

Knot Floer homology and rational surgeries

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Let K be a rationally null-homologous knot in a three-manifold Y . We construct a version of knot Floer homology in this context, including a description of the Floer homology of a three-manifold obtained as Morse surgery on the knot K . As an application, we express the Heegaard Floer homology of rational surgeries on Y along a null-homologous knot K in terms of the filtered homotopy type of the knot invariant for K . This has applications to Dehn surgery problems for knots in S^3 . In a different direction, we use the techniques developed here to calculate the Heegaard Floer homology of an arbitrary Seifert fibered three-manifold with even first Betti number.

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1 Introduction

Heegaard Floer homology (see our paper [21]) is an invariant for closed, oriented three-manifolds Y , taking the form of a collection of homology groups which are functorial under cobordisms. This invariant is extended by the authors [19] and Rasmussen [28] to an invariant for null-homologous knots K . (Here, we say that a knot is null-homologous if its induced homology class in $H_1(Y; \mathbb{Z})$ is trivial. If the induced homology class of a knot $K \subset Y$ in $H_1(Y; \mathbb{Q})$ is trivial, we call the knot *rationally null-homologous*.) The knot invariant takes the form of a $\mathbb{Z} \oplus \mathbb{Z}$ -filtration of the chain complex whose homology calculates the Heegaard Floer complex for Y . It is the filtered chain homotopy type of this filtered complex which depends on the particular knot K .

The knot filtration gives rise to collection of chain complexes $\{A_s^+(K)\}_{s \in \mathbb{Z}}$ and chain maps $\{v_s^+: A_s^+(K) \rightarrow B^+\}_{s \in \mathbb{Z}}$ and $\{h_s^+: A_s^+(K) \rightarrow B^+\}_{s \in \mathbb{Z}}$, where here $B^+ = CF^+(Y)$ is a chain complex whose homology is the Heegaard Floer homology $HF^+(Y)$. Indeed, the homology groups of the chain complex $A_s^+(K)$ represent the homology HF^+ of sufficiently large integer surgeries on Y along K , in a sense which can be made precise (cf Theorem 4.4 of [19] and also [28]). (These complexes are defined in a more general setting in Section 3.)

Suppose that K is a null-homologous knot in a three-manifold Y . Given a rational number r , let $Y_r(K)$ denote the three-manifold obtained by Dehn filling $Y - \text{nd}(K)$ with a solid torus with slope r (with respect to the canonical Seifert framing of K). In the case where r is an integer, the Heegaard Floer homology of $Y_r(K)$ can be described in terms of the above-mentioned data coming from the knot filtration, according to our paper [27].

The primary aim of this article is to generalize this construction to the case of Morse surgery on a knot $K \subset Y$ which is only rationally null-homologous. (By Morse surgery, we mean here Dehn surgery on a knot which can be realized as the boundary of a single two-handle addition to $[0, 1] \times Y$; in the case where K is null-homologous, this corresponds to Dehn surgery with an integral slope). This construction has new consequences even in the case of null-homologous knots in a three-manifold: since the result of Dehn surgery on a null-homologous knot $K \subset Y$ can be viewed as Morse surgery on a knot in the connected sum of Y with a lens space, we obtain a description of the Heegaard Floer homology of $Y_r(K)$ in terms of the original knot Floer homology of K .

Rather than introducing the generalization of the knot package to knots which are only rationally null-homologous in this introduction, which will require some additional material in its statement (cf Sections 3, 5 and 6 below), we focus now on the description of the Floer homology of $Y_r(K)$ when r is a rational number, and $K \subset Y$ is null-homologous.

As a preliminary point, recall that the Heegaard Floer homology of Y admits a direct sum splitting indexed by the set of Spin^c structures over Y , which in turn is an affine space for $H^2(Y; \mathbb{Z})$. In particular, if $K \subset Y$ is a knot in an integral homology sphere, then there is a splitting

$$HF^+(Y_{p/q}(K)) \cong \bigoplus_{i \in \mathbb{Z}/p\mathbb{Z}} HF^+(Y_{p/q}(K), i).$$

Fix an integer i , and consider the chain complexes

$$\mathbb{A}_i^+ = \bigoplus_{s \in \mathbb{Z}} (s, A_{\lfloor (i+ps)/q \rfloor}^+(K)) \quad \text{and} \quad \mathbb{B}_i^+ = \bigoplus_{s \in \mathbb{Z}} (s, B^+),$$

where here $\lfloor x \rfloor$ denotes the greatest integer smaller than or equal to x , and all $B^+ = CF^+(Y)$. We view the above chain homomorphisms v^+ and h^+ as maps

$$v^+: (s, A_{\lfloor (i+ps)/q \rfloor}^+(K)) \rightarrow (s, B^+) \quad \text{and} \quad h^+: (s, A_{\lfloor (i+ps)/q \rfloor}^+(K)) \rightarrow (s+1, B^+).$$

Adding these up, we obtain a chain map

$$D_{i,p/q}^+ : \mathbb{A}_i^+ \rightarrow \mathbb{B}_i^+,$$

that is,

$$D_{i,p/q}^+ \{(s, a_s)\}_{s \in \mathbb{Z}} = \{(s, b_s)\}_{s \in \mathbb{Z}},$$

where here

$$b_s = v_{\lfloor (i+ps)/q \rfloor}^+(a_s) + h_{\lfloor (i+p(s-1))/q \rfloor}^+(a_{s-1}).$$

Let $\mathbb{X}_{i,p/q}^+$ denote the mapping cone of $D_{i,p/q}^+$. Note that $\mathbb{X}_{i,p/q}^+$ depends on i only through its congruence class modulo p . Note also that A_s^+ and B_s^+ are relatively \mathbb{Z} -graded, and the homomorphisms v_s^+ and h_s^+ respect this relative grading. The mapping cone \mathbb{X}_i^+ can be endowed with a relative grading, with the convention that $D_{i,p/q}^+$ drops the grading by one.

This mapping cone, whose ingredients are extracted from the knot filtration, captures Heegaard Floer homology of p/q surgeries on Y along K , according to the following:

Theorem 1.1 *Let $K \subset Y$ be a null-homologous knot, and let p, q be a pair of relatively prime integers. Then, for each $i \in \mathbb{Z}/p\mathbb{Z}$, there is a relatively graded isomorphism of groups*

$$H_*(\mathbb{X}_{i,p/q}^+) \cong HF^+(Y_{p/q}(K), i).$$

The proof of [Theorem 1.1](#) is based on generalization of the knot filtration of [\[19; 28\]](#) to the case of rationally null-homologous knots, together with a generalization of the integer surgeries description from [\[27\]](#) (where in fact [Theorem 1.1](#) is proved in the case where $q = 1$). Note that in the more general case, the knot filtration is naturally a filtration by relative Spin^c structures on the knot complement (rather than integers). Rational surgeries on $K \subset Y$ can be realized as Morse surgeries on a knot in the connected sum of Y with a lens space. The resulting knot is gotten by forming the connected sum of K with a model (homologically nontrivial) knot in the lens space. [Theorem 1.1](#) is then realized as a combination of a straightforward calculation involving this model knot, combined with a Künneth principle for connected sums, followed by the general Morse surgeries description.

We turn now to various applications of [Theorem 1.1](#).

1.1 Applications to knots with L -space surgeries

In [Section 8](#), we give applications of [Theorem 1.1](#) to knots in S^3 which admit L -space surgeries.

Recall that an L -space is a rational homology three-sphere Y whose Floer homology HF^+ in each Spin^c structure is isomorphic (as a relatively-graded $\mathbb{Z}[U]$ -module)

to $HF^+(S^3)$. This is equivalent to the condition that $\widehat{HF}(Y, \mathfrak{s}) \cong \mathbb{Z}$ for each $\mathfrak{s} \in \text{Spin}^c(Y)$. Recall also from our papers [26; 15] that if Y is a rational homology three-sphere, then $\widehat{HF}(Y, \mathfrak{s})$ is a \mathbb{Q} -graded group. Thus, the Heegaard Floer homology of an L -space is determined by the “correction term” function

$$d: \text{Spin}^c(Y) \rightarrow \mathbb{Q}$$

which associates to each $\mathfrak{s} \in \text{Spin}^c(Y)$ the degree in which $\widehat{HF}(Y, \mathfrak{s})$ is supported; compare also Frøyshov [6].

The set of L -spaces includes all lens spaces and indeed all three-manifolds with elliptic geometry (cf our paper [24]); for more examples; see also Némethi [13]. Another interesting family is given by the branched double-covers of alternating knots in S^3 ; cf our paper [25].

Let $K \subset S^3$ be a knot in the three-sphere. Write its symmetrized Alexander polynomial as

$$\Delta_K(T) = a_0 + \sum_{i>0} a_i(T^i + T^{-i}),$$

and let

$$(1) \quad t_i(K) = \sum_{j=1}^{\infty} j a_{|i|+j}.$$

Note that for any knot C in S^3 , there is a canonical affine map $\mathbb{Z}/p\mathbb{Z} \cong \text{Spin}^c(S^3_{p/q}(C))$.

Theorem 1.2 *Let $K \subset S^3$ be a knot which admits an L -space surgery, for some $r = p/q \in \mathbb{Q}$ with $r \geq 0$. Then, for all integers i with $|i| \leq p/2$ we have that*

$$(2) \quad d(S^3_{p/q}(K), i) - d(S^3_{p/q}(O), i) = -2t_{\lfloor |i|/q \rfloor}(K),$$

while for all $|j| > p/(2q)$, we have that $t_j(K) = 0$.

In the case where $q = 1$, a version of the above theorem is established in [15, Theorem 7.2]. Theorem 1.2 (in the case where $q = 2$) also gives the symmetry used in [23] to find an obstruction to a knot having unknotting number equal to one.

The following is a quick consequence of this result, together with our result that knot Floer homology distinguishes the unknot (cf [18]) (though alternative proofs could be given which model the proof by Kronheimer, Mrowka and the authors [10] more closely):

Corollary 1.3 *If $S^3_{p/q}(K) \cong S^3_{p/q}(O)$ as oriented three-manifolds, then $K = O$.*

The above result, which was conjectured by Gordon in [8], was first established using Seiberg–Witten monopoles by Kronheimer, Mrowka and the authors [10]. Thanks to a theorem of Eliashberg [4] and Etnyre [5], it is now possible to prove results of this type purely in the context of Heegaard Floer homology; see also our work [18].

Theorem 1.2, together with results from [18], has the following corollary:

Corollary 1.4 *If K is a knot with the property that $S_r^3(K)$ is an L -space for some rational number r , then $|r| \geq 2g - 1$ where g is the Seifert genus of K .*

1.2 On cosmetic surgeries

Let Y be a closed, oriented three-manifold, and $K \subset Y$ be a framed knot. Given a rational number r , let $Y_r(K)$ denote the three-manifold obtained by Dehn surgery along K with slope r (with respect to the initial framing). If there are two distinct rational numbers r and s with the property that $Y_r(K)$ and $Y_s(K)$ are homeomorphic (but the homeomorphism is not required to preserve the orientation inherited from Y), then the surgeries are called *cosmetic*. A pair of surgeries on K with $r \neq s$ is called *truly cosmetic* if $Y_r(K) \cong Y_s(K)$ as oriented manifolds.

Amphicheiral knots have cosmetic surgeries; specifically, if K is an amphicheiral knot, then $S_r^3(K) \cong -S_{-r}^3(K)$. The unknot O admits infinitely many truly cosmetic surgeries: $S_{p/q}^3(O) = S_{p/p+q}^3(O)$. Lackenby [11] has shown that under general conditions on a knot $K \subset Y$, there are at most finitely many cosmetic surgeries; see also Bleiler, Hodgson and Weeks [2]. It is conjectured [2] that if $Y_r(K) \cong Y_s(K)$, then $Y - \text{nd}(K)$ admits an automorphism which carries the slope r to the slope s .

The present state of Heegaard Floer homology – and specifically the surgery formulas given here – works best for excluding cosmetic surgeries on knots in S^3 . For example, we have the following result:

Theorem 1.5 *If K is a knot with Seifert genus equal to one, and $S_r^3(K) \cong S_s^3(K)$ with $r \neq s$, then $S_r^3(K)$ is an L -space.*

The conclusion of the above theorem places severe restrictions on K . In particular, according to results of [24], it follows that K must have the same knot Floer homology (and in particular the same Alexander polynomial) as the trefoil knot T , and thus according to **Theorem 1.1**, $S_r^3(K)$ and $S_r^3(T)$ have the same (graded) Floer homology groups. Indeed, this result, combined with other methods, can rule out genus one cosmetic surgeries, as in Wang [32].

For particular integers p , the existence of a truly cosmetic surgery on such a knot K with specified numerator p can be ruled out by an explicit, finite search.

Theorem 1.6 *Let $K \subset S^3$ and suppose that $S_r^3(K) \cong \pm S_s^3(K)$. Then either $S_r^3(K)$ is an L -space or r and s have opposite signs.*

For the above theorem, both possible conclusions can hold. The simplest example is the unknot which admits cosmetic surgeries with positive slopes. A more interesting example of cosmetic surgeries with positive slopes is provided by the trefoil knot K , which has the property that $S_9^3(K) \cong -S_{9/2}^3(K)$; cf Mathieu [12]. Examples where r and s have opposite signs are given by amphicheiral knots.

Our methods can be refined to exclude cosmetic surgeries for certain numerators p . We study here the case where $p = 3$.

Theorem 1.7 *Suppose that $K \subset S^3$ is a knot with the property that $S_{p/q}^3(K) \cong S_{p/q'}^3(K)$ as oriented manifolds. In the case where $p = 3$, we can conclude that $q = q'$.*

1.3 Heegaard Floer homology of Seifert fibered spaces

We give some other applications of the general surgeries description along a rationally null-homologous knot. In Section 10, we use it to describe the Heegaard Floer homology with $\mathbb{Z}/2\mathbb{Z}$ coefficients of any Seifert fibered space whose first Betti number is even.

1.4 Organization

This paper is organized as follows. In Section 2, we review some of the topological preliminaries required by the knot filtrations, including the notion of *relative Spin^c structures* for three-manifolds with torus boundary. In Section 3, we give the construction of the knot filtration. In the next two sections, we turn to some properties of the knot Floer homology which are rather straightforward adaptations of the corresponding results for null-homologous knots [19; 28]: the relationship between knot Floer homology and “large” surgeries on a rationally null-homologous knot (Section 4), and the Künneth principle for connected sums of knots (Section 5). The first result is an ingredient in the Morse surgery formula from Section 6. In Section 7, we show how the Künneth principle, together with the Morse surgery formula, give the rational surgery formula described in this introduction. In Section 8, we turn to knots which admit L -space surgeries. In Section 9, we give the applications of the rational surgery formula to the problem of cosmetic surgeries on a knot in S^3 . In Section 10, we turn to the Heegaard Floer homology groups of Seifert fibered spaces with even first Betti number.

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2 Preliminaries

We recall some of the background material and notation which will be required for the construction of the knot filtration. The bulk of this material is about the constructions relating (doubly-pointed) Heegaard diagrams for knots, and relative Spin^c structures for three-manifolds with boundary. We include also a few key properties of Heegaard Floer homology which will be used repeatedly throughout; see also [21].

2.1 Surgeries

Let $K \subset Y$ be a knot. The boundary of a tubular neighborhood of the knot K is a torus T equipped with a canonical choice of (isotopy class of) embedded curve μ , a *meridian* for K . A *longitude* for K is any embedded curve λ in T which meets a meridian transversally in a single point.

Given a homologically nontrivial, embedded curve γ in T , we can form the three-manifold $Y_\gamma(K)$ which is gotten by attaching a solid torus to $Y - \text{nd}(K)$ so that γ bounds a disk in the attached solid torus. We say that $Y_\gamma(K)$ is obtained from Y by *Dehn surgery along K* . In the special case where γ is a longitude for K , there is a canonical two-handle cobordism from Y to $Y_\gamma(K)$, and we say that $Y_\gamma(K)$ is obtained from Y by *Morse surgery on K* . A choice of longitude for K is also called a *framing* of K .

Let $K \subset Y$ be a rationally null-homologous knot in a closed, oriented three-manifold, equipped with a framing λ . Since $K \in Y$ has finite order, there is a pair of integers n and d with minimal absolute value which satisfy the property that

$$(3) \quad d \cdot \lambda = n \cdot \mu \in H_1(Y - K; \mathbb{Z}).$$

In particular, it follows that the induced homology class $[K_\lambda] \in H_1(Y; \mathbb{Z})$ has order $|d|$. Here, and in the future, K_λ denotes a copy of K displaced into $Y - \text{nd}(K)$ using the framing λ .

2.2 Relative Spin^c structures for three-manifolds with boundary

Following Turaev [31], we say that two nowhere-vanishing vector fields v_1 and v_2 on a closed, oriented three-manifold are *homologous* if they are homotopic in the complement of a ball in Y . The set of homology classes of nowhere-vanishing vector fields can be given the structure of an affine space for $H^2(Y; \mathbb{Z})$, and indeed, it can be identified with the space of Spin^c structures over Y , $\text{Spin}^c(Y)$.

This construction can be readily generalized to the case of a three-manifold with torus boundary; cf Chapter I.4 of [31]. Specifically, on an oriented three-manifold M with torus boundary, two vector fields v_1 and v_2 on M which vanish nowhere and point outwards at ∂M are said to be *homologous* if they are homotopic in the complement of a ball in the interior of M , and throughout the vector fields point outwards along the homotopy. The set of homology classes of nowhere vanishing vector fields can be naturally given the structure of an affine space for the relative cohomology $H^2(M, \partial M; \mathbb{Z})$. The homology classes of nowhere vanishing vector fields are called *relative Spin^c structures on M* , and are denoted $\underline{\text{Spin}}^c(M, \partial M)$.

Let $K \subset Y$ be a knot in a closed, oriented three-manifold, we can construct the three-manifold with torus boundary $M = Y - \text{nd}(K)$. In this case, we denote the relative Spin^c structures over M by $\underline{\text{Spin}}^c(Y, K)$.

Orienting the core of the solid torus, we obtain a canonical isotopy class of nowhere vanishing vector field which points inward at the boundary, and which has a closed orbit which is the core of the solid torus (with its given orientation). More explicitly, the isotopy class of this vector field on $D^2 \times S^1$ is characterized by the property that in the interior of the solid torus, w is everywhere transverse to the tangent planes to D^2 .

Giving K an orientation (and denoting this oriented knot by \bar{K}), we glue in this vector field w to obtain a natural map

$$G_{Y, \bar{K}}: \underline{\text{Spin}}^c(Y, K) \longrightarrow \text{Spin}^c(Y)$$

which is equivariant with respect to the action by $H^2(Y, K; \mathbb{Z})$; ie letting

$$\iota: H^2(Y, K; \mathbb{Z}) \longrightarrow H^2(Y; \mathbb{Z})$$

be the natural map, we have for each $k \in H^2(Y, K; \mathbb{Z})$,

$$G_{Y, \bar{K}}(\xi + k) = G_{Y, \bar{K}}(\xi) + \iota(k).$$

As the notation suggests, $G_{Y, \bar{K}}$ depends on the orientation for K ; indeed, for any $\xi \in \underline{\text{Spin}}^c(Y, K)$,

$$(4) \quad G_{Y, \bar{K}}(\xi) = G_{Y, -\bar{K}}(\xi) - \text{PD}[\bar{K}].$$

Note that if Y is a three-manifold, and $K \subset Y$ is a knot with meridian μ , then the fibers of the map $G_{Y, \bar{K}}$ are the orbits of $\text{Spin}^c(Y, K)$ under the action by $\mathbb{Z} \cdot \text{PD}[\mu]$, where we think of $\text{PD}[\mu] \in H^2(Y, K; \mathbb{Z})$; ie $G_{Y, \bar{K}}$ realizes an identification

$$\text{Spin}^c(Y) \cong \frac{\text{Spin}^c(Y, K)}{\mathbb{Z} \cdot \text{PD}[\mu]}.$$

2.3 Doubly-pointed Heegaard diagrams

A *doubly-pointed Heegaard diagram* is a collection of data $(\Sigma, \alpha, \beta, w, z)$ where Σ is an oriented surface of genus g , $\alpha = \{\alpha_1, \dots, \alpha_g\}$ is a g -tuple of homologically linearly independent, pairwise disjoint, embedded curves in Σ (a g -tuple of attaching circles), and $\beta = \{\beta_1, \dots, \beta_g\}$ is another g -tuple of attaching circles, and w and z are two distinct points in $\Sigma - \alpha_1 - \dots - \alpha_g - \beta_1 - \dots - \beta_g$.

A doubly-pointed Heegaard diagram gives rise to an oriented three-manifold Y together with an oriented knot $\bar{K} \subset Y$. The orientation on Σ induces an orientation on U_α (so that $\partial U_\alpha = \Sigma$), which can then be uniquely extended to an orientation over Y . The knot K is obtained as a union of two arcs, η_α and η_β . The arc η_α is gotten by connecting w and z in $\Sigma - \alpha_1 - \dots - \alpha_g$, and orienting it as a path from w to z . This arc is then pushed into U_α so that only its boundary meets Σ (at w and z). The arc η_β is obtained in an analogous manner, only reversing the roles of the circles in α and β . The oriented knot \bar{K} is gotten by the difference $\eta_\alpha - \eta_\beta$.

2.4 Relative Spin^c structures associated to intersection points

Let $(\Sigma, \alpha, \beta, w, z)$ be a doubly-pointed Heegaard diagram. Fix $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. There are paths

$$a: [0, 1] \longrightarrow \mathbb{T}_\alpha, b: [0, 1] \longrightarrow \mathbb{T}_\beta$$

with $\partial a = \partial b = \mathbf{x} - \mathbf{y}$. These paths can be viewed as arcs in Σ (supported inside the $\alpha \cup \beta$). The difference $a - b$ is a closed one-cycle in Σ which is disjoint from w and z . Indeed, since $\Sigma - w - z$ is a subset of $Y - K$, this one-cycle can be viewed as an element $\underline{\epsilon}(\mathbf{x}, \mathbf{y}) \in H_1(Y - K; \mathbb{Z})$.

We construct a map

$$\underline{\epsilon}_{w,z}: \mathbb{T}_\alpha \cap \mathbb{T}_\beta \longrightarrow \underline{\text{Spin}}^c(Y, K),$$

as follows (compare the analogous construction from [21]). A Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$ can be realized by a self-indexing Morse function $f: Y \longrightarrow [0, 3]$, with a single index zero and three critical point (and g index one and two critical points), together with a Riemannian metric g , for which Σ is the mid-level $f^{-1}(3/2)$,

α_i is the locus of points flowing out of the i -th index one critical point (via gradient flow), and β_j is the locus of points flowing into the j -th index two critical point. Thus, $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ can be thought of as a g -tuple of gradient flow-lines $\gamma_{\mathbf{x}}$ containing all the index one and two critical points.

Now, the knot K is realized as a union of two flow-lines γ_w and γ_z which connect the index zero and index three critical points, meeting Σ in the points w and z respectively. The oriented knot \bar{K} is gotten by $\gamma_z - \gamma_w$.

We construct a vector field representing the relative Spin^c structure $\mathfrak{s}_{w,z}(\xi)$ as follows. Modify the gradient vector field $\vec{\nabla}f$ in a neighborhood of the flows $\gamma_{\mathbf{x}}$ so that it has no zeros at any of the index one or two critical points. This modification involves a choice of nowhere vanishing vector field in a regular neighborhood of $\gamma_{\mathbf{x}}$, but it will follow easily from the construction that this choice will not affect the relative Spin^c structure of the induced vector field. Next, we modify the vector field in a neighborhood of γ_w to obtain a new vector field v which has no zeros at either the index zero or three-critical points. In fact, this can be achieved so that the knot K is a closed orbit of the resulting vector field, whose orientation as a flow-line agrees with the orientation induced from \bar{K} . This v modification involves a choice X of nowhere vanishing vector field on the neighborhood of γ_w , with fixed behaviour on K . When calling attention to this choice, we write $v = v(X)$. The resulting vector field v over Y is the vector field representing the Spin^c structure $\mathfrak{s}_w(\mathbf{x}) \in \text{Spin}^c(Y)$ associated to the intersection point \mathbf{x} and reference point w , as described in [21] (and this fact is independent of the choice of X).

