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Constructions of $E_{\mathcal{VC}}$ and $E_{\mathcal{FBC}}$ for groups acting on CAT(0) spaces

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If Γ is a group acting properly by semisimple isometries on a proper CAT(0) space X, then we build models for the classifying spaces $E_{\mathcal{VC}}\Gamma$ and $E_{\mathcal{FBC}}\Gamma$ under the additional assumption that the action of Γ has a well-behaved collection of axes in X. We verify that the latter assumption is satisfied in two cases: (i) when X has isolated flats, and (ii) when X is a simply connected real analytic manifold of nonpositive sectional curvature. We conjecture that Γ has a well-behaved collection of axes in the great majority of cases.

Our classifying spaces are natural variations of the constructions due to Connolly, Fehrman and Hartglass [4] of $E_{\mathcal{VC}}\Gamma$ for crystallographic groups Γ.

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1 Introduction

We say that a nonempty collection $\mathcal{F}$ of subgroups of Γ is a *family* if $\mathcal{F}$ is closed under conjugation and passage to subgroups. If Γ is a discrete group, then $E_{\mathcal{F}}\Gamma$, the *classifying complex of* Γ *with isotropy in* $\mathcal{F}$, is a Γ–CW–complex such that

(1) if c is a cell of $E_{\mathcal{F}}\Gamma$ and $\gamma \in \Gamma$ leaves c invariant as a set, then γ fixes c pointwise;

(2) if $H \in \mathcal{F}$, then the set of points fixed by H is contractible;

(3) if $H \leq \Gamma$ and $H \notin \mathcal{F}$, then the set of points fixed by H is empty.

Let $\mathcal{VC}$ denote the family of virtually cyclic subgroups of Γ. The classifying space $E_{\mathcal{VC}}\Gamma$ can be used to help compute the algebraic K–theory of the group Γ if the Farrell–Jones isomorphism conjecture has been proved for Γ. Recent work by several authors (see, for instance, Davis, Khan and Ranicki [5]) shows that one can use the space $E_{\mathcal{FBC}}\Gamma$ for similar purposes. Here $\mathcal{FBC}$ is the family of finite-by-cyclic subgroups of Γ, namely those that map onto a cyclic group with finite kernel.

Connolly, Fehrman and Hartglass [4] built classifying spaces $E_{\mathcal{VC}}\Gamma$ for crystallographic groups Γ. Their construction can be summarized as follows. We let Γ be

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an n–dimensional crystallographic group. Thus, Γ acts properly and cocompactly by isometries on Euclidean n–space $\mathbb{R}^n$. Each element $\gamma \in \Gamma$ will either fix a point in $\mathbb{R}^n$ or act by translation on some line $\ell \subseteq \mathbb{R}^n$. In the latter case, we call such a line ℓ an *axis* for γ. Given an axis ℓ, we let $\mathbb{R}^n_\ell$ be the set of all lines in $\mathbb{R}^n$ that are parallel to ℓ. It is rather clear that $\mathbb{R}^n_\ell$ is naturally isometric to $\mathbb{R}^{n-1}$. We let $f_\ell\colon \mathbb{R}^n \to \mathbb{R}^n_\ell$ be the quotient map, and let $M(f_\ell)$ be the mapping cylinder of f_ℓ. Now we let ℓ range over all possible axes of elements $\gamma \in \Gamma$, and we glue together all of the mapping cylinders $M(f_\ell)$ together along the tops, which are identical copies of $\mathbb{R}^n$. The resulting space (with the cell structure described in [4] and with respect to a natural Γ–action) is $E_{\mathcal{VC}}\Gamma$.

The goal of this paper is to show that a version of the construction from [4] is available for a wide variety of groups Γ acting by isometries on CAT(0) spaces X. Let X be a proper CAT(0) space. Suppose that Γ acts properly on X by *semisimple* isometries: that is, every $\gamma \in \Gamma$ either fixes a point in X or acts on a line ℓ (again called an *axis*) by translation. For instance, if Γ acts properly and cocompactly by isometries on X, then the action of Γ is automatically by semisimple isometries; see Bridson and Haefliger [2, Proposition 6.10(2), page 233]. We let $\mathcal{A}(X;\Gamma)$ denote the *space of axes*; thus, $\mathcal{A}(X;\Gamma)$ is the set of all lines $\ell \subseteq X$ for which there is $\gamma \in \Gamma$ acting by translations on ℓ. We write $d(\ell_1,\ell_2) = K$ if the axes ℓ_1 and ℓ_2 are parallel and bound a flat strip of width K; we write $d(\ell_1,\ell_2) = \infty$ if ℓ_1 and ℓ_2 are not parallel. The function d is a metric on $\mathcal{A}(X;\Gamma)$, and makes $\mathcal{A}(X;\Gamma)$ into a disjoint union of CAT(0) spaces. The space $E_{\mathcal{VC}}\Gamma$ that we build in this paper is (roughly) the join $X * \mathcal{A}(X;\Gamma)$, although there are complications as we now explain.

The main difficulty comes from the requirement that $E_{\mathcal{VC}}\Gamma$ have a CW structure. There is no reason, in general, to expect that the CAT(0) space X (let alone $\mathcal{A}(X;\Gamma)$) will have such a structure, so we are forced to produce our own. Our method is to replace both X and $\mathcal{A}(X;\Gamma)$ by nerves of suitable covers. Our procedure for producing these covers requires that $\mathcal{A}(X;\Gamma)$ be a disjoint union of proper CAT(0) spaces. Unfortunately, this is not always the case, even if the action of Γ is cocompact, so we add a hypothesis.

We need some definitions first for background. We let $\mathcal{L}(X)$ denote the space of all lines in X. The space $\mathcal{L}(X)$ is given the metric d (as defined above for $\mathcal{A}(X;\Gamma)$). This metric makes $\mathcal{L}(X)$ a disjoint union of proper CAT(0) spaces. We say that a subspace K of $\mathcal{L}(X)$ is *component-wise convex* if the intersection of K with each connected component of $\mathcal{L}(X)$ is convex in the ordinary sense.

We can now describe the hypothesis that we need. If $\mathcal{A} \subseteq \mathcal{A}(X;\Gamma)$, we say that $\mathcal{A}$ is *a Γ–well-behaved space of axes* if $\mathcal{A}$ is closed and component-wise convex as a subset of $\mathcal{L}(X)$, $\Gamma \cdot \mathcal{A} = \mathcal{A}$, and, for any $\ell \in \mathcal{A}(X;\Gamma)$, $\mathcal{A}$ contains at least one axis parallel to ℓ. (In fact, the space $\mathcal{A}(X;\Gamma)$ satisfies all of these conditions, except that it need not be closed.) We have the following theorem:

Theorem 1.1 *If Γ acts properly by semisimple isometries on a proper* CAT(0) *space X and there is some Γ–well-behaved space of axes A, then there are Connolly–Fehrman–Hartglass constructions of $E_{\mathcal{VC}}\Gamma$ and $E_{\mathcal{FBC}}\Gamma$ based on the action of Γ on X.*

This statement of our main theorem is a place holder. The precise statements appear in Theorem 5.1 and Corollary 6.5. One builds the $E_{\mathcal{FBC}}\Gamma$ complexes by using a slight modification of our techniques for building $E_{\mathcal{VC}}\Gamma$ complexes.

The hypothesis that there exists a Γ–well-behaved space of axes is clearly undesirable, but I believe that the simplicity of the construction compensates for this flaw somewhat. Moreover, it seems reasonable to guess that the hypothesis will be satisfied in the great majority of cases. In Example 3.11 we give an example of a space X and a group Γ such that $\mathcal{A}(X;\Gamma)$ is not a Γ–well-behaved space of axes. The example is rather artificial, and it is unclear whether such badly behaved examples occur "naturally". In Section 7, we show that CAT(0) spaces with isolated flats and simply connected real analytic manifolds of nonpositive sectional curvature always have well-behaved spaces of axes. The construction of the main theorem also appears to be economical and useful for computations (like the construction of $E_{\mathcal{VC}}\Gamma$ for crystallographic groups from [4]).

While this paper was being completed, Lück [10] described a construction of $E_{\mathcal{VC}}\Gamma$ for any group Γ acting properly and cocompactly on a CAT(0) space X. He is able to build classifying spaces for all CAT(0) groups (without our additional hypothesis) and he gives bounds for the topological dimension of his construction. His construction is also different from ours. I hope that readers will see virtues in both approaches.

I would like to thank Qayum Khan for suggesting the problem of building complexes $E_{\mathcal{FBC}}\Gamma$ to me while I was visiting Vanderbilt in 2008. I thank Ross Geoghegan for suggesting the method of using nerves of covers. I thank Ivonne Ortiz for her help in revising this paper. I thank the referee for suggesting valuable revisions, and especially for suggesting the classes of examples in Section 7.2 and Section 7.3.

2 A general construction

Definition 2.1 If Γ is any group, then a *family* $\mathcal{F}$ of subgroups of Γ is a nonempty collection of subgroups that is closed under conjugation by elements of Γ and passage to subgroups. If Γ is any group, then we let $\mathcal{FIN}$, $\mathcal{VC}$, and $\mathcal{FBC}$ denote (respectively) the families of finite, virtually cyclic, and *finite-by-cyclic groups*, ie, groups which map onto a cyclic group with finite kernel. We will also let $\mathcal{VC}_\infty$ denote the difference $\mathcal{VC} - \mathcal{FIN}$ and $\mathcal{FBC}_\infty$ denote $\mathcal{FBC} - \mathcal{FIN}$. (Note that neither $\mathcal{VC}_\infty$ nor $\mathcal{FBC}_\infty$ is a family, since neither collection contains the trivial group.)

