On free discrete subgroups of Diff(I)

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We prove that the free group $\mathbb{F}_2$ admits a faithful discrete representation into $\mathrm{Diff}_+^1[0,1]$. We also prove that $\mathbb{F}_2$ admits a faithful discrete representation of bi-Lipschitz class into $\mathrm{Homeo}_+[0,1]$. Some properties of these representations are studied.

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Introduction

In recent decades and especially in recent years, some remarkable papers devoted to the study of finitely generated subgroups of $\mathrm{Diff}_+^1[0,1]$ have appeared (see Bergman [1], Calegari [2], Farb and Franks [3; 4], Farb and Shalen [5], Ghys [6], Navas [7; 8; 9], Tsuboi [12] and Yoccoz [13] for some of the most current developments). In contrast, discrete subgroups of $\mathrm{Diff}_+^1[0,1]$ are much less studied. Very little is known in this area especially in comparison with the very rich theory of discrete subgroups of Lie groups started in the works of F Klein and H Poincaré in the 19th century, and expanded enormously in the works of A Selberg, A Borel, G Mostow, G Margulis and many others in the 20th century. Many questions which are either very easy or were studied a long time ago for (discrete) subgroups of Lie groups remain open in the context of the infinite-dimensional group $\mathrm{Diff}_+^1[0,1]$ and its relatives. In this paper, we address a question about the existence of discrete faithful representations of nonabelian free groups into the group $\mathrm{Diff}_+^1[0,1]$.

We assume the usual topology on the group $\mathrm{Diff}_+^1[0,1]$ given by the standard metric of $C^1[0,1]$. We will denote this metric by d_1.

Theorem 1 *A free group $\mathbb{F}_2$ admits a faithful discrete representation into* $\mathrm{Diff}_+^1[0,1]$.

We will also be interested in discrete subgroups of $\mathrm{Homeo}_+[0,1]$ – the group of orientation preserving homeomorphisms of the closed interval. Here, the metric comes from the sup norm of the Banach space $C[0,1]$. For $f \in C[0,1]$ we will denote $\|f\|_0 = \sup_{x\in[0,1]} |f(x)|$.

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Theorem 2 *A free group $\mathbb{F}_2$ admits a faithful discrete representation into* $\mathrm{Homeo}_+[0,1]$. *Moreover,*

(a) *the representation can be chosen from the class $C^1(0,1) \cap \mathrm{BiLip}[0,1]$.*

(b) *for any nonempty open neighborhood Ω of the identity in $\mathrm{Homeo}_+[0,1]$, the generators of the faithful discrete representation of $\mathbb{F}_2$ can be chosen from Ω.*

Here, $\mathrm{BiLip}[0,1]$ denotes the set of all bi-Lipschitz functions from the closed interval $[0,1]$ into itself.

Proofs of main theorems

In this section we will prove Theorems 1 and 2.

In the free group $\mathbb{F}_2$ we will fix the left-invariant Cayley metric with respect to standard generating set, and denote it by $|\cdot|$. The following notions will be useful.

Definition 1 Let W be a reduced word in the alphabet of the standard generating set of the free group $\mathbb{F}_2$. We say that a reduced word U is a *suffix* of W, if $W = U_1U$ where U_1 is a reduced word, and $|W| = |U_1| + |U|$. We also say that a reduced word V is a *prefix* of W, if $W = VV_1$ where V_1 is a reduced word, and $|W| = |V| + |V_1|$.

Proof of Theorem 1 Let $I_n = (1/(2n+1), 1/(2n))$ for any $n \in \mathbb{N}$ and let $C > 0$.

We will build two maps $f, g \in \mathrm{Diff}^1_+[0,1]$ such that the group $\Gamma_{f,g}$ generated by them is isomorphic to $\mathbb{F}_2$ and satisfies the following condition:

($\star$) For all $g_1, g_2 \in \Gamma_{f,g}, g_1 \neq g_2$, the inequality $\sup_{t\in[0,1]} |g_1'(t) - g_2'(t)| > C$ is satisfied.

