colon B_n \to \mathrm{GL}_N(K)$ a representation afforded by $\mathrm{BMW}_n(s,\alpha)$ for α, s algebraically independent over $\mathbb{Q}$, or for $\alpha = s^m$ with s transcendant over $\mathbb{Q}$ and m outside a finite set of integer values.*

(1) *If R is irreducible and does not factor through $H_n(s)$, then $R(B_n)$ is Zariski-dense in $\mathrm{GL}_N(K)$.*

(2) *If $R = R_0 \oplus R_1 \oplus \cdots \oplus R_k\colon B_n \to \mathrm{GL}_N(K)$ with $R_i\colon B_n \to \mathrm{GL}_{N_i}(K_i)$ satisfying (1) when $i \geq 1$, $N = N_0 + N_1 + \cdots + N_k$, R_0 a (not necessarily irreducible) representation factoring through $H_n(s)$, and $R_i \not\simeq R_j$ for $i \neq j$, then the image $R(B_n')$ of the commutator subgroup of B_n is Zariski-dense in $G_0 \times \mathrm{SL}_{N_1}(K) \times \cdots \times \mathrm{SL}_{N_k}(K)$, where G_0 is the closure of $R_0(B_n')$.*

Another natural question is whether the representations of B_n obtained this way are unitarizable, when $K = \mathbb{C}$. In this case, the determination of the Zariski closure done above is more or less equivalent to the determination of the topological closure for the usual topology of $\mathbb{C}$.

This question makes sense only for $|\alpha| = |s| = 1$, because of the spectrum of the Artin generator. Even in this case, it is known that the representations afforded by $H_n(s)$ are not unitarizable in general, but they are so if in addition s is close enough to 1. This was first proved by H Wenzl, who exhibited in [16] explicit unitary matrix models for these representations. For α, s close to 1 and some additional constraints, this was also proved for the Krammer representation by R Budney, who followed the method of C Squier (for the Burau representation) of constructing an explicit sesquilinear form preserved in the usual matrix models of this representation.

We showed in previous works [12; 14] how to obtain new proofs of these two results by making use of Drinfel′d theory of associators. The idea is that all these representations appear as monodromy of so-called KZ–systems, and that these KZ–systems have for coefficients real matrices which are compatible with certain natural bilinear forms. Then, from the choice of a Drinfel′d associator with rational (or real) coefficients, whose existence was proved by Drinfel′d, one can built a representation of B_n over the ring $\mathbb{R}[[h]]$ of formal series with image in some formal unitary group (with respect to the automorphism $f(h) \mapsto f(-h)$). By specializing the matrices in h a purely imaginary complex number we then get unitary representations of B_n, and this provides a natural path for explaining the unitarisability of this kind of representations. A technical problem however is that we do not have (so far) any insurance that the series involved have nonzero convergence radius. For the Hecke algebra representations [12], one can explicitly compute the matrix models obtained this way, which turn out to be the same than the ones obtained earlier by Wenzl, and check that they are indeed convergent. Another way, that we already used in [14] for the Krammer representation, is to approximate the representation over $\mathbb{R}[[h]]$ by equivalent representations over the ring of convergent power series. This is the device we use here, giving in the above spirit a new proof of the following result, which was originally proved by Wenzl [18] by using the Jones construction.

Theorem 1.2 (Wenzl) *Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. There exists an open subset U of $S^1 \times S^1$ whose closure contains $(1,1)$ such that, for $(s,\alpha) \in U$, the representations of B_n induced by $\mathrm{BMW}_n(s,\alpha)$ are unitarizable.*

Putting the two theorems together, they determine up to isomorphism the topological closure of all representations of B_n that factor through $\mathrm{BMW}_n(s,\alpha)$ for $(s,\alpha) \in U$ with s, α algebraically independent over $\mathbb{Q}$, since the compact Zariski-dense subgroups of $\mathrm{GL}_N(\mathbb{C})$ are its maximal compact subgroups. This generalizes (part of) the work of Freedman, Larsen and Wang in [6] on the representations of $H_n(s)$.

Finally note that the open set U of Theorem 1.2 contains (a dense set of) couples (s,α) which are algebraically independent. In particular, every irreducible representation R as in Theorem 1.1 (not factorizing through $H_n(s)$) enables, when faithful, to embed B_n in the corresponding unitary group as a dense subgroup.

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2 Brauer diagrams and Brauer algebra

We refer to Wenzl [17] or Goodman and Wallach [7] for the definition and basic properties of the algebra Br_n of Brauer diagrams (in short: Brauer algebra) as a finite-dimensional algebra over $\mathbb{Q}[m]$, and its specialization $\mathrm{Br}_n(m)$ over some field $\Bbbk$ of characteristic 0, where $m \in \Bbbk$. They are algebras spanned by so-called Brauer diagrams, where ab means composing the diagram b below the diagram a with additional relations $p_{ij}^2 = mp_{ij}$. These algebras contain $\mathfrak{S}_n$, and in particular the transpositions s_{ij}. In this section we assume $\Bbbk \subset \mathbb{R}$, and in particular $m \in \mathbb{R}$.

We first define an involutive linear automorphism τ, defined at the level of the diagrams by reflecting the diagram "upside-down" (see Figure 1). For $w \in \mathfrak{S}_n$ we have $\tau(w) = w^{-1}$; moreover $\tau(p_{ij}) = p_{ij}$. This antiautomorphism thus leaves the generators s_{ij} and p_{ij} invariant, and can alternatively be defined from this property. For $m > n$ or $m \notin \mathbb{Q}$ there exists a nondegenerate trace tr_M such that $\mathrm{tr}_M(b) = \mathrm{tr}_M(\tau(b))$ for every diagram b, for instance the Markov trace defined in [17]. We assume from now on that tr_M is this Markov trace. We then let $\langle D_1, D_2\rangle = \mathrm{tr}_M(D_1\tau(D_2))$ on the Brauer diagrams and extend it by linearity. Consequences of the assumptions on tr_M are the following:

- For all a, b we have
$$\langle a,b\rangle = \mathrm{tr}_M(a\tau(b)) = \mathrm{tr}_M(\tau(a\tau(b)) = \mathrm{tr}_M(b\tau(a)) = \langle b,a\rangle.$$
- For all a, b, and $w \in \mathfrak{S}_n$,
$$\begin{aligned}\langle wa, wb\rangle = \mathrm{tr}_M(wa\tau(wb)) &= \mathrm{tr}_M(wa\tau(b)\tau(w)) \\ &= \mathrm{tr}_M(\tau(w)wa\tau(b)) = \mathrm{tr}_M(a\tau(b)) = \langle a,b\rangle.\end{aligned}$$
- For all a, b, and p_{ij},
$$\langle p_{ij}a, b\rangle = \mathrm{tr}_M(p_{ij}a\tau(b)) = \mathrm{tr}_M(a\tau(b)\tau(p_{ij})) = \mathrm{tr}_M(a\tau(p_{ij}b)) = \langle a, p_{ij}b\rangle.$$

In particular, the endomorphism $s_{ij} - p_{ij}$ is selfadjoint with respect to $\langle\ ,\ \rangle$ and the elements of $\mathfrak{S}_n$ act orthogonally. For tr_M the Markov trace of [17], the bilinear form

$(a,b) \mapsto \mathrm{tr}_M(ab)$ on $\mathrm{Br}_n(m)$ is nondegenerate for $m > n$ or $m \notin \mathbb{Q}$. The Brauer algebra is then semisimple and decomposes as a sum of matrix algebras. The set Irr_n of irreducible representations of Br_n is in 1-1 correspondence with the partitions of r with $n-r$ a nonnegative even integer. When no confusion can arise, we denote them by the corresponding partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$, or by λ_n if the number of strands is not implicit. We let $|\lambda| = \lambda_1 + \lambda_2 + \cdots = r \leq n$.

For $\lambda \in \mathrm{Irr}_n$, let $\mathrm{tr}_\lambda \colon x \mapsto \mathrm{tr}(\lambda(x))$ denote the matrix trace on the corresponding factor of $\mathrm{Br}_n(m)$. By [17], we have

$$\mathrm{tr}_M(b) = \sum_{\lambda \in \mathrm{Irr}_n} \frac{P_\lambda(m)}{m^n} \mathrm{tr}_\lambda(b).$$

for some rational polynomials $P_\lambda(m)$ of m. For $m > n$ it is possible to choose the representations $\lambda \in \mathrm{Irr}_n$ defined over $\mathbb{R}$ and such that $\lambda(\tau(b)) = {}^t\lambda(b)$. Indeed, this means that the generators $s_{k,k+1}$ and p_{12} have real entries and are symmetric in some basis, and it possible to find such a basis by [15, (3.11) (see the proof of Theorem 3.12)]. Actually, the argument in [15] proves this under the additional condition that m is an integer. This additional condition can be readily dropped, as the symmetric formulas obtained there make sense for a real number $m > n$ hence define representations of the corresponding $\mathrm{Br}_n(m)$, the defining relations of $\mathrm{Br}_n(m)$ being polynomial in m and the matrix entries being (square roots of) rational fractions of m.

We then have

$$\begin{aligned}\langle a,b \rangle &= \sum_{\lambda \in \mathrm{Irr}_n} \frac{P_\lambda(m)}{m^n} \mathrm{tr}(\lambda(a\tau(b))) \\ &= \sum_{\lambda \in \mathrm{Irr}_n} \frac{P_\lambda(m)}{m^n} \mathrm{tr}(\lambda(a)\lambda(\tau(b))) = \sum_{\lambda \in \mathrm{Irr}_n} \frac{P_\lambda(m)}{m^n} \mathrm{tr}(\lambda(a)\,{}^t\lambda(b)).\end{aligned}$$

It follows that $\langle\ ,\ \rangle$ is positive definite, for $m \notin \mathbb{Q}$ or $m > n$, if and only if all the $P_\lambda(m)/m^n$ are positive. From [17] we have an explicit combinatorial description of $P_\lambda(m)$, and we know that $P_\lambda(m)$ coincides with the dimension of a representation for m a sufficiently large integer. In particular, $P_\lambda(m) > 0$ for $m \gg 0$. We let $\mathcal{S}_n = \{m \mid \exists \lambda \vdash r \leq n, P_\lambda(m) = 0\} \subset \mathbb{Z}$.