Definition 2.2 Let X be a Γ–CW–complex. Suppose that, if $c \subseteq X$ is a cell of X, then $\gamma \cdot c = c$ if and only if γ fixes c pointwise. Let $\mathcal{F}$ be a family of subgroups of Γ. We say that X is an $E_{\mathcal{F}}\Gamma$*–complex* if

(1) X is contractible;

(2) whenever $H \in \mathcal{F}$, the fixed set $\mathrm{Fix}(H) = \{x \in X \mid \gamma \cdot x = x \text{ for all } \gamma \in H\}$ is contractible;

(3) whenever $H \notin \mathcal{F}$, $\mathrm{Fix}(H)$ is empty.

Let $\mathcal{F}'$ be a family of subgroups of Γ containing $\mathcal{F}$. We say that X is an $I_{\mathcal{F}'-\mathcal{F}}\Gamma$*–complex* if

(1) whenever $H \in \mathcal{F}' - \mathcal{F}$, $\mathrm{Fix}(H)$ is contractible;

(2) whenever $H \notin \mathcal{F}'$, $\mathrm{Fix}(H)$ is empty.

Remark 2.3 Note that the complex $I_{\mathcal{F}'-\mathcal{F}}\Gamma$ isn't necessarily contractible. In our applications, $I_{\mathcal{F}'-\mathcal{F}}\Gamma$ won't even be connected.

Proposition 2.4 *Let Γ be a group. Let $\mathcal{F} \subseteq \mathcal{F}'$ be families of subgroups of Γ. The join*

$$(E_{\mathcal{F}}\Gamma) * (I_{\mathcal{F}'-\mathcal{F}}\Gamma)$$

is an $E_{\mathcal{F}'}\Gamma$–complex.

Proof For any two topological spaces X and Y with Γ–actions, we make the following two observations: (i) the join $X * Y$ is contractible if at least one of the spaces X and Y is contractible; (ii) $\mathrm{Fix}_{X*Y}(H) = \mathrm{Fix}_X(H) * \mathrm{Fix}_Y(H)$, for any $H \leq \Gamma$.

We let $Z = (E_{\mathcal{F}}\Gamma) * (I_{\mathcal{F}'-\mathcal{F}}\Gamma)$ and give Z the natural cell structure. We need to show that Z satisfies (1)-(3) with respect to the family $\mathcal{F}'$. The contractibility of Z follows from (i), since $E_{\mathcal{F}}\Gamma$ is contractible. Let $H \leq \Gamma$ satisfy $H \notin \mathcal{F}'$. Since $\mathrm{Fix}_{E_{\mathcal{F}}\Gamma}(H) = \varnothing = \mathrm{Fix}_{I_{\mathcal{F}'-\mathcal{F}}\Gamma}(H)$, it follows from (ii) that $\mathrm{Fix}_Z(H) = \varnothing$.

We now need to check (2). Suppose first that $H \in \mathcal{F}' - \mathcal{F}$. Since $\mathrm{Fix}_{E_{\mathcal{F}}\Gamma}(H) = \varnothing$ and $\mathrm{Fix}_{I_{\mathcal{F}'-\mathcal{F}}\Gamma}(H)$ is contractible, $\mathrm{Fix}_Z(H)$ is contractible by (ii). Suppose finally that $H \in \mathcal{F}$. Since $\mathrm{Fix}_{E_{\mathcal{F}}\Gamma}(H)$ is contractible, $\mathrm{Fix}_Z(H)$ is contractible by (ii).

Finally, we note that, for any cell $c \subseteq Z$, an element $\gamma \in \Gamma$ leaves c invariant if and only if γ fixes c pointwise, since the cells of $E_{\mathcal{F}}\Gamma$ and $I_{\mathcal{F}'-\mathcal{F}}\Gamma$ have this property. $\square$

3 Preliminaries about CAT(0) Spaces

All of the material in this section is either taken directly from Bridson and Haefliger [2] or closely based on that source.

3.1 General definitions and basic facts about CAT(0) spaces

Definition 3.1 Let X be a metric space. We say that X is a *geodesic metric space* if, for any x, $y \in X$, there is a map, called a *geodesic segment*, $\ell_{x,y}\colon [0, d(x, y)] \to X$ such that, $\ell_{x,y}(0) = x$, $\ell_{x,y}(d(x, y)) = y$, and for any t_1, $t_2 \in [0, d(x, y)]$, we have $d(\ell_{x,y}(t_1), \ell_{x,y}(t_2)) = |t_1 - t_2|$. We will also call the image of such a map a geodesic segment. Similarly, a *geodesic line* is an isometric embedding $\ell\colon \mathbb{R} \to X$. We will also call the image of such an embedding a geodesic line.

Now suppose that X is a geodesic metric space. Let $x, y, z \in X$. Choose geodesic segments $[x, y]$, $[y, z]$, and $[x, z]$ (these need not be unique, a priori). We choose three points $\overline{x}, \overline{y}, \overline{z} \in \mathbb{E}^2$ such that $d_X(x, y) = d_{\mathbb{E}^2}(\overline{x}, \overline{y})$, $d_X(y, z) = d_{\mathbb{E}^2}(\overline{y}, \overline{z})$, and $d_X(x, z) = d_{\mathbb{E}^2}(\overline{x}, \overline{z})$. The segments $[x, y]$, $[y, z]$, and $[x, z]$ are in natural isometric one-to-one correspondence with the segments $[\overline{x}, \overline{y}]$, $[\overline{y}, \overline{z}]$, and $[\overline{x}, \overline{z}]$ (respectively). If $a \in [x, y]$, $[y, z]$, or $[x, z]$, then the corresponding point $\overline{a}$ on (respectively) $[\overline{x}, \overline{y}]$, $[\overline{y}, \overline{z}]$, or $[\overline{x}, \overline{z}]$ is called a *comparison point*. We say that the geodesic triangle $\Delta = [x, y] \cup [y, z] \cup [x, z]$ satisfies the CAT(0) *condition* if, for any pair of points a, $b \in \Delta$ and pair of comparison points $\overline{a}$, $\overline{b}$, we have

$$d_X(a, b) \leq d_{\mathbb{E}^2}(\overline{a}, \overline{b}).$$

The geodesic metric space X is CAT(0) if every geodesic triangle satisfies the CAT(0) condition.

We say that a metric space X is *proper* if each closed metric ball $\overline{B}_\epsilon(x) = \{y \in X \mid d(x, y) \leq \epsilon\}$ $(0 < \epsilon < \infty)$ is compact.

Remark 3.2 It will be helpful if we allow two points in a metric space to be an infinite distance apart. We will always allow this in what follows. In particular, if X is a disjoint union of metric spaces X_i, then we metrize X by letting d_X agree with the metrics on the X_i, and setting $d_X(x, y) = \infty$ if $x \in X_i$ and $y \in X_j$ for $i \neq j$.

Definition 3.3 If X is a geodesic metric space (respectively, if X is a disjoint union of geodesic metric spaces), then we say that a subset $K \subseteq X$ is *convex* (respectively, *component-wise convex*) if, for any two points x, $y \in K$ satisfying $d(x, y) < \infty$, every geodesic segment connecting x to y is contained in K.

Definition 3.4 If $\ell_1\colon \mathbb{R}\to X$ and $\ell_2\colon \mathbb{R}\to X$ are geodesic lines in a complete CAT(0) space X, then we say that ℓ_1 and ℓ_2 are *parallel* (or *asymptotic*) if there is some constant K such that $d_X(\ell_1(t),\ell_2(t))\le K$, for all $t\in\mathbb{R}$.

Definition 3.5 If Y is a bounded subset of a metric space X, then the *radius* of Y, denoted r_Y, is the infimum of the positive numbers r such that $Y\subseteq B_r(x)$ for some $x\in X$. A *center* of Y is a point $c\in X$ such that $Y\subseteq \overline{B}_{r_Y}(c)$.

Proposition 3.6 *Let X be a complete* CAT(0) *space.*

(1) *Any two points in X can be connected by a unique geodesic segment (ie, X is* uniquely geodesic*).*
(2) *X is contractible. Every open or closed ball of finite radius in X is convex, and therefore* CAT(0) *and contractible.*
(3) *Every bounded subset B in X has a unique center c. If an isometry γ leaves the set B invariant, then γ fixes c. If B is a ball and γ fixes c, then B is invariant under the action of γ.*
(4) *If C is a nonempty closed convex subset of X and $x\in X$, then there is a unique point $\pi(x)\in C$ that is closest to x, ie, $d(x,\pi(x))=d(x,C):=\inf_{y\in C} d(x,y)$.*
(5) (Flat Strip Theorem) *If ℓ_1 and ℓ_2 are parallel geodesic lines in X, then, for some $K\ge 0$, there is an isometric embedding $\rho\colon [0,K]\times\mathbb{R}\to X$, where $\ell_1=\rho(\{0\}\times\mathbb{R})$ and $\ell_2=\rho(\{K\}\times\mathbb{R})$.*

Proof These facts are proved on pages 160, 161, 179, 176 and 182 (respectively) of Bridson and Haefliger [2]. □

3.2 Spaces of lines and axes

Definition 3.7 Let X be a proper CAT(0) space endowed with a proper Γ–action by semisimple isometries. The *space of lines* in X, denoted $\mathcal{L}(X)$, is defined as follows:

$$\mathcal{L}(X)=\{\ell\subseteq X \mid \ell \text{ is a geodesic line}\}.$$

If $\ell_1,\ell_2\in\mathcal{L}(X)$, then we write

$$d(\ell_1,\ell_2)=\begin{cases} k & \text{if } \ell_1 \text{ and } \ell_2 \text{ bound a flat strip of width } k,\\ \infty & \text{if } \ell_1 \text{ and } \ell_2 \text{ are not parallel.}\end{cases}$$

The given assignment d is a function by the Flat Strip Theorem (Proposition 3.6(5)). It follows easily from the Product Decomposition Theorem (Theorem 3.8, below) that d is a metric.