Let $\pi_n = (U_n, V_n), n \geq 1$ be a sequence of pairs of words (elements) in $\mathbb{F}_2$ satisfying the following conditions:

(a1) $U_n \neq V_n$ for all $n \geq 1$.

(a2) $|U_n| \geq |V_n|$ for all $n \geq 1$.

(a3) If $m > n$ then $|U_m| \geq |U_n|$.

(a4) If $m > n$, $|U_m| = |U_n|$ then $|V_m| \geq |V_n|$.

(a5) $U_n \neq 1$ for all $n \geq 1$.

(a6) If $U, V \in \mathbb{F}_2$, $U \neq 1$, $|U| \geq |V|$ then there exists $n \in \mathbb{N}$ such that $U = U_n, V = V_n$.

(a7) If $m \neq n$ then $\pi_m \neq \pi_n$.

For every $n \in \mathbb{N}$, the longest common suffix of U_n and V_n will be denoted by W_n and we let $s_n = |W_n|$.

Let also $m_n = \mathrm{Card}\{k \mid \pi_k = (U_k, V_k), |U_k| = n\}$ for all $n \geq 1, m_0 = 0$. Notice that m_n grows exponentially as $n \to \infty$.

Let $\alpha = (\alpha_1, \alpha_2, \dots)$ be a sequence of positive real numbers such that

(b1) $\lim_{r\to\infty} \alpha_r = 0$.

(b2) for every $r \in \mathbb{N}, s \in \{0, 1, \dots, r-1\}$, the inequality

$$(1+\alpha_r)^s((1+\alpha_r)^{r-s} - 1) > C$$

is satisfied.

(Notice that such a sequence α exists, eg $\alpha_1 = C+1$, $\alpha_r = \sqrt{(C+1)/(r-1)}, r \geq 2$.)

Let also $\beta = (\beta_1, \beta_2, \dots)$ be a sequence such that $\beta_i = \alpha_j$ for all $m_1 + \cdots + m_{j-1} < i \leq m_1 + \cdots + m_{j-1} + m_j$. We notice that $\lim_{n\to\infty} \beta_n = 0$; moreover, for every $n \in \mathbb{N}$, we have $\beta_n = \alpha_{i(n)}$ where $i(n) \to \infty$ as $n \to \infty$.

Now, for any natural n, let x_0^n be the midpoint of the interval I_n, $s = s_n$, and let f, g be defined in the interval I_n such that

(c1) $f(x) = g(x) = x$ for all $x \in \{1/(2n+1), 1/(2n)\}$.

(c2) $f'(x) \in [1/(1+\beta_n+1/n), 1+\beta_n+1/n]$, for all $x \in I_n$.

(c3) $f'(x) = g'(x) = 1$ for all $x \in \{1/(2n+1), 1/(2n)\}$.

(c4) if $|U_n| = r$, where $U_n = a_r a_{r-1} \cdots a_s \cdots a_1$ for $a_i \in \{f, g, f^{-1}, g^{-1}\}, 1 \leq i \leq r$, and if $U_n(k) = a_k \cdots a_1$, $0 \leq k \leq r-1$, then $a'_{k+1}(U_n(k)(x_0^n)) = 1+\beta_n$.

(c5) if $|V_n| = m$, where $V_n = b_m b_{m-1} \cdots b_1$, for $b_i \in \{f, g, f^{-1}, g^{-1}\}, 1 \leq i \leq m$, and if $V_n(k) = b_k \cdots b_1$, $m-1 \geq k \geq s$ then $b'_{k+1}(V_n(k)(x_0^n)) = 1$.

Now, if $x \in [0,1] \backslash (\bigsqcup_{n\in\mathbb{N}} I_n)$, we set $f(x) = g(x) = x$ (hence $f'(x) = g'(x) = 1$).