3 Convergent approximation

In this section we prove a technical result concerning finitely generated subfields of the field $\mathbb{R}(\{h\})$ of convergent Laurent series (ie the field of fractions of the ring $\mathbb{R}\{\{h\}\}$ of formal power series with nonzero convergence radius), and apply it to the twisting

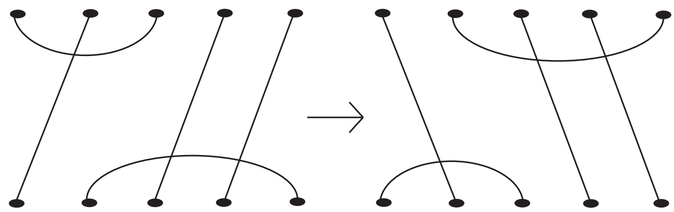

Figure 1: The “upside-down” automorphism τ

of representations by adequate field automorphisms. We denote by $\mathbb{R}(\!(h)\!)$ the field of formal Laurent series, $\mathbb{R}(h)$ the field of rational fractions.

Proposition 3.1 *Let $K = \mathbb{R}(\!(h)\!)$, $K^* = \mathbb{R}(\{h\})$, $\epsilon \in \mathrm{Aut}(K)$ defined by $f(h) \mapsto f(-h)$, and $L \subset K$ a finitely generated extension of $\mathbb{R}(h)$. There exists $L^* \subset K^*$ such that $L^* \simeq L$ and an isomorphism $\Omega\colon L \to L^*$ with $L \cap K^* \subset L^*$ and $\Omega(L \cap \mathbb{R}[\![h]\!]) \subset L^* \cap \mathbb{R}\{\{h\}\}$. If $\epsilon(L) = L$, then L^* can be chosen such that $\epsilon(L^*) = L^*$ and the isomorphism Ω can be chosen such that $\epsilon \circ \Omega = \Omega \circ \epsilon$. Moreover, for any finite set $\{u_1, \dots, u_t\} \subset L$ and $M > 0$, Ω can be chosen such that $\Omega(u_i) \equiv u_i$ modulo h^M.*

Proof Let $L_0 = L \cap K^*$ and L_1 a purely transcendental extension of L_0 in L. Since L is finitely generated, L/L_0 has finite transcendence degree r, and L/L_1 is finite, that it $L = L_1[\alpha]$ for some $\alpha \in L$ algebraic over L_1. Since $L_0 \supset \mathbb{R}(h)$ we can assume $\alpha \in h\mathbb{R}[\![h]\!]$ and also choose a transcendence basis $f_1, \dots, f_r$ of L_1/L_0 with $f_i \in \mathbb{R}[\![h]\!]$. Since L_0 is finitely generated and $\mathbb{R}(\{h\})$ has infinite transcendence degree over $\mathbb{R}(h)$ (consider eg the family $\exp(h^d), d \geq 0$), there exists $g_1, \dots g_r \in \mathbb{R}\{\{h\}\}$ algebraically independent over L_0. Since $L_0 \supset \mathbb{R}(h)$, for any $P_1, \dots, P_r \in \mathbb{R}[h]$, the family $g_1 + P_1 + \dots + g_r + P_r$ is also algebraically independent over L_0, hence for any given N we can choose $g_i \equiv f_i$ modulo h^N. Let then $L_1^* = L_0(g_1, \dots, g_r) \simeq L_1$ and $P \in L_1[X]$ a minimal polynomial for $\alpha \in L$. By not requiring P to be monic, since $L_0 \supset \mathbb{R}(h)$ we can assume that the coefficients of P belong to $L_0[f_1, \dots, f_r] \cap \mathbb{R}[\![h]\!]$. Through $L_1^* \simeq L_1$, we define $P_g \in L_0[g_1, \dots, g_r]$. We have $P(\alpha) = 0$, $P'(\alpha) \equiv \beta h^s$ modulo h^{s+1} for some $s \geq 0$ and $\beta \in \mathbb{R}^\times$. By choosing N large enough, we get $P'_g(\alpha) \equiv \beta h^s$ modulo h^{s+1} and $P_g(\alpha) \equiv 0$ modulo h^{2s+1}. In particular $P_g(\alpha) \in P'_g(\alpha)^2 h\mathbb{R}[\![h]\!]$. Hensel's lemma then asserts (see eg Eisenbud [5, Theorem 7.3]) the existence of $\gamma \in \mathbb{R}[\![h]\!]$ such that $P_g(\gamma) = 0$ and $\gamma - \alpha \in P'_g(\alpha) h\mathbb{R}[\![h]\!] \subset h\mathbb{R}[\![h]\!]$. It follows that $\gamma \in h\mathbb{R}[\![h]\!]$. Then M Artin's approximation theorem [1] states that there exists a root $\tilde{\gamma} \in \mathbb{R}\{\{h\}\}$ of P_g that can be chosen arbitrarily close to γ, hence $\gamma \in \mathbb{R}\{\{h\}\}$ (since P_g admits finitely many roots), and $L^* = L_1[\gamma] \simeq L$ is a subfield of K^* that satisfies the wanted properties.

For any finite subset $u_1, \dots, u_t \in L = L_0(f_1, \dots, f_r)[\alpha]$, by choosing N large enough we can assume that $s \geq M$, hence $\gamma \equiv \alpha$ modulo h^M, and that $\Omega(u_i) \equiv u_i$ modulo h^M.

Finally, assume that $\epsilon(L) = L$ and let $L^\epsilon = \{f \in L \mid \epsilon(f) = f\}$. We have $\mathbb{R}(h^2) \subset L^\epsilon \subset K^\epsilon = \mathbb{R}((h^2))$, $L = L^\epsilon \oplus hL^\epsilon \simeq L^\epsilon[X]/(X^2 - h)$. Let Φ: $K^\epsilon \to K$ be the isomorphism defined by $f(h^2) \mapsto f(h)$ and $\Lambda = \Phi(L^\epsilon)$. Clearly Φ and Φ^{-1} map convergent series to convergent series. We have $L \simeq \Lambda_+ = \Lambda[X]/(X^2 - h)$ and $\mathrm{Gal}(\Lambda_+/\Lambda) \simeq \mathbb{Z}/2$ is generated by the action of ϵ on $L \simeq \Lambda_+$. We have already show how to construct an extension $\mathbb{R}(h) \subset \Lambda^* \subset K^*$ with $\Lambda^* \simeq \Lambda$ over $\Lambda \cap K^* \supset \mathbb{R}(h)$. Therefore, there exists an isomorphism between the extensions Λ_+/Λ and Λ_+^*/Λ^* with $\Lambda_+^* = \Lambda[X]/(X^2 - h)$. Letting $L_-^* = \Phi^{-1}(\Lambda^*)$ and $L^* = L_1^*(h) \subset K^*$ this defines Ω: $L \to L^*$ with $\Omega \circ \epsilon = \epsilon \circ \Omega$. Moreover, if $f \in L \cap K^*$, then $f = f_1 + hf_2$ with $f_i \in L^\epsilon \cap K^*$, hence $\Phi(f_i) \in \Lambda \cap K^*$, $\Phi(f_i) = f_i$ and $\Omega(f) = f$. The verification that Ω can be chosen with $\Omega(u_i)$ close to u_i in this case too, is straightforward and left to the reader. $\square$

For L a subfield of $\mathbb{R}((h))$ such that $\epsilon(L) = L$, we let $U_N^\epsilon(L) = \{X \in \mathrm{GL}_N(L) \mid X^{-1} = {}^t\epsilon(X)\}$.

Corollary 3.2 *Let G be a finitely generated group, R: $G \to \mathrm{GL}_N(\mathbb{R}[[h]])$ be a linear representation, $g_1, \dots, g_r \in G$ such that $R(g_1), \dots, R(g_r)$ have entries in $\mathbb{R}\{\{h\}\}$, and $M > 0$. Then there exists a linear representation R^*: $G \to \mathrm{GL}_N(\mathbb{R}\{\{h\}\}$ such that $R^*(g_i) = R(g_i)$ and, for all $g \in G$, $R^*(g) \equiv R(g)$ modulo h^M. Moreover, if $R(G) \subset U_N^\epsilon(\mathbb{R}((h)))$, then we can assume $R^*(G) \subset U_N^\epsilon(\mathbb{R}(\{h\}))$.*

Proof Let $u_1, \dots, u_n$ be generators of G, and L the subfield of $\mathbb{R}((h))$ generated over $\mathbb{R}(h)$ by the entries of the $R(u_i)$ and $R(u_i^{-1})$. We have $R(G) \subset \mathrm{GL}_N(L)$ and L is finitely generated over $\mathbb{R}(h)$. Using an isomorphism Ω: $L \to L^* \subset \mathbb{R}(\{h\})$ provided by the proposition, we let $R^* = \Omega \circ R$. If Ω is chosen so that $R^*(u_i) \equiv R(u_i) \bmod h^M$ and $R^*(u_i^{-1}) \equiv R(u_i^{-1}) \bmod h^M$ then $R^*(g) \equiv R(g) \bmod h^M$ for every $g \in G$, which concludes the proof. In case $R(G) \subset U_N^\epsilon(\mathbb{R}((h)))$, then L clearly satisfies $\epsilon(L) = L$ and the condition $\Omega \circ \epsilon = \epsilon \circ \Omega$ implies $R^*(G) \subset U_N^\epsilon(\mathbb{R}(\{h\}))$. $\square$