Our representative v has been constructed so that there is a neighborhood $D_2 \times S^1$ of the closed flow-line $\{0\} \times S^1 \cong \bar{K}$ (where here D_2 is a disk of radius two centered at the origin) with the property that the disks $D_2 \times \theta$ for any $\theta \in S^1$ are transverse to the vector field v . Consider a concentric disk $D_1 \subset D_2$. We can continuously extend $v_2 = v|_{Y - D_2 \times S^1}$ to a new vector field $v_1 = v_1(\mathbf{x}, X)$ over $Y - D_1 \times S^1$, by using a vector field over $(D_2 - D_1) \times S^1$ which is everywhere transverse to the annuli $(D_2 - D_1) \times \theta$, and which point towards the origin at $\partial D_1 \times S^1$. Thus, the vector field v_1 over $Y - (D_1 \times S^1)$ inherits the vector field v' which points outwards at the boundary. It is easy to see that the isotopy class of v_1 is uniquely determined by the isotopy class of our initial vector field v .

The induced relative Spin^c structure v_1 over $Y - \text{nd}(K) = Y - (D_1 \times S^1)$ depends, of course, on w, z, \mathbf{x} , and our choice X , and we write it correspondingly as $v_1(w, z, \mathbf{x}, X)$. It is easy to see that

$$G_{Y, \bar{K}}(v_1(w, z, \mathbf{x}, X)) = \mathfrak{s}_w(\mathbf{x}).$$

It is easy to see that the induced relative Spin^c structures over $Y - \text{nd}(K)$ depends on the choice of X by

$$v_1(w, z, \mathbf{x}, X) - v_1(w, z, \mathbf{x}, X') = a \cdot \text{PD}[\mu],$$

where here $a \in \mathbb{Z}$ is a universal constant (depending on only X and X' ; in fact, it is even independent of the ambient three-manifold Y). We choose X now to satisfy a normalization condition as follows.

Consider the unknot $K \subset S^3$. An orientation \bar{K} specifies a canonical relative Spin^c structure $\underline{s}_0 \in \underline{\text{Spin}}^c(S^3, \bar{K})$. This is the relative Spin^c structure represented by a vector field v over $S^3 - \text{nd}(\bar{K}) \cong D^2 \times S^1$ which is everywhere transverse to the disks D^2 . The direction is specified by the condition that v can be represented by a vector field with closed orbits which have linking number one with our original knot \bar{K} . Our condition on X now is that for the standard genus one doubly-pointed diagram for the oriented unknot with a single intersection point \mathbf{x} , the relative Spin^c structure induced by $v_1(w, z, \mathbf{x}, X)$ is the canonical Spin^c structure for the oriented unknot.

With this choice for X , it is easy now to see that the relative Spin^c structure underlying $v_1(w, z, \mathbf{x}, X)$ depends only on w, z , and \mathbf{x} , inducing the required map $\underline{s}_{w,z}: \mathbb{T}_\alpha \cap \mathbb{T}_\beta \rightarrow \underline{\text{Spin}}^c(Y, K)$.

We investigate its dependence on w, z , and \mathbf{x} in the following lemma. Continuing notation from [21], letting $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, let $\pi_2(\mathbf{x}, \mathbf{y})$ denote the space of homotopy classes of Whitney disks connecting \mathbf{x} and \mathbf{y} , and for fixed

$$p \in \Sigma - \alpha_1 - \cdots - \alpha_g - \beta_1 - \cdots - \beta_g$$

and $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, let $n_p(\phi)$ denote the intersection number of ϕ with the submanifold $\{p\} \times \text{Sym}^{g-1}(\Sigma) \subset \text{Sym}^g(\Sigma)$.

Lemma 2.1 *Given $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, we have that*

$$(5) \quad \underline{s}_{w,z}(\mathbf{y}) - \underline{s}_{w,z}(\mathbf{x}) = \text{PD}[\underline{\epsilon}(\mathbf{x}, \mathbf{y})].$$

In particular, if there is some $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, then

$$(6) \quad \underline{s}_{w,z}(\mathbf{x}) - \underline{s}_{w,z}(\mathbf{y}) = (n_z(\phi) - n_w(\phi)) \cdot \text{PD}[\mu].$$

Proof We begin by establishing Equation (5). Vector fields representing $\underline{s}_{w,z}(\mathbf{x})$ and $\underline{s}_{w,z}(\mathbf{y})$ can be chosen so that they agree everywhere except in a regular neighborhood of $\underline{\epsilon}(\mathbf{x}, \mathbf{y})$. It follows that $\underline{s}_{w,z}(\mathbf{x})$ and $\underline{s}_{w,z}(\mathbf{y})$ differ by some multiple of the Poincaré dual of this curve. The fact that this multiple is one follows from the analogous property of $\mathfrak{s}_w(\mathbf{x})$ established in Lemma 2.19 of [21].

We turn to Equation (6). The homotopy class $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ gives rise to a null-homotopy of $\underline{\epsilon}(\mathbf{x}, \mathbf{y})$ inside Y . This null-homology meets the knot K with intersection number $n_z(\phi) - n_w(\phi)$. Thus, it can be modified to give a homology of $\underline{\epsilon}(\mathbf{x}, \mathbf{y})$ with $(n_z(\phi) - n_w(\phi)) \cdot \text{PD}[\mu]$ in $Y - K$. Equation (6) now follows from Equation (5). \square

2.5 Heegaard triples and relative Spin^c structures

Suppose $K \subset Y$ is a knot with framing λ . We can form a four-manifold $W_\lambda(K)$, which is obtained by attaching a two-handle to $[0, 1] \times Y$ along K (thought of as a subset of $\{1\} \times Y$) with framing λ .

A *Heegaard triple* is a closed, oriented two-manifold Σ , equipped with three g -tuples of attaching circles, $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$; cf [21]. This gives rise to a four-manifold $X_{\alpha\beta\gamma}$ which has three boundary components $-Y_{\alpha\beta}$, $Y_{\beta\gamma}$, and $Y_{\alpha\gamma}$. (Of course, $Y_{\alpha\beta}$ here denotes the three-manifold described by the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$; $Y_{\alpha\gamma}$ and $Y_{\beta\gamma}$ are defined analogously.)

Given a knot $K \subset Y$ with framing λ , we can construct a Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$, where $Y_{\alpha\beta}$ represents Y , $Y_{\alpha\gamma}$ represents framed surgery along K , and $Y_{\beta\gamma}$ is a connected sum of $g - 1$ copies of $S^2 \times S^1$. Filling in $Y_{\beta\gamma}$ by a boundary connected sum of $g - 1$ copies of $B^3 \times S^1$, we obtain a four-manifold which is diffeomorphic to $W_\lambda(K)$.

The Heegaard triple is constructed by first writing down a Heegaard diagram for Y with the property that K is supported entirely inside U_β , as the core of one of the handles in the handlebody, and β_g represents its meridian. Next, realize the framing λ as a curve γ_g which is disjoint from the β_i for $i = 1, \dots, g - 1$. We then let γ_i for $i = 1, \dots, g - 1$ be small, isotopic translates of the corresponding β_i . We would like to choose convenient reference points. These are points w and z are chosen so that there is an arc from z to w which crosses none of the α_i or γ_i for $i = 1, \dots, g$, or β_j for $j = 1, \dots, g - 1$, so that an arc from z to w crosses β_g transversally once. An ordering on w and z is specified by an orientation on K . We call such a doubly-pointed Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, w, z)$ to be a *doubly-pointed Heegaard triple subordinate to the framed, oriented knot $\bar{K} \subset Y$ with framing λ* .

By constructing the diagram carefully, we can arrange for there to be a unique intersection point $\Theta \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$ representing the generator Θ of the top-most nontrivial Floer homology group of $\widehat{HF}(\#^{g-1}(S^2 \times S^1))$. We denote this intersection point, and its corresponding homology class, by Θ .

As in [21], we let $\pi_2(\mathbf{x}, \Theta, \mathbf{y})$ denote the space of homotopy classes of Whitney triangles, ie continuous maps of the triangle into $\text{Sym}^g(\Sigma)$ which map the three edges

to \mathbb{T}_α , \mathbb{T}_β , and \mathbb{T}_γ , and the vertices to the given points. In Section 8 of [21], it is shown that homotopy classes of triangles give rise to Spin^c structures over $X_{\alpha\beta\gamma}$.

Proposition 2.2 *Let $(\Sigma, \alpha, \beta, \gamma, w, z)$ be a doubly-pointed Heegaard triple subordinate to an oriented knot $\bar{K} \subset Y$, equipped with framing λ . The map*

$$\psi \in \pi_2(\mathbf{x}, \Theta, \mathbf{y}) \mapsto \underline{s}_{w,z}(\mathbf{x}) + (n_w(\psi) - n_z(\psi))\mu$$

descends to give a well-defined map

$$E_{Y,\lambda,\bar{K}}: \text{Spin}^c(W_\lambda(K)) \longrightarrow \underline{\text{Spin}}^c(Y, \bar{K})$$

(ie independent of the choice of Heegaard triple). Moreover, letting $\hat{F} \in H_2(W_\lambda(K), Y) \cong \mathbb{Z}$ denote a generator, we have that

$$(7) \quad E_{Y,\lambda,\bar{K}}(\mathfrak{s} + \text{PD}[\hat{F}]) = E_{Y,\lambda,\bar{K}}(\mathfrak{s}) + \text{PD}[K_\lambda].$$

Proof Recall [21] that two triangles $\psi \in \pi_2(\mathbf{x}, \Theta, \mathbf{y})$ and $\psi' \in \pi_2(\mathbf{x}', \Theta, \mathbf{y}')$ induce the same Spin^c structure on $W_\lambda(K)$ if and only if there are $\phi_1 \in \pi_2(\mathbf{x}, \mathbf{x}')$ (relative to the subspaces \mathbb{T}_α and \mathbb{T}_β) and $\phi_2 \in \pi_2(\mathbf{y}', \mathbf{y})$ (relative to the subspaces \mathbb{T}_α and \mathbb{T}_γ) with the property that

$$\mathcal{D}(\psi) - \mathcal{D}(\psi') - \mathcal{D}(\phi_1) - \mathcal{D}(\phi_2) = n \cdot [\Sigma]$$

for some integer n . (Here, as in [21], $\mathcal{D}(\psi)$ denotes the two-chain in Σ induced by the homotopy class ψ , whose multiplicity at some point p is the intersection number of ψ with $p \times \text{Sym}^{g-1}(\Sigma)$.) But now $n_w(\phi_2) = n_z(\phi_2)$ (since w and z lie in the same component of $\Sigma - \alpha_1 - \dots - \alpha_g - \gamma_1 - \dots - \gamma_g$); combining this with the fact that

$$\underline{s}_{w,z}(\mathbf{x}') = \underline{s}_{w,z}(\mathbf{x}) + (n_w(\phi_1) - n_z(\phi_1)) \cdot \mu$$

(Lemma 2.1), it follows that in this case

$$\underline{s}_{w,z}(\mathbf{x}) + (n_w(\psi) - n_z(\psi))\mu = \underline{s}_{w,z}(\mathbf{x}') + (n_w(\psi') - n_z(\psi'))\mu.$$

Independence of the Heegaard triple is a routine consequence that any two Heegaard triples can be connected by a sequence of stabilizations and handleslides among the α_i, β_j for $j = 1, \dots, g - 1$, and the distinguished meridian β_g handlesliding over the other β_j .

We now verify Equation (7). Suppose that we have $\psi \in \pi_2(\mathbf{x}, \Theta, \mathbf{y})$ and $\psi' \in \pi_2(\mathbf{x}', \Theta, \mathbf{y}')$. The difference $\mathcal{D}(\psi) - \mathcal{D}(\psi')$ gives a two-chain C in Σ , which has no corners on γ_i for $i = 1, \dots, g$. The corners of C occur at $x'_i \in \mathbf{x}'$ and $x_i \in \mathbf{x}$. Indeed,

the boundary of the two-chain consists of $\underline{\epsilon}(\mathbf{x}, \mathbf{x}')$ and some number of copies of the γ_i . As such, it can be thought of as inducing a relation

$$\underline{\epsilon}(\mathbf{x}, \mathbf{x}') = \ell \cdot \gamma_g + (n_w(\psi - \psi') - n_z(\psi - \psi'))\mu$$

in the first homology of the knot complement (we are thinking of γ_g as a curve in the Heegaard surface; of course, it is identified with K_λ thought of as a curve in the knot complement). Indeed, $\psi - \psi'$ corresponds to a surface-with-boundary Z in the four-manifold underlying the Heegaard triple, whose boundary is supported entirely Y . The integer ℓ corresponds to the multiplicity with which Z meets the cocore of the attaching two-handle. Thus, $\mathfrak{s}_w(\psi) - \mathfrak{s}_w(\psi') = \ell \cdot \hat{F}$, and the equation follows. \square

For a pointed Heegaard triple $(\Sigma, \alpha, \beta, \gamma, z)$, the group of two-dimensional homology classes is identified with the group of relations $a + b + c = 0$ in $H_1(\Sigma; \mathbb{Z})$ where a resp. b resp. c lies in the span of $\{\alpha_i\}_{i=1}^g$ resp. $\{\beta_i\}_{i=1}^g$ resp. $\{\gamma_i\}_{i=1}^g$. These relations correspond to two-chains \mathcal{P} in Σ with boundary a formal linear combination of the attaching circles, and with $n_z(\mathcal{P}) = 0$. Given a Whitney triangle $\psi \in \pi_2(\mathbf{x}, \mathbf{y}, \mathbf{w})$, there is a formula for the evaluation $\langle c_1(\mathfrak{s}_z(\psi)), H(\mathcal{P}) \rangle$, where $H(\mathcal{P})$ denotes the second homology class (in $H_2(X_{\alpha\beta\gamma}; \mathbb{Z})$) corresponding to \mathcal{P} , in terms of the Heegaard triple; cf Proposition 6.3 of [26]. To describe this, we need two notions, the *Euler measure* of the periodic domain and the *dual spider number* of the triangle with respect to the triply-periodic domain.

Let $\Phi: F \rightarrow \Sigma$ be a representative for \mathcal{P} , where here Φ is an immersion in a neighborhood of ∂F . The line bundle $\Phi^*(T\Sigma)$ has a canonical trivialization over ∂F , since Φ induces an isomorphism

$$TF \cong \Phi^*(T\Sigma),$$

and TF is canonically trivialized near ∂F (using the outward normal orientation on F). We define $\hat{\chi}(\mathcal{P})$ to be the Euler number of $\Phi^*(T\Sigma)$, relative to this trivialization at ∂F ,

$$\hat{\chi}(\mathcal{P}) = \langle c_1(\Phi^*(T\Sigma), \partial F), F \rangle.$$

Note first that the orientation of Σ , and the orientations on all the attaching circles α , β and γ naturally induce “inward” normal vector fields to the attaching circles (ie if $\gamma: S^1 \rightarrow \Sigma$ is a unit speed immersed curve, this inward normal vector is given by $J(d\gamma/dt)$). Let α'_i, β'_i and γ'_i denote copies of the corresponding attaching circles α'_i, β'_i and γ'_i , translated slightly in these normal directions. Let α', β' and γ' denote the corresponding g -tuples, and $\mathbb{T}'_\alpha, \mathbb{T}'_\beta$ and \mathbb{T}'_γ be the corresponding tori in $\text{Sym}^g(\Sigma)$. By construction, then, $u(e_\alpha)$ misses \mathbb{T}'_α , $u(e_\beta)$ misses \mathbb{T}'_β , and $u(e_\gamma)$ misses \mathbb{T}'_γ .

Let $x \in \Delta$ be a point in the interior, chosen in general position, so that the g -tuple $u(x)$ misses all of α' , β' and γ' . Choose three paths a , b and c from x to e_0 , e_1 and e_2 respectively. The central point x and the three paths a , b and c is called a *dual spider*. We can think of the paths a , b and c as one-chains in Σ . Recall that $\partial\mathcal{P}$ has three types of boundaries: the α , β and γ boundaries, which we denote $\partial_\alpha\mathcal{P}$, $\partial_\beta\mathcal{P}$ and $\partial_\gamma\mathcal{P}$. Let $\partial'_\alpha\mathcal{P}$, $\partial'_\beta\mathcal{P}$ and $\partial'_\gamma\mathcal{P}$ respectively denote the one-chains obtained by translating the corresponding boundary components using the induced normal vector fields. Let $u: \Delta \rightarrow \text{Sym}^g(\Sigma)$ be a representative for a homotopy class ψ of Whitney triangle. The dual spider number of u and \mathcal{P} is defined by

$$\sigma(u, \mathcal{P}) = n_{u(x)}(\mathcal{P}) + \#(a \cap \partial'_\alpha\mathcal{P}) + \#(b \cap \partial'_\beta\mathcal{P}) + \#(c \cap \partial'_\gamma\mathcal{P}).$$

According to Proposition 6.3 of [26],

$$(8) \quad \langle c_1(\mathfrak{s}_z(u)), H(\mathcal{P}) \rangle = \widehat{\chi}(\mathcal{P}) + \#(\partial\mathcal{P}) - 2n_z(\mathcal{P}) + 2\sigma(u, \mathcal{P}),$$

where here $\#(\partial\mathcal{P})$ represents the number of boundary components of \mathcal{P} , with multiplicity.

2.6 Filtered complexes

A $\mathbb{Z} \oplus \mathbb{Z}$ -filtered complex is a chain complex C whose underlying Abelian group decomposes as

$$C = \bigoplus_{(i,j) \in \mathbb{Z} \oplus \mathbb{Z}} C\{i, j\},$$

and whose boundary operator ∂ carries $C\{i_0, j_0\}$ into the subset

$$C\{i \leq i_0 \text{ and } j \leq j_0\} = \bigoplus_{\{i', j' \mid i' \leq i \text{ and } j' \leq j\}} C\{i', j'\} \subset C.$$

A map $f: C \rightarrow C'$ between filtered complexes is called a *filtered map* if it carries $C\{i_0, j_0\}$ into $C'\{i \leq i_0 \text{ and } j \leq j_0\}$. Two filtered chain maps

$$f_0, f_1: C \rightarrow C'$$

are called *filtered homotopic* if there is a filtered map $H: C \rightarrow C'$ with

$$f_0 - f_1 = \partial' \circ H + H \circ \partial.$$

Two filtered chain maps C and C' are called *filtered chain homotopy equivalent* if there are filtered chain maps $f: C \rightarrow C'$ and $g: C' \rightarrow C$ with the property that $f \circ g$ and $g \circ f$ are filtered homotopic to the corresponding identity maps.

A $\mathbb{Z} \oplus \mathbb{Z}$ -filtered $\mathbb{Z}[U]$ -complex C is a filtered chain complex equipped with an endomorphism $U: C \rightarrow C$ whose restriction to $C\{i, j\}$ maps to $C\{i - 1, j - 1\}$.

If C_1 and C_2 are $\mathbb{Z} \oplus \mathbb{Z}$ -filtered $\mathbb{Z}[U]$ -chain complexes, then we can form their tensor product $C_1 \otimes_{\mathbb{Z}[U]} C_2$. This can be given a $\mathbb{Z} \oplus \mathbb{Z}$ -filtration by

$$(C \otimes C')\{i, j\} = \frac{\bigoplus_{\{(i_1, j_1), (i_2, j_2) \mid (i_1, j_1) + (i_2, j_2) = (i, j)\}} C\{i_1, j_1\} \otimes C\{i_2, j_2\}}{U_1 \cdot \xi_1 \otimes \xi_2 \sim \xi_1 \otimes U_2 \cdot \xi_2}.$$

If C is a filtered complex, and $(a, b) \in \mathbb{Z} \oplus \mathbb{Z}$, let $C[(a, b)]$ denote the filtered complex whose underlying chain complex is isomorphic, but whose filtration is shifted by (a, b) ; ie

$$C[(a, b)]\{i, j\} = C\{a + i, b + j\}.$$

A relatively filtered map $f: C \rightarrow C'$ is a chain map which respects the filtration on C and the filtration $C'[(a, b)]$ for some $(a, b) \in \mathbb{Z} \oplus \mathbb{Z}$.

2.7 Absolute gradings

Heegaard Floer homology is natural under cobordisms. Indeed, if W is a smooth, connected, oriented cobordism with $\partial W = -Y_1 \cup Y_2$ which is equipped with a Spin^c structure \mathfrak{s} whose restriction $\mathfrak{t}_i = \mathfrak{s}|_{Y_i}$ for $i = 1, 2$ has torsion first Chern class, then there is an induced chain map

$$f_{W, \mathfrak{s}}^+ : CF^+(Y_1, \mathfrak{t}_1) \rightarrow CF^+(Y_2, \mathfrak{t}_2)$$

which is homogeneous of degree

$$(9) \quad \frac{c_1(\mathfrak{s})^2 - 2\chi(W) - 3\sigma(W)}{4}$$

(cf Theorem 7.1 of [26]).

Let \mathcal{T}^+ denote the module $\mathbb{Z}[U, U^{-1}]/U \cdot \mathbb{Z}[U]$. Recall [21] that $HF^+(S^3) \cong \mathcal{T}^+$. The \mathbb{Q} -grading on Floer homology is characterized by Equation (9), together with the normalization that $HF_d^+(S^3)$ is trivial for all $d < 0$, nontrivial in degree $d = 0$.

Following our usual notational conventions, we write $\mathcal{T}_{(d)}^+$ for the module \mathcal{T}^+ , thought of as a graded $\mathbb{Z}[U]$ -module, where multiplication by U lowers absolute degree by 2, and the nonzero homogeneous elements of lowest degree have degree d . In this notation, then, $HF^+(S^3) \cong \mathcal{T}_{(0)}^+$.

2.8 Approximating CF^+

Following [27], it is useful to have the following approximation to $CF^+(Y)$: fix an integer $\delta \geq 0$, let $CF^\delta(Y) \subset CF^+(Y)$ denote the subcomplex which is annihilated by multiplication by $U^{\delta+1}$, and let $HF^\delta(Y)$ denote the homology of $CF^\delta(Y)$. In particular, for $\delta = 0$, this construction gives $\widehat{HF}(Y)$. Note that for all $\delta \geq 0$, $HF^\delta(Y)$ is a three-manifold invariant.

3 Construction of the knot filtration

The aim of this section is to construct the knot filtration: a filtration of $CF^\infty(Y)$ induced by a rationally null-homologous knot $K \subset Y$. Most of this discussion is a straightforward generalization of the corresponding constructions for null-homologous knots $K \subset Y$ described in [19] and [28].

Let $K \subset Y$ be a rationally null-homologous knot, and let $(\Sigma, \alpha, \beta, w, z)$ be a corresponding doubly-pointed Heegaard diagram.

For fixed $\xi \in \underline{\text{Spin}}^c(Y, K)$, let $\mathfrak{T}(\xi) \subset (\mathbb{T}_\alpha \cap \mathbb{T}_\beta) \times \mathbb{Z} \times \mathbb{Z}$ be the subset of elements $[\mathbf{x}, i, j]$ satisfying

$$(10) \quad \underline{s}_{w,z}(\mathbf{x}) + (i - j) \cdot \text{PD}[\mu] = \xi.$$

According to Lemma 2.1, if $[\mathbf{x}, i, j] \in \mathfrak{T}(\xi)$ and $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, then

$$[\mathbf{y}, i - n_w(\phi), j - n_z(\phi)] \in \mathfrak{T}(\xi).$$

Let $CFK^\infty(\Sigma, \alpha, \beta, w, z, \xi)$ be the chain complex generated by $[\mathbf{x}, i, j] \in \mathfrak{T}(\xi)$, endowed with the differential

$$\partial^\infty[\mathbf{x}, i, j] = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \mid \mu(\phi) = 1\}} \# \left(\frac{\mathcal{M}(\phi)}{\mathbb{R}} \right) \cdot [\mathbf{y}, i - n_w(\phi), j - n_z(\phi)],$$

where as usual, $\mathcal{M}(\phi)$ is the moduli space of pseudo-holomorphic representatives of ϕ , and $\mu(\phi)$ denotes its expected dimension.