We say that a line $\ell \subseteq X$ is an *axis* for $\gamma \in \Gamma$ if γ acts on ℓ by translation. The *space of axes* for elements of Γ in X, denoted $\mathcal{A}(X;\Gamma)$, is defined as follows:

$$\mathcal{A}(X;\Gamma) = \{\ell \in \mathcal{L}(X) : \ell \text{ is an axis for some } \gamma \in \Gamma\}.$$

Theorem 3.8 (Product Decomposition Theorem) *Let X be a proper* CAT(0) *space and let $c\colon \mathbb{R} \to X$ be a geodesic line.*

(1) *The union of the images of all geodesic lines $c'\colon \mathbb{R} \to X$ parallel to c is a closed convex subspace X_c of X.*
(2) *Let p be the restriction to X_c of the projection from X to the complete convex subspace $c(\mathbb{R})$. Let $X_c^0 = p^{-1}(c(0))$. Then, X_c^0 is closed and convex (in particular, it is a proper* CAT(0) *space) and X_c is canonically isometric to the product $X_c^0 \times \mathbb{R}$.*
(3) *Any (image of a) geodesic line parallel to c has the form $\{x\} \times \mathbb{R}$ for some x in X_c^0. (Conversely, any subspace of the form $\{x\} \times \mathbb{R}$ is the image of a geodesic line parallel to c.)*

Proof This theorem is exactly the same as the Product Decomposition Theorem from [2, page 183], except that the latter theorem contains neither the word "proper", nor the word "closed", nor statement (3). Statement (3) follows directly from the correspondence j as defined in [2, page 183]. (In fact, (3) is the canonical identification which was alluded to in (2).) Therefore, it is enough to prove (1) and (2).

Statement (1) is proved in [2, page 183], except for the statement that X_c is closed. The latter follows easily from the fact that X is proper and from Ascoli's Theorem as in Munkres [11, page 290]. The details of the argument are routine, and are omitted. Statement (2) follows directly from (1) and the Product Decomposition Theorem from [2]. $\square$

Proposition 3.9 *If X is a proper* CAT(0) *space and Γ acts by isometries on X, then the space of lines $\mathcal{L}(X)$ is a disjoint union of proper* CAT(0) *spaces on which Γ acts isometrically.*

Proof It is clear that Γ acts isometrically on $\mathcal{L}(X)$. We need to show that $\mathcal{L}(X)$ is a disjoint union of proper CAT(0) spaces.

Fix a geodesic line $c\colon \mathbb{R} \to X$. We consider X_c, the union of all lines parallel to c. By Theorem 3.8(2) $X_c = X_c^0 \times \mathbb{R}$. Let $\mathcal{L}_c(X) = \{\ell \in \mathcal{L}(X) \mid \ell \text{ is parallel to } c(\mathbb{R})\}$. The space $\mathcal{L}_c(X)$ is naturally isometric to X_c^0 by Theorem 3.8(3). Therefore, $\mathcal{L}_c(X)$ is a proper CAT(0) space by Theorem 3.8(2). The space $\mathcal{L}(X)$ is the disjoint union of all $\mathcal{L}_c(X)$, as c ranges over a maximal collection of pairwise nonparallel lines. $\square$

The following Proposition isn't needed in order to understand the main construction of this paper (although we will refer to it in Section 7). It shows that the space of axes comes very close to being "Γ–well-behaved" (see Definition 3.12). Example 3.11 shows how the space of axes can fail to be well-behaved.

Proposition 3.10 *If Γ acts properly by semisimple isometries on the proper* CAT(0) *space X, then Γ acts discretely by isometries on $\mathcal{A}(X;\Gamma)$. The space $\mathcal{A}(X;\Gamma)$ is a Γ–equivariant, component-wise convex subset of $\mathcal{L}(X)$.*

Proof Let $\ell \in \mathcal{A}(X;\Gamma)$. By definition, there is some $\gamma \in \Gamma$ such that γ acts on ℓ by translations. Now choose an arbitrary $\gamma_1 \in \Gamma$. We claim that $\gamma_1 \cdot \ell$ is an axis for $\gamma_1\gamma\gamma_1^{-1}$:

$$(\gamma_1\gamma\gamma_1^{-1}) \cdot \gamma_1 \cdot \ell = \gamma_1\gamma\ell = \gamma_1\ell.$$

Therefore $\gamma_1\ell$ is invariant under the action of $\gamma_1\gamma\gamma_1^{-1}$. Since $\gamma_1\gamma\gamma_1^{-1}$ has infinite order and the action of Γ is proper, $\gamma_1\gamma\gamma_1^{-1}$ must act without a fixed point, and therefore $\gamma_1\gamma\gamma_1^{-1}$ acts by translation on $\gamma_1\ell$. This shows that Γ acts on $\mathcal{A}(X;\Gamma)$. The action is by isometries since Γ acts on $\mathcal{L}(X)$ by isometries.

We now need to show that the action of Γ on $\mathcal{A}(X;\Gamma)$ has discrete orbits. Suppose otherwise. There must exist a sequence $\ell_1, \dots, \ell_n, \dots$ of axes (all in the same orbit) converging to another axis ℓ (in the same orbit as the others). We can assume that any two of the axes are parallel, and that $d(\ell_n, \ell)$ is a strictly decreasing sequence, after passing to a subsequence if necessary. It follows that $\ell, \ell_1, \dots, \ell_n, \dots$ all lie in the same component $\mathcal{L}_c(X)$ of $\mathcal{L}(X)$. By Theorem 3.8(2), $\mathcal{L}_c(X)$ is isometric to $Y \times \mathbb{R}$, where Y is a proper CAT(0) space. By Theorem 3.8(3), $\ell, \ell_1, \dots, \ell_n, \dots$ are identified with subspaces $\{y\} \times \mathbb{R}$, $\{y_1\} \times \mathbb{R}$, $\dots, \{y_n\} \times \mathbb{R}, \dots$ respectively.

We consider the point $(y, 0) \in \ell$. Since ℓ is an axis, there is some isometry γ that acts by translation on $\ell = \{y\} \times \mathbb{R}$. Since γ acts on $Y \times \mathbb{R}$ and preserves the line ℓ, Proposition 5.3(4) from [2, page 56] implies that $\gamma_{|Y\times\mathbb{R}} = \gamma_1 \times \gamma_2$, where $\gamma_1\colon Y \to Y$, $\gamma_2\colon \mathbb{R} \to \mathbb{R}$ are isometries, γ_1 fixes y, and $\gamma_2(t) = t + K$ for some $K \in \mathbb{R}$.

Since ℓ, $\ell_1, \dots, \ell_n, \dots$ are all in the same orbit under Γ, there is a sequence (y_1, t_1), $(y_2, t_2), \dots, (y_n, t_n), \dots$ of points in the orbit of $(y, 0)$ under Γ, where $(y_i, t_i) \in \ell_i$, for all $i \in \mathbb{N}$. Note that $d(\ell_i, \ell) = d(y_i, y)$, so $d_Y(y_i, y)$ is a decreasing sequence. We can find integral powers $m_1, m_2, \dots, m_n, \dots$ where $|\gamma_2^{m_i}(t_i)| \leq K$, for each $i \in \mathbb{N}$. Now consider the points

$$(\gamma_1^{m_1}(y_1), \gamma_2^{m_1}(t_1)), (\gamma_1^{m_2}(y_2), \gamma_2^{m_2}(t_2)), \dots.$$

All of these points are in the orbit $(y,0)$ according to our assumption. Moreover, they form a bounded set, since

$$\begin{aligned} d((\gamma_1^{m_i}(y_i), \gamma_2^{m_i}(t_i)), (y,0)) &= d((\gamma_1^{m_i}(y_i), \gamma_2^{m_i}(t_i)), (\gamma_1^{m_i}(y), 0)) \\ &\leq \sqrt{d_Y(y_i, y)^2 + K^2}. \end{aligned}$$

Finally, we note that all of the points $(\gamma_1^{m_i}(y_1),\ \gamma_2^{m_1}(t_1)), \dots$ are distinct since the points $\gamma_1^{m_1}(y_1), \gamma_1^{m_2}(y_2), \dots$ are all distinct. (Indeed, the distances $d_Y(\gamma_1^{m_i}(y_i), y) = d_Y(y_i, y)$ are all distinct.)

We've found an infinite, bounded collection of points in a single orbit. This violates properness of the action of Γ. Thus Γ acts on $\mathcal{A}(X;\Gamma)$ with discrete orbits, as claimed.

Lastly, we need to show that if $\ell_1, \ell_2 \in \mathcal{A}(X;\Gamma)$ and $[\ell_1, \ell_2]$ is a geodesic segment connecting ℓ_1 to ℓ_2 in $\mathcal{L}(X)$, then $[\ell_1, \ell_2] \subseteq \mathcal{A}(X;\Gamma)$. We choose a point $\ell_3 \in [\ell_1, \ell_2]$. We will show first that the orbit of ℓ_3 is discrete, and then that ℓ_3 is an axis. Suppose, for a contradiction, that ℓ_3 has a nondiscrete orbit. Let $\gamma_1, \gamma_2, \dots, \gamma_n, \dots$ be a sequence of isometries of X such that $\gamma_1 \cdot \ell_3, \gamma_2 \cdot \ell_3, \dots$ are all distinct and lie inside a ball of radius 1 in $\mathcal{L}(X)$. Let us call the center of this ball ℓ_c. The lines $\gamma_1 \cdot \ell_1, \gamma_2 \cdot \ell_1, \dots$ all lie within $d(\ell_1, \ell_2) + 1$ units of ℓ_c. Since the component of $\mathcal{L}(X)$ containing ℓ_c is proper and Γ acts discretely on $\mathcal{A}(X;\Gamma)$, the set $L_1 = \{\gamma_1 \cdot \ell_1, \gamma_2 \cdot \ell_1, \dots, \gamma_n \cdot \ell_1, \dots\}$ is finite. The same reasoning applies to the set $L_2 = \{\gamma_1 \cdot \ell_2, \gamma_2 \cdot \ell_2, \dots, \gamma_n \cdot \ell_2, \dots\}$, so it is finite as well. Consider the points $\ell \in \mathcal{L}(X)$ with the following properties:

(1) ℓ lies on a geodesic $[\hat{\ell}_1, \hat{\ell}_2]$, where $\hat{\ell}_i \in L_i$ for $i = 1, 2$.