Proposition 3.3 *In the situation of the corollary, if R is absolutely irreducible, then R^* can be chosen absolutely irreducible.*

Proof Since R is absolutely irreducible, the image of the group algebra $\mathbb{R}((h))G$ inside $\mathrm{Mat}_N(\mathbb{R}((h)))$ is full, that is there exists $g_1, \dots, g_{N^2} \in \mathbb{R}[[h]]G$ such that $R(g_1), \dots, R(g_{N^2})$ is linearly independent over $\mathbb{R}((h))$. The determinant of this family is thus congruent to βh^s modulo h^{s+1}, for some $\beta \in \mathbb{R}^\times$ and $s \geq 0$. Replacing the coefficients of $g_1, \dots, g_{N^2}$ by their approximation modulo h^{s+1} we can assume that $g_1, \dots, g_{N^2}$ have coefficients in $\mathbb{R}[h]$. Let $u_1, \dots, u_r$ denote the elements of G that appear in the $g_1, \dots, g_r$. Choosing R^* such that $R^*(u_i) \equiv R(u_i)$ modulo h^{s+1} we get that $R^*(g_i) \equiv R(g_i) \bmod h^{s+1}$, whence the family $R^*(g_1), \dots, R^*(g_N)$ is also linearly independent over $\mathbb{R}(\{h\})$, which concludes the proof. $\square$

4 Unitarisability of the representations of the BMW algebras

We first construct representations of the braid groups over $\mathbb{R}[\![h]\!]$ which are formally unitary, then approximate these by convergent series. By an adequate specialization this affords unitary representations which are shown to be equivalent to representations of the BMW algebras.

Recall from eg Drinfel′d [4] that the Lie algebra of (pure) infinitesimal braids $\mathcal{T}_n$, or horizontal chord diagrams, is defined by generators t_{ij}, $1 \leq i, j \leq n$ with relations $t_{ii} = 0, t_{ij} = t_{ji}$, $[t_{ij}, t_{ik} + t_{kj}] = 0$, and $[t_{ij}, t_{kl}] = 0$ for $\#\{i, j, k, l\} = 4$. It is endowed with an action of $\mathfrak{S}_n$ by $w.t_{ij} = t_{w(i),w(j)}$, and a grading given by $\deg t_{ij} = 1$. Drinfel′d theory of associators [4] defines, for $\Bbbk$ a field of characteristic 0 and $\mu \in \Bbbk^\times$, a set $M_\mu(\Bbbk)$ of formal series in two noncommuting variables, such that any $\Phi \in M_\mu(\Bbbk)$ provides an (injective) morphism $B_n \to \mathfrak{S}_n \ltimes \exp \widehat{\mathcal{T}}_n$ where $\widehat{\mathcal{T}}_n$ denotes the completion of $\mathcal{T}$ with respect to its natural grading. Such a morphism maps σ_1 to $(1\ 2)\exp(\mu t_{12})$ (our $M_\mu(\Bbbk)$ is Drinfel′d's $M_{2\mu}(\Bbbk)$). Drinfel′d then exhibits a special element $\Phi_{KZ} \in M_{\mathrm{i}\pi}(\mathbb{C})$ and deduces abstractly from this that $M_\mu(\Bbbk) \neq \varnothing$ for every μ and $\Bbbk$. We refer to [4] for the main properties of such elements.

Let $\rho\colon \mathfrak{S}_n \to \mathrm{GL}_N(\mathbb{R})$ be a representation of the symmetric group $\mathfrak{S}_n$ and $\varphi\colon \mathcal{T}_n \to \mathfrak{gl}_N(\mathbb{R})$ be a representation of the Lie algebra of infinitesimal braids compatible with ρ, ie $\rho(w)\varphi(t_{ij})\rho(w)^{-1} = \varphi(t_{w(i),w(j)})$. We can extend φ into a representation $\widehat{\mathcal{T}}_n \to \mathfrak{gl}_N(\mathbb{R}[\![h]\!])$ through $t_{ij} \mapsto h\varphi(t_{ij})$. The choice of a real Drinfel′d associator $\Phi \in M_1(\mathbb{R})$ provides a representation $R\colon B_n \to \mathrm{GL}_N(\mathbb{R}[\![h]\!])$ whose image lies in $\rho(\mathfrak{S}_n)\exp h\,(\varphi(\mathcal{T}) \otimes \mathbb{R}[\![h]\!])$. We let as before $K = \mathbb{R}(\!(h)\!)$.

Proposition 4.1 *If $\mathbb{R}^N$ is endowed with its canonical Euclidean structure, $\rho(\mathfrak{S}_n) \subset O_N(\mathbb{R})$ and ${}^t\varphi(t_{ij}) = \varphi(t_{ij})$, then $R(B_n) \subset U_N^\epsilon(K)$.*

Proof We only need to check that, for every Artin generator σ_i, we have $R(\sigma_i) \subset U_N^\epsilon(K)$. Recall from [4] that $R(\sigma_i)$ is conjugate to $\exp h\varphi(t_{i,i+1}) \in U_N^\epsilon(K)$ by some element of the form $\Phi(hx, hy)$ where x, y are linear combination of the $\varphi(t_{ij})$, hence are symmetric matrices. Now Φ is the exponential of a Lie series Ψ, and $u \mapsto -{}^t u$ is an automorphism of $\mathfrak{gl}_N(\mathbb{R})$. We thus get $-{}^t\Psi(hx,hy) = \Psi(-{}^thx, -{}^thy) = \epsilon(\Psi(hx,hy))$, hence ${}^t\Phi(hx,hy)^{-1} = \epsilon(\Phi(hx,hy))$. It follows that $\Phi(hx,hy) \in U_N^\epsilon(K)$ whence $R(B_n) \subset U_N^\epsilon(K)$. □

We will also need the following.

Proposition 4.2 (1) *If φ is absolutely irreducible, then R is absolutely irreducible.*

(2) *In general, the Lie algebra of the Zariski closure of $R(B_n)$ inside $\mathrm{GL}_N(K)$ contains $\varphi(\mathcal{T}) \otimes_{\mathbb{R}} K$.*

Proof The proof of (1) basically follows from the fact that $R(\gamma_{ij})$ is congruent to $1 + 2h\varphi(t_{ij})$ modulo h^2, where γ_{ij} denote the standard generator of the pure braid group, and that $\varphi(\mathsf{U}\mathcal{T})$ is generated by the $\varphi(t_{ij})$. Indeed, by Burnside theorem there exists a basis of $\mathrm{Mat}_N(\mathbb{R}) = \mathbb{R}^{N^2}$ of noncommutative polynomials $P_1, \dots, P_{N^2}$ in the $\varphi(t_{ij})$, that is $P_k = P_k((\varphi(t_{ij})_{i,j})$. Then the $P_k(((\gamma_{ij}-1)/2h)_{ij})$ define elements of KB_n whose image under R is congruent modulo h to a basis of $\mathrm{Mat}_N(\mathbb{R})$. It follows that $R(KB_n)$ generates $\mathrm{Mat}_N(K)$, that is that R is absolutely irreducible. This concludes the proof of (1).

With no assumption on φ, we have $R(\gamma_{ij}) = \exp(hx_{ij})$ for some $x_{ij} \in \varphi(\mathcal{T}) \otimes \mathbb{R}[\![h]\!]$, such that $x_{ij} \equiv 2\varphi(t_{ij}) \bmod h$ since $R(\gamma_{ij}) \equiv 1 + 2h\varphi(t_{ij}) \bmod h^2$. It follows that the Lie subalgebra $\mathfrak{g}$ of $\varphi(\mathcal{T}) \otimes \mathbb{R}[\![h]\!]$ generated by the x_{ij} reduces to $\varphi(\mathcal{T})$ modulo h. Thus $\dim_K \mathfrak{g} \otimes K \geq \dim_{\mathbb{R}} \varphi(\mathcal{T})$ and, since $\mathfrak{g} \subset \varphi(\mathcal{T}) \otimes \mathbb{R}[\![h]\!]$, we get $\mathfrak{g} \otimes K = \varphi(\mathcal{T}) \otimes K$. The fact that $\mathfrak{g} \otimes K$ is contained in the Lie algebra of the Zariski closure of $R(B_n)$ is then an elementary consequence of Chevalley's formal exponentiation theory (see Marin [13, Lemme 21]). This proves (2). □

We then approximate R by a convergent $R^*\colon B_n \to \mathrm{GL}_N(K^*)$ as in the previous section. A first remark is that R^* can be chosen such that $R^*(\sigma_i)$ is conjugate to $R(\sigma_i)$ in $\mathrm{GL}_N(K)$, at least when $R(\sigma_i)$ is diagonalizable with eigenvalues in K^*. Indeed, if this is the case, there exists a polynomial $P \in K^*[X]$ with simple roots in K^* such that $P(R(\sigma_i)) = 0$, hence $P(R^*(\sigma_i)) = \Omega(P)(\Omega(R(\sigma_i))) = \Omega(0) = 0$. Moreover, the traces of the $R(\sigma_i)^k$, which belong to K^*, equal the traces of the $R^*(\sigma_i)^k$, hence $R(\sigma_i)$ and $R^*(\sigma_i)$ have the same spectrum with multiplicities. Notice that this assumption is satisfied in our case as soon as $\varphi(t_{12})$ is semisimple, since $R(\sigma_1) = \rho(s_{12})\exp(h\varphi(t_{12}))$.