The map $\mathcal{F}: \mathfrak{T}(\xi) \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$

given by $\mathcal{F}[\mathbf{x}, i, j] = (i, j)$ induces a $\mathbb{Z} \oplus \mathbb{Z}$ filtration on $CFK^\infty(\Sigma, \alpha, \beta, w, z)$.

Theorem 3.1 *The filtered chain homotopy type of $CFK^\infty(\Sigma, \alpha, \beta, w, z, \xi)$ is an invariant of the underlying oriented knot $\bar{K} \subset Y$ and choice of relative Spin^c structure $\xi \in \underline{\text{Spin}}^c(Y, K)$.*

Proof This is a routine adaptation of the corresponding statement in Theorem 3.1 of [19]. □

We denote this complex by $CFK^\infty(Y, \bar{K}, \xi)$.

Proposition 3.2 Let $C_\xi = CFK^\infty(Y, \bar{K}, \xi)$. We have that

$$C_{\xi+\text{PD}[\mu]} = C_\xi[(0, -1)]$$

(where the right-hand-side denotes the shift of C_ξ by $(0, -1)$, as in Section 2.6). The short exact sequences of $\mathbb{Z}[U]$ -chain complexes

$$0 \longrightarrow C_\xi\{i < 0\} \longrightarrow C_\xi \longrightarrow C_\xi\{i \geq 0\} \longrightarrow 0,$$

$$0 \longrightarrow C_\xi\{i = 0\} \longrightarrow C_\xi\{i \geq 0\} \xrightarrow{U} C_\xi\{i \geq 0\} \longrightarrow 0$$

are isomorphic to the short exact sequences

$$0 \longrightarrow CF^-(Y, \mathfrak{s}) \longrightarrow CF^\infty(Y, \mathfrak{s}) \longrightarrow CF^+(Y, \mathfrak{s}) \longrightarrow 0,$$

$$0 \longrightarrow \widehat{CF}(Y, \mathfrak{s}) \longrightarrow CF^+(Y, \mathfrak{s}) \xrightarrow{U} CF^+(Y, \mathfrak{s}) \longrightarrow 0,$$

respectively, where here $\mathfrak{s} = G_{Y, \bar{K}}(\xi)$. Similarly, the short exact sequences of $\mathbb{Z}[U]$ -chain complexes

$$0 \longrightarrow C_\xi\{j < 0\} \longrightarrow C_\xi \longrightarrow C_\xi\{j \geq 0\} \longrightarrow 0,$$

$$0 \longrightarrow C_\xi\{j = 0\} \longrightarrow C_\xi\{j \geq 0\} \xrightarrow{U} C_\xi\{j \geq 0\} \longrightarrow 0$$

are isomorphic to the short exact sequences

$$0 \longrightarrow CF^-(Y, \mathfrak{s}') \longrightarrow CF^\infty(Y, \mathfrak{s}') \longrightarrow CF^+(Y, \mathfrak{s}') \longrightarrow 0,$$

$$0 \longrightarrow \widehat{CF}(Y, \mathfrak{s}') \longrightarrow CF^+(Y, \mathfrak{s}') \xrightarrow{U} CF^+(Y, \mathfrak{s}') \longrightarrow 0,$$

where here $\mathfrak{s}' = G_{Y, -\bar{K}}(\xi)$.

The set of relative Spin^c structures $G_{Y, \bar{K}}^{-1}(\mathfrak{s})$ inducing a fixed Spin^c structure over Y is a well-ordered set, under the rule that $\xi_1 \leq \xi_2$ if $\xi_2 = \xi_1 + j \cdot \text{PD}[\mu]$ for some $j \geq 0$.

This ordering on relative Spin^c structures gives rise to an ordering of the generators of $\widehat{CF}(Y, \mathfrak{s})$ by $\mathbf{x} \mapsto \mathfrak{s}_{w,z}(\mathbf{x})$. It is easy to see that this ordering gives rise to a filtration of the complex $\widehat{CF}(Y) = \bigoplus_{\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \mathbb{Z}\mathbf{x}$ endowed with the usual differential

$$\partial \mathbf{x} = \sum_{\{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \mid n_w(\phi) = 0, \mu(\phi) = 1\}} \# \left(\frac{\mathcal{M}(\phi)}{\mathbb{R}} \right) \cdot \mathbf{y}.$$

The homology of the associated graded object is the knot Floer homology

$$\widehat{HFK}(Y, \bar{K}) = \bigoplus_{\xi \in \underline{\text{Spin}}^c(Y, \bar{K})} \widehat{HFK}(Y, \bar{K}, \xi),$$

where $\widehat{HFK}(Y, \bar{K}, \xi)$ is generated by \mathbf{x} with $\mathfrak{s}_{w,z}(\mathbf{x}) = \xi$, and whose differential counts holomorphic disks with $n_w(\phi) = n_z(\phi) = 0$.

3.1 Relationship with knot Floer homology for null-homologous knots

If $K \subset Y$ is a null-homologous knot, then a choice of Seifert surface for Y gives an identification $\text{Spin}^c(Y, K) \cong \mathbb{Z} \oplus \text{Spin}^c(Y)$. Thus, with this additional choice, we identify the $\text{Spin}^c(Y, K)$ -filtration of $\widehat{CF}(Y, \mathfrak{s})$ with a \mathbb{Z} -filtration. This identification is used, for example, in [19], where the knot filtration is described as a filtration by \mathbb{Z} , rather than by relative Spin^c structures.

For example, if \bar{K} is a knot in S^3 and s is an integer, then $\widehat{HFK}(S^3, \bar{K}, \xi)$ in the present notation corresponds to $\widehat{HFK}(Y, \bar{K}, s)$ for $s \in \mathbb{Z}$ in the notation from the introduction or [19], where here s and ξ are related by $c_1(\xi) = 2s \text{PD}[\mu]$.

4 Knot Floer homology and large surgeries

We describe here the result of forming “large surgeries” on a rationally null-homologous, framed knot $K \subset Y$. This result generalizes a corresponding result in the null-homologous case [19; 28], and it is a special case of [Theorem 6.1](#)

Let $K \subset Y$ be a knot given with framing λ . We can form a three-manifold $Y_{m\mu+\lambda}(K)$ obtained by filling the curve $m \cdot \mu + \lambda$ in $Y - \text{nd}(K)$. Let

$$W'_m(K): Y_{m\mu+\lambda}(K) \longrightarrow Y$$

denote the two-handle cobordism obtained by turning around the two-handle cobordism from $-Y$ to $-Y_{m\cdot\mu+\lambda}(K)$. Note that

$$H_2(W'_m(K), Y; \mathbb{Z}) \cong \mathbb{Z},$$

and let $\widehat{F} \subset W'_m(K)$ be a surface-with-boundary representing a generator. Clearly, $\text{PD}[\widehat{F}]|_Y = \text{PD}[K] \in H^2(Y; \mathbb{Z})$. Note that for sufficiently large m , the self-intersection number of \widehat{F} is negative.

Let $\bar{K} \subset Y$ be an oriented knot in a three-manifold Y , whose induced homology class is trivial in rational homology. We fix also a framing λ of K . For a fixed relative Spin^c structure $\xi \in \text{Spin}^c(Y, K)$, let C_ξ be the $\mathbb{Z} \oplus \mathbb{Z}$ -filtered chain complex $CFK^\infty(Y, \bar{K}, \xi)$. There are two projection maps

$$(11) \quad C_\xi\{\max(i, j) \geq 0\} \longrightarrow C_\xi\{i \geq 0\} \quad \text{and} \quad C_\xi\{\max(i, j) \geq 0\} \longrightarrow C_\xi\{j \geq 0\}.$$

Denoting

$$(12) \quad A_\xi^+(Y, \bar{K}) = C_\xi\{\max(i, j) \geq 0\} \quad \text{and} \quad B_\xi^+(Y, \bar{K}) = CF^+(Y, G_{Y, \bar{K}}(\xi))$$

and using the identifications (from Proposition 3.2)

$$C_\xi \{i \geq 0\} \cong CF^+(Y, G_{Y, \bar{K}}(\xi)),$$

$$C_\xi \{j \geq 0\} \cong CF^+(Y, G_{Y, -\bar{K}}(\xi)) \cong CF^+(Y, G_{Y, \bar{K}}(\xi + \text{PD}[K_\lambda])),$$

where here K_λ is the push-off of K inside $Y - K$ using the framing λ , we can view the canonical projection maps of Equation (11) as maps

$$(13) \quad v_\xi^+ : A_\xi^+(Y, \bar{K}) \longrightarrow B_\xi^+(Y, \bar{K}) \quad \text{and} \quad h_\xi^+ : A_\xi^+(Y, \bar{K}) \longrightarrow B_{\xi + \text{PD}[K_\lambda]}^+(Y, \bar{K}).$$

Theorem 4.1 *Let $K \subset Y$ be a rationally null-homologous knot in a closed, oriented three-manifold, equipped with a framing λ . Then, for all sufficiently large m , there is a map*

$$\Xi : \text{Spin}^c(Y_{m, \mu + \lambda}(K)) \longrightarrow \underline{\text{Spin}}^c(Y, K)$$

with the property that for all $\mathfrak{t} \in \text{Spin}^c(Y_{m, \mu + \lambda}(K))$, the group $CF^+(Y_{m, \mu + \lambda}(K), \mathfrak{t})$ is represented by the chain complex

$$A_{\Xi(\mathfrak{t})}^+ = C_{\Xi(\mathfrak{t})} \{\max(i, j) \geq 0\},$$

in the sense that there are isomorphisms (of relatively \mathbb{Z} -graded $\mathbb{Z}[U]$ -complexes)

$$\Psi_{\mathfrak{t}, m}^+ : CF^+(Y_{m, \mu + \lambda}(K), \mathfrak{t}) \longrightarrow A_{\Xi(\mathfrak{t})}^+(Y, \bar{K}).$$

Furthermore, fix $\mathfrak{t} \in \text{Spin}^c(Y_{m, \mu + \lambda}(K))$, and let $\xi = \Xi(\mathfrak{t})$. There are Spin^c structures $\mathfrak{x}, \eta \in \text{Spin}^c(W'_m(K))$ with $E_{Y, m, \mu + \lambda, \bar{K}}(\mathfrak{x}) = \xi$, and $\eta = \mathfrak{x} + \text{PD}[\hat{F}]$ with the property that the maps v_ξ^+ and h_ξ^+ correspond to the maps induced by the cobordism $W'_m(K)$ equipped with \mathfrak{x} and η respectively. More precisely, the following squares commute:

$$\begin{array}{ccc} CF^+(Y_{m, \mu + \lambda}(K), \mathfrak{t}) & \xrightarrow{f_{W'_m(K), \mathfrak{x}}^+} & CF^+(Y, G_{Y, \bar{K}}(\xi)) \\ \Psi_{\mathfrak{t}, m}^+ \downarrow & & \downarrow = \\ A_\xi^+(Y, \bar{K}) & \xrightarrow{v_\xi^+} & B_\xi^+(Y, \bar{K}) \\ CF^+(Y_{m, \mu + \lambda}(K), \mathfrak{t}) & \xrightarrow{f_{W'_m(K), \eta}^+} & CF^+(Y, G_{Y, -\bar{K}}(\xi)) \\ \Psi_{\mathfrak{t}, m}^+ \downarrow & & \downarrow = \\ A_\xi^+(Y, \bar{K}) & \xrightarrow{h_\xi^+} & B_{\xi + \text{PD}[\bar{K}]}^+(Y, \bar{K}) \end{array}$$

The following result is also an easy consequence of the proof:

Proposition 4.2 *Let $K \subset Y$ be a rationally null-homologous knot in a closed, oriented three-manifold, equipped with framing λ . For any $\delta > 0$, there is an integer N with the property that for all $m \geq N$ and all $\mathfrak{t} \in \text{Spin}^c(Y_{m \cdot \mu + \lambda}(K))$, there are at most two Spin^c structures in $\text{Spin}^c(W'_m(K))$ with restriction to $Y_{m \cdot \mu + \lambda}(K)$ equal to \mathfrak{t} for which the induced map*

$$F_{W'_m(K),s}^\delta: HF^\delta(Y_{m \cdot \mu + \lambda}(K), \mathfrak{t}) \longrightarrow HF^\delta(Y)$$

is nontrivial. These are the Spin^c structures \mathfrak{x} and \mathfrak{y} associated to \mathfrak{t} from Theorem 4.1 above.

We return to the proofs of Theorem 4.1 and Proposition 4.2 after some preliminary discussion and lemmas.

We work with a family of doubly-pointed Heegaard triples $(\Sigma, \alpha, \beta, \gamma, w, z)$ for the framed knot \bar{K} , so that there are identifications $Y_{\alpha\gamma} \cong Y_{m\mu + \lambda}(K)$, $Y_{\beta\gamma} \cong \#^{g-1}(S^2 \times S^1)$, $Y_{\alpha\beta} \cong Y$. We can give Heegaard triples for all of the W'_m which differ only in γ_g , which winds along the meridian μ . We call this region the “winding region” (cf Figure 1 for an illustration of a winding region; in this picture, the subscript for γ_g is dropped).

Definition 4.3 An intersection point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ is said to be *supported in the winding region* if its component along γ_g lies in the winding region. Given $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ (both supported in the winding region), we say that $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ is *supported in the winding region* if the segment of $\partial(\mathcal{D}(\phi))$ in γ_g is a subset of the winding region. (Note that if $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ is supported in the winding region, then so is any $\phi' \in \pi_2(\mathbf{x}, \mathbf{y})$.) An equivalence class of intersection point for $Y_{m \cdot \mu + \lambda}$ is said to be supported in the winding region if every intersection point in the equivalence class is supported in the winding region, and any two intersection points can be connected by Whitney disks supported in the winding region. Finally, $\psi \in \pi_2(\mathbf{x}, \Theta, \mathbf{y})$ is said to be a *small triangle* if the γ_g arc in $\partial\mathcal{D}(\psi)$ is supported in the winding region. Note that in this case, \mathbf{y} is the “closest point” in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ to \mathbf{x} .

Definition 4.4 The *depth* of a basepoint z in the winding region is the minimum absolute value of the algebraic intersection number of γ_g with any arc $A \subset \Sigma - \alpha_1 - \dots - \alpha_g$ which connects z with some basepoint z_0 outside the winding region with z . Given an integer $\epsilon > 0$, we say that z is ϵ -centered if its depth is less than or equal to $(m - \epsilon)/2$. Similarly, a choice of meridian μ is called ϵ -centered if each point $z \in \mu$ is ϵ -centered.

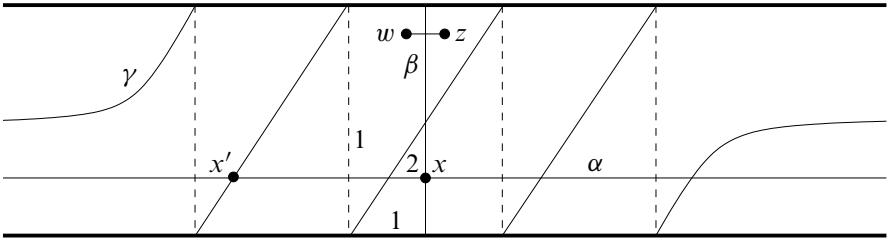


Figure 1: Illustration of the Heegaard triple. The integers denote (nonzero) local multiplicities of the “small triangle” connecting x and x' . This picture is taking place in a cylindrical region in Σ .

Fix a Heegaard diagram with an ϵ -centered meridian, with basepoints w and z on either side of μ . Then, for each small triangle ψ , we have that at least one of $n_w(\psi)$ or $n_z(\psi) = 0$, and also

$$(14) \quad \max(|n_w(\psi)|, |n_z(\psi)|) \leq \frac{m - \epsilon}{2}.$$

This is an immediate consequence of the definitions.

Lemma 4.5 *There is an integer $\epsilon > 0$ with the property that for all sufficiently large m , each $t \in \text{Spin}^c(Y_{m\mu+\lambda}(K))$ can be represented by an equivalence class of intersection points supported in the winding region and an ϵ -centered choice of meridian.*

Proof If an equivalence class of intersection points in $\mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ is not supported in the winding region, we say it is *bad*. It is clear that for all large m , the set of bad equivalence classes is bounded. We choose ϵ so that 2ϵ is greater than this number. Now move the basepoint as needed. \square

Given $\xi \in \text{Spin}^c(Y, K)$, define

$$\Psi_\xi^\infty: CF^\infty(Y_{m\cdot\mu+\lambda}(K), t) \longrightarrow CFK^\infty(Y, \bar{K}, \xi)$$

$$\text{by } \Psi_\xi^\infty[\mathbf{x}, i] = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\{\psi \in \pi_2(\mathbf{x}, \Theta, \mathbf{y}) \mid E_{W'_m(K)}(\mathfrak{s}_w(\psi)) = \xi\}} (\#\mathcal{M}(\psi)) \cdot [\mathbf{y}, i - n_w(\psi), i - n_z(\psi)],$$

where here \mathbf{x} represents $\mathfrak{s}_w(\psi)|_{Y_{m\mu+\lambda}(K)}$. We claim that this is a chain map.

Given $t \in \text{Spin}^c(Y_{m\mu+\lambda}(K))$, we realize t by an equivalence class of intersection points supported in the winding region. Define $\xi = \Xi(t)$ by

$$\xi = \mathfrak{s}_{w,z}(\mathbf{x}) + (n_w(\psi) - n_z(\psi)) \cdot \mu,$$

where $\psi \in \pi_2(\mathbf{x}', \Theta, \mathbf{x})$ is any small triangle connecting an intersection point $\mathbf{x}' \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ representing \mathfrak{t} with its “nearest point” $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. The restriction of Ψ_ξ^∞ to $CF^+(Y_{m\mu+\lambda}(K), \mathfrak{t})$ induces a chain map

$$\Psi_\xi^+ : CF^+(Y_{m\mu+\lambda}(K), \mathfrak{t}) \longrightarrow A_\xi^+(Y, \bar{K}).$$

Proof of Theorem 4.1 The argument from [19] shows that Ψ^+ induces an isomorphism. Postcomposing Ψ^+ with the projection $C_\xi\{\max(i, j) \geq 0\}$ to $C\{i \geq 0\}$ (ie v^+), we obtain the map induced by the cobordism $W'_m(K)$ equipped with the Spin^c structure \mathfrak{r} ; ie the first square in the statement of the theorem commutes. Commutativity of the second square follows similarly. \square

Lemma 4.6 *There is a constant c with the property that for all sufficiently large m , for any of the Spin^c structures $\{\mathfrak{r}(\mathfrak{t}), \mathfrak{r}(\mathfrak{t})\}_{\mathfrak{t} \in \text{Spin}^c(Y_{m\mu+\lambda}(K))}$ appearing in Theorem 4.1, we have that*

$$\begin{aligned} -c &\leq \langle c_1(\mathfrak{r}), [\hat{F}] \rangle \leq 2m + c, \\ -2m - c &\leq \langle c_1(\mathfrak{r}), [\hat{F}] \rangle \leq c. \end{aligned}$$

Proof There is a constant C (depending on $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, but independent of m) with the property that for any small triangle $\psi \in \pi_2(\mathbf{x}', \Theta, \mathbf{x})$,

$$(15) \quad \langle c_1(\mathfrak{s}_w(\psi)), [\hat{F}] \rangle + [\hat{F}] \cdot [\hat{F}] = C + 2(n_w(\psi) - n_z(\psi)).$$

This can be seen as follows. Consider the function

$$f(\mathbf{x}) = \langle c_1(\mathfrak{s}_w(\psi)), [\hat{F}] \rangle + [\hat{F}] \cdot [\hat{F}] - 2(n_w(\psi) - n_z(\psi)),$$

where here $\psi \in \pi_2(\mathbf{x}', \Theta, \mathbf{x})$ is a small triangle. We claim that this is independent of the choice of \mathbf{x}' . This can be seen by varying \mathbf{x}' , and appealing to Equation (8). Similarly, to verify its independence of m , it suffices to consider a fixed small triangle $\psi \in \pi_2(\mathbf{x}, \Theta, \mathbf{x}')$, and observe that $\langle c_1(\mathfrak{s}_w(\psi)), [\hat{F}] \rangle$ increases by one as the number m is increased by one, and furthermore $[\hat{F}] \cdot [\hat{F}]$ also decreases by one. Note here that by “fixed small triangle”, we mean that \mathbf{x} is fixed, as is $n_w(\psi) - n_z(\psi)$, though, of course, γ_g is varied. This assertion is an easy consequence of Equation (8). More specifically, with the integer d chosen as in Equation (3), we see that $d\hat{F}$ represents a generator of $H_2(W'_m(K); \mathbb{Z})$. Specifically, let \mathcal{P}_m be a generator for the space of triply-periodic domains, and write $\partial[\mathcal{P}_m] = A + B + C$, where A , B and C are first homology classes in the spans of α , β , and γ respectively. In view of Equation (3), $\partial\mathcal{P}_m$ has C component given by $d(m\mu + \lambda)$ and B component $-(dm + n)\mu$, modulo the other β_i for $i < g$. Thus, $\#(\partial\mathcal{P}_m) = dm + c$ for some constant c . It is easy to see that the other quantities in Equation (8) are independent of m .

Thus, we have seen that for any small triangle $\psi \in \pi_2(\mathbf{x}', \Theta, \mathbf{x})$,

$$\langle c_1(\mathfrak{s}_w(\psi)), [\widehat{F}] \rangle + [\widehat{F}] \cdot [\widehat{F}] - 2(n_w(\psi) - n_z(\psi))$$

depends only on $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Since there are only finitely many intersection points $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$, it follows that there is some constant C' with the property that

$$(16) \quad \left| \langle c_1(\mathfrak{s}_w(\psi)), [\widehat{F}] \rangle + [\widehat{F}] \cdot [\widehat{F}] - 2(n_w(\psi) - n_z(\psi)) \right| \leq C'.$$

By restricting to ϵ -centered base points for some integer ϵ independent of m (which we can do in light of Lemma 4.5), we can arrange for $2|n_w(\psi) - n_z(\psi)| \leq m - \epsilon$ for all small triangles (cf Equation (14)). The result now follows, bearing in mind that $\mathfrak{x} = s_w(\psi)$, and $\eta = \mathfrak{s}_z(\psi) = \mathfrak{x}(\psi) - \text{PD}[\widehat{F}]$. \square

Proof of Proposition 4.2 We choose m large enough that $W'_m(K)$ has negative-definite intersection form. Given $\mathfrak{t} \in \text{Spin}^c(Y)$, let $\mathfrak{S}(\mathfrak{t}) \in \text{Spin}^c(W'_m(K))$ denote the set of Spin^c structures whose restriction to Y is \mathfrak{t} . This set $\mathfrak{S}(\mathfrak{t})$ has the form $\{\mathfrak{s}_0 + n \cdot \text{PD}[\widehat{F}]\}_{n \in \mathbb{Z}}$ for some fixed $\mathfrak{s}_0 \in \text{Spin}^c(W'_m(K))$. The function $n \mapsto c_1(\mathfrak{s}_0 + n \cdot \text{PD}[\widehat{F}])^2$ is a quadratic function of n which is bounded above.

Choosing m larger than the constant c from Lemma 4.6, it follows easily that $c_1(\eta)^2 \geq c_1(\eta + \text{PD}[\widehat{F}])^2$, while $c_1(\mathfrak{x})^2 \geq c_1(\mathfrak{x} - \text{PD}[\widehat{F}])^2$. Since $\eta = \mathfrak{x} + \text{PD}[\widehat{F}]$, it follows readily that at least one of $c_1(\mathfrak{x})^2$ or $c_1(\eta)^2$ is a maximum of $c_1(\mathfrak{s})^2$ for $\mathfrak{s} \in \mathfrak{S}(\mathfrak{t})$.