(2) $d_{\mathcal{L}(X)}(\ell, \ell_i) = d_{\mathcal{L}(X)}(\ell_3, \ell_i)$ for $i = 1, 2$.

The set of all such ℓ is finite (indeed, the collection of such geodesics $[\hat{\ell}_1, \hat{\ell}_2]$ is finite by the finiteness of L_1 and L_2, and uniqueness of geodesics in $\mathcal{L}(X)$). This is a contradiction, because each element of $\{\gamma_1 \cdot \ell_3, \gamma_2 \cdot \ell_3, \dots, \gamma_n \cdot \ell_3, \dots\}$ is such an ℓ. We conclude that each $\ell_3 \in [\ell_1, \ell_2]$ has a discrete orbit.

We now show that ℓ_3 is the axis of some isometry, using the fact that ℓ_3 has a discrete orbit and that ℓ_3 is parallel to an axis ℓ_1. We write $\ell_1 = \{y_1\} \times \mathbb{R}$ and $\ell_3 = \{y_3\} \times \mathbb{R}$. Since ℓ_1 is an axis, there is some γ acting on ℓ_1 by translation, and this γ preserves the factorization of $X_{\ell_1} \subseteq X$, which, by definition is the union of all lines parallel to ℓ_1. Therefore $\gamma(\hat{y}, t) = (\gamma_1(\hat{y}), \gamma_2(t))$, where $\gamma_1\colon Y \to Y$ fixes y_1, and $\gamma_2\colon \mathbb{R} \to \mathbb{R}$ is defined by $\gamma_2(t) = t + K$, for some $K \in \mathbb{R}$. Since ℓ_3 has a discrete orbit and Y is proper by Theorem 3.8(2), the orbit of y_3 under the action of $\langle \gamma_1 \rangle$ must be finite. Therefore, there is some power N such that $\gamma_1^N(y_3) = y_3$. This γ_1^N acts on ℓ_3 by translation. $\square$

Example 3.11 We now give an example of a proper CAT(0) space X and an isometric action by Γ on X which demonstrates that the space $\mathcal{A}(X;\Gamma)$ can fail to be a disjoint union of proper CAT(0) spaces.

Let T be a rooted infinite binary tree. We say that the root of the tree T lies at the 0–*th level* of T. If v is a node in the tree at the n–th level, then its children lie at the $(n+1)$–st level. We assign edge lengths to the tree T as follows: an edge connecting a node of level $n-1$ to one of level n has length 2^{-n}. The space T is CAT(0) with respect to its length metric (ie, the metric d such that $d(x,y)$ is the length of the shortest path from x to y). We compactify T by adding all points at infinity. The resulting space $\overline{T}$ is compact and CAT(0); each point in $\partial T = \overline{T} - T$ is precisely 1 unit from the root.

Every graph automorphism $\phi\colon T \to T$ induces an isometry of $\overline{T}$. We choose some automorphism $\phi\colon T \to T$ of infinite order. (One way to produce such an automorphism is as follows. Identify the set of nodes at level n with the set $\mathbb{Z}/2^n\mathbb{Z}$ of left cosets of $2^n\mathbb{Z}$ in $\mathbb{Z}$. An edge connects a node $a + 2^{n+1}\mathbb{Z}$ at level $n+1$ to a node $b + 2^n\mathbb{Z}$ at level n if and only if $a + 2^{n+1}\mathbb{Z} \subseteq b + 2^n\mathbb{Z}$. With these identifications, addition by 1 induces an automorphism $\phi_1\colon T \to T$ of infinite order.) We let $\widehat{\phi}\colon \overline{T} \to \overline{T}$ denote the unique extension of $\phi\colon T \to T$ to the boundary.

Consider the action of $\mathbb{Z} = \langle \gamma \rangle$ on $\overline{T} \times \mathbb{R}$ defined by $\gamma \cdot (x,t) = (\widehat{\phi}(x), t+1)$. This is a proper, cocompact, isometric action of $\mathbb{Z}$ on the proper CAT(0) space $\overline{T} \times \mathbb{R}$. The space of lines $\mathcal{L}(\overline{T} \times \mathbb{R})$ is $\{\{x\} \times \mathbb{R} \mid x \in \overline{T}\}$; this space can be identified with $\overline{T}$. All of the lines $\{x\} \times \mathbb{R}$, for $x \in T$, are axes for subgroups $\langle \gamma^N \rangle$ (where N varies with x). If $x \in \partial T$ has an infinite orbit with respect to $\langle \widehat{\phi} \rangle$, then $\{x\} \times \mathbb{R}$ isn't an axis. It follows that $\mathcal{A}(\overline{T} \times \mathbb{R}; \langle \gamma \rangle)$ isn't closed as a subset of $\mathcal{L}(X)$, so it cannot be proper.

Definition 3.12 Let X be a proper CAT(0) space, and let Γ act properly on X by semisimple isometries. We say that $\mathcal{A} \subseteq \mathcal{A}(X;\Gamma)$ is a *well-behaved space of axes for* Γ (or $\mathcal{A}$ is a Γ*–well-behaved space of axes*) if $\mathcal{A}$ is closed and component-wise convex as a subset of $\mathcal{L}(X)$, Γ–equivariant, and, whenever a component C of $\mathcal{L}(X)$ contains an element of $\mathcal{A}(X;\Gamma)$, C also contains an element of $\mathcal{A}$.

Remark 3.13 Note that the space $\overline{T} \times \mathbb{R}$ has many Γ–well-behaved spaces of axes. For instance, the root of T crossed with $\mathbb{R}$ is one such well-behaved space of axes. I don't know of a proper CAT(0) space X admitting a proper isometric action by a group Γ such that Γ has no well-behaved space of axes.

4 Good covers for proper CAT(0) Γ–spaces

Definition 4.1 Let X be a proper CAT(0) space or disjoint union of countably many proper CAT(0) spaces, endowed with an isometric Γ–action having discrete orbits. A Γ–*good* cover $\mathcal{U}$ of X is a cover of X by open balls such that

(1) $\Gamma \cdot \mathcal{U} = \mathcal{U}$;

(2) if $B \in \mathcal{U}$ and $\gamma \cdot B \cap B \neq \varnothing$, then $\gamma \cdot B = B$.

Proposition 4.2 *If X is a proper* CAT(0) *space or a disjoint union of countably many proper* CAT(0) *spaces endowed with an isometric Γ–action having discrete orbits, then X has a locally finite Γ–good cover $\mathcal{U}$.*

Proof We sketch the proof for the case in which X is a proper CAT(0) space. If X is a disjoint union of countably many proper CAT(0) spaces, then essentially the same argument works with only minor changes.

For each $x \in X$, we choose $0 < \epsilon_x < 1/2$ so that, for any $\gamma \in \Gamma$, $\gamma \cdot B_{\epsilon_x}(x) \cap B_{\epsilon_x}(x) = \varnothing$ if $\gamma \cdot x \neq x$. We let $W_x = \bigcup_{\gamma \in \Gamma} \gamma \cdot B_{\epsilon_x}(x)$ and let $\mathcal{W} = \{W_x \mid x \in X\}$.

Choose a basepoint $* \in X$. We inductively define a subcover $\mathcal{W}_\infty \subseteq \mathcal{W}$ as follows. We let $\mathcal{W}_1 \subseteq \mathcal{W}$ be a finite cover of $\overline{B}_1(*)$. (This closed ball is compact by properness of X.) We let $\mathcal{W}_2 \subseteq \mathcal{W}$ be a finite cover of $\overline{B}_2(*)$. If $\mathcal{W}_i$ has been defined, then we let $\mathcal{V}_i = \mathcal{W}_1 \cup \cdots \cup \mathcal{W}_i$. Now suppose that $\mathcal{W}_n \subseteq \mathcal{W}$ has been defined, where $n \geq 2$. We let $\mathcal{W}_{n+1} \subseteq \mathcal{W}$ be a finite cover of

$$\overline{B}_{n+1}(*) - \bigcup_{W_x \in \mathcal{V}_n} W_x,$$

where $\mathcal{W}_{n+1}$ is chosen to be disjoint from $\overline{B}_{n-1}(*)$. (Any element of $\mathcal{W}$ meeting $\overline{B}_{n-1}(*)$ must be completely covered by the collection $\mathcal{V}_n$, so it is possible to satisfy the latter condition.) This completes the inductive definition of $\mathcal{W}_n$. We let

$$\mathcal{W}_\infty = \bigcup_{n=1}^{\infty} \mathcal{W}_n.$$

Finally, we replace the cover $\mathcal{W}_\infty$, which consists of orbits of balls, by the individual constituent balls. The result is locally finite, and a Γ–good cover. □

Remark 4.3 One can probably control the dimension of the nerve of $N(\mathcal{U})$ (see Definition 4.4 below) with sufficient care, although it may be easier to do this in particular examples than it would be to do in full generality. It is highly desirable to have such control when attempting calculations of (for instance) the lower algebraic K–theory of a group.