Let $\alpha > 0$ such that the entries of the $R(\sigma_i)$ and $R(\sigma_i^{-1})$ all have convergence radius at least α. By specialization of h to a purely imaginary number $\mathrm{i}u$ of modulus less than α, we get unitary representations $R_{\mathrm{i}u}\colon B \to U_N \subset \mathrm{GL}_N(\mathbb{C})$.

We now apply this to an irreducible representation of the Brauer algebra $\lambda \in \mathrm{Irr}_n$, given in matrix form over $\mathbb{R}$ such that $\lambda(\tau(b)) = {}^t\lambda(b)$ for all b. Recall from Section 2 that this is possible for $m > n$. Then ρ restricts to an orthogonal representation of $\mathfrak{S}_n$, and $\varphi(t_{ij}) = \rho(s_{ij}) - \rho(p_{ij})$ defines a compatible representation of $\mathcal{T}_n$ with ${}^t\varphi(t_{ij}) = \varphi(t_{ij})$. We thus get representations R and R^* of B_n.

Proposition 4.3 *Assume $m \notin \mathbb{Q}$ or $m > n$. For m outside a finite set of other values, the representations R and R^* factor through the Birman–Wenzl–Murakami algebra $\mathrm{BMW}_n(e^h, e^{(1-m)h})$ and correspond to the same partition λ.*

Proof The assertion about R is well-known, and can be proved eg along the lines of [11, Proposition 4]. Since the defining relations of the BMW algebra have coefficients in K^*, then R^* also factors through it. Introduce the elements $\delta_2 = \sigma_1^2, \delta_3 = \sigma_2\sigma_1^2\sigma_2, \dots, \delta_{n-1} = \sigma_{n-1}\cdots\sigma_2\sigma_1^2\sigma_2\cdots\sigma_{n-1}$. We have $R(\delta_k) = \exp h\varphi(Y_k)$, where $Y_2 = t_{12}, \dots, Y_n = t_{1n} + \cdots + t_{n-1,n}$. We recall that, for $m > n$ or $m \notin \mathbb{Q}$, and possibly outside a finite set of other values of m, the values of $\varphi(Y_2), \dots, \varphi(Y_n)$ determine λ [11, Section 9]. Since the $R(\delta_k)$ have entries in K^*, we have $R^*(\delta_k) = R(\delta_k)$ and the conclusion follows. □

We denote $\log z$ the determination of the complex logarithm over $\mathbb{C} \setminus \mathbb{R}_-$ such that $\log 1 = 0$. The following corollary proves Theorem 1.2, for U drawn in Figure 2.

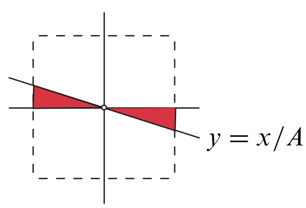

Figure 2: $s = e^{\mathrm{i}\pi y}, \alpha = e^{\mathrm{i}\pi x}$

Corollary 4.4 *Assume $s, \alpha \in \mathbb{C}$ with $|\alpha| = |s| = 1$, and $s, \alpha \notin \{1, -1\}$. There exists $A < -1$ and $0 < \eta < 2$ such that, for $0 < |s-1| < \eta$, $0 < |\alpha - 1| < \eta$ and $\log\alpha/\log s < A$, all the representations of B_n originating from $\mathrm{BMW}_n(s, \alpha)$ are unitarizable.*

Proof Let $m \in \mathbb{R}$ defined by $1 - m = (\log\alpha)/(\log s)$. We can choose A and η_0 such that $m > n$ and does not belong to the exceptions of the proposition for any $\lambda \in \mathrm{Irr}_n$, and such that, for $0 < |s-1| < \eta_0$ and $0 < |\alpha - 1| < \eta_0$, the algebra $\mathrm{BMW}_n(s, \alpha)$ is split semisimple and has for irreducible representations the ones deduced from the generic algebras BMW_n over $\mathbb{R}(\widetilde{s},\widetilde{\alpha})$, where $\widetilde{s}$ and $\widetilde{\alpha}$ are formal parameters. We mean by that the given irreducible representation R_0 of B_n over $\mathbb{C}$ factorizing through $\mathrm{BMW}_n(s,\alpha)$ is deduced from the corresponding representation R_g of B_n over $\mathbb{R}(\widetilde{s},\widetilde{\alpha})$ by specialization of matrices, where the smallness of η_0 ensures that s,α, lie outside the poles of the entries.

The representation R^* of B_n over $K^*=\mathbb{R}(\{h\})$ afforded by the proposition satisfies $R^* \simeq R_g^*$ over K^* where R_g^* denotes the specialization of R_g to $\widetilde{s}=e^h$ and $\widetilde{\alpha}=e^{(1-m)h}$ (here again we can change A so that e^h, $e^{(1-m)h}$ lie outside the poles, since e^h and $e^{(1-m)h}$ are algebraically independent over $\mathbb{R}$ for $m \notin \mathbb{Q}$). It follows that there exists $P \in \mathrm{GL}_N(K^*)$ such that $R^*(b)=PR_g^*(b)P^{-1}$ for all $b \in B_n$. Let now $\beta>0$ such that, for $0<|h|<\beta$, the entries of the matrices $R^*(\sigma_i)$, $R_g(\sigma_i)$, P and P^{-1} are convergent. We choose η such that $0<\eta<\eta_0$ and $|e^{\mathrm{i}u}-1|<\eta \Rightarrow |u|<\beta$ for $u \in]-\pi,\pi[$. Then R_0 is isomorphic to $R_{\mathrm{i}u}^*$ which is a unitary representation of B_n. Since Irr_n is finite we can choose A,η uniformly in λ and this concludes the proof. □

5 Infinitesimal Brauer algebras

Let $\mathbb{k}$ be a field of characteristic 0, $m \in \mathbb{k}$ and $\mathrm{Br}_n(m)$ be defined over $\mathbb{k}$. In this section we study the Lie subalgebra of $\mathrm{Br}_n(m)$, with bracket given by $[a,b]=ab-ba$, which is generated by the elements $t_{ij}=s_{ij}-p_{ij}$, that is the images of the generators of $\mathcal{T}_n$ also denoted t_{ij} in Section 4. We call it the infinitesimal Brauer algebra and denote it $\mathcal{B}_n(m)$. The purpose of this section is to show that it is a reductive Lie algebra for generic values of m and to determine its structure in this case. We recall from Section 2 that $\mathcal{S}_n$ denotes the set of values of m for which $\mathrm{Br}_n(m)$ is not "generic". We have $\mathcal{S}_n \subset \mathbb{Z} \cap]-\infty,n]$.

5.1 Reductiveness and center

We let $t=t_{12}=s_{12}-p_{12}$ and $s=s_{12}$. A straightforward computation in $\mathrm{Br}_2(m)$ shows that $t^2-1=(m-2)p_{12}$ and $(t^2-1)(t+(m-1))=0$. It follows that, for $m \neq 2$, s_{ij} and p_{ij} are polynomials of t_{ij}. Since these elements generate $\mathrm{Br}_n(m)$ as an algebra, it follows that every irreducible representation ρ of $\mathrm{Br}_n(m)$ induces an irreducible representation $\rho_{\mathcal{B}}$ of $\mathcal{B}_n(m)$. In particular, when $m \notin \mathcal{S}_n$, $\mathcal{B}_n(m)$ is a reductive Lie algebra, as it admits a faithful semisimple representation (take the direct sum of all irreducible representations of $\mathrm{Br}_n(m)$).

For $m \notin \mathcal{S}_n$, $\mathrm{Br}_n(m)$ is reductive as a Lie algebra. We denote $p\colon \mathrm{Br}_n(m) \twoheadrightarrow Z(\mathrm{Br}_n(m))$ the natural projection, with kernel $[\mathrm{Br}_n(m),\mathrm{Br}_n(m)]$. Let $\mathbb{T}$ denote the subspace of

$\mathrm{Br}_n(m)$ spanned by the t_{ij}, and $T = \sum t_{ij} \in \mathbb{T}$. We have $T \in Z(\mathcal{B}_n(m)) \subset Z(\mathrm{Br}_n(m))$ for $m \neq 2$. Let $\mathbb{T}'$ denote the subspace of $\mathbb{T}$ spanned by the $t_{ij} - t_{kl}$. Since the t_{ij} are linearly independent we have $\mathbb{T} = \Bbbk T \oplus \mathbb{T}'$. Since $\mathfrak{S}_n$ is 2–transitive, there exists $\sigma \in \mathfrak{S}_n$ with $t_{kl} = t_{\sigma(i)\sigma(j)} = \sigma t_{ij} \sigma^{-1}$. Now an explicit formula for p can be given in terms of the primitive idempotents j_λ of $\mathrm{Br}_n(m)$ for $\lambda \in \mathrm{Irr}_n$, as $p(x) = \sum d(\lambda) tr_\lambda(x) j_\lambda$ where $d(\lambda)$ is a scalar coefficient. Then $p(sxs^{-1}) = p(x)$ for any invertible $s \in \mathrm{Br}_n(m)$, hence $p(\mathbb{T}') = 0$. Finally, p acts by 1 on the center of $\mathrm{Br}_n(m)$ hence $Z(\mathcal{B}_n(m)) = p(Z(\mathcal{B}_n(m))) \subset p(\mathcal{B}_n(m)) = p(\Bbbk T + \mathbb{T}' + [\mathcal{B}_n(m), \mathcal{B}_n(m)]) = p(\Bbbk T) = \Bbbk T$. We thus proved the following.