As in the proof of Lemma 4.6 (cf Inequality (16)), there is a constant B independent of m with the property that if $\langle c_1(\mathfrak{x}), [\widehat{F}] \rangle + [\widehat{F}] \cdot [\widehat{F}] \geq B$, then for any corresponding small triangle ψ representing \mathfrak{x} , we have $n_w(\psi) > 0$. Moreover, in this case, h^+ , which corresponds to the Spin^c structure η , induces an isomorphism, while η maximizes $c_1(\mathfrak{s})^2$ among all $\mathfrak{s} \in \mathfrak{S}(\mathfrak{t})$. In the same way, when $\langle c_1(\mathfrak{x}), [\widehat{F}] \rangle + [\widehat{F}] \cdot [\widehat{F}] \leq -B$, then \mathfrak{x} induces an isomorphism, and it maximizes $c_1(\mathfrak{s})^2$ among all $\mathfrak{s} \in \mathfrak{S}(\mathfrak{t})$.

In either case, in view of Equation (9), we see that the degree of any element of $\widehat{HF}(Y_{m\mu+\lambda}(K), \mathfrak{t})$ lies within a bounded distance (independent from m and \mathfrak{t}) from $-c_1(\mathfrak{s})^2/4$, where $\mathfrak{s} \in \mathfrak{S}(\mathfrak{t})$ minimizes $c_1(\mathfrak{s})^2$. The claimed result follows. \square

Proposition 4.2 has the following consequence:

Corollary 4.7 *Let $K \subset Y$ be a null-homologous knot in an integer homology three-sphere, and fix an integer $\delta \geq 0$. There is a constant C with the property that for all sufficiently large m and any $\mathfrak{t} \in \text{Spin}^c(Y_{m\mu+\lambda}(K))$, there is a chain complex for $CF^\delta(Y_{m\mu+\lambda}(K), \mathfrak{t})$ with the property that if $CF^\delta_d(Y_{m\mu+\lambda}(K), \mathfrak{t})$ is nontrivial, then*

$$\left| d - \frac{m}{4} \right| \leq C.$$

Proof It is easy to see that for fixed $\mathfrak{t} \in \text{Spin}^c(Y)$, if we consider $\mathfrak{S}(\mathfrak{t})$, the element \mathfrak{s}_0 which maximizes $c_1(\mathfrak{s})^2$ has $c_1(\mathfrak{s}_0)^2 = -m + c$, where here c is some constant (independent of m). Thus, according to Equation (9), the map $f_{\mathcal{W}'_\lambda(K), \mathfrak{s}_0}^\delta$ carries an element of degree d to an element of $d - m/4 + c$. Now, according to Theorem 4.1, we have a chain complex representing $CF^\delta(Y_{m\mu+\lambda}(K), \mathfrak{t})$ (for any choice of \mathfrak{t}) whose breadth is constant, independent of m and the choice of \mathfrak{t} . According to the proof of Proposition 4.2, $f_{\mathcal{W}'_\lambda(K), \mathfrak{s}_0}^\delta$, the map of degree $d - m/4 + c$, carries some element of this complex nontrivially to $CF^\delta(Y)$. \square

5 Künneth principle

Let $\bar{K}_1 \subset Y_1$ and $\bar{K}_2 \subset Y_2$ be a pair of oriented three-manifolds equipped with oriented knots. Then, we can form the connected sum to obtain an oriented knot $\bar{K}_1 \# \bar{K}_2 \subset Y_1 \# Y_2$. Indeed, given $\xi_i \in \text{Spin}^c(Y_i, K_i)$, we can form their connected sum $\xi_1 \# \xi_2$. This induces a gluing map

$$\text{Spin}^c(Y_1, K_1) \times \text{Spin}^c(Y_2, K_2) \longrightarrow \text{Spin}^c(Y_1 \# Y_2, K_1 \# K_2),$$

written $\xi_1, \xi_2 \mapsto \xi_1 \# \xi_2$, which is equivariant with respect to the natural map

$$H^2(Y_1, K_1; \mathbb{Z}) \oplus H^2(Y_2, K_2; \mathbb{Z}) \longrightarrow H^2(Y_1 \# Y_2, K_1 \# K_2; \mathbb{Z}).$$

More explicitly, we can think of ξ_i ($i = 1, 2$) as specified by a nonvanishing vector field which contains \bar{K}_i as a closed orbit. We realize the connected sum as attaching a one-handle to $Y_1 \amalg Y_2$ along a pair of three-balls B_i supported on \bar{K}_i . We can assume that on each sphere $S_i = \partial B_i$, there are exactly two points where the vector field is normal to S_i , the two points where S_i meets \bar{K}_i : the “in-going” and “out-going points”. Here, a point in S_i is “in-going” if ξ_i points into $Y_i - B_i$. We can then match the in-going point on $Y_1 - B_1$ with the out-going point on $Y_2 - B_2$ and vice versa, to construct a nowhere vanishing vector field on $Y_1 \# Y_2$. This new vector field has $K_1 \# K_2$ as a closed orbit, and it has prescribed form in the connected sum region. This vector field gives rise to a relative Spin^c structure $\text{Spin}^c(Y_1 \# Y_2, \bar{K}_1 \# \bar{K}_2)$, inducing the gluing map above.

The gluing map can be described in terms of Heegaard diagrams as follows. Let $(\Sigma_1, \alpha_1, \beta_1, w_1, z_1)$ and $(\Sigma_2, \alpha_2, \beta_2, w_2, z_2)$ be the doubly-pointed Heegaard diagrams compatible with the oriented knots $\bar{K}_1 \subset Y_1$ and $\bar{K}_2 \subset Y_2$. Consider the oriented two-manifold Σ_3 obtained as the connected sum of Σ_1 with Σ_2 , identifying neighborhoods of z_1 with w_2 . Then, the doubly-pointed Heegaard diagram $(\Sigma_3, \alpha_1 \cup \alpha_2, \beta_1 \cup \beta_2, w_1, z_2)$ is compatible with $\bar{K}_1 \# \bar{K}_2 \subset Y_1 \# Y_2$. Now, given

$\mathbf{x}_1 \in \mathbb{T}_{\alpha_1} \cap \mathbb{T}_{\beta_1}$ and $\mathbf{x}_2 \in \mathbb{T}_{\alpha_2} \cap \mathbb{T}_{\beta_2}$, we can think of $\mathbf{x}_1 \times \mathbf{x}_2$ as an intersection point in $\mathbb{T}_{\alpha_1 \cup \alpha_2} \cap \mathbb{T}_{\beta_1 \cup \beta_2}$. Then,

$$\underline{s}_{w_1, z_2}(\mathbf{x}_1 \times \mathbf{x}_2) = \underline{s}_{w_1, z_1}(\mathbf{x}_1) \# \underline{s}_{w_2, z_2}(\mathbf{x}_2).$$

The following is a straightforward generalization of the connected sum principle for knot homology in the case of null-homologous knot;; cf [19; 28]. A proof can be found, for example, in Theorem 7.1 of [19]; we omit it here.

Theorem 5.1 Fix $\xi_i \in \underline{\text{Spin}}^c(Y_i, K_i)$ for $i = 1, 2$. There is a filtered chain homotopy equivalence

$$\begin{aligned} CFK^\infty(Y_1, K_1, \xi_1) \otimes_{\mathbb{Z}[U, U^{-1}]} CFK^\infty(Y_2, K_2, \xi_2) \\ \longrightarrow \bigoplus_{\{\xi_3 \mid \mathbf{x}_1 \# \xi_2 = \xi_3\}} CFK^\infty(Y_1 \# Y_2, K_1 \# K_2, \xi_3). \end{aligned}$$

Of course, the tensor product in the above theorem is to be taken in a suitable manner: $CFK^\infty(Y_1, K_1, \xi_1)$ and $CFK^\infty(Y_2, K_2, \xi_2)$ are $\mathbb{Z} \oplus \mathbb{Z}$ -filtered chain complexes over $\mathbb{Z}[U, U^{-1}]$. Their tensor product (over $\mathbb{Z}[U, U^{-1}]$) can then be equipped with a filtration gotten as the sum of the filtrations of two tensor factors.

Definition 5.2 A U -knot is a knot in a rational homology three-sphere Y with the property that the induced filtration of $CF^\infty(Y, \mathfrak{s})$ is trivial. More precisely, given any $\xi \in \underline{\text{Spin}}^c(Y, K)$, $CFK^\infty(Y, K, \xi)$ is chain homotopy equivalent via a relatively $\mathbb{Z} \oplus \mathbb{Z}$ -filtered chain homotopy to the chain complex R which is a free, rank one $\mathbb{Z}[U, U^{-1}]$ module with the trivial differential.

The unknot in S^3 is a U -knot. Indeed, the results of [18] can be interpreted as saying that the only U -knot in S^3 is the unknot. Other U -knots will be described in Lemma 7.1.

Corollary 5.3 If $K_2 \subset Y_2$ is a U -knot, then for each $\xi_1 \in \underline{\text{Spin}}^c(Y_1, \bar{K}_1)$ and $\mathfrak{s}_2 \in \underline{\text{Spin}}^c(Y_2)$, there is some $\xi_2 \in \underline{\text{Spin}}^c(Y_2)$ representing \mathfrak{s}_2 , with the property that

$$CFK^\infty(Y_1, \bar{K}_1, \xi_1) \cong CFK^\infty(Y_1 \# Y_2, \bar{K}_1 \# \bar{K}_2, \xi_1 \# \xi_2)$$

as $\mathbb{Z} \oplus \mathbb{Z}$ -filtered chain complexes.

Proof Let $\xi \in \underline{\text{Spin}}^c(Y_2, K_2)$ be any relative Spin^c structure representing \mathfrak{s}_2 . Then, $CFK^\infty(Y_2, \bar{K}_2, \xi)$ is quasi-isomorphic to the $\mathbb{Z} \oplus \mathbb{Z}$ -filtered chain complex $\mathbb{Z}[U, U^{-1}]$ which contains a nontrivial element in filtration $(0, n)$ for some n . Letting

$$\xi_2 = \xi + n \text{PD}[\mu]$$

(where μ of course is a meridian for K_2), we see that $CFK^\infty(Y_2, \bar{K}_2, \xi_2)$ is quasi-isomorphic to $\mathbb{Z}[U, U^{-1}]$ with a generator in filtration level $(0, 0)$ (cf Proposition 3.2). For this choice of ξ_2 , the corollary is an immediate consequence of Theorem 5.1. \square

6 The Morse surgery formula

We describe the Morse surgery formula, which expresses the Heegaard Floer homologies of Morse surgery along a rationally null-homologous knot in terms of its knot Floer homology.

Given $\mathfrak{s} \in \text{Spin}^c(Y_\lambda(K))$, let

$$\begin{aligned} \mathbb{A}_\mathfrak{s}^+(Y, \bar{K}) &= \bigoplus_{\{\xi \in \underline{\text{Spin}}^c(Y_\lambda(K), \bar{K}) \mid G_{Y_\lambda(K), \bar{K}}(\xi) = \mathfrak{s}\}} A_\xi^+(Y, \bar{K}), \\ \mathbb{B}_\mathfrak{s}^+(Y, \bar{K}) &= \bigoplus_{\{\xi \in \underline{\text{Spin}}^c(Y_\lambda(K), \bar{K}) \mid G_{Y_\lambda(K), \bar{K}}(\xi) = \mathfrak{s}\}} B_\xi^+(Y, \bar{K}), \end{aligned}$$

where here $A_\xi^+(Y, \bar{K})$ and $B_\xi^+(Y, \bar{K})$ are defined as in Equation (12).

In the above definition, we view K also as a knot in $Y_\lambda(K)$. An orientation on $K \subset Y$ naturally induces also an orientation on the induced knot in $Y_\lambda(K)$ in a natural way: an orientation on K corresponds to an orientation on a meridian for K , which can be thought of as supported in $Y - \text{nd}(K)$, which in turn can be thought of as a subset of $Y_\lambda(K)$, where it in turn induces an orientation of the induced knot in $Y_\lambda(K)$.

Theorem 6.1 *Let $K \subset Y$ be a knot in a rational homology three-sphere, and let λ be a framing on K with the property that $Y_\lambda(K)$ is also a rational homology three-sphere. The Heegaard Floer homology $HF^+(Y_\lambda(K), \mathfrak{s})$ is calculated by the homology of the mapping cone $\mathbb{X}_\mathfrak{s}^+(Y, \bar{K}, \lambda)$ of the chain map*

$$D_\mathfrak{s}^+ : \mathbb{A}_\mathfrak{s}^+(Y, \bar{K}) \longrightarrow \mathbb{B}_\mathfrak{s}^+(Y, \bar{K})$$

defined by $D_\mathfrak{s}^+(\{a_\xi\}_{\xi \in G_{Y_\lambda(K), \bar{K}}^{-1}(\mathfrak{s})}) = \{b_\xi\}_{\xi \in G_{Y_\lambda(K), \bar{K}}^{-1}(\mathfrak{s})}$,

where $b_\xi = h_{\xi - \text{PD}[K_\lambda]}^+(a_{\xi - \text{PD}[K_\lambda]}) - v_\xi^+(a_\xi)$.

The proof is modeled on the proof of the main result from [27]. Specifically, it is based on two key ingredients: the relationship between the knot Floer homology of $K \subset Y$ and the Heegaard Floer homology of three-manifolds obtained as sufficiently large surgeries on K from Section 4, and also an exact triangle which relates, for all

integers m , the Heegaard Floer homologies of $Y_\lambda(K)$, $Y_{m\cdot\mu+\lambda}(K)$ and Y (the latter taken with twisted coefficients).

We have not strived for maximal generality in [Theorem 6.1](#). As a technical tool, we make heavy use of the rational grading on HF^+ which is defined on the Heegaard Floer homology of any three-manifold equipped with a Spin^c structure whose first Chern class is torsion; cf [Section 7](#) of [\[26\]](#). Thus, [Theorem 6.1](#) holds – and indeed the proof we give below applies – whenever we consider Spin^c structures over $Y_\lambda(K)$ whose first Chern class is torsion.

6.1 A surgery exact sequence

The following is essentially a restatement of [Theorem 3.1](#) of [\[27\]](#):

Theorem 6.2 *Let Y be a closed, oriented three-manifold, and $K \subset Y$ be a rationally null-homologous knot with framing λ . Then, for all positive integers m , there is a long exact sequence*

$$\dots \rightarrow HF^+(Y_\lambda(K)) \xrightarrow{F_1^+} HF^+(Y_{m\cdot\mu+\lambda}(K)) \xrightarrow{F_2^+} \bigoplus^m HF^+(Y) \xrightarrow{F_3^+} \dots,$$

where here μ denotes the meridian of K .

We describe the maps appearing in the above theorem, and then make a few comments on the proof.

Let $K \subset Y$ be a rationally null-homologous knot, and (Σ, α, β) be a corresponding Heegaard diagram, where β_g represents a meridian for the knot K (pushed into the U_β handlebody). We place a basepoint p on β_g , and consider twisted homology with coefficients in $\mathbb{Z}/m\mathbb{Z}$; ie write $\mathbb{Z}[\mathbb{Z}/m\mathbb{Z}] = \mathbb{Z}[T]/(T^m - 1)$, and consider the chain complex $CF^+(Y) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}]$ endowed with the differential

$$\partial^+[\mathbf{x}, i] = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \mid \mu(\phi) = 1\}} \# \left(\frac{\mathcal{M}(\phi)}{\mathbb{R}} \right) \cdot T^{m_p(\phi)} \cdot [\mathbf{y}, i - n_z(\phi)]$$

where as usual here $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $i \geq 0$, $\pi_2(\mathbf{x}, \mathbf{y})$ denotes the space of homotopy classes of Whitney disks connecting \mathbf{x} and \mathbf{y} , $\mu(\phi)$ denotes the Maslov index of ϕ , and terms in the above equation for which $i - n_z(\phi) < 0$ are to be dropped. Moreover, $m_p(\phi)$ denotes the multiplicity of the basepoint p in the boundary of ϕ ; ie p determines a codimension one submanifold $\beta_1 \times \dots \times \beta_{g-1} \times \{p\} \subset \mathbb{T}_\beta$, and $m_p(\phi)$ denotes the intersection number with the restriction of the boundary of ϕ with this subset. We denote the complex by $CF^+(Y; \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}])$. (In the terminology of [\[20\]](#), this is the chain complex for Y with twisted coefficients in $\mathbb{Z}[\mathbb{Z}/m\mathbb{Z}]$, where it is denoted

$CF^+(Y; \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}])$, however, as in [27], we drop the underline here in the interest of notational simplicity.) Recall that there is an isomorphism of chain complexes of modules over $\mathbb{Z}[\mathbb{Z}/m\mathbb{Z}]$,

$$(17) \quad \theta: CF^+(Y; \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}]) \xrightarrow{\cong} CF^+(Y) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}],$$

where here the right-hand-side is endowed with the differential which is the original differential on $CF^+(Y)$ tensored with the identity map on $\mathbb{Z}[\mathbb{Z}/m\mathbb{Z}]$. There is a corresponding identification

$$HF^+(Y; \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}]) \cong HF^+(Y) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}] \cong \bigoplus^m HF^+(Y).$$

The quantity $m_p(\phi)$ has an expression more in the spirit of the constructions from Section 3. Fix two basepoints, w and z on either side of β_g , so that there is an arc in Σ connecting them, but disjoint from all the attaching circles except for β_g , which it meets transversally in the single intersection point p . Then,

$$(18) \quad m_p(\phi) = n_w(\phi) - n_z(\phi).$$

We now define the chain maps f_1^+ , f_2^+ and f_3^+ inducing maps on homology F_1^+ , F_2^+ and F_3^+ appearing in Theorem 6.1.

We consider the Heegaard quadruple (in the sense of [21, Section 8.4]; see also the proof of [27, Theorem 3.1]) $(\Sigma, \alpha, \beta, \gamma, \delta, z)$, where here the g -tuples of circles β , γ and δ differ only in the last factor, where γ_g is a meridian for K , γ_g represents the framing curve λ , and δ_g represents the resolution of the curve $m \cdot \beta_g + \gamma_g$.

The map f_1^+ is defined by counting pseudo-holomorphic triangles between \mathbb{T}_α , \mathbb{T}_γ and \mathbb{T}_δ . More precisely, note that the Heegaard triple $(\Sigma, \alpha, \gamma, \delta, z)$ determines a four-manifold $X_{\alpha\gamma\delta}$ with three boundary components

$$(19) \quad Y_{\alpha\gamma} \cong Y_\lambda(K), \quad Y_{\alpha\delta} \cong Y_{m \cdot \mu + \lambda}(K) \quad \text{and} \quad Y_{\gamma\delta} \cong \#^{g-1}(S^2 \times S^1) \# L(m, 1).$$

Definition 6.3 Let $\ell \in \text{Spin}^c(\#^{g-1}(S^2 \times S^1) \# L(m, 1))$ denote the *canonical Spin^c structure*, ie this is the one which extends over the tubular neighborhood N of a sphere with self-intersection number m (after attaching a two-handle and a collection of $g - 1$ three-handles) in such a way that its first Chern class evaluates as $\pm m$ on the two-sphere generator of the two-dimensional homology of N .

Let $\Theta_{\gamma\delta}$ denote the Floer homology class corresponding to the generator (over the ring $\Lambda^* H_1(Y_{\gamma\delta}) \otimes \mathbb{Z}[U]$) of

$$HF^{\leq 0}(Y_{\gamma\delta}, \ell) \cong \Lambda^* H^1(Y_{\gamma\delta}) \otimes \mathbb{Z}[U]$$

in its canonical Spin^c structure ℓ . (As usual, we arrange for the homology class $\Theta_{\gamma\delta}$ to be represented by a single intersection point in $\mathbb{T}_\gamma \cap \mathbb{T}_\delta$, which we also denote by $\Theta_{\gamma\delta}$.)

We then define

$$(20) \quad f_1^+([\mathbf{x}, i]) = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\delta} \sum_{\{\psi \in \pi_2(\mathbf{x}, \Theta_{\gamma\delta}, \mathbf{y}) \mid \mu(\psi) = 0\}} \# \mathcal{M}(\psi) \cdot [\mathbf{y}, i - n_z(\psi)].$$

Similarly, we define $f_2^+ : CF^+(Y_{m \cdot \mu + \lambda}(K)) \longrightarrow CF^+(Y; \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}])$ by

$$(21) \quad f_2^+([\mathbf{y}, i]) = \sum_{\mathbf{w} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\{\psi \in \pi_2(\mathbf{y}, \Theta_{\delta\beta}, \mathbf{w}) \mid \mu(\psi) = 0\}} \# \mathcal{M}(\psi) \cdot [\mathbf{w}, i - n_z(\psi)] \cdot T^{m_p(\psi)},$$

where $m_p(\psi)$ is the natural extension of m_p to triangles. (In particular, in the case where we consider doubly-pointed Heegaard diagrams, $m_p(\psi) = n_w(\psi) - n_z(\psi)$ as in Equation (18).)

To define f_3^+ , we proceed as follows. Fix $\psi \in \pi_2(\Theta_{\gamma\beta}, \Theta_{\delta\beta}, \Theta_{\gamma\delta})$. The congruence class c of $m_p(\psi)$ modulo m is independent of the choice of ψ . The map

$$f_3^+ : CF^+(Y; \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}]) \longrightarrow CF^+(Y_\lambda(K))$$

is given by the formula

$$(22) \quad f_3^+(T^s \cdot [\mathbf{x}, i]) = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma} \sum_{\{\psi \in \pi_2(\mathbf{x}, \Theta_{\beta\gamma}, \mathbf{y}) \mid \begin{smallmatrix} \mu(\psi) = 0, \\ s + m_p(\psi) \equiv c \pmod{m} \end{smallmatrix}\}} \# \mathcal{M}(\psi) \cdot [\mathbf{y}, i - n_z(\psi)].$$

Proof of Theorem 6.2 Let H_1^+ and H_2^+ denote the null-homotopies of $f_2^+ \circ f_1^+$ and $f_3^+ \circ f_2^+$ respectively. With this notation, the proof of Theorem 3.1 of [27] applies to show that the chain maps

$$\phi^+ : CF^+(Y_\lambda(K)) \longrightarrow M(f_2^+) \quad \text{and} \quad \psi^+ : M(f_2^+) \longrightarrow CF^+(Y_\lambda(K))$$

defined by

$$(23) \quad \phi^+(\xi) = (f_1^+(\xi), H_1^+(\xi)),$$

$$(24) \quad \psi^+(x, y) = H_2^+(x) + f_3^+(y)$$

respectively are quasi-isomorphisms, and hence establishing Theorem 6.2. □

6.2 The analogue of Theorem 6.1 for the case of CF^δ

Given an integer $\delta \geq 0$, let CF^δ denote the approximation of CF^+ described in Section 2.8. We state and prove analogue of Theorem 6.1 for this group, following the pattern of proof from [27].

Given $\delta \geq 0$, let

$$(25) \quad A_\xi^\delta(Y, \bar{K}) = C_\xi \{0 \leq \max(i, j) \leq \delta\} \quad \text{and} \quad B_\xi^\delta(Y, \bar{K}) = CF^\delta(Y, G_{Y, \bar{K}}(\xi)).$$

Given $\mathfrak{s} \in \text{Spin}^c(Y_\lambda(K))$, let

$$\begin{aligned} \mathbb{A}_\mathfrak{s}^\delta(Y, \bar{K}) &= \bigoplus_{\{\xi \in \underline{\text{Spin}}^c(Y_\lambda(K), K) \mid G_{Y_\lambda(K), \bar{K}}(\xi) = \mathfrak{s}\}} A_\xi^\delta(Y, \bar{K}), \\ \mathbb{B}_\mathfrak{s}^\delta(Y, \bar{K}) &= \bigoplus_{\{\xi \in \underline{\text{Spin}}^c(Y_\lambda(K), K) \mid G_{Y_\lambda(K), \bar{K}}(\xi) = \mathfrak{s}\}} B_\xi^\delta(Y, \bar{K}). \end{aligned}$$

Consider the map

$$D_\mathfrak{s}^\delta: \mathbb{A}_\mathfrak{s}^\delta(Y, \bar{K}) \longrightarrow \mathbb{B}_\mathfrak{s}^\delta(Y, \bar{K})$$

defined by

$$D_\mathfrak{s}^\delta(\{a_\xi\}_{\xi \in G_{Y_\lambda(K), \bar{K}}^{-1}(\mathfrak{s})}) = \{b_\xi\}_{\xi \in G_{Y_\lambda(K), \bar{K}}^{-1}(\mathfrak{s})},$$

where

$$b_\xi = h_{\xi - \text{PD}[K_\lambda]}^\delta(a_{\xi - \text{PD}[K_\lambda]}) - v_\xi^\delta(a_\xi).$$

Our aim in the present subsection is to prove the following analogue of Theorem 6.1 for CF^δ .