Then the functions f, g will belong to $\mathrm{Diff}_+^1[0,1]$. Moreover, for any $n \geq 1$, by the Chain Rule, we have

$$U_n'(x_0^n) = (1+\beta_n)^r, \quad V_n'(x_0^n) = (1+\beta_n)^s 1^{m-s} = (1+\beta_n)^s.$$

Since $\beta_n = \alpha_{i(n)}$ and $i(n) = r$, the inequality $|(U_n(f,g))'(x_0^n) - (V_n(f,g))'(x_0^n)| > C$ follows from condition (b2). □

Remark 1 We indeed prove more than discreteness; the inequality

$$\sup_{\substack{t\in[0,1],\\ g\in\mathbb{F}_2\setminus\{1\}}} |g'(t)-1| \geq C > 0$$

would suffice for discreteness. By proving more general inequality

$$\sup_{\substack{t\in[0,1],\\ g_1,g_2\in\mathbb{F}_2,\, g_1\neq g_2}} |g_1'(t)-g_2'(t)| \geq C > 0\,,$$

we show that the representation is *uniformly discrete*. Since the metric in $\mathrm{Diff}^1_+[0,1]$ is not left-invariant, discreteness does not necessarily imply uniform discreteness.

Remark 2 It is clear from the proof that the functions $f(t)$ and $g(t)$ can be chosen from an arbitrary nonempty open neighborhood of the identity. This is contrary to the case of connected Lie groups: the *Margulis Lemma* states that any connected Lie group G possesses a nonempty open neighborhood U of the identity such that any discrete subgroup of G generated by elements from U is nilpotent (see Raghunathan [10]). Thus we have shown that the Margulis Lemma does not hold for the group $\mathrm{Diff}^1_+[0,1]$.

It is easy to put the main idea of the proof of Theorem 1 in words: we take all pairs (U_n, V_n) in the free group $\mathbb{F}_2$ that are interesting to us and enumerate them with some care (conditions (a1)–(a7)). For simplicity, let us also assume that $V_n = 1, n \geq 1$. Then we choose countable pairwise disjoint open subintervals $I_1, I_2, \dots, I_n, \dots$ of $[0,1]$ which are accumulating to the left endpoint of $[0,1]$, (I_i is on the left side of I_j for all $i > j$). Then, on each of the subintervals we arrange the maps f, g such that $\sup_{x\in I_n} |f'(x)-1|$ and $\sup_{x\in I_n} |g'(x)-1|$ converge to zero as $n\to\infty$ while for each midpoint $x_n \in I_n$ we have $U_n'(x_n) > C$.

To satisfy this condition, one notices that the word U_n has length at least $\log(n)$ which goes to infinity as n grows. Then, since $U_n'(x_n)$ is the product of $\log(n)$ derivatives we can have this product to be bigger than C yet each of the factor stay close to 1. (and converge to 1 as n goes to infinity). For fixed n, each of these conditions imposes only finitely many conditions on f and g in I_n, and for the next pair we go to a different interval I_{n+1}, hence we have no obstruction left to the existence of discrete $\mathbb{F}_2$ of C^1 class.

However, because of the slow growth of $\log(n)$, and because the lengths of intervals of I_n converge to zero faster than $1/n$, it is easy to see that this construction will not work in C^2 class. In fact, as Danny Calegari pointed out, it will not work in any $C^{1+\epsilon}$ class for any $\epsilon > 0$; imposing the same condition will blow-up the Holder norm. So one cannot achieve higher regularity of representations by taking care of different pairs in disjoint areas of the closed interval $[0,1]$. If we want to mix fields of actions for different pairs, we need to take some cautions.

Now we will prove Theorem 2. We need the following definitions.

Definition 2 For open subintervals $I, J \subset (0,1)$ we say $I < J$ if any element I is less than any element of J.

Definition 3 A two-sided sequence $\{I_n\}_{n\in\mathbb{Z}}$ of open subintervals of $(0,1)$ is called a *chain* if $I_n < I_{n+1}$ for all $n \in \mathbb{Z}$.

Proof of Theorem 2 Let $\epsilon > 0$, and let $A_n, B_n,\ n \in \mathbb{Z}$ be open subintervals of $(0,1)$ such that

(i) the two sided sequence $\{A_n, B_n\}_{n\in\mathbb{Z}}$ is a chain of subintervals (that is, we have $\cdots < A_{-1} < B_{-1} < A_0 < B_0 < A_1 < B_1 < A_2 < \cdots$).