Proposition 5.1 *For $m \notin \mathcal{S}_n$ and $m \neq 2$, the Lie algebra $\mathcal{B}_n(m)$ is reductive, and its center is spanned by T.*

Lemma 5.2 *Let $\lambda \in \mathrm{Irr}'_n$. For m outside a finite set of rational values, $\rho_\lambda(T) \neq 0$.*

Proof Since $\rho_\lambda(T)$ is scalar, $(\dim \lambda)\rho_\lambda(T) = \mathrm{tr}\,\rho_\lambda(T) = (n(n-1)/2)\,\mathrm{tr}\,\rho(t_{12})$. But $\mathrm{Sp}\,\rho(t_{12}) \subset \{-1, 1, 1-m\}$ and $1 - m \in \mathrm{Sp}\,\rho(t_{12})$ since $\lambda \in \mathrm{Irr}'_n$. The conclusion is immediate. □

5.2 Representations of $\mathrm{Br}_n(m)$

We will need a few technical results on the representations of $\mathrm{Br}_n(m)$ that we gather here. We assume $m \notin \mathcal{S}_n$.

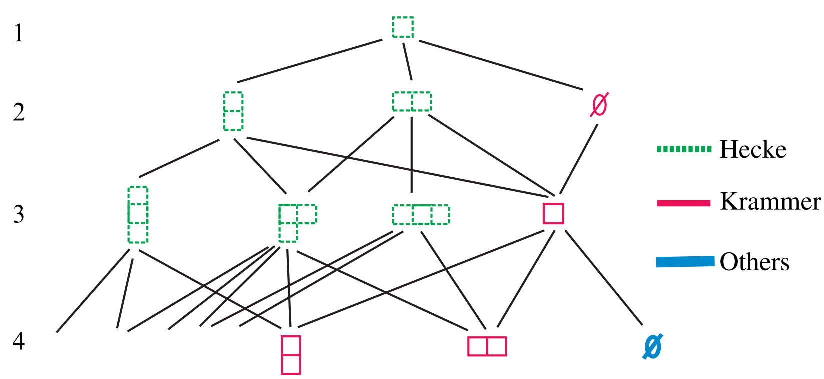

Figure 3: Bratteli diagram for the Brauer algebra

Lemma 5.3 *Let $n \geq 3$ and $\rho \in \mathrm{Irr}_n$. Either $\mathrm{Sp}\,\rho(t_{12}) \subset \{-1, 1\}$ or $\rho \in \mathrm{Irr}'_n$ and $\mathrm{Sp}\,\rho(t_{12}) = \{-1, 1, 1-m\}$.*

Proof If ρ factors through $\mathfrak{S}_n$ then clearly $\mathrm{Sp}\,\rho(t_{12}) = \mathrm{Sp}\,\rho(s_{12}) \subset \{-1, 1\}$. Otherwise $\rho(p_{12}) \neq 0$, which implies that the restriction of ρ to $\mathrm{Br}_3(m)$ contains $[1]_3$, over which t_{12} acts with spectrum $\{-1, 1, 1-m\}$. This proves the statement. □

Lemma 5.4 *Let* $n \geq 5$. *If* $\lambda \in \mathrm{Irr}(\mathfrak{S}_n)$ *and* $\dim\lambda > 1$, *then* $\dim\lambda \geq n-1$. *If* $\lambda \in \mathrm{Irr}'_n$ *then* $\dim\lambda \geq n(n-1)/2$.

Proof The first part is well-known (see eg [13, Lemme 8]) and easy to check, so we leave it to the reader. We check the second inequality directly for $n = 5$ and proceed by induction, assuming $n \geq 6$. We thus assume $\lambda \in \mathrm{Irr}'_n$, and denote $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0)$ with $|\lambda| < n$.

We first deal with the cases $|\lambda| \leq 2$, proving $\dim[1]_n \geq n(n+1)/2$ and $\dim\lambda \geq n(n-1)/2$ for $\lambda \in \{\varnothing, [2], [1,1]\}$, by induction on $n \geq 5$. We have $\dim[1]_5 = 15 = 5 \times 6/2$. Let then $n \geq 6$. If n is even, then $\lambda \in \{\varnothing, [2], [1,1]\}$ and the restriction to $\mathrm{Br}_{n-1}(m)$ admits for component $[1]_{n-1}$; it follows that $\dim\lambda \geq \dim[1]_{n-1} \geq n(n-1)/2$. If n is odd, then the restriction of $\lambda = [1]_n$ is $\varnothing_{n-1} + [2]_{n-1} + [1,1]_{n-1}$, hence $\dim[1]_n \geq 3(n-1)(n-2)/2 \geq n(n+1)/2$ for $n \geq 10$, and we check that $\dim[1]_9 = 945$ and $\dim[1]_7 = 105$ also satisfy our assumption.

We now assume $|\lambda| = 2$. Since $\lambda \neq \varnothing$ there exists $\mu_1 \in \mathrm{Irr}'_{n-1}$ such that $\mu_1 \nearrow \lambda$. Let $\mu_2 = (\lambda_1 + 1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0)_{n-1}$ and $\mu_3 = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1 > 0)_{n-1}$. We have $\mu_2, \mu_3 \in \mathrm{Irr}_{n-1}$ and $\lambda \nearrow \mu_2$, $\lambda \nearrow \mu_3$. If $\dim\mu_2 = 1$ then $r = 1$, and in this case $\dim\mu_3 = 1$ implies $\lambda_1 = \lambda_r = 1$, that is $\lambda = [1]$ which has been excluded. It follows that $\dim\mu_2 > 1$ or $\dim\mu_3 > 1$, hence $\dim\mu_2 + \dim\mu_3 \geq 1 + (n-2) = n-1$ (note that $(n-1)(n-2)/2 \geq n-2$), therefore $\dim\lambda \geq n-1+(n-1)(n-2)/2 = n(n-1)/2$, and this concludes the proof. □

5.3 Isomorphic representations

Let $\mathrm{X} = \{1, -1, m-1\}$. For every representation ρ, we have $\mathrm{Sp}\,\rho(t) \subset \mathrm{X}$. We assume $m \notin \mathcal{S}_n$ and $m \neq 2$ in what follows.

Since $\mathcal{B}_n(m)$ generates $\mathrm{Br}_n(m)$ as an algebra, for every two irreducible representations $\rho^1, \rho^2 \in \mathrm{Irr}_n$, we have $\rho^1 \simeq \rho^2$ if and only if $\rho^1_{\mathcal{B}} \simeq \rho^2_{\mathcal{B}}$.

For such irreducible representations, we denote $\rho^1_{\mathcal{B}'}$, $\rho^2_{\mathcal{B}'}$ the irreducible representations of $\mathcal{B}'_n(m) = [\mathcal{B}_n(m), \mathcal{B}_n(m)]$ that they induce. If $\rho^1 \simeq \rho^2$ then $\rho^1_{\mathcal{B}'} \simeq \rho^2_{\mathcal{B}'}$. We are interested in the converse, and assume that we have $\rho^1_{\mathcal{B}'} \simeq \rho^2_{\mathcal{B}'}$. We let

$$t'_{ij} = t_{ij} - 2T/n(n-1) = \frac{2}{n(n-1)} \sum_{k,l} t_{ij} - t_{kl}.$$

We saw in Section 5.1 that $p(t'_{ij}) = 0$. Since p is a projector onto $Z(\mathcal{B}_n(m))$ whose kernel contains $\mathcal{B}'_n(m)$, it induces the canonical projection of $\mathcal{B}_n(m)$ onto its center, whence $t'_{ij} \in \mathcal{B}'_n(m)$. From this it is clear that $\mathcal{B}'_n(m) = [\mathcal{B}_n(m), \mathcal{B}_n(m)]$ is generated by the t'_{ij}.

Let $t' = t'_{12}$, and assume without loss of generality that ρ^1, ρ^2 share the same underlying vector space V. We have $P \in \mathrm{GL}(V)$ such that $\rho^1(t') = P\rho^2(t')P^{-1}$, that is $\rho^1(t) = P\rho^2(t)P^{-1} + \alpha$, with $\alpha = (1/N)(\rho^1(T) - \rho^2(T)) \in \Bbbk$ and $N = n(n-1)/2$. Since $\mathrm{Sp}\,\rho^i(t) \subset \mathrm{X}$ we have $\alpha \in \mathrm{X} - \mathrm{X}$, that is $\alpha = 0$ or $\alpha \in \{2, 2-m, -m, -2, m-2, m\}$. This latter set has 6 elements, except when $m \in S = \{-2, 0, 1, 2, 4\}$. We assume $m \notin S$. This imposes that, either $\alpha = 0$, or both $\rho^1(t)$ and $\rho^2(t)$ have a single eigenvalue. Since the $\rho^i(t)$ are semisimple endomorphisms, at least for $m \notin \{0, 2\}$, the $\rho^i(t)$ have then to be scalars. Through conjugation under $\mathfrak{S}_n$ this has for consequence that all the $\rho^i(t_{kl})$ are scalars, that is $\dim \rho^1 = \dim \rho^2 = 1$, by irreducibility of ρ^1 and ρ^2. This proves the following.