Theorem 6.4 *The group $CF^\delta(Y_\lambda(K), \mathfrak{s})$ is quasi-isomorphic to the mapping cone $\mathbb{X}_\mathfrak{s}^\delta(\lambda)$ of*

$$D_\mathfrak{s}^\delta: \mathbb{A}_\mathfrak{s}^\delta(Y, \bar{K}) \longrightarrow \mathbb{B}_\mathfrak{s}^\delta(Y, \bar{K}).$$

We deduce Theorem 6.4 from a combination of Theorems 6.2 and 4.1. Theorem 6.1 will follow from Theorem 6.4, together with some further observations about gradings; see Section 6.3.

We turn our attention now towards proving Theorem 6.4. We assume that both d and n have the same sign, and hence without additional loss of generality that both are positive, returning to the case where their signs are opposite in Section 6.4.

It will be useful to have the following lemma. In the statement, ‘‘sufficiently large’’ is meant with respect to the ordering on $\underline{\text{Spin}}^c(Y, K)$ with fixed filling \mathfrak{t} over Y described earlier. Specifically, recall that $\xi_1 \leq \xi_2$ if $\xi_2 = \xi_1 + j \cdot \text{PD}[\mu]$ for some

$j \geq 0$. In particular, we say that a condition P holds for all sufficiently large relative Spin^c structures $\text{Spin}^c(Y, \bar{K})$ if there is a finite collection $\Xi \subset \text{Spin}^c(Y, \bar{K})$ with the properties that each $\mathfrak{t} \in \text{Spin}^c(Y)$ can be represented by some $\eta \in \underline{\text{Spin}}^c(Y, \bar{K})$ with $\eta \in \Xi$, and also for any $\xi \in \underline{\text{Spin}}^c(Y, \bar{K})$ which has the property that $\xi \geq \eta$ for some $\eta \in \Xi$ then P holds for ξ .

Lemma 6.5 *If ξ is sufficiently large, then for any $\delta \geq 0$, the map*

$$v_\xi^\delta: A_\xi^\delta(Y, \bar{K}) \longrightarrow B_\xi^\delta(Y, \bar{K})$$

is an isomorphism, while

$$h_\xi^\delta: A_\xi^\delta(Y, \bar{K}) \longrightarrow B_{\xi+\text{PD}[K_\lambda]}^\delta(Y, \bar{K})$$

is trivial; moreover, if ξ is sufficiently small, then h_ξ^δ is an isomorphism, while v_ξ^δ is trivial.

Proof Since there are finitely many $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, there is a maximal ξ_0 among all $\underline{\mathfrak{x}}_{w,z}(\mathbf{x})$ with $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Clearly, for any $\xi \geq \xi_0$, there are no generators $[\mathbf{x}, i, j]$ of $CFK^\infty(Y, K, \xi)$ with $i < 0$ and $j \geq 0$; thus, in fact, $A_\xi^+(Y, \bar{K}) = C_\xi\{i \geq 0\}$, ie v_ξ^+ is the identity map, and the corresponding statement for v_ξ^δ follows at once.

The other assertions follow from similar reasoning. □

It will help to work in a finite piece of $\mathbb{X}_s^+(Y, \bar{K}, \lambda)$, by using a suitable truncation (as in [27]). We describe this truncation presently in cases where the coefficients n and d appearing in Equation (3) have the same sign; see Section 6.4 for the other case.

Fix a relative Spin^c structure ξ . Given an integer b and $\delta > 0$, we can form b -truncated complexes

$$(26) \quad \left\{ \begin{array}{l} \mathbb{A}_\xi^+(Y, \bar{K}; b) = \bigoplus_{s=-b}^b A_{\xi+s \cdot \text{PD}[K_\lambda]}^+(Y, \bar{K}), \\ \mathbb{B}_\xi^+(Y, \bar{K}; b) = \bigoplus_{s=-b+1}^b B_{\xi+s \cdot \text{PD}[K_\lambda]}^+(Y, \bar{K}), \end{array} \right.$$

$$(27) \quad \left\{ \begin{array}{l} \mathbb{A}_\xi^\delta(Y, \bar{K}; b) = \bigoplus_{s=-b}^b A_{\xi+s \cdot \text{PD}[K_\lambda]}^\delta(Y, \bar{K}), \\ \mathbb{B}_\xi^\delta(Y, \bar{K}; b) = \bigoplus_{s=-b+1}^b B_{\xi+s \cdot \text{PD}[K_\lambda]}^\delta(Y, \bar{K}), \end{array} \right.$$

where here $A_{\xi}^+(Y, \bar{K})$ and $B_{\xi}^+(Y, \bar{K})$ are defined as in Equation (12), and $A_{\xi}^{\delta}(Y, \bar{K})$ and $B_{\xi}^{\delta}(Y, \bar{K})$ are their analogues, as in Equation (25). The maps v_{ξ}^+ and h_{ξ}^+ can be added up to define a chain map between b -truncated complexes

$$D_{\xi, \lambda; b}^+ : \mathbb{A}_{\xi}^+(Y, \bar{K}; b) \longrightarrow \mathbb{B}_{\xi}^+(Y, \bar{K}; b),$$

from which we construct a mapping cone $\mathbb{X}_{\xi}^+(Y, \lambda, \bar{K}; b)$. We can also specialize to $U^{\delta} = 0$ to get a map

$$D_{\xi, \lambda; b}^{\delta} : \mathbb{A}_{\xi}^{\delta}(Y, \bar{K}; b) \longrightarrow \mathbb{B}_{\xi}^{\delta}(Y, \bar{K}; b),$$

from which we construct a mapping cone $\mathbb{X}_{\xi}^{\delta}(Y, \lambda, \bar{K}; b)$. We call $\mathbb{X}_{\xi}^+(Y, \lambda, \bar{K}; b)$ and $\mathbb{X}_{\xi}^{\delta}(Y, \lambda, \bar{K}; b)$ the b -truncated mapping cones. (Note that the untruncated mapping cones depend on the framing parameter λ and a Spin^c structure over $Y_{\lambda}(K)$; by contrast, the truncated mapping cones also depend on a relative Spin^c structure ξ , which can be thought of as a lift of the Spin^c structure over $Y_{\lambda}(K)$.)

Lemma 6.5 has the following easy consequence:

Proposition 6.6 Fix ξ , $K \subset Y$, and a framing λ . Suppose moreover that the coefficients n and d from Equation (3) have the same sign. Then, for all sufficiently large b , $\mathbb{X}_{\xi}^+(Y, \lambda, \bar{K}; b)$ is quasi-isomorphic to $\mathbb{X}_{\mathfrak{s}}^+(Y, \lambda, \bar{K})$ (where \mathfrak{s} extends ξ over $Y_{\lambda}(K)$). Similarly, for b as above, we have that $\mathbb{X}_{\xi}^{\delta}(\lambda; b)$ is quasi-isomorphic to $\mathbb{X}_{\mathfrak{s}}^{\delta}(\lambda)$ for any $\delta > 0$.

Proof $\mathbb{X}_{\xi}^+(Y, \lambda, \bar{K}; b)$ can be viewed as a quotient complex of $\mathbb{X}_{\mathfrak{s}}^+(Y, \lambda, \bar{K})$, by a subcomplex, which is acyclic according to Lemma 6.5 □

Proof of Theorem 6.4 when d and n have the same sign Choose $m = nk$, where n is as in Equation (3). It is easy to see that $\text{PD}[K_{\lambda}]$ has order $dk + 1$ in $H^2(Y_{m \cdot \mu + \lambda})$ whereas $\text{PD}[K_{\lambda}]$ has order dk in the quotient group $H^2(Y, Y - K; \mathbb{Z})/m \text{PD}[\mu]$.

By Theorem 6.2, we may express $CF^{\delta}(Y_{\lambda}(K))$ as the mapping cone of a map from $CF^{\delta}(Y_{m \cdot \mu + \lambda}(K))$ to $CF^{\delta}(Y, \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}])$. We can think of $CF^{\delta}(Y, \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}])$ more invariantly, as a sum

$$\bigoplus_{\{[\xi] \in \underline{\text{Spin}}^c(Y, K)/\mathbb{Z}m \cdot \mu\}} ([\xi], CF^{\delta}(Y, G_{Y, \bar{K}}(\xi))),$$

where, as the notation suggests, the index set consists of $m \cdot \mu$ -orbits in $\underline{\text{Spin}}^c(Y, K)$ (each of which induces the same Spin^c structure over Y , of course). The identification is induced by the map

$$T^i \otimes \xi \mapsto (\underline{\mathfrak{s}}_{w, z}(\mathbf{x}) + i \cdot \text{PD}[\mu], \mathbf{x}).$$

Now, the map

$$f_2^\delta: CF^\delta(Y_{m\mu+\lambda}(K)) \longrightarrow CF^\delta(Y; \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}])$$

can be written as

$$a \mapsto \sum_{\mathfrak{s} \in \text{Spin}^c(W'_m(K))} ([E_{Y, \bar{K}}(\mathfrak{s})], f_{\mathfrak{s}}^\delta(a)),$$

where here $W'_m(K)$ denotes the natural cobordism from $Y_{m\mu+\lambda}(K)$, and $E_{Y, \bar{K}}$ is the map from Spin^c structures over $W'_m(K)$ to $\text{Spin}^c(Y, \bar{K})$ as in Proposition 2.2.

Choosing k sufficiently large, (where $m = nk$) according to Theorem 4.1, we have that the summand $CF^\delta(Y_{m\mu+\lambda}, \mathfrak{t}) \subset CF^\delta(Y_{m\mu+\lambda})$ is identified with $A_\xi^\delta(Y, \bar{K})$ for $\xi = \Xi(\mathfrak{t})$. In fact, there are only two homotopically nontrivial components of $f_2^\delta|_{CF^\delta(Y_{m\mu+\lambda}(K), \mathfrak{t})}$, according to Proposition 4.2: these are the components belonging to \mathfrak{r} and $\mathfrak{\eta}$, whose corresponding maps are identified with v_ξ^δ and h_ξ^δ respectively. In view of Equation (7),

$$E_{Y, \bar{K}}(\mathfrak{r} + \text{PD}[\hat{F}]) = E_{Y, \bar{K}}(\mathfrak{r}) + m\mu + \lambda;$$

and hence, if $\Xi(\mathfrak{t}) = \xi$ and $\Xi(\mathfrak{t} + \text{PD}[K_\lambda]) = \xi + \text{PD}(K_\lambda)$, then the ranges of v_ξ^δ and $h_{\xi - \text{PD}[K_\lambda]}^+$ coincide.

In effect, we have shown that $CF^\delta(Y_\lambda(K))$ is quasi-isomorphic to the mapping cone of a map which splits according to K_λ orbits in $\text{Spin}^c(Y, K)$ or equivalently, Spin^c structures over $Y_\lambda(K)$. In each such orbit, the map has the form

$$(f_2^\delta)': \bigoplus_{s \in \mathbb{Z}/(dk+1)\mathbb{Z}} A_{\xi+s \cdot \text{PD}[K_\lambda]}^\delta(Y, \bar{K}) \longrightarrow \bigoplus_{s \in \mathbb{Z}/dk\mathbb{Z}} B_{\xi+s \cdot \text{PD}[K_\lambda]}^\delta(Y, \bar{K}),$$

defined by adding

$$v_s^\delta: A_{\xi+s \cdot \text{PD}[K_\lambda]}^\delta(Y, \bar{K}) \longrightarrow B_{\xi+s \cdot \text{PD}[K_\lambda]}^\delta(Y, \bar{K})$$

and
$$h_s^\delta: A_{\xi+s \cdot \text{PD}[K_\lambda]}^\delta(Y, \bar{K}) \longrightarrow B_{\xi+(s+1) \cdot \text{PD}[K_\lambda]}^\delta(Y, \bar{K}).$$

According to Lemma 6.5 and our hypothesis on the sign of d , we see that for any ξ , if s is sufficiently large, then letting $\xi' = \xi + s \cdot \text{PD}[K_\lambda]$, we have that $h_{\xi'}^\delta$ is null-homotopic, and $v_{\xi'}^\delta$ is an isomorphism. It follows from this (together with the analogous statement for s sufficiently small) that the mapping cone of $(f_2^\delta)'$ is identified with the mapping cone $\mathbb{X}_\xi^\delta(Y, \bar{K}; b)$ of

$$D_{\lambda; b}^\delta: \mathbb{A}_\xi^\delta(Y, \bar{K}; b) \longrightarrow \mathbb{B}_\xi^\delta(Y, \bar{K}; b).$$

The theorem now follows from Proposition 6.6. □

The case where d and n have opposite signs is handled in Section 6.4.

6.3 Gradings and the proof of Theorem 6.1 (in the case where d and n have the same sign)

In the proof of Theorem 6.4, for each $\delta \geq 0$, we establish quasi-isomorphisms

$$\phi^\delta: CF^\delta(Y_\lambda(K), \mathfrak{s}) \longrightarrow \mathbb{X}_s^\delta(\lambda).$$

We wish to conclude that

$$HF^+(Y_\lambda(K), \mathfrak{s}) \cong H_*(\mathbb{X}_s^+(Y, \lambda, \bar{K})).$$

To this end, we must establish that $\mathbb{X}_s^+(Y, \lambda, \bar{K})$ is a relative \mathbb{Z} -graded complex, which can be given an absolute grading so that the map ϕ^δ is homogeneous of degree zero.

Clearly, the groups $A_\xi^+(Y, \bar{K})$ and $B_\xi^+(Y, \bar{K})$ are relatively \mathbb{Z} -graded, and the maps v_ξ^+ and h_ξ^+ are all relatively \mathbb{Z} -graded maps. It is now a formal consequence of the shape of

$$D_s^+: A_s^+(Y, \bar{K}) \longrightarrow B_s^+(Y, \bar{K})$$

that $A_s^+(Y, \bar{K})$ and $B_s^+(Y, \bar{K})$ can be given absolute \mathbb{Z} -gradings so that D_s^+ drops this grading by one. This naturally endows the mapping cone $\mathbb{X}_s^+(Y, \lambda, \bar{K})$ with an absolute \mathbb{Z} -grading. In fact, this grading is uniquely determined up to an overall shift.

In the proof of Theorem 6.2, we considered maps

$$\begin{aligned} f_1^\delta: CF^\delta(Y_\lambda(K)) &\longrightarrow CF^\delta(Y_{nk \cdot \mu + \lambda}(K)), \\ f_2^\delta: CF^\delta(Y_{nk \cdot \mu + \lambda}(K)) &\longrightarrow CF^\delta(Y; \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}]). \end{aligned}$$

Fix $\xi \in \text{Spin}^c(Y, K)$. Given $s \in \mathbb{Z}$, we have a projection map

$$\Pi_s^A: CF^+(Y_{nk \cdot \mu + \lambda}(K)) \longrightarrow CF^+(Y_{nk \cdot \mu + \lambda}(K), G_{Y_{nk \cdot \mu + \lambda}(K), \bar{K}}(\xi + s \cdot \text{PD}[K_\lambda])).$$

We prove the following analogue of Proposition 4.6 of [27]. The proposition considers $K \subset Y$ be an oriented knot in a rational homology sphere, and λ a framing for which $Y_\lambda(K)$ is also a rational homology sphere. In this case, both $CF^+(Y, \mathfrak{t})$ and $CF^+(Y_\lambda, \mathfrak{t})$ are relatively \mathbb{Z} -graded groups.

Proposition 6.7 *Let $K \subset Y$ be an oriented knot in a rational homology sphere, and λ a framing for which $Y_\lambda(K)$ is also a rational homology sphere. Fix an absolute lift of the relative \mathbb{Z} -grading on $CF^+(Y_\lambda(K), G_{Y_\lambda(K), \bar{K}}(\xi))$, and an integer $\delta \geq 0$. Then, given an integer $b > 0$, any sufficiently large k has the property that for each s with $|s| \leq b$, there are absolute lifts of the relative \mathbb{Z} -gradings on both*

$$A_{\xi + s \cdot \text{PD}[K_\lambda]}^\delta(Y, \bar{K}) \cong CF^\delta(Y_{nk \cdot \mu + \lambda}(K), \mathfrak{t}) \subset CF^\delta(Y_{nk \cdot \mu + \lambda}(K))$$

and $CF^\delta(Y, G_{Y, \bar{K}}(\xi + s \cdot \text{PD}[K_\lambda]))$ for all $|s| \leq b$, with the property that $\Pi_s^A \circ f_1^\delta$ and also the restriction of $f_2^\delta|_{A_{\xi + s \cdot \text{PD}[K_\lambda]}^\delta(Y, \bar{K})}$ have degree zero.

We return to a proof of [Proposition 6.7](#) after two lemmas.

Recall (cf [Equation \(19\)](#)) that the map f_1^+ was defined by counting holomorphic triangles for a Heegaard triple describing a four-manifold which we denoted $X_{\alpha\gamma\delta}$, whose boundaries consist of $Y_\lambda(K)$, $Y_{(nk)\mu+\lambda}(K)$ and $\#^{g-1}(S^2 \times S^1) \# L(nk, 1)$. We denote this four-manifold here by $X(k)$, to call attention to its dependence on k .

The following lemma is an adaptation of the proof of [Lemma 4.7](#) from [\[27\]](#).

Lemma 6.8 *Fix a constant C_0 . Then, for all sufficiently large k , the following statement holds. Each Spin^c structure over $Y_{nk\cdot\mu+\lambda}(K)$ has at most one extension \mathfrak{s} over $X(k)$ whose restriction to $\#^{g-1}(S^2 \times S^1) \# L(nk, 1)$ is the canonical Spin^c structure (in the sense of [Definition 6.3](#), and for which*

$$(28) \quad C_0 \leq c_1(\mathfrak{s})^2 + nk.$$

Proof Observe that $H_2(X(k); \mathbb{Z}) \cong \mathbb{Z}$ is generated by a surface Σ with

$$\Sigma^2 = -nk d^2 (dk + 1).$$

This can be seen, for example, by observing that the group of triply-periodic domains in $X(k)$ is generated by a relation of the form

$$-(dk + 1)d \cdot \lambda + d \cdot (nk \cdot \mu + \lambda) + a + b,$$

where a is a sum of curves among the α_i with $i = 1, \dots, g$, and b a sum of curves among the β_j with $j = 1, \dots, g - 1$. The intersection number of the first two curves – which gives the self-intersection number of the homology class corresponding to the triply-periodic domain – is $-nk d^2 (dk + 1)$.

Thus, if \mathfrak{s} satisfies [Inequality \(28\)](#), then writing $c_1(\mathfrak{s}) = \alpha \cdot \text{PD}[\Sigma]$, we see that

$$|\alpha| = \sqrt{\frac{nk - C_0}{nk d^2 (dk + 1)}} \leq \frac{1}{2}$$

for large enough k .

Now, any other Spin^c structure over $X(k)$ which interpolates between the same two Spin^c structures on $Y_\lambda(K)$ and $Y_{nk\cdot\mu+\lambda}(K)$ (and whose restriction to the remaining boundary component $\#^{g-1}(S^2 \times S^1) \# L(nk, 1)$ is the canonical Spin^c structure) has the form $\mathfrak{s} + j \text{PD}[\Sigma]$, for some integer $j \neq 0$; thus,

$$\begin{aligned} c_1(\mathfrak{s} + j \cdot \text{PD}[\Sigma])^2 - c_1(\mathfrak{s})^2 &= 4(j^2 \Sigma \cdot \Sigma + j \langle c_1(\mathfrak{s}), [\Sigma] \rangle) \\ &\leq -4j^2 nk d^2 (dk + 1) \left(1 - \frac{|\alpha|}{j} \right) \leq -2nk d^2 (dk + 1). \end{aligned}$$

Of course, if k is sufficiently large, then [Inequality \(28\)](#) is violated. □

Lemma 6.9 Fix $K \subset Y$ with framing λ . Fix also $\xi \in \text{Spin}^c(Y, K)$ and $t \in \mathbb{Z}$. Then, for sufficiently large δ , there is a cycle $\zeta = \{\zeta_s\}_{\{-b \leq s \leq b\}} \in \mathbb{A}_\xi^\delta(Y, \bar{K}; b)$ with the following properties:

- Each component a_s is homogeneous with respect to the relative grading on

$$H_*(A_{\xi+s \cdot \text{PD}[K_\lambda]}^\delta(Y, \bar{K})).$$

- For each $s = -b + 1, \dots, b$, the homology class of $v_s^\delta(\zeta_s)$ is nontrivial.
- The element a is in the kernel of the map

$$D_{\xi, \lambda; b}^\delta: \mathbb{A}_\xi^\delta(Y, \bar{K}; b) \longrightarrow \mathbb{B}_\xi^\delta(Y, \bar{K})$$

(from Equation (27)).

Proof We have a chain map

$$\Pi_b^\delta: \mathbb{X}_\xi^\delta(Y, \bar{K}; b) \longrightarrow \mathbb{B}_\xi^\delta(Y, \bar{K}; b)$$

defined by

$$\Pi^\delta(\{a_s\}_{-b \leq s \leq b}) = \{v_s^\infty(a_s)\}_{-b+1 \leq s \leq b}.$$

Note that both maps

$$\begin{aligned} v_s^\infty: A_{\xi+s \cdot \text{PD}[K_\lambda]}^\infty(Y, \bar{K}) &\longrightarrow B_{\xi+s \cdot \text{PD}[K_\lambda]}^\infty(Y, \bar{K}) \\ h_s^\infty: A_{\xi+s \cdot \text{PD}[K_\lambda]}^\infty(Y, \bar{K}) &\longrightarrow B_{\xi+s \cdot \text{PD}[K_\lambda]}^\infty(Y, \bar{K}) \end{aligned}$$

are isomorphisms. It follows easily that $H_*(\mathbb{X}_\xi^\infty(Y, \bar{K}; b)) \cong \mathbb{Z}[U, U^{-1}]$, and indeed the homology group is represented as a cycle ζ^∞ in $\mathbb{A}_\xi^\infty(Y, \bar{K}; b)$, which has the property that each component of $\Pi^\infty(\zeta^\infty) \in \mathbb{B}_\xi^\infty(Y, \bar{K}; b)$ is nonzero. By multiplying ζ^∞ through by a sufficiently large power of U^{-i} , we can assume that ζ^∞ is the image of a cycle ζ^+ in $\mathbb{X}_\xi^+(Y, \lambda, \bar{K}; b)$ supported in $\mathbb{A}_\xi^+(Y, \bar{K}; b)$ which that the property that each component of $\Pi^+(\zeta^+) \in \mathbb{B}_\xi^+(Y, \bar{K}; b)$ is nonzero. For $\delta > 0$, we can consider the image ζ^δ of ζ^+ in $\mathbb{X}_\xi^\delta(Y, \bar{K}; b)$. For suitably large $\delta > 0$, we can assume that each component of $\Pi^\delta(\zeta^\delta)$ is also nonzero. This element ζ^δ is the desired cycle. \square

Proof of Proposition 6.7 Fix an absolute lift of the relative \mathbb{Z} -grading on the complex $CF^+(Y_\lambda(K), \mathfrak{s})$, and fix some $\delta \geq 0$. Now, f_1^δ decomposes as a sum of homogeneous terms, indexed by those $\mathfrak{s} \in \text{Spin}^c(X(k))$ whose restriction to the boundary component $\#^{g-1}(S^2 \times S^1) \# L(nk, 1)$ is the canonical Spin^c structure. By a suitable adaptation of Equation (9), we see that each term is homogeneous of degree $(c_1(\mathfrak{s})^2 + nk)/4$ (in this application, note that $X(k)$ has three, rather than two, boundary components, and we are considering the pairing with a fixed homology class on the third term). It

follows readily from [Lemma 6.8](#) that there is at most one such Spin^c structure which can induce a nontrivial map from $CF^\delta(Y_\lambda(K), G_{Y_\lambda(K)}(\xi))$ to $CF^\delta(Y_{nk \cdot \mu + \lambda}(K))$.