Nevertheless, the argument of this section does not require any control on the dimension. In fact, Hatcher [6, Corollary 4G.3, page 459] proves Lemma 4.5 without even the assumption that $\mathcal{U}$ is locally finite.

Definition 4.4 Let $\mathcal{U}$ be a cover of a topological space X. The *nerve* $N(\mathcal{U})$ of the cover $\mathcal{U}$ is a simplicial complex on the set $\mathcal{U}$. A finite collection $\{U_1, U_2, U_3, \dots, U_n\} \subseteq \mathcal{U}$ is a simplex of $N(\mathcal{U})$ if and only if $U_1 \cap \cdots \cap U_n \neq \varnothing$.

Lemma 4.5 [6, Corollary 4G.3, page 459] *If $\mathcal{U}$ is an open cover of a paracompact space X such that every nonempty intersection of finitely many sets in $\mathcal{U}$ is contractible, then X is homotopy equivalent to the nerve $N(\mathcal{U})$.* □

The first part of the following Proposition was essentially proved by Ontaneda in [12, Proposition A, 2.1]. The second part of Proposition 4.6 is a straightforward adaptation of his argument.

Proposition 4.6 *Let X be a proper* $\mathrm{CAT}(0)$ *space or a disjoint union of proper* $\mathrm{CAT}(0)$ *spaces. Suppose that Γ acts isometrically on X with discrete orbits. Let $\mathcal{U}$ be a Γ–good cover for X.*

1. *If X is a proper* $\mathrm{CAT}(0)$ *space and Γ acts properly on X, then $N(\mathcal{U})$ is an $E_{\mathcal{FIN}}\Gamma$–complex.*
2. *Suppose X is a disjoint union of proper* $\mathrm{CAT}(0)$ *spaces. Suppose that, for each $\gamma \in \Gamma$ of infinite order, $\mathrm{Fix}_X(\gamma)$ is path connected (and thus nonempty). Suppose that the stabilizer of each $x \in X$ is in $\mathcal{VC}$. The nerve $N(\mathcal{U})$ is an $I_{\mathcal{VC}_\infty}\Gamma$ complex.*

Proof If $\mathcal{U}$ is a Γ–good cover for X, then it is fairly clear that an element $\gamma \in \Gamma$ leaves a simplex $c \subseteq N(\mathcal{U})$ invariant if and only if γ fixes c pointwise. We concentrate here on establishing the other conditions from Definition 2.2.

(1) Every nonempty intersection of open balls in $\mathcal{U}$ is a convex subset of X, and therefore a $\mathrm{CAT}(0)$ space by Proposition 3.6(2). It now follows from Lemma 4.5 that $N(\mathcal{U})$ is homotopy equivalent to X, and therefore contractible by Proposition 3.6(2).

Let us suppose that H is an infinite subgroup of Γ. The fixed set $\mathrm{Fix}_{N(\mathcal{U})}(H)$ is the full subcomplex on the set $\{U \mid h \cdot U = U, \text{ for all } h \in H\}$. We claim that the latter set is empty. For suppose $h \cdot U = U$, for all $h \in H$. Since U has compact closure and H is infinite, this violates the properness of the action of Γ. Therefore $\mathrm{Fix}_{N(\mathcal{U})}(H) = \varnothing$.

Now suppose that H is a finite subgroup of Γ. The space $\mathrm{Fix}_X(H)$ is a nonempty closed convex subspace of X (by [2, Corollary 2.8, page 179]), and therefore a proper

CAT(0) space. We let $\mathcal{U}_{\mathrm{res}} = \{U \cap \mathrm{Fix}_X(H) \mid U \in \mathcal{U} \text{ and } U \cap \mathrm{Fix}_X(H) \neq \varnothing\}$. The nerve $N(\mathcal{U}_{\mathrm{res}})$ is homotopy equivalent to $\mathrm{Fix}_X(H)$ for exactly the reasons that $N(\mathcal{U})$ is homotopy equivalent to X. It follows that $N(\mathcal{U}_{\mathrm{res}})$ is contractible.

Consider the subcomplex $\mathrm{Fix}_{N(\mathcal{U})}(H)$ for our finite H. We claim that $\mathrm{Fix}_{N(\mathcal{U})}(H)^0 = \{U \in \mathcal{U} \mid U \cap \mathrm{Fix}_X(H) \neq \varnothing\}$ (and therefore $\mathrm{Fix}_{N(\mathcal{U})}(H)$ is the full subcomplex of $N(\mathcal{U})$ on the latter set). Suppose that $U \in \mathcal{U}$ and $U \cap \mathrm{Fix}_X(H) \neq \varnothing$. Let $x \in \mathrm{Fix}_X(H) \cap U$, and let $h \in H$. Since $h \cdot x = x$, we have that $x \in h \cdot U \cap U \neq \varnothing$, so $h \cdot U = U$. Therefore $\{U \in \mathcal{U} \mid U \cap \mathrm{Fix}_X(H) \neq \varnothing\} \subseteq \mathrm{Fix}_{N(\mathcal{U})}(H)^0$.

Now suppose that $U \in \mathcal{U}$ and $U \cap \mathrm{Fix}_X(H) = \varnothing$. It follows that some $h \in H$ moves the center c of U: $h \cdot c \neq c$. This implies that $h \cdot U \neq U$, so $U \notin \mathrm{Fix}_{N(\mathcal{U})}(H)^0$. This proves the claim.

We define a simplicial map $\rho\colon \mathrm{Fix}_{N(\mathcal{U})}(H) \to N(\mathcal{U}_{\mathrm{res}})$ on vertices, sending U to $U \cap \mathrm{Fix}_X(H)$. We claim that ρ is a homotopy equivalence, because, indeed, the inverse image of each simplex is a simplex ([12, pages 50–51] uses this principle). We prove the latter statement. Let $\{U_1 \cap \mathrm{Fix}_X(H), U_2 \cap \mathrm{Fix}_X(H), \ldots, U_n \cap \mathrm{Fix}_X(H)\}$ be a simplex of $N(\mathcal{U}_{\mathrm{res}})$. Thus, in particular

$$(*) \qquad \bigcap_{i=1}^{n} (U_i \cap \mathrm{Fix}_X(H)) \neq \varnothing.$$

We note that $\rho^{-1}\{U_1 \cap \mathrm{Fix}_X(H), \ldots, U_n \cap \mathrm{Fix}_X(H)\} = \{U \in \mathcal{U} \mid U \cap \mathrm{Fix}_X(H) = U_i \cap \mathrm{Fix}_X(H), \text{ for some } i \in \{1, \ldots, n\}\}$. (This is immediate from the definition of ρ.) The latter set is indeed a simplex of $\mathrm{Fix}_{N(\mathcal{U})}(H)$ by $(*)$. This proves the claim. It now follows that $\mathrm{Fix}_{N(\mathcal{U})}(H)$ is contractible. This completes the proof of (1).

(2) Suppose that $H \leq \Gamma$ is not virtually cyclic. We have that $\mathrm{Fix}_{N(\mathcal{U})}(H)^0 = \{U \mid h \cdot U = U, \text{ for all } h \in H\}$. Let $U \in \mathcal{U}$. We let c be the center of U. Since H is not virtually cyclic, there is some $h \in H$ such that $h \cdot c \neq c$. It follows that $h \cdot U \neq U$ (since $\mathcal{U}$ is a Γ–good cover, and $h \cdot U = U$ if and only if h fixes c). It follows that $\mathrm{Fix}_{N(\mathcal{U})}(H)^0 = \varnothing$, so $\mathrm{Fix}_{N(\mathcal{U})}(H) = \varnothing$.

Now suppose that $H \leq \Gamma$ is infinite virtually cyclic. Let $\langle \alpha \rangle$ be a cyclic normal subgroup of finite index in H. By our assumptions, $\mathrm{Fix}_X(\alpha)$ is path-connected. In particular, $\mathrm{Fix}_X(\alpha) \subseteq Y$, where Y is a (proper, CAT(0)) path component of X. Since $\langle \alpha \rangle \lhd H$, H must leave the fixed set $\mathrm{Fix}_X(\alpha)$ invariant. It follows that H acts by isometries on Y. Let $h_1\langle \alpha \rangle$, $h_2\langle \alpha \rangle$, $\ldots$, $h_m\langle \alpha \rangle$ be a complete list of the left cosets of $\langle \alpha \rangle$ in H. Let $* \in \mathrm{Fix}_X(\alpha)$. The orbit of $*$ under H is $\{h_1 \cdot *, \ldots, h_m \cdot *\}$. The center of this orbit invariant under all of H by Proposition 3.6(3). In particular, $\mathrm{Fix}_X(H)$ is nonempty.

We summarize:
$$\mathrm{Fix}_X(H) = \bigcap_{h\in H} \mathrm{Fix}_X(h)$$
is a nonempty subset of the path component Y, and $\mathrm{Fix}_X(H)$ is therefore a closed convex subset of Y. In particular, it is a proper CAT(0) space and therefore contractible.

We let $\mathcal{U}_{\mathrm{res}} = \{U \cap \mathrm{Fix}_X(H) \mid U \in \mathcal{U} \text{ and } U \cap \mathrm{Fix}_X(H) \neq \varnothing\}$ (exactly as in (1)). As in (1), $N(\mathcal{U}_{\mathrm{res}})$ is homotopy equivalent to $\mathrm{Fix}_X(H)$, so $N(\mathcal{U}_{\mathrm{res}})$ is contractible.

One can show that $\mathrm{Fix}_{N(\mathcal{U})}(H)^0 = \{U \in \mathcal{U} \mid U \cap \mathrm{Fix}_X(H) \neq \varnothing\}$ exactly as in (1). It follows that $\mathrm{Fix}_{N(\mathcal{U})}(H)$ is the full subcomplex of $N(\mathcal{U})$ on the latter set.