Proposition 5.5 *Let $S = \{-2, 0, 1, 2, 4\}$ and assume $m \notin \mathcal{S}_n \cup S$. For $\rho^1, \rho^2 \in \mathrm{Irr}_n$ with $\dim \rho^i > 1$, $\rho^1 \simeq \rho^2$ if and only if $\rho^1_{\mathcal{B}'} \simeq \rho^2_{\mathcal{B}'}$.*

We now assume that $\rho^2_{\mathcal{B}'}$ is isomorphic to the dual $(\rho^1_{\mathcal{B}'})^*$ of $\rho^1_{\mathcal{B}'}$. Let $P \in \mathrm{GL}(V)$ be an intertwinner. We have $-{}^t\rho^1(t') = P\rho^2(t')P^{-1}$ hence $-{}^t\rho^1(t) = P\rho^2(t)P^{-1} + \alpha$ with $\alpha = (-\rho^1(T) - \rho^2(T))/N$. Up to conjugation of the matrix model of ρ^1 we can assume that $\rho^1(t)$ is diagonal, with eigenvalues $\lambda_1, \dots, \lambda_r \in \mathrm{X}$. Then $\alpha - \lambda_i = \mu_i \in \mathrm{Sp}\,\rho^2(t) \subset \mathrm{X}$ and $\rho^2(t)$ is diagonalizable with eigenvalues $\mu_1, \dots, \mu_r$. In particular $\alpha \in \mathrm{X} + \mathrm{X}$. We write down the addition table for X:

	1	-1	$m-1$
1	2	0	m
-1	0	-2	$m-2$
$m-1$	m	$m-2$	$2(m-2)$

The values in the upper triangular corner of this table are distinct provided that $m \notin S^* = \{0, 2, -2, 4, 3, 1\}$. In this case, $\alpha \in \{2, -2, 2(m-2)\}$ implies that for all i, $\lambda_i = \mu_i = c$ is independent of i, namely that the $\rho^i(t_{kl})$ are scalars and $\dim \rho^i = 1$. Excluding that case (hence assuming $n \geq 3$), we thus have $\alpha \in \{0, m, m-2\}$, which corresponds to $\mathrm{Sp}\,\rho^i(t) \in \{\{1, -1\}, \{1, m-1\}, \{-1, m-1\}\}$. By Lemma 5.3, we have $\mathrm{Sp}\,\rho(t) \notin \{\{1, m-1\}, \{-1, m-1\}\}$ (when $m-1 \notin \{1, -1\}$). It follows that $\mathrm{Sp}\,\rho^1(t) = \mathrm{Sp}\,\rho^2(t) = \{1, -1\}$, meaning that the ρ^i factor through $\Bbbk\mathfrak{S}_n$, which is the quotient of $\mathrm{Br}_n(m)$ by p_{12} (from $\rho(t)^2 = 1$ and $\rho(t^2 - 1) = (m-2)\rho(p_{12})$ we indeed get $\rho(p_{12}) = 0$ for $m \neq 2$). But this case is known by [13, Proposition 2 and Section 5.1] (see also [10, Proposition 2.7]), so this concludes the proof of the following proposition:

Proposition 5.6 *Let $S^* = \{-2, 0, 1, 2, 3, 4\}$ and assume $m \notin \mathcal{S}_n \cup S^*$. For $\rho^1, \rho^2 \in \mathrm{Irr}_n$, $\rho^2_{\mathcal{B}'} \simeq (\rho^1_{\mathcal{B}'})^*$ if and only if, either $\dim \rho^1 = \dim \rho^2 = 1$, or ρ^1 and ρ^2 factor through $\Bbbk\mathfrak{S}_n$ and $\rho^2 = \rho^1 \otimes \epsilon$, where ϵ is the sign character of $\mathfrak{S}_n$.*

5.4 Infinitesimal Hecke algebra

Let $\mathcal{H}_n$ denote the Lie subalgebra of $\Bbbk\mathfrak{S}_n$ generated by the transpositions. This Lie subalgebra, which is a special case of the infinitesimal Hecke algebras dealt with in [10], is reductive and has been decomposed in [13]. The quotient map $\mathrm{Br}_n(m) \to \Bbbk\mathfrak{S}_n$ induces a Lie algebra morphism $\mathcal{B}_n(m) \to \mathcal{H}_n$, that is onto and maps center to center isomorphically. It thus induces $\mathcal{B}'_n(m) \twoheadrightarrow \mathcal{H}'_n$. Let then $\mathrm{Irr}'_n = \{\lambda \in \mathrm{Irr}_n \mid |\lambda| < n\}$, where $|\lambda|$ denotes the number of boxes of the Young diagram associated to λ. We still assume $m \notin \mathcal{S}_n$ and consider the isomorphism $\mathrm{Br}_n(m)' \to \bigoplus_{\lambda \in \mathrm{Irr}_n} \mathfrak{sl}(V_\lambda)$, where V_λ is the underlying vector space of λ. Since $\lambda \in \mathrm{Irr}_n \setminus \mathrm{Irr}'_n$ means that λ factors through $\Bbbk\mathfrak{S}_n$, this isomorphism can be written $\mathrm{Br}_n(m)' \to (\Bbbk\mathfrak{S}_n)' \times \bigoplus_{\lambda \in \mathrm{Irr}'_n} \mathfrak{sl}(V_\lambda)$, hence induces an injective map

$$\mathcal{B}'_n(m) \hookrightarrow \mathcal{H}'_n \times \bigoplus_{\lambda \in \mathrm{Irr}'_n} \mathfrak{sl}(V_\lambda)$$

Our goal here is to show that this map is surjective, namely:

Theorem 5.7 *Let $m \notin \mathcal{S}_n \cup S \cup S^*$. Then $\mathcal{B}_n(m)$ is a reductive algebra with 1–dimensional center, whose derived Lie algebra is naturally isomorphic to $\mathcal{H}'_n \times \bigoplus_{\lambda \in \mathrm{Irr}'_n} \mathfrak{sl}(V_\lambda)$.*

For the convenience of the reader, we recall from [13] the simple ideals of $\mathcal{H}'_n$. For $\rho \in \mathrm{Irr}(\mathfrak{S}_n)$, identified with a partition of n or a Young diagram of size n, we denote V_ρ the underlying vector space and $\rho_{\mathcal{H}'}\colon \mathcal{H}'_n \to \mathfrak{sl}(V_\rho)$ the induced representation of $\mathcal{H}'_n$. Then the orthogonal $\mathcal{H}(\rho)$ of $\mathrm{Ker}\, \rho_{\mathcal{H}'}$ for the Killing form is a simple ideal of $\mathcal{H}'_n$, and all simple ideals are obtained this way. A nonoverlapping list is given by $\mathcal{H}([n-1, 1]) \simeq \mathfrak{sl}_{n-1}(\mathbb{C})$ and, for ρ *not a hook*, letting $\epsilon\colon \mathfrak{S}_n \twoheadrightarrow \{\pm 1\}$ denote the sign character:

- $\mathcal{H}(\rho) \simeq \mathfrak{so}(V_\rho)$ when the symmetric square $S^2\rho$ contains ϵ.
- $\mathcal{H}(\rho) \simeq^{(} V_\rho)$ when the symmetric square $S^2\rho$ contains ϵ.
- $\mathcal{H}(\rho) = \mathcal{H}(\rho \otimes \epsilon) \simeq \mathfrak{sl}(V_\rho)$ otherwise.

For a given n, the decomposition of $\mathcal{B}'_n(m)$ given by the theorem is equivalent to the property that $\rho_\lambda(\mathcal{B}'_n(m)) = \mathfrak{sl}(V_\lambda)$ for all $\lambda \in \mathrm{Irr}'_n$. Indeed, if we have this property, then the semisimple Lie algebra $\mathcal{B}'_n(m)$ contains simple Lie ideals isomorphic to $\mathfrak{sl}(V_\lambda)$ for every $\lambda \in \mathrm{Irr}'_n$, which do not intersect the simple Lie ideals inherited from $\mathcal{H}'_n$, and do not intersect each other: indeed, if there was such an intersection this would mean that this two simple ideals coincide, meaning that two $\lambda, \mu \in \mathrm{Irr}'_n$ of the same dimension would factor through the same ideal. Since $\mathfrak{sl}_N(\mathbb{C})$ admits at most two irreducible representations of dimension N, this would imply $\lambda_{\mathcal{B}'} \simeq \mu_{\mathcal{B}'}$ or $\lambda_{\mathcal{B}'} \simeq (\mu_{\mathcal{B}'})^*$, which is excluded by the propositions above; the case $\lambda \in \mathrm{Irr}'_n, \mu \in \mathrm{Irr}_n \setminus \mathrm{Irr}'_n$ is dealt with similarly. This property for any given n thus implies the theorem for this n.

5.5 Induction step

We prove here that $\rho(\mathcal{B}'_n(m)) = \mathfrak{sl}(V_\lambda)$ for every $\lambda \in \mathrm{Irr}'_n$ by induction on n, assuming the theorem true for $n-1$. Note that no confusion should arise in the notation between Irr_n and Irr_{n-1} as, if $\lambda \in \mathrm{Irr}_n$ and $\mu \in \mathrm{Irr}'_n$, then $|\lambda|$ and $|\mu$ do not have the same parity. We let V_λ denote the underlying space of $\lambda \in \mathrm{Irr}_n$ or $\lambda \in \mathrm{Irr}_{n-1}$. For two partitions λ, μ, we use the notation $\lambda \nearrow \mu$ for $|\mu| = |\lambda| + 1$ and $\mu_i = \lambda_i$ for all but one i. When confusion could arise, we let ρ_λ denote the representation of $\mathrm{Br}_n(m)$ or $\mathrm{Br}_{n-1}(m)$ associated to the partition λ.

The branching rule for $\mathcal{B}_{n-1}(m) \subset \mathcal{B}_n(m)$ can be written as

$$\mathrm{Res}\, \rho_\lambda = \sum_{\mu \nearrow \lambda} \rho_\mu + \sum_{\lambda \nearrow \mu} \rho_\mu .$$

We consider several cases. First note that we can assume $n \geq 5$. Indeed, for $n \leq 4$, an element of Irr'_n is either the infinitesimal Krammer representation (or its transformed under the automorphism $s_{ij} \mapsto -s_{ij}$, $p_{ij} \mapsto -p_{ij}$ of $\mathrm{Br}_n(m)$), which has been studied separately in [14], or $[\varnothing]_4$ (see Figure 3). In this last case, its restriction to $\mathrm{Br}_3(m)$ is the irreducible representation $[1]_3$ which is a Krammer representation, hence $\rho_{[\varnothing]_4}(\mathcal{B}'_4(m)) \supset \rho_{[1]_3}(\mathcal{B}'_3(m)) = \mathfrak{sl}(V_{[1]_3})$.