(ii) for all $n \in \mathbb{Z}$ and all $i \in \{1,2,3,4\}$ we have $f^i(A_n) \subseteq B_n,\ f^{-i}(A_n) \subseteq B_{n-1}$.

(iii) for all $n \in \mathbb{Z}$, we have $g(B_n) \subseteq A_{n+1},\ g^{-1}(B_n) \subseteq A_n$.

(iv) for all $n \in \mathbb{Z}$, the inequality $\sup_{x\in A_n, y\in A_{n+2}} |x-y| < \epsilon$ holds.

It is straightforward to choose $f, g \in \mathrm{Homeo}_+[0,1]$ satisfying conditions (i)–(iv).

Now, let $A = \bigcup_{n\in Z} A_n,\ B = \bigcup_{n\in Z} B_n$. Notice that by conditions (i)–(ii),

$f^i(A) \subseteq B$ for all $i \in \{-4,-3,-2,-1,1,2,3,4\}$ and $g^i(B) \subseteq A$ for all $i \in \{-1,1\}$.

This allows us to use a ping-pong argument.

The ping-pong argument is usually used to guarantee existence of free subgroups, here we will be using it also to satisfy discreteness (which is natural). Using the ping-pong lemma, we will show the following: Assume conditions (i)–(iv), and suppose $U(f,g), V(f,g)$ are reduced words satisfying two conditions:

(1) $U(f,g) = f^2 U_0(f,g) f^2,\ V(f,g) = f V_0(f,g) f$ where $U_0(f,g), V_0(f,g)$ are both nonempty reduced words starting and ending in letter g.

(2) None of the letters $\{f,g\}$ occur with exponent other than $\{-1,1\}$ in $U_0(f,g)$ and in $V_0(f,g)$.

Then $U(f,g)$ and $V(f,g)$ actually generate a free subgroup isomorphic to $\mathbb{F}_2$ in $\mathrm{Homeo}_+[0,1]$. We will have that this subgroup (which we will denote by Γ) is discrete.

Let $W(U,V)$ be any reduced nontrivial word in the alphabet $\{U = U(f,g), V = V(f,g), U^{-1} = U(f,g)^{-1}, V^{-1} = V(f,g)^{-1}\}$. Then in the alphabet $\{f, g, f^{-1}, g^{-1}\}$ the word W ends with either f or f^{-1}.

Let x_0 be the midpoint of A_0.

We notice that $f^i(A) \subseteq B$ for all $i \in \{-4,-3,-2,-1,1,2,3,4\}$. Furthermore, $g^{\pm 1}(B) \subseteq A$. Then by a standard ping-pong argument, we have that

$$W(x_0) = W(U(f,g), V(f,g))(x_0) \notin A_0,$$

hence $W \neq 1$ in Γ, and $\|W\|_0 \geq |A_0|/2$.

We now consider the general case of arbitrary distinct $h_1, h_2 \in \Gamma$. Let $h_1 = W_1(U,V)$, $h_2 = W_2(U,V)$ be two distinct reduced words in the alphabet $\{U, V, U^{-1}, V^{-1}\}$. Then we can write $W_2 = W W_1$ where $W = W(U,V) = W(U(f,g), V(f,g))$.

Since $W_1 = W_1(U(f,g), V(f,g))$ is a bijective map from $[0,1]$ onto $[0,1]$, there exists $z \in [0,1]$ such that $W_1(z) = x_0$. Then $W_2(z) = W(W_1(z)) = W(x_0) \notin A_0$.

Then we have $|W_1(z) - W_2(z)| = |x_0 - W(x_0)| > |A_0|/2$. Thus we established that the nonabelian free subgroup generated by U and V is discrete.

For claim (b), suppose Ω contains a ball of radius r, and $M = \max\{|U|, |V|\}$. Then by condition (iv), $\max\{\|U\|_0, \|V\|_0\} < \epsilon M$. Since ϵ is arbitrary we can choose it to be such that $M\epsilon < r$, and hence we obtain claim (b).