We define a simplicial map $\rho\colon \mathrm{Fix}_{N(\mathcal{U})}(H) \to N(\mathcal{U}_{\mathrm{res}})$ on vertices exactly as before, sending U to $U \cap \mathrm{Fix}_X(H)$. This map is a homotopy equivalence for the same reasons as in (1). It follows that $\mathrm{Fix}_{N(\mathcal{U})}(H)$ is contractible. □

5 A construction of $E_{\mathcal{VC}}\Gamma$ for CAT(0) groups

Theorem 5.1 *Let X be a proper* CAT(0) *space, let Γ be a group acting properly on X by semisimple isometries, and let $\mathcal{A}$ be a Γ–well-behaved collection of axes in X. There are Γ–good covers $\mathcal{U}$ of X, and $\mathcal{V}$ of $\mathcal{A}$. The space $N(\mathcal{U}) * N(\mathcal{V})$ is an $E_{\mathcal{VC}}\Gamma$–complex.*

Proof By Proposition 4.2, X has a Γ–good cover $\mathcal{U}$. By Proposition 4.2 and Definition 3.12, $\mathcal{A}$ has a Γ–good cover $\mathcal{V}$. By Proposition 4.6(1), $N(\mathcal{U})$ is an $E_{\mathcal{FIN}}\Gamma$–complex.

We want to apply Proposition 4.6(2). Thus, we check that, for each $\gamma \in \Gamma$ of infinite order, $\mathrm{Fix}_{\mathcal{A}}(\gamma)$ is path connected, and that the stabilizer of each $\ell \in \mathcal{A}$ is virtually cyclic. So suppose first that $\gamma \in \Gamma$ has infinite order. Since Γ acts on X by semisimple isometries, there is some line $\ell \subseteq X$ on which γ acts by translation. Therefore $\gamma \cdot \ell = \ell$. Since $\mathcal{A}$ is a Γ–well-behaved space of axes, there is some $\ell_1 \in \mathcal{A}$ parallel to ℓ. The orbit of ℓ_1 under the action of $\langle\gamma\rangle$ is bounded, since $d(\ell_1, \ell) < \infty$ and all translates of ℓ_1 under $\langle\gamma\rangle$ are equidistant from ℓ. Since $\langle\gamma\rangle \cdot \ell_1$ is bounded, there is a center $\ell_c \in \mathcal{A}$ of $\langle\gamma\rangle \cdot \ell_1$ which is invariant under the action of $\langle\gamma\rangle$ (Proposition 3.6(3)). This shows that $\mathrm{Fix}_{\mathcal{A}}(\gamma)$ is nonempty. Now since any two axes of γ are parallel, $\mathrm{Fix}_{\mathcal{A}}(\gamma)$ is contained in a single connected component of $\mathcal{L}(X)$, namely $\mathcal{L}_{\ell_c}(X)$. We take two axes $\widehat{\ell}_1, \widehat{\ell}_2 \in \mathrm{Fix}_{\mathcal{A}}(\gamma)$. Since $\mathcal{A}$ is a convex subset of $\mathcal{L}(X)$, the geodesic segment $[\widehat{\ell}_1, \widehat{\ell}_2]$ lies in $\mathcal{A}$. Since the endpoints $\widehat{\ell}_1, \widehat{\ell}_2$ are fixed by γ and $\mathcal{L}_{\ell_c}(X)$ is uniquely geodesic, it must be that $[\widehat{\ell}_1, \widehat{\ell}_2] \subseteq \mathrm{Fix}_{\mathcal{A}}(\gamma)$. This shows that $\mathrm{Fix}_{\mathcal{A}}(\gamma)$ is path connected.

Now let $\ell \in \mathcal{A}$ be arbitrary. We consider the stabilizer $\Gamma_\ell = \{\gamma \in \Gamma \mid \gamma \cdot \ell = \ell\}$. The homomorphism $\rho\colon \Gamma_\ell \to \mathrm{Isom}(\ell)$ must have finite kernel by the properness of the action of Γ on X. Since the orbit of a point $x \in \ell$ is discrete, it must be that $\rho(\Gamma_\ell)$ is isomorphic to $\{1\}$, D_∞ (the infinite dihedral group), or $\mathbb{Z}$. It follows that Γ_ℓ fits into an exact sequence

$$F \to \Gamma_\ell \to C,$$

where F is finite and C is isomorphic to $\{1\}$, D_∞, or $\mathbb{Z}$. It follows that Γ_ℓ is virtually cyclic.

It now follows from Proposition 4.6(2) that $N(\mathcal{V})$ is an $I_{\mathcal{VC}_\infty}\Gamma$–complex. Proposition 2.4 implies that $N(\mathcal{U}) * N(\mathcal{V})$ is an $E_{\mathcal{VC}}\Gamma$–complex. □

6 A construction of $E_{\mathcal{FBC}}\Gamma$ for CAT(0) groups

Definition 6.1 Let $c, c'\colon \mathbb{R} \to X$ be geodesic lines in a CAT(0) space X. We write $c \sim c'$ if there is a constant K such that $c'(t) = c(t+K)$. An equivalence class $[c]$ is called a *directed geodesic line*.

If $\mathcal{A}$ is an arbitrary subset of the space of lines $\mathcal{L}(X)$, we set

$$\mathcal{DA} = \{[c] \mid c(\mathbb{R}) \in \mathcal{A}\}.$$

We metrize $\mathcal{DA}$ as follows:

$$d([c_1],[c_2]) = \inf\{d(c_1', c_2') \mid c_i' \sim c_i \text{ for } i = 1, 2\},$$

where $d(c_1', c_2') = \inf\{d(c_1'(t), c_2'(t)) \mid t \in \mathbb{R}\}$. It is not difficult to see that d is indeed a metric on $\mathcal{DA}$.

Remark 6.2 Another way to describe the metric d is as follows: The distance between the directed lines $[c_1], [c_2]$ is the same as the distance between $c_1(\mathbb{R})$ and $c_2(\mathbb{R})$, if the latter distance is finite (in the sense of Definition 3.7) and if c_1 and c_2 are asymptotic. Otherwise, the distance is infinite.

Lemma 6.3 *Let $\mathcal{A} \subseteq \mathcal{L}(X)$ be a Γ–well-behaved space of axes. The map $\pi\colon \mathcal{DA} \to \mathcal{A}$, given by sending $[c]$ to $c(\mathbb{R})$, is a 2–to–1 covering map and $\mathcal{DA}$ is a stack-of-pancakes covering space, ie, $\mathcal{DA}$ is isometric to the product $\mathcal{A} \times \{0,1\}$, where $\{0,1\}$ is given the metric in which $d(0,1) = \infty$. In particular, $\mathcal{DA}$ is a disjoint union of proper* CAT(0) *spaces.*

There is an involution $i\colon \mathcal{DA} \to \mathcal{DA}$ sending $[c]$ to $[-c]$, where $-c(t) = c(-t)$. The map i is a covering transformation of the covering map $\pi\colon \mathcal{DA} \to \mathcal{A}$. Moreover, for any $\gamma \in \Gamma$, $\gamma \cdot (i([c])) = i(\gamma \cdot [c])$.

Let $\mathcal{U}$ be a Γ–good cover of $\mathcal{A}$. If $B \in \mathcal{U}$, we let B^+ and B^- denote the two sheets over B. The collection

$$\overline{\mathcal{U}} = \bigcup_{B \in \mathcal{U}} \{B^+, B^-\}$$

is a Γ–good cover of $\mathcal{DA}$, and the involution i acts cellularly on $N(\overline{\mathcal{U}})$.

Proof The group $\langle i \rangle \cong \mathbb{Z}/2\mathbb{Z}$ acts freely and isometrically on $\mathcal{DA}$. Since the group $\langle i \rangle$ is finite, the action is therefore by covering transformations. The quotient of $\mathcal{DA}$ by the action of $\langle i \rangle$ is $\mathcal{A}$. We let $\pi\colon \mathcal{DA} \to \mathcal{A}$ denote the quotient projection. The map π is a 2–to–1 covering map. Since each connected component of $\mathcal{A}$ is simply connected (indeed, contractible), $\mathcal{DA}$ is a stack-of-pancakes covering space.

It is straightforward to check that $\gamma \cdot (i([c])) = i(\gamma \cdot [c])$, and that $\overline{\mathcal{U}}$ is a Γ–good cover of $\mathcal{DA}$. $\square$

In what follows, we equip the infinite-dimensional sphere $\mathbb{S}^\infty$ with a cell structure such that the antipodal map $a\colon \mathbb{S}^\infty \to \mathbb{S}^\infty$ acts in a cell-permuting way, and $a \cdot c \cap c = \varnothing$ for each cell $c \subseteq \mathbb{S}^\infty$.

Theorem 6.4 *Suppose that $\mathcal{A}$ is a Γ–well-behaved collection of axes, $\mathcal{U}$ is a Γ–good cover of $\mathcal{A}$, and let $\overline{\mathcal{U}}$ be the Γ–good cover of $\mathcal{DA}$ determined by the previous lemma.*

The space $$K = \left(N(\overline{\mathcal{U}}) \times \mathbb{S}^\infty\right)/\sim.$$

is an $I_{\mathcal{FBC}_\infty}\Gamma$–complex, where $(x,p) \sim (i(x), a(p))$, and, for any $\gamma \in \Gamma$, $\gamma \cdot [(x,p)] = [(\gamma \cdot x, p)]$.