Thus, we can grade $CF^\delta(Y_{nk \cdot \mu + \lambda}(K))$ so that each component of f_1^δ has degree zero.

Similarly, we can endow $([s], CF^\delta(Y, \xi + s \cdot \text{PD}[K_\lambda]) \subset CF^\delta(Y, \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}]))$ with the grading for which

$$v_s^\delta: A_{\xi+s \cdot \text{PD}[K_\lambda]}^\delta(Y, \bar{K}) \cong CF^\delta(Y_{nk \cdot \mu + \lambda}(K), \xi + s \cdot \text{PD}[K_\lambda]) \longrightarrow ([s], CF^\delta(Y, \xi + s \cdot \text{PD}[K_\lambda]))$$

has degree zero. We must check that this is compatible with the grading for which

$$h_s^\delta: A_{\xi+s \cdot \text{PD}[K_\lambda]}^\delta(Y, \bar{K}) \cong CF^\delta(Y_{nk \cdot \mu + \lambda}(K), \xi + s \cdot \text{PD}[K_\lambda]) \longrightarrow ([s], CF^\delta(Y, \xi + (s + 1) \cdot \text{PD}[K_\lambda]))$$

has degree zero. But this follows from [Lemma 6.9](#).

Specifically, let $\zeta \in \mathbb{A}_\xi^\delta(Y, \bar{K}; b)$ the element given by [Lemma 6.9](#), now thought of as a homology class in $\mathbb{X}_\xi^\delta(Y, \bar{K}; b)$. Let $\eta \in CF^\delta(Y_\lambda)$ be an element mapped to ζ under the quasi-isomorphism q from $CF^\delta(Y_\lambda)$ to $\mathbb{X}_\xi^\delta(Y, \bar{K}; b)$. We argue that η can be taken to be homogeneous with respect to the grading on $CF^\delta(Y_\lambda)$. To this end, we write out η as a sum of terms $\eta_0 + \eta_1$, where η_0 consists of the component of η in a fixed grading (to be determined in a moment), and η_1 consists of all components of η in other gradings. Here, we choose η_0 so that the component of $q(\eta_0)$ in $H_*(A_{\xi+s \cdot \text{PD}[K_\lambda]}^\delta(Y, \bar{K}))$ for some fixed s is homologous to a_s (as in [Lemma 6.9](#)). One can readily verify that η_0 , which is clearly a cycle as well, projects under q to an element as in [Lemma 6.9](#). \square

With the above proposition in place, the proof of [Theorem 6.1](#) follows from [Theorem 6.4](#) by following the pattern from [\[27\]](#).

Specifically, according to [Proposition 6.7](#), an absolute grading on $CF^\delta(Y_\lambda(K))$ induces absolute gradings on both

$$\bigoplus_{-b \leq s \leq b} A_{\xi+s \cdot \text{PD}[K_\lambda]}^\delta$$

and
$$\bigoplus_{-b+n \leq s \leq b} ([s], CF^\delta(Y, G_{Y, \bar{K}}(\xi + s \cdot \text{PD}[K_\lambda]))) \subset CF^\delta(Y; \mathbb{Z}[\mathbb{Z}/nk\mathbb{Z}]),$$

so that f_1^δ and f_2^δ both are graded maps with degree zero. Let

$$\Pi_s^B: CF^\delta(Y; \mathbb{Z}[\mathbb{Z}/nk\mathbb{Z}]) \longrightarrow CF^\delta(Y, G_{Y, \bar{K}}(\xi + s \cdot \text{PD}[K_\lambda]))$$

denote the projection onto the summand $([s], CF^\delta(Y, \xi + s \cdot \text{PD}[K_\lambda]))$.

Lemma 6.10 *With respect to the above gradings, given $\delta \geq 0$, we have that for any sufficiently large k and $|s| \leq b$, the map $\Pi_s^B \circ H_1^\delta$ is homogeneous of degree $+1$.*

Proof This follows exactly the proof of Lemma 4.8 of [27] □

Proof of Theorem 6.1, in the case where d and n have the same sign This now follows exactly as in [27]. Specifically, Theorem 6.4 provides, for any $\delta > 0$, a quasi-isomorphism between $CF^\delta(Y_\lambda(K), \mathfrak{s})$ and $M(D_s^\delta)$.

There is an integer b with the property that for all $s \geq b$, v_s^+ and h_{-s}^+ are isomorphisms. Thus, we can truncate the mapping cone at this level to obtain a quasi-isomorphic complex. According to Proposition 6.7, the truncated mapping cone inherits a grading, and according to Lemma 6.10, the quasi-isomorphism from Theorem 6.4 respects these gradings. In effect, we have shown that for any $\delta \geq 0$, there is a graded isomorphism of $HF^\delta(Y_\lambda(K))$ with $HF^\delta(M(D_s^\delta))$. It is now a formal consequence (see Lemma 2.7 of [27] for details) that $HF^+(Y_\lambda(K)) \cong HF^+(M(D_s^+))$, as well. □

6.4 The signs of d and n are opposite

In the case where the signs of d and n in Equation (3) do not coincide, we write instead

$$(29) \quad d \cdot \lambda = -n \cdot \mu \in H_1(Y - K; \mathbb{Z})$$

where in this new notation now, both d and n are positive. Letting $m = nk$, we see that in this case, $PD[K_\lambda]$ has order $dk - 1$ in $H^2(Y_{m \cdot \mu + \lambda})$.

Theorem 6.4 is obtained as before, with minor notational changes.

For example, the precise description of the b -truncated complexes (Equation (27)) needs to be modified. Specifically, for fixed relative Spin^c structure ξ , and integers b and $\delta > 0$, we define in this case b -truncated complexes

$$\begin{aligned} \mathbb{A}_\xi^+(Y, \bar{K}; b) &= \bigoplus_{s=-b+1}^b A_{\xi+s \cdot PD[K_\lambda]}^+(Y, \bar{K}), & \mathbb{B}_\xi^+(Y, \bar{K}; b) &= \bigoplus_{s=-b}^b B_{\xi+s \cdot PD[K_\lambda]}^+(Y, \bar{K}), \\ \mathbb{A}_\xi^\delta(Y, \bar{K}; b) &= \bigoplus_{s=-b+1}^b A_{\xi+s \cdot PD[K_\lambda]}^\delta(Y, \bar{K}), & \mathbb{B}_\xi^\delta(Y, \bar{K}; b) &= \bigoplus_{s=-b}^b B_{\xi+s \cdot PD[K_\lambda]}^\delta(Y, \bar{K}). \end{aligned}$$

Again, we can add up v_ξ^+ and h_ξ^+ or v_ξ^δ and h_ξ^δ to define chain maps

$$\begin{aligned} D_{\lambda; b}^+ &: \mathbb{A}_s^+(Y, \bar{K}; b) \longrightarrow \mathbb{B}_s^+(Y, \bar{K}; b), \\ D_{\xi, \lambda; b}^\delta &: \mathbb{A}_s^\delta(Y, \bar{K}; b) \longrightarrow \mathbb{B}_s^\delta(Y, \bar{K}; b), \end{aligned}$$

from which we construct mapping cones $\mathbb{X}_s^+(\lambda; b)$ and $\mathbb{X}_s^\delta(\lambda; b)$.

Proof of Theorem 6.4 when d and n have opposite sign Applying Theorem 6.2 as before, we have a description of $HF^\delta(Y_\lambda(K), \mathfrak{s})$ as the homology of a mapping cone of a map from $CF^\delta(Y_{m \cdot \mu + \lambda}(K))$ to $CF^\delta(Y, \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}])$, which decomposes as

$$(f_2^\delta)': \bigoplus_{s \in \mathbb{Z}/(dk-1)\mathbb{Z}} A_{\xi+s \cdot \text{PD}[K_\lambda]}^\delta(Y, \bar{K}) \longrightarrow \bigoplus_{s \in \mathbb{Z}/dk\mathbb{Z}} B_{\xi+s \cdot \text{PD}[K_\lambda]}^\delta(Y, \bar{K}),$$

defined by adding

$$v_s^\delta: A_{\xi+s \cdot \text{PD}[K_\lambda]}^\delta(Y, \bar{K}) \longrightarrow B_{\xi+s \cdot \text{PD}[K_\lambda]}^\delta(Y, \bar{K})$$

and

$$h_s^\delta: A_{\xi+s \cdot \text{PD}[K_\lambda]}^\delta(Y, \bar{K}) \longrightarrow B_{\xi+(s+1) \cdot \text{PD}[K_\lambda]}^\delta(Y, \bar{K}).$$

Combining Lemma 6.5 with the positivity of d and n in Equation (29), we see that for any ξ , if s is sufficiently large and $\xi' = \xi + s \cdot \text{PD}[K_\lambda]$, we have that $h_{\xi'}^\delta$ is null-homotopic, and $v_{\xi'}^\delta$ is an isomorphism. It follows from this (together with the analogous statement for s sufficiently small) that the mapping cone of $(f_2^\delta)'$ is identified with the mapping cone of

$$(f_2^\delta)'': \bigoplus_{s \in \mathbb{Z}/(dk-1)\mathbb{Z}} A_{\xi+s \cdot \text{PD}[\mu]}^\delta(Y, \bar{K}) \longrightarrow \bigoplus_{s \in \mathbb{Z}/dk\mathbb{Z}} B_{\xi+s \cdot \text{PD}[K_\lambda]}^\delta(Y, \bar{K}),$$

gotten by adding all the maps v_s^δ and h_s^δ . Lemma 6.5 and our hypothesis on the sign of d ensures that this mapping cone is, in turn, identified with the truncated mapping cone $\mathbb{X}_\xi^\delta(Y, \lambda, \bar{K}; b)$. Theorem 6.4 in the present case now follows from an analogue of Proposition 6.6, for negative truncation (whose proof, in turn, follows similarly: note, however that now $\mathbb{X}_\xi^\delta(Y, \lambda, \bar{K}; b)$ is a subcomplex, rather than quotient complex, of the untruncated version $\mathbb{X}_s^\delta(Y, \lambda, \bar{K})$, and it is now the quotient of this complex which is acyclic). \square

To complete the proof of Theorem 6.1, we must verify that the quasi-isomorphism between $CF^\delta(Y_\lambda, \mathfrak{s})$ and the truncated mapping cone $\mathbb{X}_s^\delta(Y, \lambda, \bar{K}; b)$ respects relative gradings. We are using here the same relative grading on the mapping cone $\mathbb{X}_s^+(Y, \lambda, \bar{K})$, as before, and this gives the truncated version $\mathbb{X}_\xi^\delta(Y, \lambda, \bar{K}; b)$ an analogous relative grading. We find it more convenient to work with negative-definite cobordisms so we will be working with the map from $\mathbb{X}_s^\delta(\lambda; b)$ to $CF^\delta(Y_\lambda(K))$ with component f_3^δ (induced from the map ψ^+ in the proof of Theorem 6.2).

Thus, instead of studying the grading properties of f_1^+ (as we did in Lemma 6.8), we study f_3^+ , the map associated to the cobordism W_λ from Y to Y_λ . To this end, we have the following:

Lemma 6.11 Fix an integer $\delta \geq 0$ and constant C_0 . For all sufficiently large k , the following condition holds. For each $\mathfrak{s}_0 \in \text{Spin}^c(W_\lambda(K))$, there is at most one Spin^c structure of the form $\mathfrak{s} = \mathfrak{s}_0 + jk \text{PD}[\hat{F}]$ with $j \in \mathbb{Z}$ for which

$$C_0 \leq c_1(\mathfrak{s})^2 + nk.$$

Here, \hat{F} denotes a generator of $H^2(W_\lambda(K), Y)$.

Proof This follows along the lines of Lemma 6.8, bearing in mind that now W_λ is a negative-definite cobordism. \square

The relevance of the lemma is the following: $f_3^\delta(\xi \otimes T^i)$ is a sum of the maps associated to Spin^c structures \mathfrak{s} over the cobordism $W_\lambda(K)$ which differ by addition of $nk \text{PD}[\mu] = dk \text{PD}[\lambda]$, under the identification $H^2(Y, K; \mathbb{Z}) \cong H^2(W_\lambda(K))$. By Equation (7), the latter in turn correspond to $dk\mathbb{Z} \cdot \hat{F}$ orbits.

It follows from Lemma 6.11 that if k is sufficiently large, then given $\mathfrak{t} \in \text{Spin}^c(Y)$ and $i \in \mathbb{Z}/nk\mathbb{Z}$, there is a unique Spin^c structure on W_λ which contributes nontrivially to $f_3^\delta(\xi \cdot T^i)$ for any $\xi \in HF^\delta(Y, \mathfrak{t})$. Specifically, according to Equation (9), the nontriviality of the map places a lower bound (independent of k) on the square of the first Chern class of any such Spin^c structure (cf Corollary 4.7).

We are now able to use the absolute grading on $Y_\lambda(K)$ to fix an absolute grading on $\mathbb{B}_\xi^\delta(Y, \bar{K}; b)$; ie it is the one for which f_3^δ has degree zero. We check that this compatible with the relative grading on $\mathbb{X}^\delta(Y, \bar{K}; b)$ in the spirit of Proposition 6.7. In fact, we wish to show that the map

$$\psi^\delta: \mathbb{X}_\xi^\delta(Y, \bar{K}; b) \simeq M(f_2^\delta) \longrightarrow CF^\delta(Y_\lambda(K))$$

induced by $\psi^\delta(x, y) = H_2^\delta(x) + f_3^\delta(y)$ is a relatively graded map.

In the present case, the map $D_{\xi, \lambda; b}^\delta$ is injective (rather than surjective, as it was before). Indeed, we can always find a homogeneous cycle representing a nontrivial homology class in the cokernel of $D_{\xi, \lambda; b}^\delta$ (using nontriviality of HF^∞ ; see the proof of Lemma 6.9); indeed, we can find a representative of this homology class supported in any of the $b + 1$ summands of $\mathbb{B}_\xi^\delta(Y, \bar{K}; b)$. It follows now that any of these representatives is taken to a fixed, homogeneous homology class in $CF^\delta(Y_\lambda(K))$. This guarantees compatibility of the relative gradings of the relative gradings on $\mathbb{B}_\xi^\delta(Y, \bar{K}; b)$ induced from the mapping cone with its relative gradings induced from the condition that f_3^δ is homogeneous of degree 0.

The preceding discussion shows that the component f_3^δ of the map ψ^δ is homogeneous. Seeing that the map H_2^δ also respects the relative grading follows similarly: it, too,

is induced by a negative definite cobordism, where at most one Spin^c structure can contribute, and the above argument shows that its grading shift is compatible with all the other gradings.

Having established that ψ^δ is a graded quasi-isomorphism, the proof of [Theorem 6.1](#) proceeds as before.

7 The proof of [Theorem 1.1](#)

If $K \subset Y$ is a null-homologous knot, then $Y_{p/q}(K)$ can be realized by surgery with coefficient a inside the knot $K \# O_{q/r} \subset Y \# (-L(q, r))$, where a is the greatest integer smaller than or equal to p/q , $a = \lfloor p/q \rfloor$, and

$$\frac{p}{q} = a + \frac{r}{q},$$

and $O_{q/r} \subset -L(q, r)$ is the knot which is obtained by viewing one component of the Hopf link as a knot inside the lens space $-L(q, r)$, thought of as $-q/r$ surgery on the other component of the Hopf link; cf [Figure 2](#).

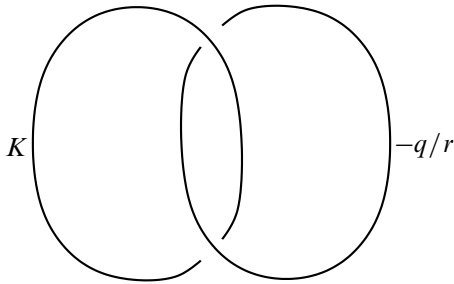


Figure 2: Thinking of K as a knot in the lens space obtained by performing $-q/r$ on the other component of this link, we obtain the knot $O_{q/r}$.

In view of these remarks, [Theorem 1.1](#) is proved by combining a model calculation of the knot Floer homology of $O_{q/r}$, the Künneth principle for connected sums, together with the surgery formula of [Theorem 6.1](#).

Of course, the complement of $O_{q/r} \subset -L(q, r)$ is a solid torus, and consequently, there is an affine identification $\underline{\text{Spin}}^c(-L(q, r), O_{q/r}) \cong \mathbb{Z}$.

Lemma 7.1 *The knot $O_{q/r} \subset -L(q, r)$ is a U -knot. Indeed, there is an affine identification ϕ fitting into a commutative diagram*

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\phi} & \underline{\text{Spin}}^c(L(q, r), O_{q/r}) \\ \downarrow & & \downarrow G_{L(q, r), O_{q/r}} \\ \mathbb{Z}/q\mathbb{Z} & \xrightarrow{\cong} & \text{Spin}^c(L(q, r)) \end{array}$$

(where the left vertical arrow is the natural quotient map) with the property that

$$\widehat{\text{HFK}}(L(q, r), O_{q/r}, \phi(i)) \cong \begin{cases} \mathbb{Z} & \text{if } 0 \leq i \leq q-1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof Consider the standard genus one Heegaard decomposition of the lens space $-L(q, r)$, where α is a curve of slope $0/1$ and β is a curve of slope $-q/r$. Placing two basepoints w and z , separated by an arc which is disjoint from β and meets α exactly once, we obtain a doubly-pointed Heegaard diagram for $O_{q/r} \subset -L(q, r)$. Of course $\alpha \cap \beta$ consists of exactly q points $\{x_0, \dots, x_{q-1}\}$, each one representing a different Spin^c structure over $L(q, r)$. See Figure 3 for an illustration.

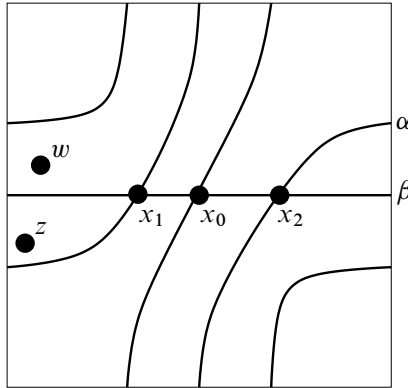


Figure 3: Heegaard diagram for $O_{3/2}$. The intersection points x_0, x_1, x_2 are ordered so that $\epsilon(x_0, x_1) = \epsilon(x_1, x_2)$ is a generator of the homology of $L(3, 2) - O_{3/2}$.

Now, we have that

$$(30) \quad H_1(-L(q, r) - O_{q/r}) \cong \frac{H_1(T^2 - w - z)}{\mathbb{Z} \cdot \alpha + \mathbb{Z} \cdot \beta} \cong \frac{H_1(T^2)}{\mathbb{Z} \cdot \beta} \cong \mathbb{Z}.$$

We order the intersection points $\{x_i\}_{i=0}^{q-1}$ in the order they are encountered as we traverse the α circle in the complement of the arc from w to z . It is easy to see that

with respect to this ordering, we have that $\epsilon(x_i, x_{i+1}) = \gamma$, where here γ is some fixed generator of $H_1(-L(q, r) - O_{q/r}) \cong \mathbb{Z}$. Let $\phi(i)$ correspond to $\underline{x}_{w,z}(x_i)$. The lemma follows. \square

Suppose that K is a null-homologous knot in an integral homology three-sphere Y . In this case, $H^2(Y, K) \cong \mathbb{Z}$, and also $H^2(Y \# -L(q, r), K \# O_{q/r}) \cong \mathbb{Z}$.

Lemma 7.2 *Suppose that $K \subset Y$ is a null-homologous knot in an integer homology three-sphere. Under the connected sum*

$$(Y, K) \# (-L(q, r), O_{q/r}) \longrightarrow (Y \# -L(q, r), K \# O_{q/r}),$$

the following diagram commutes:

$$\begin{array}{ccc} \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{f} & \mathbb{Z} \\ \cong \downarrow & & \downarrow \cong \\ H^2(Y, K) \oplus H^2(-L(q, r), O_{q/r}) & \longrightarrow & H^2(Y \# -L(q, r), K \# O_{q/r}), \end{array}$$

where $f(x, y) = qx + y$. Moreover, under this correspondence, if K_λ is the push-off of K with respect to the integral framing a , then $\text{PD}[K_\lambda]$ represents the element $p \in \mathbb{Z} \cong H^2(Y \# -L(q, r), K \# O_{q/r})$, where $p/q = a + q/r$.

Proof Let m and ℓ denote the meridian and the longitude of $O_{q/r}$. We have an isomorphism $H^2(-L(q, r), O_{q/r}) \cong \mathbb{Z}$ under which m and ℓ are mapped to q and r respectively. Now in $Y \# -L(q, r) - K \# O_{q/r}$, clearly the meridian of K is homologous to a meridian for $O_{q/r}$, and hence it is mapped to q under the isomorphism. Moreover, the push-off of K_λ is homologous $a \cdot m + \ell$, which in turn is mapped to $aq + r = p$, under this above map. \square

Proof of Theorem 1.1 Fix an identification $\text{Spin}^c(Y \# -L(q, r), K \# O_{q/r}) \cong \mathbb{Z}$, and correspondingly think of $A_s^+(Y \# -L(q, r), K \# O_{q/r})$ as indexed by $s \in \mathbb{Z}$. Since $O_{q/r}$ is a U -knot (cf Lemma 7.1), the Künneth principle for connected sums, in the form of Corollary 5.3 applies to show that

$$A_s^+(Y \# -L(q, r), K \# O_{q/r}) \simeq A_{g(s)}^+(Y)$$

for some $g: \mathbb{Z} \rightarrow \mathbb{Z}$ (where here \simeq means filtered quasi-isomorphic). Indeed, according to Lemma 7.2, the formula $g(s)$ satisfies $s = qg(s) + j$ where, according to Lemma 7.1, $0 \leq j < q$; ie

$$g(s) = \left\lfloor \frac{s}{q} \right\rfloor.$$

Recall also that $\text{PD}[K_\lambda]$ represents p times a generator of $H^2(Y\#-L(q, r), K\#O_{q/r})$.

With these remarks in place, [Theorem 1.1](#) is obtained as a direct application of [Theorem 6.1](#). \square

7.1 The case of \widehat{HF}

Some of the algebra is simpler when one considers HF^+ , rather than \widehat{HF} . To this end, we let

$$\widehat{D}_{i,p/q}: \widehat{A}_i \longrightarrow \widehat{B}_i,$$

that is,
$$\widehat{D}_{i,p/q}\{a_{\lfloor(i+ps)/q\rfloor}\}_{s\in\mathbb{Z}} = \{b_{\lfloor(i+ps)/q\rfloor}\}_{s\in\mathbb{Z}},$$

where here

$$b_{\lfloor(i+ps)/q\rfloor} = \widehat{v}_{\lfloor(i+ps)/q\rfloor}(a_{\lfloor(i+ps)/q\rfloor}) + \widehat{h}_{\lfloor(i+p(s-1))/q\rfloor}(a_{\lfloor(i+p(s-1))/q\rfloor}).$$

The proof of [Theorem 1.1](#) adapts readily to this context to give the following:

Theorem 7.3 *Let $K \subset Y$ be a null-homologous knot. There is a relatively graded isomorphism of groups*

$$H_*(\widehat{\mathbb{X}}_{i,p/q}) \cong \widehat{HF}(Y_{p/q}(K), i)$$

for each $i \in \mathbb{Z}/p\mathbb{Z}$.

7.2 Absolute gradings

In fact, [Theorem 1.1](#) (and indeed [Theorem 6.1](#)) actually determines $HF^+(Y_{p/q}(K), i)$ as an absolutely graded group.