For claim (a), we may choose a sufficiently large natural number N, and further assume that

$$A_n = \left(\frac{1}{5(|n|+1)}, \frac{1}{5|n|+4}\right),\ B_n = \left(\frac{1}{5|n|+4}, \frac{1}{5|n|}\right) \qquad \text{for all } n \leq -N,$$
$$A_n = \left(1 - \frac{1}{5n}, 1 - \frac{1}{5n+1}\right),\quad B_n = \left(1 - \frac{1}{5n+1}, 1 - \frac{1}{5(n+1)}\right) \quad \text{for all } n \geq N$$

(and we choose $A_{-N+1}, B_{-N+1}, \ldots, A_{N-1}, B_{N-1}$ to be arbitrary open nonempty intervals such that conditions (i) and (iv) hold). Then it is straightforward to choose $f, g \in \mathrm{Homeo}_+[0,1]$ such that $f \in C^1[0,1], g \in C^1(0,1)$, and g is a bi-Lipschitz function with Lipschitz constant at most 5 in $[0, 1/(5N)]$ and in $[1-1/(5N), 1]$. Then for any word W in the free group $\mathbb{F}_2$, the function $W(U(f,g), V(f,g))$ will be a bi-Lipschitz function of class $C^1(0,1)$. □

Remark 3 We would like to point out what goes wrong if one applies the idea of the proof to Theorem 2 directly to obtain a faithful discrete representation of $\mathbb{F}_2$ in $\mathrm{Diff}^1_+[0,1]$:

Let $A_n, B_n, n \in \mathbb{Z}$ be mutually disjoint open subintervals in $(0,1)$ satisfying conditions (i), (ii) and (iii).

We will show that it is impossible to have the maps differentiable (C^1 class) under these conditions (i)–(iii); there are obstructions easily obtained from the Mean Value Theorem.

Without loss of generality we may assume that A_n, B_n converge to 1 as $n \to \infty$. Let $\lim_{x\to 1^-} f'(x) = p$. (Then $p > 0$.)

Let p_1, p_2 be positive real numbers such that

$$p_1 < p < p_2, \quad p_1 > \frac{99}{100}p, \quad p_2 < \frac{101}{100}p.$$

So by the Mean Value Theorem, from condition (ii) we obtain that

$$|B_n| > (p_1 + p_1^2 + p_1^3)|A_n| \quad \text{and} \quad |B_n| > \left(\frac{1}{p_2} + \frac{1}{p_2^2} + \frac{1}{p_2^3}\right)|A_{n+1}|$$

for sufficiently big positive n. Then

$$\begin{aligned} \frac{|g(B_n)|}{|B_n|} &\leq \frac{|A_{n+1}|}{|B_n|} < \frac{1}{1/p_2 + 1/p_2^2 + 1/p_2^3}, \\ \frac{|g^{-1}(B_n)|}{|B_n|} &\leq \frac{|A_n|}{|B_n|} < \frac{1}{p_1 + p_1^2 + p_1^3}. \end{aligned}$$

Then, by the Mean Value Theorem, we obtain that for sufficiently big positive n, there exists $u_n, v_n \in B_n$ such that

$$g'(u_n) < \frac{1}{1/p_2 + 1/p_2^2 + 1/p_2^3} \quad \text{and} \quad (g^{-1})'(v_n) < \frac{1}{p_1 + p_1^2 + p_1^3}.$$

However, since $\lim_{x\to 1^-} g'(x) = 1/\lim_{x\to 1^-}(g^{-1})'(x)$, we obtain a contradiction because

$$\frac{1}{1/p_2 + 1/p_2^2 + 1/p_2^3}\,\frac{1}{p_1 + p_1^2 + p_1^3} < \frac{1}{p_1/p_2 + p_1^2/p_2^2 + p_1^3/p_2^3} < \frac{1}{2} < 1.$$

Remark 4 In the proof of Theorem 2, by slightly changing conditions (1)–(2), it is possible to replace condition (ii) by the following weaker version:

(ii)$'$ for all $i \in \{1,2\}, n \in \mathbb{Z}$, we have $f^i(A_n) \subseteq B_n,\ f^{-i}(A_n) \subseteq B_{n-1}$.