Proof We first note that K has a natural cell structure, since it is the quotient of the CW–complex $N(\overline{\mathcal{U}}) \times \mathbb{S}^\infty$ by a free cell-permuting action of $\mathbb{Z}/2\mathbb{Z} \cong \langle t \rangle$, where $t \cdot (x,p) = (i(x), a(p))$.

Now we show that if a cell of K is left invariant as a set by some $\gamma \in \Gamma$, then the cell is actually fixed pointwise by γ. A cell e of K has the form

$$\left[(c_1 \times c_2) \cup (i(c_1) \times a(c_2))\right]/\sim.$$

If this cell is invariant under the action of γ, it must be that the cell $\pi(c_1) \subseteq N(\mathcal{U})$ (where π is induced by the projection $\pi\colon \mathcal{DA} \to \mathcal{A}$) is fixed pointwise by the action of γ. It follows that the action of γ on c_1 either agrees with that of i, or $\gamma_{|c_1} = id_{|c_1}$. We choose $(x,y) \in c_1 \times c_2$ to represent a typical element of e. If γ agrees with i on c_1, then $\gamma \cdot (x,y) = (i(x), y)$. This point is not an element of e. This is a contradiction, and shows that γ agrees with the identity on c_1. Therefore $\gamma \cdot (x, y) = (x, y)$, as we wished to show.

The same line of reasoning establishes the identity

$$\mathrm{Fix}_K(H) = \big(\mathrm{Fix}_{N(\overline{\mathcal{U}})}(H) \times \mathbb{S}^\infty\big)/\!\sim.$$

We verify properties (1) and (2) of $I_{\mathcal{FBC}_\infty}\Gamma$–complexes from Definition 2.2. Let $H \in \mathcal{FBC}_\infty$. We use the following fact about infinite finite-by-cyclic groups H: if H acts properly and isometrically on a line ℓ, then the image of the associated homomorphism $\rho\colon H \to \mathrm{Isom}(\ell)$ is a group of translations. Since H is infinite and virtually cyclic, it follows from Proposition 3.9(2) that $\mathrm{Fix}_{N(\mathcal{U})}(H)$ is a contractible subcomplex of $N(\mathcal{U})$.

We claim that $\pi^{-1}(\mathrm{Fix}_{N(\mathcal{U})}(H)) = \mathrm{Fix}_{N(\overline{\mathcal{U}})}(H)$. Suppose that c is a cell of $N(\overline{\mathcal{U}})$, and $c \subseteq \pi^{-1}(\mathrm{Fix}_{N(\mathcal{U})}(H))$. We have that $\pi(c)$ is fixed by all of H, so each $h \in H$ either acts as the identity on c, or as the inversion i. However, if $h \in H$ acts as inversion on c, then h must act as the inversion on some line $\ell \subseteq X$. This contradicts the fact that $\rho(H) \subseteq \mathrm{Isom}(\ell)$ must be a group of translations. It follows that each $h \in H$ fixes c, so $\pi^{-1}(\mathrm{Fix}_{N(\mathcal{U})}(H)) \subseteq \mathrm{Fix}_{N(\overline{\mathcal{U}})}(H)$. The reverse inclusion is clear.

It now follows that

$$\mathrm{Fix}_K(H) = \big(\pi^{-1}(\mathrm{Fix}_{N(\mathcal{U})}(H)) \times \mathbb{S}^\infty\big)/\!\sim.$$

Thus, $\mathrm{Fix}_K(H)$ is the quotient of two contractible sets by an involution $(x, y) \to (i(x), a(y))$ which identifies these components by a homeomorphism. Therefore, $\mathrm{Fix}_K(H)$ is contractible.

If $H \in \mathcal{VC} - \mathcal{FBC}$, then whenever H acts properly and isometrically on a line ℓ, the image of the associated homomorphism $\rho\colon H \to \mathrm{Isom}(\ell)$ has an involution. The group H has a subgroup of index two, H^+, where $H^+ \in \mathcal{FBC}$. We let α be an element of order 2 such that $H = \langle H^+, \alpha\rangle$. By the above argument, $\pi^{-1}(\mathrm{Fix}_{N(\mathcal{U})}(H^+) = \mathrm{Fix}_{N(\overline{\mathcal{U}})}(H^+)$, and this space is a disjoint union of two contractible pieces which are swapped by the covering transformation i. We note that the element α must act as an inversion on any line ℓ on which H acts properly and isometrically, for otherwise it would necessarily act as the identity, and then the entire group H would map to a group of translations of ℓ under $p\colon H \to \mathrm{Isom}(\ell)$, and this is impossible. It follows that α swaps the components of $\pi^{-1}(\mathrm{Fix}_{N(\mathcal{U})}(H^+))$; in particular $\mathrm{Fix}_{N(\overline{\mathcal{U}})}(H) = \varnothing$, so $\mathrm{Fix}_K(H) = \varnothing$.

We note finally that $\mathrm{Fix}_{N(\overline{\mathcal{U}})}(H) = \varnothing$ if $H \notin \mathcal{VC}$, and so $\mathrm{Fix}_K(H) = \varnothing$ if $H \notin \mathcal{VC}$. This completes the proof. □

Corollary 6.5 *If Γ acts properly and isometrically by semisimple isometries on the proper* $\mathrm{CAT}(0)$ *space X, $\mathcal{A}$ is a Γ–well-behaved space of axes, $\mathcal{U}$ is a Γ–good cover of X, and $\mathcal{V}$ is a Γ–good cover of $\mathcal{A}$, then the space*

$$N(\mathcal{U}) * K$$

is a $E_{\mathcal{FBC}}\Gamma$–complex, where K is as defined in the previous theorem.

Proof This follows immediately from the Theorem 6.4 and from Proposition 2.4. □

7 Examples

Suppose that Γ acts properly by semisimple isometries on a proper $\mathrm{CAT}(0)$ space X. It seems difficult to prove, in general, that there will always exist a Γ–well-behaved collection of axes. In this section, we will consider two general cases in which it is possible to establish the existence of such a collection of axes. Both cases were suggested by the referee. These cases are not intended to exhaust the range of possible spaces to which our techniques apply.

In Section 7.2, we show that there is always a Γ–well-behaved collection of axes in the event that X is a $\mathrm{CAT}(0)$ space with isolated flats. Examples of groups acting on $\mathrm{CAT}(0)$ spaces with isolated flats include crystallographic groups and $\mathrm{CAT}(-1)$ groups (both in a fairly trivial sense), and groups acting properly, isometrically and cocompactly on truncated hyperbolic spaces; see Hruska and Kleiner [7]. We note that every group Γ acting isometrically, properly, and cocompactly on a $\mathrm{CAT}(0)$ space with isolated flats is relatively hyperbolic by results from [7], so an explicit construction of $E_{\mathcal{VC}}\Gamma$ for such groups has already appeared in Lafont and Ortiz [9]. The special case in which X is a $\mathrm{CAT}(-1)$ space was handled in [8], where Juan-Pineda and Leary built $E_{\mathcal{VC}}\Gamma$ for all hyperbolic groups Γ.

In Section 7.3, we show that our constructions work if X is a simply connected real analytic Riemannian manifold of nonpositive sectional curvature.

7.1 A lemma

In this subsection, we need to assume some background from Riemannian geometry. We need only very well-known facts, which can be found, for instance, in do Carmo [3]. Lemma 7.1 is used in the applications in Section 7.2 and Section 7.3.

Lemma 7.1 *Let Γ act properly by semisimple isometries on a proper* $\mathrm{CAT}(0)$ *space X. Suppose $\mathcal{L}(X)$ (the space of lines) contains a closed, component-wise convex, Γ–equivariant subset L such that*

(1) *for every component C of $\mathcal{L}(X)$, $L \cap C \neq \varnothing$;*

(2) *each component of L is a geodesically complete C^∞ Riemannian manifold of nonpositive sectional curvature.*

The set
$$L \cap \mathcal{A}(X;\Gamma)$$
is a Γ–well-behaved collection of axes.

Proof It follows from Proposition 3.10 that $L \cap \mathcal{A}(X;\Gamma)$ is component-wise convex and Γ–equivariant. We need to show that $L \cap \mathcal{A}(X;\Gamma)$ is closed, and that, whenever a component C of $\mathcal{L}(X)$ contains an element of $\mathcal{A}(X;\Gamma)$, C also contains an element of $L \cap \mathcal{A}(X;\Gamma)$.

We prove the latter statement first. Let C be a component containing an element $\ell \in \mathcal{A}(X;\Gamma)$. Since $L \cap C$ is a nonempty closed convex subset of C, there is a unique point $\pi(\ell) \in L \cap C$ such that $d(\ell, \pi(\ell)) = d(\ell, L \cap C)$, by Proposition 3.6(4). Let $\gamma \in \Gamma$ be a translation having the axis ℓ. Now
$$d(\ell, \pi(\ell)) = d(\gamma \cdot \ell, \gamma \cdot \pi(\ell)) = d(\ell, \gamma \cdot \pi(\ell)).$$
Since L is Γ–invariant, we have that $\gamma \cdot \pi(\ell) \in L \cap C$. It now follows that $\gamma \cdot \pi(\ell) = \pi(\ell)$, by the uniqueness of the closest point $\pi(\ell)$. Thus, $\pi(\ell)$ is a line that is invariant under the translation γ, so it must be an axis, and so $\pi(\ell) \in L \cap \mathcal{A}(X;\Gamma)$.