5.5.1 $|\lambda| < n-2$ In this case, for every $\mu \nearrow \lambda$ or $\lambda \nearrow \mu$, we have $\mu \in \mathrm{Irr}'_{n-1}$. By the induction assumption, $\rho_\lambda(\mathcal{B}'_n(m))$ contains

$$\left(\bigoplus_{\mu \nearrow \lambda} \mathfrak{sl}(V_\mu) \right) \oplus \left(\bigoplus_{\lambda \nearrow \mu} \mathfrak{sl}(V_\mu) \right)$$

which has (semisimple) rank

$$\sum_{\mu \nearrow \lambda} (\dim V_\mu - 1) + \sum_{\lambda \nearrow \mu} (\dim V_\mu - 1).$$

For $\mu \in \mathrm{Irr}'_{n-1}$ and $n-1 \geq 3$, that is $n \geq 4$, we have $\dim V_\mu \geq 3$ hence $\dim V_\mu > (1/2)\dim V_\mu$ hence $\mathrm{rk}\,\rho_\lambda(\mathcal{B}'_n(m)) > (1/2)\dim V_\lambda$. This implies $\rho_\lambda(\mathcal{B}'_n(m)) = \mathfrak{sl}(V_\lambda)$ (see [14, Proposition 3.8]).

5.5.2 $|\lambda| = n-2$ **and no** μ **with** $\lambda \nearrow \mu$ **is a hook** In this section, the original assumption $n \geq 5$ implies $n \geq 6$, because for $n = 5$ and $|\lambda| = 3$ there is always a hook μ with $\lambda \nearrow \mu$.

First notice that $\lambda \nearrow \mu$ and $\lambda \nearrow \mu'$, where μ' is the transposed partition of μ, implies that either $\mu = \mu'$, in which case μ is uniquely determined, or $\lambda = \lambda'$ and $\lambda \nearrow \nu \Rightarrow \lambda \nearrow \nu'$ for every ν. If there is no such μ, for $n \geq 6$ we get again $\dim V_\mu \geq 3$ for all irreducible component μ of the restriction (as $\dim V_\mu = 1$ would imply that λ is a hook, and if λ is a hook there is a hook μ with $\lambda \nearrow \mu$, contradicting the assumption), and we conclude as in the previous case that $\rho_\lambda(\mathcal{B}'_n(m)) = \mathfrak{sl}(V_\lambda)$.

If there is such a μ with $\mu = \mu'$, it is uniquely determined, we have $\mathrm{rk}\,\rho_\mu(\mathcal{B}_{n-1}(m)') = (\dim V_\mu)/2$, and the same computation as before shows $\mathrm{rk}\,\rho_\lambda(\mathcal{B}'_n(m)) > (1/2)\dim V_\lambda$.

So the only case that we have to deal with is $\lambda = \lambda'$. Then $\lambda \nearrow \mu \Rightarrow \mu \neq \mu'$ and $\lambda \nearrow \mu'$. Moreover,

$$\mathrm{rk}\,\rho_\lambda(\mathcal{B}_{n-1}(m)') = \sum_{\mu \nearrow \lambda} (\dim V_\mu - 1) + \frac{1}{2}\sum_{\lambda \nearrow \mu} (\dim V_\mu - 1).$$

If $\dim V_\mu \geq 3$ (always the case for $n \geq 6$), we have $(1/2)(\dim V_\mu - 1) \geq (1/3)\dim V_\mu$ hence

$$r = \mathrm{rk}\,\rho_\lambda(\mathcal{B}_{n-1}(m)') > \frac{1}{3}\left(\sum_{\mu \nearrow \lambda} \dim V_\mu + \sum_{\lambda \nearrow \mu} \dim V_\mu\right) = \frac{1}{3}\dim V_\lambda = d/3.$$

Then $r > d/3$ implies $d < 3r < (r+1)^2$, because $d \geq 3$. This implies that $\rho_\lambda(\mathcal{B}')$ is simple [10, Lemma 3.3 (I)]. Moreover $\mathrm{rk}\,\rho_\lambda(\mathcal{B}') > (\dim V_\lambda)/3 > (\dim V_\lambda)/4$. Note that, for every $\mu \in \mathrm{Irr}'_{n-1}$, in particular for every $\mu \nearrow \lambda$, we have $\dim V_\mu \geq (n-1)(n-2)/2$. If $|\lambda| \geq 1$ we thus get $\mathrm{rk}\,\rho_\lambda(\mathcal{B}') \geq (n-1)(n-2)/2 - 1 \geq 9$ for $n \geq 6$. These two inequalities imply $\rho_\lambda(\mathcal{B}') = \mathfrak{sl}(V_\lambda)$ [10, Lemma 3.4]. If $|\lambda| = 0$, then $\rho_\lambda(\mathcal{B}') = \rho_{[1]_{n-1}}(\mathcal{B}') = \mathfrak{sl}(V_{[1]_{n-1}}) = \mathfrak{sl}(V_\lambda)$ by the induction assumption.

5.5.3 $|\lambda| = n-2$ **and there exists a hook** μ **with** $\lambda \nearrow \mu$ In that case λ has the shape of a hook $[n-k, 1^{k-2}]$ and we can assume that $k \geq 3$, $n-k-2 \geq 2$ (and $n \geq 5$). Indeed, it is otherwise isomorphic to the infinitesimal Krammer representation (or its transformed under the automorphism $s_{ij} \mapsto -s_{ij}$, $p_{ij} \mapsto -p_{ij}$ of $\mathrm{Br}_n(m)$).

We denote $C(n,k)$ the dimension of this representation. The μ such that $\lambda \nearrow \mu$ are the partitions $A=[n-k,1^{k-1}]$, $B=[n-k+1,1^{k-2}]$, $C=[n-k,2,1^{k-3}]$. The μ such that $\mu \nearrow \lambda$ are the partitions $E=[n-k-1,1^{k-2}]$, $F=[n-k,1^{k-3}]$. We have

$$\dim A = \binom{n-2}{k-1}, \quad \dim B = \binom{n-2}{k-2}$$

and, by [13, Section 6.3],

$$\dim C = \frac{k-2}{n-k}\binom{n-3}{n-k-2}(n-1) = \binom{n-1}{k-1}\frac{(n-k-1)(k-2)}{n-2}.$$

Moreover, $\dim E = C(n-1,k)$ and $\dim F = C(n-1,k-1)$. We have

$$C(n,k) = \dim V_\lambda = C(n-1,k) + C(n-1,k-1) + \binom{n-2}{k-1} + \binom{n-2}{k-2} + \dim C$$

hence $$C(n,k) > \binom{n-1}{k-1} + C(n-1,k) + C(n-1,k-1).$$

We show by induction on n that $C(n,k) > (n-2)\binom{n-1}{k-1}$. This holds true for $n=5$, because $C(5,3)=\dim[2,1]_5=20$. Then

$$\binom{n-1}{k-1} + C(n-1,k) + C(n-1,k-1) > \binom{n-1}{k-1} + (n-3)\left(\binom{n-2}{k-1} + \binom{n-2}{k-2}\right)$$

and the last term is equal to $(n-2)\binom{n-1}{k-1}$, which proves by induction the inequality, except for $k=3$. In that case, $C(n,3) = \binom{n-1}{2} + C(n-1,3) + \dim[n-3]_{n-1} + \dim[n-3,2]_{n-1}$ and $\dim[n-3]_{n-1} = \binom{n-1}{2}$. Then, by the induction assumption,

$$C(n,3) > \binom{n-1}{2} + (n-3)\binom{n-2}{2} + \binom{n-1}{2} + \frac{(n-1)(n-4)}{2}$$

hence $C(n,3) > \dfrac{(n-1)(n-2)}{2}\left(n-2+1-\dfrac{n^2-5n+8}{n^2-3n+2}\right) > \dfrac{(n-1)(n-2)}{2}(n-2).$

In particular, we have

$$\begin{aligned}\dim E = C(n-1,k) &> (n-3)\binom{n-2}{k-1} = (n-3)\dim A\\ \dim F = C(n-1,k-1) &> (n-3)\binom{n-2}{k-2} = (n-3)\dim B\end{aligned}$$

if $k \geq 4$ ($\dim B = n-2$ if $k=3$), and

$$\dim E + \dim F > (n-3)\binom{n-1}{k-1} \geq 4\dim C$$

if and only if $(n-3)(n-2) \geq 4(n-k-1)(k-2)$, which holds true. As a consequence, for $k \geq 4$,

$$\begin{aligned}\operatorname{rk} \rho_\lambda(\mathcal{B}') &\geq \dim E - 1 + \dim F - 1 + (n-2) + (\dim C)/2 \\ &\geq \dim E + \dim F - 2 + (n-2) + (\dim C)/2.\end{aligned}$$

Moreover,
$$\begin{aligned}\dim V_\lambda &= \dim E + \dim F + \dim A + \dim B + \dim C \\ &< (\dim E + \dim F)(1 + 1/(n-3) + 1/4)\end{aligned}$$

hence $\dim V_\lambda < ((5n-11)/(4n-12))(\dim E + \dim F)$. We thus have

$$\operatorname{rk} \rho_\lambda(\mathcal{B}') \geqslant \dim E + \dim F > \frac{4n-12}{5n-11} \dim V_\lambda > \frac{1}{2} \dim V_\lambda$$

for $n \geq 5$, hence $\rho_\lambda(\mathcal{B}') = \mathfrak{sl}(V_\lambda)$ for $n \geq 5$ and $k \geq 4$.