However, a similar argument shows that there are no $f, g \in \mathrm{Diff}^1_+[0,1]$ satisfying conditions (i), (ii)$'$ and (iii). It also follows from the criterion of Calegari [2] that no C^1–class diffeomorphisms exist which satisfy conditions (i), (ii)$'$ and (iii).

Remark 5 The metric in $C^1[0,1]$ is given by the norm $\|f\| = \|f\|_0 + \|f\|_1$ where $\|f\|_0 = \sup_{x\in[0,1]}|f(x)|$, $\|f\|_1 = \sup_{x\in[0,1]}|f'(x)|$. If $\|f\|_1$ is small and $f(0)=0$, then by Mean Value Theorem $\|f\|_0$ cannot be big. However, $\|f\|_1$ can be big even if $\|f\|_0$ is small. In the proof of Theorem 1, taking $f(x) = W(x) - x$, we actually show that $\|f\|_1$ stays big for all $W \neq 1$; we do not show that $\|f\|_0$ is big. However, in the proof of Theorem 2, we indeed show a stronger fact that $\|f\|_0$ remains big.

Questions

In this section, we raise several questions. We will address these questions in our next article.

The regularity of the representation is a very interesting question; if a finitely generated group Γ admits a faithful discrete representation in $\mathrm{Diff}_+^1[0,1]$ or in $\mathrm{Homeo}_+[0,1]$, it is interesting to know if one can achieve faithful discrete representations of higher ($C^k, k>1,\ C^\infty$, analytic, etc) regularity.

Question 1 Does a free group $\mathbb{F}_2$ admit a faithful discrete representation into $\mathrm{Diff}_+^1[0,1]$

(a) of C^k regularity for some $k > 1$?

(b) of C^k regularity for any $k \geq 1$?

(c) of C^∞ regularity?

(d) of analytic regularity?

Let Γ be a finitely generated group, and $\pi\colon \Gamma \to \mathrm{Diff}_+^1[0,1]$ be a faithful discrete representation of it.

Definition 4 The representation π is called $\|\cdot\|_0$*–discrete* if there exists $C>0$ such that $\|\pi(g)\|_0 > C$ for all $g \in \Gamma\backslash\{1\}$.

By Remark 5, $\|\cdot\|_0$–discreteness of the representation implies its discreteness in $\mathrm{Diff}_+^1[0,1]$. Also, a $\|\cdot\|_0$–discrete representation of a group into $\mathrm{Diff}_+^1[0,1]$ is just a discrete representation into $\mathrm{Homeo}_+[0,1]$ of C^1–regularity.

Question 2 Does $\mathbb{F}_2$ admit a faithful $\|\cdot\|_0$–discrete representation into $\mathrm{Diff}_+^1[0,1]$?

Definition 5 The representation π is called *strongly discrete* if there exists $C>0$ and $x_0 \in (0,1)$ such that $\|\pi(g)(x_0)\|_1 > C$ for all $g \in \Gamma\backslash\{1\}$.

Question 3 Does $\mathbb{F}_2$ admit a faithful strongly discrete representation into $\mathrm{Diff}_+^1[0,1]$?

Similarly, we say that a faithful representation $\pi\colon \Gamma \to \mathrm{Homeo}_+[0,1]$ is *strongly discrete* (in $\mathrm{Homeo}_+[0,1]$) if there exists $C>0$ and $x_0 \in (0,1)$ such that $\|\pi(g)(x_0)\|_0 > C$ for all $g \in \Gamma \backslash \{1\}$. Notice that in the proof of Theorem 2, the representation of $\mathbb{F}_2$ into $\mathrm{Homeo}_+[0,1]$ is indeed strongly discrete.