Next we show that $L \cap \mathcal{A}(X;\Gamma)$ is closed. Fix a component M of L. We may assume that $M \cap \mathcal{A}(X;\Gamma)$ contains at least one point, say ℓ. Let Γ_ℓ denote the stabilizer subgroup of $\ell \in \mathcal{L}(X)$. We consider the tangent space $T_\ell(M)$ of M at ℓ. Let $V \subseteq T_\ell(M)$ denote the collection of all tangent vectors to geodesic segments $[\ell, \ell']$, where $\ell' \in M \cap \mathcal{A}(X;\Gamma)$. (Here "geodesic segment" means "geodesic segment *in the Riemannian sense*". In effect, this simply means in the present context that we are talking about *linearly reparametrized geodesics*, as defined in [2, page 4]. Thus, here our geodesic segments are traced with constant speed, rather than unit speed, as we had originally assumed in Definition 3.1.) We claim (a) that V is a vector subspace of $T_\ell(M)$, and (b) that $\exp_\ell(V) = M \cap \mathcal{A}(X;\Gamma)$, where $\exp_\ell\colon T_\ell(M) \to M$ denotes the exponential map. Since $\exp_\ell$ is a diffeomorphism by the Cartan–Hadamard Theorem [3, Theorem 3.1], it will directly follow that $M \cap \mathcal{A}(X;\Gamma)$ is closed.

We prove (a). Consider a maximal linearly independent subset $\{\vec{v}_1, \ldots, \vec{v}_p\}$ of V. The derivative of each $\gamma \in \Gamma_\ell$ acts as a linear transformation on $T_\ell(M)$. The vector $\vec{v}_i \in \{\vec{v}_1, \ldots, \vec{v}_p\}$ is the tangent vector to a geodesic segment $[\ell, \ell_i] \in M$, where $\ell_i \in M \cap \mathcal{A}(X;\Gamma)$. If an orbit $\Gamma_\ell \cdot \vec{v}_i \subseteq T_\ell(M)$ is infinite, then there are infinitely many elements in the set $\Gamma_\ell \cdot \ell_i$, which means, by the properness of M, that the orbit of $\ell_i \in \mathcal{A}(X;\Gamma)$ is not discrete, contradicting Proposition 3.10. It follows that each orbit $\Gamma_\ell \cdot \vec{v}_i$ is finite. Thus, there is a finite index subgroup $\Gamma'_\ell \leq \Gamma_\ell$ fixing $\{\vec{v}_1, \ldots, \vec{v}_p\}$ pointwise. Since Γ'_ℓ acts as the identity on $\operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_p\}$, the orbit $\Gamma_\ell \cdot \vec{v}$ of each $\vec{v} \in \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_p\}$ is finite.

We claim now that $V = \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_p\}$. Clearly, $V \subseteq \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_p\}$. Let $\vec{v} \in \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_p\}$. Let $[\ell, \ell']$ be a geodesic segment with tangent vector $\vec{v}$. Let γ be a translation acting on ℓ (so $|\gamma| = \infty$ and $\gamma \cdot \ell = \ell$). Since $\langle \gamma \rangle \cdot \vec{v}$ is finite, there is some power of γ, γ^k say, so that $\gamma^k \cdot \vec{v} = \vec{v}$. It follows that $\gamma^k \cdot \ell' = \ell'$, so $\ell' \in \mathcal{A}(X;\Gamma)$ and (therefore) $\vec{v} \in V$. This establishes (a).

We now prove (b). Let $\ell' \in \exp_\ell(V)$. This means that there is a geodesic segment $[\ell, \ell']$ such that its tangent vector $\vec{v} \in T_\ell(M)$ lies in V. The tangent vector $\vec{v}$ defines a unique geodesic line $\widehat{L}$ in M. Since $\vec{v} \in V$, there is some $\ell'' \in \widehat{L} \cap \mathcal{A}(X;\Gamma)$. We let $\gamma \in \Gamma$ be a translation acting on ℓ''. Since the orbit of ℓ is discrete, $\mathcal{L}(X)$ is proper, and $\gamma \cdot \ell'' = \ell''$, the orbit $\langle \gamma \rangle \cdot \ell$ is finite. Therefore, there is some $\gamma^k \in \langle \gamma \rangle$ such that $\gamma^k \cdot \ell = \ell$. Since γ^k fixes two points on $\widehat{L}$, it fixes the entire line $\widehat{L}$. Therefore, $\gamma^k \cdot \ell' = \ell'$, so γ^k acts as a translation on ℓ'. It follows that $\exp_\ell(V) \subseteq M \cap \mathcal{A}(X;\Gamma)$.

Now suppose that $\ell' \in M \cap \mathcal{A}(X;\Gamma)$. It follows that the tangent vector $\vec{v} \in T_\ell(M)$ to $[\ell, \ell']$ is in V, and it follows easily that $\ell' \in \exp_\ell(V)$. This proves (b). □

7.2 CAT(0) spaces with isolated flats

In this subsection, we assume some knowledge of the Tits boundary of a CAT(0) space. See [2, pages 289–291] for the relevant background.

Definition 7.2 Let X be a proper CAT(0) space. Let Γ act properly, isometrically, and cocompactly on X. We say X has *isolated flats* if each connected component of the Tits boundary $\partial_T X$ is an isolated point or a standard Euclidean sphere S^n $(n \geq 1)$.

If X has isolated flats, then there exists a family $\mathcal{F}$ of *flats* $F \subseteq X$ (ie, isometrically embedded copies of $\mathbb{R}^n$, where $n \geq 2$) such that (i) $\mathcal{F}$ is Γ–invariant, and (ii) each Euclidean sphere of dimension at least 1 in $\partial_T X$ is the Tits boundary of some $F \in \mathcal{F}$.

Remark 7.3 Our definition of CAT(0) spaces with isolated flats is not the usual one, but is equivalent to it by Theorem 5.2.5 of [7]. We note that, as in [7], geodesic lines are not considered flats for the purposes of the above definition.

Theorem 7.4 *If X is a proper* CAT(0) *space with isolated flats, and Γ acts properly, isometrically, and cocompactly on X, then there is a Γ–well-behaved collection of axes.*

Proof First, we note that Γ must act by semisimple isometries on X [2, Proposition 6.10(2)]. We want to apply Lemma 7.1. The set $L \subseteq \mathcal{L}(X)$ from that lemma is the union of two pieces, L_1 and L_2. We begin by letting $L_1 = \{\ell \in \mathcal{L}(X) \mid \ell \subseteq F \in \mathcal{F}\}$.

A line $\ell \subseteq X$ determines two points in $\partial_T X$, which we denote $\ell(\infty)$ and $\ell(-\infty)$. Consider

$$L_2' = \{\ell \in \mathcal{L}(X) \mid \ell(\infty) \text{ and } \ell(-\infty) \text{ are in separate components of } \partial_T X\}.$$

We claim that each $\ell \in L_2'$ is contained in a compact component C_ℓ of $\mathcal{L}(X)$. In view of Proposition 3.9, it is enough to show that C_ℓ is bounded. If not, then there is a point $\zeta \in \partial_T C_\ell$. It follows that there is a geodesic ray r in C_ℓ issuing from ℓ. Therefore, ℓ bounds a flat half-plane in X. Proposition 9.21(3) from [2] says that $d_T(\ell(\infty), \ell(-\infty)) = \pi$ (where d_T denotes the Tits metric on $\partial_T X$). This is a contradiction, since our assumption that $\ell(\infty)$ and $\ell(-\infty)$ are in separate components of $\partial_T X$ implies that $d_T(\ell(\infty), \ell(-\infty)) = \infty$. We continue to let C_ℓ denote the connected component of $\mathcal{L}(X)$ containing the line ℓ, and write

$$L_2 = \{\ell \in \mathcal{L}(X) \mid C_\ell \text{ is compact and } \ell \text{ is the center of } C_\ell\}.$$

We claim that $L = L_1 \cup L_2$ satisfies the hypotheses of Lemma 7.1. It is straightforward to check that L is closed, component-wise convex, and Γ–equivariant. Clearly, each component is isometric to $\mathbb{R}^k$ (for some $k \geq 0$), so the second hypothesis of Lemma 7.1 is satisfied. We need to establish (1). Let C be a component of $\mathcal{L}(X)$. Since any two lines in C are parallel, C determines two points $\zeta, \zeta' \in \partial_T X$ such that $d_T(\zeta, \zeta') \geq \pi$ (the latter statement about distance comes from [2, Proposition 9.21(3)]). It is not difficult to see, moreover, that the lines $\ell \in C$ are completely characterized by the property that $\{\ell(\infty), \ell(-\infty)\} = \{\zeta, \zeta'\}$. There are just two possibilities for ζ, ζ': either both are in the same component of $\partial_T X$, or they are in different components. If they are in different components, then the above argument shows that C is compact, and therefore the center of C is in L_2. If they are in the same component, then there is some Euclidean sphere $S^k \subseteq \partial_T X$ ($k \geq 1$) containing the points ζ, ζ', which must be antipodal by Proposition 9.21(3) from [2]. It follows that there is a line ℓ in the associated flat $\mathbb{R}^{k+1}$ such that $\{\ell(\infty), \ell(-\infty)\} = \{\zeta, \zeta'\}$, and this line is in $L_1 \cap C$ by the definition of L_1. □

7.3 Real analytic manifolds of nonpositive sectional curvature

Theorem 7.5 *Suppose that X is a simply connected real analytic Riemannian manifold (without boundary) having nonpositive sectional curvature, and that Γ acts by semi-simple isometries on X. The space $\mathcal{A}(X;\Gamma)$ of all axes is Γ–well-behaved.*

Proof This is an application of Lemma 2.4 of [1]. If we choose an arbitrary line $\ell \subseteq X$ and set $Y = \ell$ in that lemma, then the union P_ℓ of all totally geodesic submanifolds parallel to ℓ factors as $\ell \times N$, where N is a complete totally geodesic submanifold of X. It follows that the space of lines $\mathcal{L}(X)$ is a disjoint union of complete C^∞ Riemannian manifolds, so that Lemma 7.1 applies with $L = \mathcal{L}(X)$. □

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