The only remaining case is when $k = 3$, ie λ corresponds to the partition $[n-3, 1]$. We still have $\dim E > (n-3) \dim A$, but this time $\dim F = (n-1)(n-2)/2$, $\dim B = n-2$,

$$\dim C = \binom{n-1}{2} \frac{n-4}{n-2} = \frac{(n-1)(n-4)}{2}.$$

We have $\dim A = (n-2)(n-3)/2$ and $\dim E > (n-3) \dim A = (n-2)(n-3)^2/2$. Since $n-3 \geq (n-1)/2$ when $n \geq 5$, we also have $\dim E > ((n-1)/2) \dim A$. From $\dim F = (n-1)/2 \dim B$ we then get

$$\dim E + \dim F > \frac{n-1}{2}(\dim A + \dim B)$$

hence
$$\dim C < \frac{n-1}{n-2}\frac{n-4}{n-3}\frac{1}{n-3}(\dim E + \dim F) < \frac{1}{n-3}(\dim E + \dim F).$$

We thus have

$$\dim V_\lambda < (\dim E + \dim F)\left(1 + \frac{2}{n-1} + \frac{1}{n-3}\right) = (\dim E + \dim F)\frac{n^2-n-4}{(n-1)(n-3)}$$

and $\operatorname{rk} \rho_\lambda(\mathcal{B}') \geq \dim E + \dim F + \dim C - 3 + (n-2)$ if $n \geq 6$, $\operatorname{rk} \rho_\lambda(\mathcal{B}') \geq \dim E + \dim F + \dim C/2 - 2 + (n-2)$ if $n = 5$ (in that case $C' = C$). In both cases

$$\operatorname{rk} \rho_\lambda(\mathcal{B}') > \dim E + \dim F > \frac{(n-1)(n-3)}{n^2-n-4} \dim V_\lambda.$$

Finally, $(n-1)(n-3)/(n^2-n-4) \geq 1/2$ for $n \geq 5$, and this again proves $\rho_\lambda(\mathcal{B}') = \mathfrak{sl}(V_\lambda)$ by the same criterium [13, Proposition 3.8].

6 Proof of Theorem 1.1

We first let $K = \mathbb{C}((h))$. For $\lambda \in \mathrm{Irr}_n$, $m \in \mathbb{C}$, we denote V_λ the underlying vector space of λ: $\mathrm{Br}_n(m) \to \mathrm{End}(V_\lambda)$ over $\mathbb{C}$, and $R_\lambda^{(m)}$: $B_n \to \mathrm{GL}(V_\lambda \otimes K)$ the representation constructed in Section 4 from ρ by using some associator. Let Γ_λ denote the algebraic (Zariski) closure of $R_\lambda^{(m)}(B_n)$ inside $\mathrm{GL}(V_\lambda \otimes K)$.

Assume $\lambda \in \mathrm{Irr}'_n$ and $m \notin \mathcal{S}_n \cup S \cup S^*$. By Theorem 5.7 and Proposition 4.2, the Lie algebra $\mathrm{Lie}\,\Gamma_\lambda$ of Γ_λ contains $\mathfrak{sl}(V_\lambda) \otimes K$. Recall from the proof of Lemma 5.2 that $\lambda(T)$ is a (scalar) rational affine polynomial in m; we denote $E_{\lambda,n}$ the set of rationals m such that $\lambda(T) = 0$. The generator $\gamma_n = (\sigma_1 \cdots \sigma_{n-1})^n$ of $Z(B_n)$ is mapped to $\exp T$ through $B_n \to \mathfrak{S}_n \ltimes \exp \widehat{\mathcal{T}_n}$; thus $\mathrm{Lie}\,\Gamma_\lambda$ contains K as soon as $m \notin E_{\lambda,n}$.

It follows that $\Gamma_\lambda = \mathrm{GL}(V_\lambda \otimes K)$ if $m \notin E_{\lambda,n}$, and $\Gamma_\lambda \supset \mathrm{SL}(V_\lambda \otimes K)$. In particular the algebraic closure of $R_\lambda^{(m)}(B'_n)$ is equal to $\mathrm{SL}(V_\lambda \otimes K)$.

We now consider $\mathrm{BMW}_n(e^h, e^{(1-m)h})$ as an algebra over K, and still assume $m \notin \mathcal{S}_n \cup S \cup S^*$. Then $\mathrm{BMW}_n(e^h, e^{(1-m)h})$ is the direct sum of the Hecke algebra $H_n(e^h)$ and of $\bigoplus_{\lambda \in \mathrm{Irr}'_n} \mathrm{End}(V_\lambda \otimes K)$. Let G_0 denote the algebraic closure of B'_n inside $H_n(e^h)$. This closure has been described in full detail in [13]. We recall from there that its Lie algebra is $\mathcal{H}'_n \otimes K$, and that it is a direct sum of $\mathrm{SL}_N, \mathrm{SP}_N$ and SO_N with respect to some hyperbolic nondegenerate quadratic form. From Theorem 5.7 we get similarly that the image of B'_n inside $\mathrm{BMW}_n(e^h, e^{(1-m)h})$ is Zariski-dense inside $G_0 \times \prod_{\lambda \in \mathrm{Irr}'_n} \mathrm{SL}(V_\lambda \otimes K)$, whose Lie algebra is $\mathcal{B}'_n(m)$.

When $m \notin \mathbb{Q}$, e^h and $e^{(1-m)h}$ are algebraically independent over $\mathbb{Q}$, so we get an embedding $\mathbb{Q}(s, \alpha) \hookrightarrow K$ through $s \mapsto e^h$ and $\alpha \mapsto e^{mh}$. Similarly, embedding $\mathbb{Q}(s)$ into K through $s \mapsto e^h$ we get realizations of the representations of $\mathrm{BMW}_n(s, s^m) \otimes K$ for $m \in \mathbb{Z}$ and $m \notin \mathcal{S}_n \cup S \cup S^*$, where $\mathrm{BMW}_n(s, s^m)$ is defined over $\mathbb{Q}(s)$ through the $R_\lambda^{(m)}$, $\lambda \in \mathrm{Irr}_n$. We know the Zariski closure of the $R_\lambda = R_\lambda^{(m)}(B'_n)$ for $\lambda \notin \mathrm{Irr}'_n$ from [13], provided we know that the orthosymplectic groups involved there when $\lambda = \lambda'$ are defined over $\mathbb{Q}(s)$. The bilinear form defining them span the subspace of $R_\lambda \otimes R_\lambda$ over which B_n acts by the sign character $B_n \twoheadrightarrow \{\pm 1\}$, $\sigma_i \mapsto -1$. Since ϵ and the R_λ are defined over $\mathbb{Q}(s)$, this subspace has nonzero points over $\mathbb{Q}(s)$, so these groups are indeed defined over $\mathbb{Q}(s)$.

Theorem 1.1 is then an immediate consequence of the above for an arbitrary (characteristic 0) field, by noticing that all the algebraic groups G involved here are defined over $\mathbb{Q}(s)$, satisfy that $G(L_1)$ is Zariski-dense in $G(L_2)$ for $\mathbb{Q}(s) \subset G(L_1) \subset G(L_2)$, and browsing along the following pattern of field extensions.

$$\begin{tikzcd}
\mathbb{C}(\!(h)\!) \arrow[dr, dash] \arrow[ddr, dash] & & K \arrow[dl, dash] \arrow[ddl, dash] \\
& \mathbb{Q}(s,\alpha) \arrow[d, dash] & \\
& \mathbb{Q}(s) &
\end{tikzcd}$$

One point that remains to be clarified is the type of the orthogonal groups possibly appearing for $\lambda \not\in \mathrm{Irr}'_n$ and $\lambda = \lambda'$, when considered over $\mathbb{Q}(s)$. We know that the quadratic forms β_λ involved here are hyperbolic over $\mathbb{C}(\!(h)\!)$, but it was not proved in [13] that they are hyperbolic over $\mathbb{Q}(s)$. We prove this below.

Lemma 6.1 *For $n \geq 2$, $\lambda \vdash n$, $\lambda = \lambda'$, if $\epsilon \hookrightarrow S^2 R_\lambda$, then β_λ is hyperbolic.*

Proof We prove the lemma by induction on n, the cases $n = 2$ being clear. By Young rule the restriction to B_{n-1} of R_λ is the direct sum of the R_μ for $\mu \nearrow \lambda$. Notice that $\mu \nearrow \lambda$ implies $\mu' \nearrow \lambda$ under our assumptions. Let $\mu, \nu \nearrow \lambda$ and consider the restriction $\beta\colon V_\mu \otimes V_\nu \to \mathbb{Q}(s)$ of β_λ. First assume $\nu \neq \mu'$. If $\beta \neq 0$ it would provide an isomorphism $R_\mu \simeq R_\nu^* \otimes \epsilon \simeq R_{\nu'}$ where ϵ is the sign character of B_{n-1} and R_ν^* is the dual representation of R_ν. Since $\mu \neq \nu'$ we have $\beta = 0$. If $\nu = \mu' = \mu$, by induction we know that β is either 0 or hyperbolic, but the case $\beta = 0$ is excluded because β_λ would then be degenerate. If $\nu = \mu' \neq \mu$, we consider the restriction of β_λ to $V_\mu \oplus V_{\mu'}$. If $M\colon V_{\mu'} \to V_\mu$ affords $R_{\mu'} \simeq R_\mu^+ \otimes \epsilon$, this restriction can be written in matrix form as

$$\begin{pmatrix} 0 & M \\ {}^tM & 0 \end{pmatrix}$$

and is therefore hyperbolic. The quadratic space $(V_\lambda, \beta_\lambda)$ is thus isomorphic to a direct sum of hyperbolic spaces, and is therefore hyperbolic. □

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