Definition 6 Let G be a topological group or a group with a metric. We say that the *Weak Margulis Lemma* holds for G, if there exists an open nonempty neighborhood U of identity such that any discrete subgroup of G generated by elements from U does not contain a nonabelian free subgroup.

We will be interested in the group $\mathrm{Diff}_+^{1+\epsilon}[0,1]$ where ϵ is a fixed positive real number. On this group, we are considering the metric d_1, ie the metric which comes from the Banach norm of $C^1[0,1]$.

Question 4 Does the Weak Margulis Lemma hold for the group $\mathrm{Diff}_+^{1+\epsilon}[0,1]$ for some $\epsilon > 0$?

Remark 6 It follows from the proof of Theorem 1 and from Theorem 2 that the Weak Margulis Lemma does not hold neither for $\mathrm{Diff}_+^1[0,1]$ nor for $\mathrm{Homeo}_+[0,1]$, in respective metrics.

The study of discrete subgroups of $\mathrm{Diff}_+^1[0,1]$ is interesting beyond the existence question of discrete faithful representations of free groups or even of the groups which contain nonabelian free subgroups. The existence of a faithful representation into $\mathrm{Diff}_+^1[0,1]$ imposes some algebraic properties onto the group; for example, it is well-known that any subgroup of $\mathrm{Homeo}_+[0,1]$ is left-orderable (see Ghys [6]). Furthermore, if a group is isomorphic to a subgroup of $\mathrm{Diff}_+^1[0,1]$ then it is locally indicable, as proven by Thurston [11]. It is interesting to consider if discreteness implies further algebraic restrictions on the group. We would like to ask the following:

Question 5 Is there a finitely generated group which admits a faithful representation into $\mathrm{Diff}_+^1[0,1]$ but does not admit a faithful discrete representation?

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References

[1] **G M Bergman**, *Right orderable groups that are not locally indicable*, Pacific J. Math. 147 (1991) 243–248 MR1084707

[2] **D Calegari**, *Nonsmoothable, locally indicable group actions on the interval*, Algebr. Geom. Topol. 8 (2008) 609–613 MR2443241

[3] **B Farb**, **J Franks**, *Group actions on one-manifolds. II. Extensions of Hölder's theorem*, Trans. Amer. Math. Soc. 355 (2003) 4385–4396 MR1986507

[4] **B Farb**, **J Franks**, *Groups of homeomorphisms of one-manifolds. III. Nilpotent subgroups*, Ergodic Theory Dynam. Systems 23 (2003) 1467–1484 MR2018608

[5] **B Farb**, **P Shalen**, *Groups of real-analytic diffeomorphisms of the circle*, Ergodic Theory Dynam. Systems 22 (2002) 835–844 MR1908556

[6] **É Ghys**, *Groups acting on the circle*, Enseign. Math. (2) 47 (2001) 329–407 MR1876932

[7] **A Navas**, *Sur les groupes de difféomorphismes du cercle engendrés par des éléments proches des rotations*, Enseign. Math. (2) 50 (2004) 29–68 MR2084334

[8] **A Navas**, *Growth of groups and diffeomorphisms of the interval*, Geom. Funct. Anal. 18 (2008) 988–1028 MR2439001

[9] **A Navas**, *A finitely generated, locally indicable group with no faithful action by C^1 diffeomorphisms of the interval*, Geom. Topol. 14 (2010) 573–584 MR2602845

[10] **M S Raghunathan**, *Discrete subgroups of Lie groups*, Ergebnisse der Math. und ihrer Grenzgebiete 68, Springer, New York (1972) MR0507234

[11] **W P Thurston**, *A generalization of the Reeb stability theorem*, Topology 13 (1974) 347–352 MR0356087

[12] **T Tsuboi**, *Homological and dynamical study on certain groups of Lipschitz homeomorphisms of the circle*, J. Math. Soc. Japan 47 (1995) 1–30 MR1304186

[13] **J-C Yoccoz**, *Centralisateurs et conjugaison différentiable des difféomorphismes du cercle. Petits diviseurs en dimension* 1, Astérisque 231, Soc. Math. France (1995) MR1367354

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