An absolute grading on $\mathbb{X}_{i,p/q}^+(K)$ compatible with the relative grading is specified by fixing an absolute grading on $B_{\lfloor(i+ps)/q\rfloor}^+$ (for any s), thought of as a subcomplex of $\mathbb{X}_{i,p/q}^+(K)$ (note that it is independent of the choice of K). This absolute grading in turn is in turn determined by a grading on its homology $H_*(B_{\lfloor(i+ps)/q\rfloor}^+) \cong \mathcal{T}^+$. Finally, that datum is fixed by the requirement $H_*(\mathbb{X}_{i,p/q}^+(O)) \cong \mathcal{T}^+$ in such a way that its bottom-most nontrivial element is supported in dimension $d(S_{p/q}^3(O), i) = d(L(p, q), i)$. (Explicit, recursive formulas for these rational numbers can be found in [Proposition 4.8](#) of [\[15\]](#).)

This assertion follows easily from the proof of [Theorem 1.1](#). Specifically, as in [Section 6.3](#), the composite of

$$f_1^\delta: CF^\delta(Y_\lambda(K)) \longrightarrow CF^\delta(Y_{nk\cdot\mu+\lambda}(K))$$

with the projection onto $CF^\delta(Y_{nk \cdot \mu + \lambda}(K), \mathfrak{t}) \subset CF^\delta(Y_{nk \cdot \mu + \lambda}(K))$ is a homogeneous map (provided that n is sufficiently large). Moreover, its degree depends on the intersection form of the cobordism $X(k)$ and the first Chern class of \mathfrak{t} . Moreover, the proof of [Theorem 6.4](#) shows that this map is also identified with the projection $\mathbb{X}_s^\delta(\lambda) \rightarrow A_\xi^\delta(Y, \bar{K})$. Since the intersection form of $X(k)$ does not depend on the particular knot \bar{K} , the claim follows.

8 Knots which admit L -space surgeries

Suppose that $K \subset S^3$ is a knot which admits an L -space surgery with positive slope R . Examples of such knots include all torus knots, and also knots from Berge’s list [\[1\]](#). Moreover, an alternating knot K with unknotting number equal to one gives rise to another knot $C \subset S^3$ which admits an L -space surgery. This new knot C is obtained by performing the unknotting operation, but connecting the two strands which were crossed by an arc γ ; C then is the branched double-cover γ , compare [\[23\]](#).

[Theorem 1.2](#) says that for such a knot, the Alexander polynomial is determined from the correction terms. In this application, we find it convenient to use the group HF^+ ; to do this, recall the characterization of L -spaces in terms of this group [\[24\]](#): a rational homology three-sphere Y is an L -space if and only if for each $\mathfrak{t} \in \text{Spin}^c(Y)$,

$$HF^+(Y, \mathfrak{t}) \cong \mathcal{T}^+.$$

The correction terms $d(Y, \mathfrak{t})$ of an L -space is the minimal grading of any homogeneous element of $HF^+(Y, \mathfrak{t})$.

The symmetrized Alexander polynomial $\Delta_K(T)$ plays a role, since it is the Euler characteristic of the knot Floer homology; ie

$$\sum_s \chi(\widehat{HF}K_*(S^3, K, s)) \cdot T^s = \Delta_K(T)$$

(cf Equation (1) of [\[19\]](#) or [\[28\]](#)).

Proof of [Theorem 1.2](#) It is easy to see that both maps $v_s^+ : H_*(A_s^+) \rightarrow H_*(B^+)$ and $h_s^+ : H_*(A_s^+) \rightarrow H_*(B^+)$ are surjective, since $H_*(B^+) \cong HF^+(S^3)$, and both maps are isomorphisms in all sufficiently large degrees.

Fix $i \in \mathbb{Z}/p\mathbb{Z}$, and let $s \in \mathbb{Z}$ be an arbitrary representative. Split $A_i^+ = A_s^+ \oplus J$ where $J = \bigoplus_{t \neq s} A_{\lfloor (i+tp)/q \rfloor}^+$. From the above remarks, it is clear that the restriction of $D_{i,p/q}^+$ to $H_*(J)$ surjects onto $H_*(\mathbb{B}_i^+)$. It follows at once that there is a surjection

$$\varphi : \text{Ker}(H_*(D_{i,p/q}^+) : H_*(A_i^+) \rightarrow H_*(\mathbb{B}_i^+)) \rightarrow H_*(A_s^+).$$

Moreover, it follows from this surjectivity, combined with [Theorem 1.1](#), that

$$\text{Ker} (H_*(D_{i,p/q}^+): H_*(\mathbb{A}_i^+) \longrightarrow H_*(\mathbb{B}_i^+)) \cong HF^+(S_{p/q}^3(K), i) \cong \mathcal{T}^+.$$

The surjectivity of φ , together with the fact that $H_*(A_s^+) \cong H_*(B^+)$ in all sufficiently large degree combine to show that $H_*(A_s^+) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathcal{T}^+ \otimes_{\mathbb{Z}} \mathbb{Q}$. Applying the same argument, only taking coefficients in $\mathbb{Z}/p\mathbb{Z}$ for arbitrary p shows that $H_*(A_s^+) \cong \mathcal{T}^+$.

Having established that $H_*(A_s^+) \cong \mathcal{T}^+$, and that v_s^+ and h_s^+ are isomorphisms in all sufficiently large degrees, it follows that v_s^+ is modeled on multiplication by U^{V_s} , and h_s^+ is multiplication by U^{H_s} , where all $V_s, H_s \geq 0$.

The condition that $\text{Ker} (H_*(D_{i,p/q}^+): H_*(\mathbb{A}_i^+) \longrightarrow H_*(\mathbb{B}_i^+)) \cong \mathcal{T}^+$ ensures readily that in each $i \in \mathbb{Z}/p\mathbb{Z}$, there can be at most one integer $s \in \mathbb{Z}$ for which both $V_{\lfloor (i+ps)/q \rfloor}$ and $H_{\lfloor (i+ps)/q \rfloor} \geq 0$. Our aim now is to determine which value of s has this property.

Let $m(s) = \min(V_s, H_s)$. We claim that

$$(31) \quad m(s_1) \leq m(s_2) \quad \text{if} \quad s_1 \leq s_2 \leq 0 \text{ or } s_1 \geq s_2 \geq 0.$$

To this end, we have an exact sequence

$$0 \longrightarrow C\{i < 0 \text{ and } j \geq s\} \longrightarrow C\{\max(i, j - s) \geq 0\} \xrightarrow{v^+} C\{i \geq 0\} \longrightarrow 0.$$

Since $\chi(\widehat{HFK}(K, s)) = a_s$, it follows at once that for $s \geq 0$,

$$\chi(C\{i < 0 \text{ and } j \geq s\}) = t_s(K),$$

which in turn is the same as V_s . Similarly, $\chi(C\{i \geq 0 \text{ and } j < s\}) = t_s(K) + s$, which is H_s . In particular, this (together with a symmetric argument for $s \leq 0$) shows that

$$(32) \quad V_s \leq H_s \quad \text{for } s \geq 0 \quad \text{and} \quad V_s \geq H_s \quad \text{for } s \leq 0.$$

Incidentally, we have just established that

$$(33) \quad t_s = m(s) \geq 0.$$

Note that there is a natural quotient map $A_s^+ \longrightarrow A_{s+1}^+$, and indeed, the projection of $A_s^+ \longrightarrow B_s^+$ factors through this quotient. It follows at once that $\{V_s\}_{s \in \mathbb{Z}}$ is a nonincreasing sequence in s . Dually, $\{H_s\}_{s \in \mathbb{Z}}$ is a nondecreasing sequence in s . Combining this with [Equation \(32\)](#), [Equation \(31\)](#) follows.

Having established that for each $i \in \mathbb{Z}/p\mathbb{Z}$, there is exactly one nonzero integer among the $\{m(\lfloor (i + ps)/q \rfloor)\}_{s \in \mathbb{Z}}$, [Equation \(31\)](#) together with the fact that $m(s) = m(-s)$ (an easy consequence of [Equation \(33\)](#)) shows that whenever $m(\lfloor i/q \rfloor) \neq 0$, we have

$$\left\lfloor \left\lfloor \frac{i}{q} \right\rfloor \right\rfloor \leq \left\lfloor \left\lfloor \frac{p}{2q} \right\rfloor \right\rfloor.$$

Finally, note that the bottom-most generator of $A_{\lfloor i/q \rfloor}^+$ is an element whose degree is $2m(\lfloor i/q \rfloor)$ less than the corresponding generator for the unknot. In view of the remarks from Section 7.2, the theorem now follows. \square

Lemma 8.1 *Let $K \subset S^3$ be a knot. The map $v_s^+ : A_s^+ \rightarrow B^+$ is an isomorphism on homology for all $s \geq 0$ if and only if $\widehat{HFK}(K, s) = 0$ for all $s > 0$.*

Proof It is easy to see by descending induction on s that v_s^+ is an isomorphism on homology for all $s \geq d$ if and only if $\widehat{HFK}(K, s) = 0$ for all $s > d$. \square

Proof of Corollary 1.3 Suppose that $S_{p/q}^3(K) \cong S_{p/q}^3(O)$. Then, for some permutation $\sigma : \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z}$, we have that $HF^+(S_{p/q}^3(O), i) \cong HF^+(S_{p/q}^3(K), \sigma(i))$ for all i . It follows from the proof of Theorem 1.2 (Equation (33)) that $t_i(K) \geq 0$ for all i . Summing Equation (2) over all i and using the hypothesis that $S_{p/q}^3(K) \cong S_{p/q}^3(O)$, we conclude that $t_i(K) \equiv 0$. It follows that $V_s = 0$ for $s \geq 0$, ie $v_s^+ : H_*(A_s^+) \rightarrow HF^+(S^3)$ is an isomorphism for all $s \geq 0$. According to Lemma 8.1, $\widehat{HFK}(K, s) = 0$ for all $s \neq 0$, and hence, according to [18], K is the unknot. \square

Proof of Corollary 1.4 If $g \neq 0$, according to [18], $\widehat{HFK}(K, g) \neq 0$. Thus, according to Lemma 8.1, v_{g-1}^+ is not an isomorphism on homology, and hence $t_{g-1} \neq 0$. We can immediately conclude from the statement of Theorem 1.2 that $|r| \geq 2g - 2$. This bound can be improved to $|r| \geq 2g - 1$, by looking more carefully at the proof of Theorem 1.2.

Suppose that $|r| < 2g - 1$. In view of Equation (31) and the fact that $m(s) = m(-s)$, we see that for $i = q(g - 1)$, both $m(\lfloor i \rfloor) \neq 0$ and $m(\lfloor (i - p)/q \rfloor) \neq 0$. But in the proof of Theorem 1.2, it is shown that if $S_r^3(K)$ is an L -space, then there is at most one $s \in \mathbb{Z}$ for which both $V_{\lfloor (i+ps)/q \rfloor}$ and $H_{\lfloor (i+ps)/q \rfloor}$ is zero. This contradiction shows that for $S_r^3(K)$ to be an L -space, we must have that $|r| \geq 2g - 1$. \square

9 On cosmetic surgeries

Under favorable circumstances, the existence of an orientation-preserving homeomorphism $\widehat{HF}(S_r^3(K)) \cong \widehat{HF}(S_s^3(K))$ for distinct r and s forces the knot Floer homology of K to agree with that for the unknot, and hence for the knot to be unknotted. This is not always the case, though. For example, the knot $K = 9_{44}$ has the property that $HF^+(S_{+1}^3(K)) \cong HF^+(S_{-1}^3(K))$, although $S_{+1}^3(K) \not\cong S_{-1}^3(K)$. Specifically, according to Theorem 6.1 of [22], the knot 9_{44} has

$$\widehat{HFK}(K, i) = \mathbb{Z}^{|ai| \binom{a}{i}}$$

where the subscript indicates the degree in which the summand is supported, and a_i is the i -th coefficient of the Alexander polynomial

$$\Delta_K(T) = \sum_{i \in \mathbb{Z}} a_i \cdot T^i = T^{-2} - 4T^{-1} + 7 - 4T + T^2.$$

It is now a straightforward application of the surgery formula, either in the form given in the present paper or from [27], that

$$HF^+(S^3_{+1}(K)) \cong \mathcal{T}_{(0)}^+ \oplus \mathbb{Z}_{(0)}^2 \oplus \mathbb{Z}_{(-1)}^2 \cong HF^+(S^3_{-1}(K)).$$

On the other hand, Walter Neumann [14] informs us that the manifolds $S^3_{+1}(K)$ and $S^3_{-1}(K)$ can be distinguished (using the computer program Snap [3]; see also SnapPea [33]) by their hyperbolic volume; one has hyperbolic volume roughly 5.52, the other has hyperbolic volume roughly 5.27.

Our results here can be proved by restricting attention to Floer homology with coefficients in any field \mathbb{F} , which we suppress from the notation. For concreteness, we restrict to $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, so that eg $\widehat{HF}(S^3)$ denotes the homology $H_*(\widehat{CF}(S^3) \otimes_{\mathbb{Z}} \mathbb{F})$. Indeed, we work here throughout the present section with \widehat{HF} (rather than HF^+).

Definition 9.1 Define $\nu(K)$ by

$$\nu(K) = \min\{s \in \mathbb{Z} \mid \widehat{v}_s: \widehat{A}_s \longrightarrow \widehat{CF}(S^3) \text{ induces a nontrivial map in homology}\}.$$

Lemma 9.2 If $K \subset S^3$ and $m(K)$ denotes its mirror, then either $\nu(K)$ or $\nu(m(K))$ is nonnegative.

Proof Let $\{\mathcal{F}_s\}_{s \in \mathbb{Z}}$ be the knot filtration of $\widehat{CF}(S^3)$ (ie in the notation of Section 3 $\mathcal{F}_s = C_{\xi_0}\{j < s\}$, where $\xi_0 \in \text{Spin}^c(S^3, K)$ is the relative Spin^c structure with trivial first Chern class) and let

$$\tau(K) = \min\{s \in \mathbb{Z} \mid H_*(\mathcal{F}_s) \longrightarrow \widehat{HF}(S^3) \text{ is nontrivial}\}.$$

Recall that $\tau(K) = -\tau(m(K))$ and also that

$$(34) \quad \nu(K) = \tau(K) \text{ or } \tau(K) + 1$$

(cf Lemma 3.3 of [16] and Proposition 3.1 from the same reference respectively; or alternatively, see [28]). □

The integer $\nu(K)$ dictates the maps on homology induced by all the \widehat{h}_s and \widehat{v}_s , according to the following two lemmas.

Lemma 9.3 For all $s \geq \nu(K)$, \widehat{v}_s induces a nontrivial map in homology.

Proof This follows at once since the image of \widehat{v}_s is \mathcal{F}_s , and $\mathcal{F}_s \subseteq \mathcal{F}_t$ if $s \leq t$. □

Lemma 9.4 *If $\nu(K) \geq 0$, then for all $s > 0$, $(\widehat{h}_s)_* = 0$.*

Proof The image of $\widehat{h}_s: \widehat{A}_s \rightarrow \widehat{B}$ is identified with $\mathcal{F}_{-s} \subset \widehat{B}$, thus, if $s > 0$ and $\tau(K) \geq 0$, then the map on homology induced by \widehat{h}_s is trivial. \square

For knots with $\nu(K) = 0$, we will need also to understand the behaviour of

$$(\widehat{h}_0 \oplus \widehat{v}_0)_*: H_*(\widehat{A}_0) \longrightarrow H_*(\widehat{B} \oplus \widehat{B}) \cong \mathbb{F} \oplus \mathbb{F}.$$

Recall that the total rank of $\widehat{HF}(Y)$ is independent of the orientation of Y :

$$(35) \quad \text{rank } \widehat{HF}(-Y) = \text{rank } \widehat{HF}(Y).$$

(Indeed, in Proposition 2.5 of [20], it is shown that $\widehat{HF}^*(-Y) \cong \widehat{HF}_*(Y)$.) We apply this principle, together with the observation that

$$(36) \quad S_r^3(K) \cong -S_{-r}^3(m(K)).$$

Lemma 9.5 *Let $K \subset S^3$ be a knot, and suppose that $\nu(K) = \nu(m(K)) = 0$. Then, the image of*

$$(\widehat{h}_0 \oplus \widehat{v}_0)_*: H_*(\widehat{A}_0) \longrightarrow H_*(\widehat{B} \oplus \widehat{B}) \cong \mathbb{F} \oplus \mathbb{F}$$

is one-dimensional; indeed, it is spanned by the diagonal element.

Proof Let $\delta(K)$ denote the dimension of the image of

$$(\widehat{h}_0 \oplus \widehat{v}_0)_*: H_*(\widehat{A}_0) \longrightarrow H_*(\widehat{B} \oplus \widehat{B}) \cong \mathbb{F} \oplus \mathbb{F},$$

and let $a_0(K)$ denote the dimension of $H_*(\widehat{A}_0(K))$. We verify that $\delta(K) = 1$.

For large enough integral surgeries N , we have that

$$(37) \quad \text{rank } \widehat{HF}(S_N^3(K), 0) = a_0(K).$$

(This follows the rational surgery formula; but in fact it also follows simply from Theorem 4.1.) Similarly, $\widehat{CF}(S_{-N}^3(K), 0)$ is described as the mapping cone of

$$\widehat{h}_0 \oplus \widehat{v}_0: \widehat{A}_0(K) \longrightarrow \widehat{B} \oplus \widehat{B}.$$

Thus, we have

$$(38) \quad \text{rank } \widehat{HF}(S_{-N}^3(K), 0) = a_0(K) + 2 - 2\delta(K).$$

Consider the equations

$$\begin{aligned} \text{rank}(\widehat{HF}(S_N^3(K), 0)) &= \text{rank}(\widehat{HF}(S_{-N}^3(m(K), 0))), \\ \text{rank}(\widehat{HF}(S_{-N}^3(K), 0)) &= \text{rank}(\widehat{HF}(S_{-N}^3(m(K), 0))) \end{aligned}$$

(both of which are straightforward combinations of Equations (35) and (36)), and substitute Equations (37) and (38) to conclude that

$$\begin{aligned} a_0(K) &= a_0(m(K)) + 2 - 2\delta(m(K)), \\ a_0(m(K)) &= a_0(K) + 2 - 2\delta(K), \end{aligned}$$

from which it follows at once that

$$(39) \quad \delta(K) + \delta(m(K)) = 2.$$

Now, $\nu(K) = 0$ implies that $\delta(K) > 0$ (since \hat{v}_0 induces a nontrivial map on homology). This, together with Equation (39) implies that $\delta(K) = \delta(m(K)) = 1$.

By symmetry, we know that $H_*(\hat{B} \oplus \hat{B})$ projects nontrivially to each factor. Since $\delta(K) = 1$, we conclude that the image of $(\hat{h}_0 \oplus \hat{v}_0)_*$ is the diagonal subspace. \square

We would like to calculate the rank of $\widehat{HF}(S^3_{p/q}(K))$. It suffices to consider knots K with $\nu(K) \geq \nu(m(K))$, since $\text{rank } \widehat{HF}(S^3_{p/q}(K)) = \text{rank } \widehat{HF}(S^3_{-p/q}(m(K)))$ (by Equations (35) and (36)).

Proposition 9.6 *Let $K \subset S^3$ be a knot, and fix a pair of relatively prime integers p and q with $p \neq 0$ and $q > 0$, and suppose that $\nu(K) \geq \nu(m(K))$.*

If $\nu(K) > 0$ or $p > 0$, then

$$(40) \quad \begin{aligned} &\text{rank } \widehat{HF}(S^3_{p/q}(K)) \\ &= p + 2 \max(0, (2\nu(K) - 1)q - p) + q \left(\sum_s (\text{rank } H_*(\hat{A}_s) - 1) \right). \end{aligned}$$

If $\nu(K) = 0$ and $p < 0$, then

$$(41) \quad \text{rank } \widehat{HF}(S^3_{p/q}(K)) = |p| + q \left(\sum_s (\text{rank } H_*(\hat{A}_s) - 1) \right).$$

Proof In view of Equation (34), either $\nu(K) > 0$ or $\nu(K) = \nu(m(K)) = 0$. Using Theorem 7.3 to express $\widehat{HF}(S^3_{p/q}(K))$ in terms of $H_*(\widehat{\mathbb{X}}_{p/q})$, calculating the rank of $H_*(\widehat{\mathbb{X}}_{p/q})$ is now straightforward application of Lemma 9.3 and Lemma 9.4 when $\nu(K) > 0$, and furthermore Lemma 9.5 when $\nu(K) = 0$. Indeed, in the case where $\nu(K) > 0$, the cokernel \hat{D} has rank $\max(0, (2\nu(K) - 1)q - p)$, while its kernel has rank

$$p + \max(0, (2\nu(K) - 1)q - p) + q \left(\sum_s (\text{rank } H_*(\hat{A}_s) - 1) \right).$$

In the case where $\nu(K) = 0$ and $p > 0$, its kernel has rank the cokernel is trivial, and kernel has rank

$$p + q \left(\sum_s (\text{rank } H_*(\hat{A}_s) - 1) \right)$$

(ie these formulas agree with the formulas when $\nu(K) > 0$). In the case where $\nu(K) = 0$ and $p < 0$, the cokernel has rank $|p|$, while its kernel has rank

$$q \left(\sum_s (\text{rank } H_*(\hat{A}_s) - 1) \right). \quad \square$$

(The above proposition holds even in the case where p and q are not relatively prime, with the understanding that if $(p, q) = a$, then $\widehat{HF}(S^3_{p/q}(K))$ denotes the direct sum of a many copies of $\widehat{HF}(S^3_{p'/q'}(K))$, where here $p' = p/a, q' = q/a$.)

The following result is analogous to [Lemma 8.1](#).

Proposition 9.7 *Let K be a knot with $\nu(K) = 0$ and $\text{rank } H_*(\hat{A}_s(K)) = 1$ for all $s \in \mathbb{Z}$. Then, K is the unknot. More generally,*

$$g(K) = \max(\nu(K), \{s \in \mathbb{Z} \mid \text{rank } H_*(\hat{A}_{s-1}) > 1\}).$$

Proof Recall [\[18\]](#) that

$$g(K) = \max\{t \in \mathbb{Z} \mid \widehat{HFK}(K, t) \neq 0\}.$$

This readily implies that the map $\hat{\nu}: \hat{A}_s \rightarrow \widehat{CF}(S^3)$ induces an isomorphism in homology for all $s \geq g(K)$: \hat{A}_s has a subcomplex $C\{i < 0, j = s\}$, the kernel of $\hat{\nu}_s$, which is easily seen to be acyclic (as it is filtered by subcomplexes whose associated graded has homology isomorphic to $\bigoplus_{t>s} \widehat{HFK}(K, t)$), and moreover its image in $C\{i = 0\}$ consists of the subcomplex $C\{i = 0, j \leq s\}$, whose quotient is acyclic.

Thus, we have proved that

$$g(K) \geq \max(\nu(K), \{s \in \mathbb{Z} \mid \text{rank } H_*(\hat{A}_s) > 1\}).$$

Now consider

$$0 \rightarrow C\{i < 0, j \geq g-1\} \rightarrow C\{\max(i, j-g+1) \geq 0\} \xrightarrow{\nu_{g-1}^+} C\{i \geq 0\} \rightarrow 0,$$

where the middle term here is, of course A_{g-1}^+ . Since $HFK(K, t) = H_*(C\{(0, t)\}) = 0$ for all $t > g$, it follows from the natural filtration that $H_*(C\{i < 0, j \geq g-1\}) \cong H_*(\{(-1, g-1)\}) \cong H_*(C\{0, g\}) = \widehat{HFK}(K, g) \neq 0$. It follows at once that $\hat{\nu}_{g-1}$ is not an isomorphism. It follows at once that either the map is not surjective, in which case $\nu(K) = g-1$, or it has kernel, in which case $\text{rank } H_*(\hat{A}_{g-1}) > 1$. \square

Theorem 9.8 *Let $K \subset S^3$ be a knot, and suppose that r and s are distinct rational numbers with the property that $S_r^3(K) \cong \pm S_s^3(K)$. Then, either $S_r^3(K)$ is an L -space, or r and s have opposite signs.*

Proof Since $H_1(S_{\pm p/q}^3(K); \mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$, we can fix p throughout. Now, according to Proposition 9.6, for fixed p , and positive integral q , $\text{rank } \widehat{HF}(S_{p/q}^3(K))$ is a monotone nondecreasing function of q . In fact, the function is strictly monotone except possibly for sufficiently small q , for which the rank is p . But this ensures that $S_{p/q}^3(K)$ is an L -space. The same remarks hold for the function $\text{rank } \widehat{HF}(S_{-p/q}^3(K))$ for fixed p and positive, integral q . Since the total rank of $\widehat{HF}(Y)$ is an invariant of the underlying (unoriented) three-manifold, cf Equation (35), the result holds. \square

Proposition 9.9 *If K is a nontrivial knot with $\nu(K) = \nu(m(K)) = 0$, then if there are rational numbers $r, s \in \mathbb{Q}$ with $r \neq s$ and $S_r^3(K) \cong S_s^3(K)$, then $r = \pm s$.*

Proof According to Proposition 9.6, if $\text{rank } \widehat{HF}(S_{p/q}^3(K)) = \text{rank } \widehat{HF}(S_{p/q'}^3(K))$ for $q' \neq \pm q$, then $\text{rank } H_*(\widehat{A}_s) = 1$ for all s . Now, in view of Proposition 9.7, K is the unknot, contrary to our assumption. Thus, it follows that for a cosmetic surgery on K , $r = -s$. \square

Theorem 9.10 *Let $K \subset S^3$ be a knot with Seifert genus equal to one. Then if $S_r^3(K) \cong S_s^3(K)$ as oriented manifolds, then either $S_r^3(K)$ is an L -space or $r = s$.*

Proof In view of Theorem 9.8, we can assume $r > 0$ and $s < 0$, and $\nu(m(K)) \leq \nu(K)$. As in the proof of Proposition 9.7, it is clear that

$$\widehat{\nu}_s: H_*(\widehat{A}_s(K)) \longrightarrow \widehat{HF}(S^3)$$

is an isomorphism for all $s > 0$. Also, $\nu(K) \leq 1$.

We exclude the possibility that $\nu(K) = 1$. For $S_r^3(K)$, we consider the description of $\widehat{HF}(S_r^3(K))$. There are a priori two types of Spin^c structure: those for which the mapping cone contains a copy of $\widehat{A}_0(K)$, and those for which it does not; and there is always at least one Spin^c structure of the first type. For Spin^c structures \mathfrak{t} of the first type, $\widehat{HF}(S_r^3(K), \mathfrak{t})$ is described by

$$H_*(\widehat{A}_0(K))^m \oplus \mathbb{F}^{m-1}$$

(while for those of the second type, $\widehat{HF}(S_r^3(K), \mathfrak{t}) \cong \mathbb{F}$; but we will not use this here). Next, for any Spin^c structure \mathfrak{t}' over $S_s^3(K)$, $\widehat{HF}(S_s^3(K), \mathfrak{t}')$ has the form

$$H_*(\widehat{A}_0(K))^n \oplus \mathbb{F}^{n+1}.$$

Fix now \mathfrak{t} with $\widehat{HF}(S_r^3(K), \mathfrak{t}) \cong H_*(\widehat{A}_0(K))^m \oplus \mathbb{F}^{m-1}$, which in turn is identified with $\widehat{HF}(S_s^3(K), \mathfrak{t}')$ for suitable \mathfrak{t}' (by the hypothesis that $S_r^3(K) \cong S_s^3(K)$). Since $\text{rank } H_*(\widehat{A}_0(K)) \neq 0$ (since its Euler characteristic is 1), the equality of these two ranks forces at once that $m = n + 1$ and $\text{rank } H_*(\widehat{A}_0(K)) = 1$. But it is easy to see that a relatively graded isomorphism

$$H_*(\widehat{A}_0(K))^{n+1} \oplus \mathbb{F}_{(-1)}^n \cong H_*(\widehat{A}_0(K))^n[1] \oplus \mathbb{F}_{(0)}^{n+1}$$

cannot possibly hold.

In the case where $\nu(K) = 0$, the relatively graded isomorphisms between the Floer homologies in the various Spin^c structures give rise to

$$K^n \oplus \mathbb{F}_{(0)} \cong K[1]^m \oplus \mathbb{F}_{(0)},$$

where here K is the kernel of the map on homology

$$(\widehat{v}_0 \oplus \widehat{h}_0)_*: H_*(\widehat{A}_0) \longrightarrow \mathbb{F} \oplus \mathbb{F}.$$

Moreover, we can always choose some Spin^c structure for which $n \neq 0$. Such a relatively graded isomorphism can hold only if the rank of K is zero. In turn, this forces $\text{rank } H_*(\widehat{A}_0) = 1$. From [Proposition 9.7](#), it follows now that K is a trivial knot, contradicting our hypothesis. \square

In a different direction, the algebra can be used in some cases to exclude cosmetic surgeries with a fixed numerator p (ie first homology of the surgered manifold). We content ourselves here with a discussion of the case where $p = 3$.

Theorem 9.11 *Suppose that K is a nontrivial knot. Then, if $r, s \in \mathbb{Q}$ both with numerators having absolute value 3, and with $r \neq s$, we have that $S_r^3(K) \not\cong S_s^3(K)$ as oriented manifolds.*

Proof By reflecting if necessary, we can assume without loss of generality that $\nu(K) \geq \nu(m(K))$. Moreover, by [Lemma 9.2](#), it follows that $\nu(K) \geq 0$.

The possibility that $\nu(K) \geq 3$ is excluded by counting ranks. According to [Proposition 9.6](#), we have that

$$\text{rank } \widehat{HF}(S_{-3/q}^3(K)) = 3 + q \cdot \left(2(2\nu(K) - 1) + \sum_s (\text{rank } H_*(\widehat{A}_s) - 1) \right).$$

When $\nu(K) \geq 3$, then for all q , we have that

$$(42) \quad 3 \leq (2\nu - 1)q,$$

and hence

$$\text{rank } \widehat{HF}(S_{3/q}^3(K)) = -3 + q \cdot \left(2(2\nu(K) - 1) + \sum_s (\text{rank } H_*(\widehat{A}_s) - 1) \right).$$

It follows that

$$(43) \quad \text{rank } \widehat{HF}(S_{3/q}^3(K)) < \text{rank } \widehat{HF}(S_{-3/q}^3(K)).$$

Moreover, since

$$(44) \quad 6 < 2(2\nu(K) - 1) + \sum_s (\text{rank } H_*(\widehat{A}_s) - 1),$$

we also have that

$$(45) \quad \text{rank } \widehat{HF}(S_{-3/q}^3(K)) < \text{rank } \widehat{HF}(S_{3/q+1}^3(K)),$$

and hence the three-manifolds $\{S_{3/q}^3(K)\}_{q \in \mathbb{Z}}$ are all distinct.

Next, we turn our attention to excluding $\nu(K) = 2$. When $\nu(K) = 2$ and $q = 1$, Inequality (42) fails, but it is still the case that $\text{rank } \widehat{HF}(S_3^3(K)) < \text{rank } \widehat{HF}(S_{-3}^3(K))$. However, for $q > 1$, Inequality (42) holds. The above argument (used when $\nu(K) = 3$) can fail, however, if Inequality (44) fails. Thus, it remains to exclude the possibility that $S_{-3/q}^3(K) \cong S_{3/q+1}^3(K)$ (when $q \equiv 1 \pmod{3}$). Now, $\widehat{HF}(S_{-3/q}^3(K)) \cong \widehat{HF}(S_{3/q+1}^3(K))$ forces equality, rather than the inequality of Equation (44). In particular, this forces $\text{rank } H_*(\widehat{A}_s) = 1$ for all s and, since $\chi(\widehat{A}_s) = 1$, each $H_*(\widehat{A}_s)$ is supported in even degree, relative to the corresponding group for the unknot. Indeed, we claim that both $H_*(\widehat{A}_0)$ and $H_*(\widehat{A}_1)$ are supported in negative degrees (again, relative to the unknot). This can be seen by considering the diagram

$$(46) \quad \begin{array}{ccc} HF^+(A_s^+) & \longrightarrow & \widehat{HF}(\widehat{A}_s) \\ v_s^+ \downarrow & & \widehat{v}_s \downarrow \\ HF^+(B_s^+) & \longrightarrow & \widehat{HF}(\widehat{B}_s) \end{array}$$

for $s = 0, 1$. Since $H_*(\widehat{A}_0)$ and $H_*(\widehat{A}_1)$ have rank one, it follows that $HF^+(A_s^+) \cong \mathcal{T}^+$ for $s = 0, 1$ and that v_s^+ is multiplication U^{i_s} for some $i_s \geq 0$. Indeed, for $s = 0, 1$, we have that $i_s > 0$, in view of Diagram (46) and the hypothesis that $\nu(K) = 2$. It now follows that $H_*(\widehat{A}_0)$ and $H_*(\widehat{A}_1)$ are supported in negative degrees (relative to the unknot). We can now exclude the existence of an absolutely graded isomorphism

$$(47) \quad \widehat{HF}_*(S_{-3/q}^3(K), 0) \cong \widehat{HF}_*(S_{-3/q+1}^3(K), 0)$$

with $q \equiv 1 \pmod{3}$. To this end, it helps to recall that for $q \equiv 1 \pmod{3}$ and \mathcal{U} the unknot,

$$\widehat{HF}(S_{3/q}^3(\mathcal{U}), 0) \cong \widehat{HF}(S_{-3/q+1}^3(\mathcal{U}), 0) \cong \mathbb{F}$$

is supported in degree $1/2$. By considering the mapping cone calculation of $\widehat{HF}_*(S_{-3/(q+1)}^3(K), 0)$, we see that this group is nontrivial in degree $1/2$, while the even degree part of $\widehat{HF}_d(S_{3/q}^3(K), 0)$ with $d \equiv 1/2 \pmod{2}$ is carried by elements from $H_{d-1/2}(\widehat{A}_0)$ and $H_{d-1/2}(\widehat{A}_1)$, which we have seen already forces $d < 1/2$, and hence the impossibility of an isomorphism as in Equation (47).

We turn our attention now to the case where $\nu(K) = 1$. Let

$$C = \sum_s (\text{rank } H_*(\widehat{A}_s) - 1).$$

First, we check $q = 1, 2$, where Inequality (42) fails, and we find

$$\begin{aligned} \text{rank } \widehat{HF}(S_{3/1}^3(K)) &= 3 + C & \text{rank } \widehat{HF}(S_{-3/1}^3(K)) &= 5 + C \\ \text{rank } \widehat{HF}(S_{3/2}^3) &= 3 + 2C & \text{rank } \widehat{HF}(S_{-3/2}^3) &= 7 + 2C & \text{rank } \widehat{HF}(S_{3/4}^3) &= 5 + 4C. \end{aligned}$$

This sequence is not monotone increasing only if $C = 0$ or $C = 2$. Above this range, Inequality (42) holds, and in fact Inequality (44) holds except if $C = 0, 2$, or 4 . (Note that C is even.) In sum, the ranks of the Floer homology groups of various surgeries are distinct except when $C = 0, 2$, or 4 . In the case where $C = 2$, it follows from symmetry that $\text{rank } H_*(\widehat{A}_0) = 3$, and $\text{rank } H_*(\widehat{A}_s) = 1$ for all $s \neq 0$.

Thus, in the cases where $C = 0$ or 2 , Proposition 9.7 ensures that K is a knot with Seifert genus equal to one. According to Theorem 9.10, $S_r^3(K)$ must be an L -space, and in particular $C = 0$. But if K is a knot with Seifert genus one, some positive surgery on K gives an L -space, then by considering Proposition 9.6, we can conclude that $q = 1$ or 2 . But the \mathbb{Q}/\mathbb{Z} -valued linking form ensures that $S_{3/1}^3(K) \not\cong S_{3/2}^3(K)$.

When $\nu(K) = 1$, we are left with the remaining case that $C = 4$. Once again, since $\chi(\widehat{A}_s) = 1$, there are two ways in which the total rank C can in principle distribute over the $H_*(\widehat{A}_s)$: either $H_*(\widehat{A}_0)$ has rank 5 (and $H_*(\widehat{A}_s)$ has rank one for all other s) or there is exactly one positive s with $\text{rank } H_*(\widehat{A}_s) = H_*(\widehat{A}_{-s}) = 3$ and for all t with $|t| \neq |s|$, $\text{rank } H_*(\widehat{A}_t) = 1$. In the first case, Proposition 9.7 again ensures that $g(K) = 1$, and hence we can apply Theorem 9.10 (which in turn ensures $C = 0$, a contradiction).

We consider now the case where $\nu(K) = 1$, and $C = 4$, where the total rank is distributed so that $\text{rank } H_*(\widehat{A}_s) = \text{rank } H_*(\widehat{A}_{-s}) = 3$. The previous analysis shows

that

$$\begin{aligned} \text{rank } \widehat{HF}(S^3_{3/1}(K)) &= 7 & \text{rank } \widehat{HF}(S^3_{-3/1}(K)) &= 9 \\ \text{rank } \widehat{HF}(S^3_{3/2}) &= 11 & \text{rank } \widehat{HF}(S^3_{-3/2}) &= 15 & \text{rank } \widehat{HF}(S^3_{3/4}) &= 21. \end{aligned}$$

while for $q \geq 4$

$$\text{rank } \widehat{HF}(S^3_{3/q}(K)) < \text{rank } \widehat{HF}(S^3_{-3/q}(K)) = \text{rank } \widehat{HF}(S^3_{3/q+1}(K)).$$

Thus, we need only exclude the possibility that $S^3_{-3/q}(K) \cong S^3_{3/q+1}(K)$ when $q \geq 4$. This in turn is done by comparing $\widehat{HF}(S^3_{-3/q}(K), 0)$ and $\widehat{HF}(S^3_{3/q+1}(K), 0)$. Indeed, for any $q \geq 4$ and $q \equiv 1 \pmod{3}$, we prove the following claims (in the case at hand; ie K is a knot with $C = 4$, $\text{rank } \widehat{HF}(\widehat{A}_0) = 1$, $\nu = 1$):

- (1) There is a unique rational number d_1 such that $\text{rank } \widehat{HF}_{d_1}(S^3_{-3/q}(K), 0)$ is odd.
- (2) There is a unique rational number d_2 such that $\text{rank } \widehat{HF}_{d_2}(S^3_{3/q+1}(K), 0)$ is odd.

To prove (1) and (2), note that the mapping cone picture for both $\widehat{CF}(S^3_{-3/q}(K), 0)$ and $\widehat{CF}(S^3_{3/q+1}(K), 0)$ is symmetric, by a symmetry which exchanges \widehat{A}_s with \widehat{A}_{-s} and the maps \widehat{v} and \widehat{h} (and preserves all absolute gradings). This involution fixes a copy of \widehat{A}_0 when q is odd and a copy of \widehat{B}_0 when q is even. Indeed, in the case where $\nu(K) = 1$, we obtain splittings

$$\begin{aligned} \widehat{CF}(S^3_{-3/q}(K), 0) &\simeq \begin{cases} \widehat{A}_0[o_1] \oplus X \oplus X' & \text{if } q \text{ is odd,} \\ \widehat{B}_0[e_1] \oplus X \oplus X' & \text{if } q \text{ is even,} \end{cases} \\ \widehat{CF}(S^3_{3/q+1}(K), 0) &\simeq \begin{cases} \widehat{B}_0[o_2] \oplus X \oplus X' & \text{if } q \text{ is odd,} \\ \widehat{A}_0[e_2] \oplus X \oplus X' & \text{if } q \text{ is even,} \end{cases} \end{aligned}$$

where the complexes X and X' in the four above cases are different, but in each particular case, $X \simeq X'$ (as graded complexes). Moreover $\widehat{A}[m]$ and $\widehat{B}[m]$ denote the complexes \widehat{A} and \widehat{B} with a shift in grading by m . Claims (1) and (2) follow immediately.

We must now calculate the shifts in grading appearing above. Comparing with absolute gradings of $\widehat{HF}(S^3_{3/i}(\mathcal{U}), 0)$ (see [15]), we find that the shifts are given by

$$o_1 = \frac{1}{2} \quad e_1 = -\frac{1}{2} \quad o_2 = -\frac{3}{2} \quad e_2 = -\frac{1}{2}.$$

Next, we claim that $\nu(K) = 1$ and $\text{rank } \widehat{HF}(\widehat{A}_0) = 1$ imply that $\widehat{HF}(\widehat{A}_0)$ is supported in grading $\delta \leq -2$ (relative to the unknot). An analogous statement was established when

excluding the possibility that $\nu(K) = 2$; the present case follows very similarly (ie it is gotten easily by inspecting Diagram (46) with $s = 0$, and noting that $HF^+(A_0^+) \cong \mathcal{T}^+$ and v_0^+ is a nontrivial map).

We conclude that

$$d_1 = \begin{cases} \delta + 1/2 & \text{if } q \text{ is odd,} \\ -1/2 & \text{if } q \text{ is even,} \end{cases}$$

$$d_2 = \begin{cases} -3/2 & \text{if } q \text{ is odd,} \\ \delta - 1/2 & \text{if } q \text{ is even,} \end{cases}$$

so $d_1 = d_2$ occurs only if $\delta = -2$ and q is odd.

Thus, we need consider only $q = 6n + 1$ for $n \geq 1$. In this case, we find that (since $\nu = 1$)

$$(48) \quad \text{rank } \widehat{HF}_{-1/2}(S_{-3/6n+1}^3(K), 0) \geq 2n + 2.$$

Moreover, inspecting the grading shifts for the description of $\widehat{HF}(S_{3/6n+2}^3(K), 0)$ (and using that $\text{rank } \widehat{HF}(A_s) = 3$, we find that the components in $\widehat{HF}(S_{3/6n+2}^3(K), 0)$ coming from the various summands corresponding to \widehat{A}_s come in different gradings; the same is true for the various summands corresponding to \widehat{A}_{-s} ; finally, there are no other groups contributing in degree $-1/2$. Thus, we find, $\text{rank } \widehat{HF}_{-1/2}(S_{3/6n+2}^3(K)) \leq 2$. This, together with Inequality (48) excludes the existence of a graded isomorphism $\widehat{HF}(S_{-3/6n+1}^3(K), 0) \cong \widehat{HF}(S_{3/6n+2}^3(K), 0)$ when $\nu(K) = 1$ and $C = 4$.

Finally, we turn attention to the case where $\nu(K) = 0$. Since K is a nontrivial knot, Proposition 9.9 ensures that the only possibility is that $S_{3/q}^3(K) \cong S_{-3/q}^3(K)$. But this is easily excluded by the \mathbb{Q}/\mathbb{Z} -valued linking form. \square

9.1 Proofs of theorems in Section 1.2

Theorem 1.5 is a restatement of Theorem 9.10 proved above. Theorem 1.6 is stated and proved as Theorem 9.8 above. Theorem 1.7 is a restatement of Theorem 9.11 stated and proved above.

10 Seifert fibered spaces

Methods from this paper lead to the calculation of the Heegaard Floer homology groups of a large class of Seifert fibered spaces, with coefficients in $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. The primary ingredients here are the calculation of the knot Floer homology of the ‘‘Borromean

knot” (cf Section 9 of [19]), the knot $O_{q/r}$ considered in Section 7 and Theorem 6.1. Our aim here is to state and prove these results.

Let $h: \mathbb{Z} \rightarrow \mathbb{Z}$ be a function with the property that

$$\lim_{s \rightarrow \pm\infty} h(s) = +\infty.$$

We describe here a natural $\mathbb{F}[U]$ -module associated to h . It is interesting to compare the following construction with a construction of Némethi [13]; see also [17].

A well at height n is a pair of integers (i, j) with $i \leq j - 2$ and the property that $h(k) \leq n$ for all $i < k < j$, while $h(i) > n$ and $h(j) > n$. Let $W_n(h)$ denote the set of wells at height n , and let $M_n(h)$ denote the free Abelian group generated by $W_n(h)$.

If $x \in W_n(h)$ and $y \in W_{n-1}(h)$, we write $x > y$ if $x = (i, j)$ and $y = (i', j')$ with $i \leq i' < j' \leq j$. Define

$$U: M_n(h) \rightarrow M_{n-1}(h)$$

by the formula

$$U \cdot x = \sum_{\{y \in W_{n-1}(h) \mid x > y\}} y,$$

and let

$$\mathbb{H}\mathbb{F}^+(h) = \bigoplus_{n \in \mathbb{Z}} M_n(h)$$

be the induced module over $\mathbb{F}[U]$. Indeed, we can view this as a graded $\mathbb{F}[U]$ module by the grading which sends $W_n(h)$ to $2n$.

In the language of Némethi, the set of wells forms a root, and $\mathbb{H}\mathbb{F}^+(h)$ is the associated $\mathbb{F}[U]$ -module.

Let Y be a Seifert fibered space over a genus g orbifold with Seifert invariants $(a, r_1/q_1, \dots, r_n/q_n)$ over a genus g base; see [30; 29; 7]. The orbifold degree is the quantity

$$\text{deg}(Y) = a + \sum_i \frac{r_i}{q_i}.$$

Recall that $b_1(Y)$ is even if and only if $\text{deg}(Y) \neq 0$. By reversing the orientation on Y , we can arrange for $\text{deg}(Y) > 0$.

There is a presentation of the first homology of Y as

$$(49) \quad H_1(Y; \mathbb{Z}) \cong \frac{H_1(\Sigma; \mathbb{Z}) \oplus \mathbb{Z}m_0 \oplus \mathbb{Z}m_1 \oplus \dots \oplus \mathbb{Z}m_n}{\left(\begin{array}{l} a \cdot m_0 + \sum_{i=1}^n m_i = 0, \\ r_i \cdot m_0 - q_i \cdot m_i = 0, \quad i = 1, \dots, n \end{array} \right)}$$

(cf below for more specifics).

Theorem 10.1 *Let Y be a Seifert fibered space over a genus g orbifold with positive degree, and Seifert invariants $(a, r_1/q_1, \dots, r_n/q_n)$. There is an affine identification $\text{Spin}^c(Y) \cong H_1(Y; \mathbb{Z})$ with the following properties.*

- *The $HF^+(Y, \mathfrak{s})$ is nontrivial only to those Spin^c structures which are supported in the span of m_0, \dots, m_n (in the notation of Equation (49)).*
- *Let \mathfrak{s} be some Spin^c structure over Y , and let $\xi_0 \cdot m_0 + \xi_1 \cdot m_1 + \dots + \xi_n \cdot m_n$ be a representative with $0 \leq \xi_i \leq q_i$. For integers $-g \leq t \leq g$, let $\delta_t: \mathbb{Z} \rightarrow \mathbb{Z}$ be the function defined by*

$$\delta_t(s) = (-1)^{s+1}t + \left(\xi_0 + a \cdot s + \sum_{i=1}^n \left\lfloor \frac{\xi_i + r_i \cdot s}{q_i} \right\rfloor \right),$$

and $h_t: \mathbb{Z} \rightarrow \mathbb{Z}$ be the function:

$$h_t(s) = \begin{cases} t/2 & \text{if } t \leq \xi_0 \\ \xi_0 - t/2 & \text{if } t \geq \xi_0 \end{cases} + \begin{cases} \sum_{i=0}^{s-1} \delta_t(i) & \text{if } s > 0 \\ 0 & \text{if } s = 0 \\ -\sum_{i=s}^{-1} \delta_t(i) & \text{if } s < 0 \end{cases}$$

Then, there is a relatively \mathbb{Z} -graded isomorphism of $\mathbb{F}[U]$ -modules:

$$(50) \quad HF^+(Y, \mathfrak{s}) \cong \bigoplus_{-g \leq t \leq g} \Lambda^{g+t} H_1(\Sigma; \mathbb{F}) \otimes_{\mathbb{F}} \mathbb{H}F^+(h_t).$$

In the above statement, the factor Λ^{g+1} is not meant to affect grading (grading shifts from $H_1(\Sigma; \mathbb{Z})$ have been incorporated in the definition of $h_t(s)$). Note also that the right-hand-side is graded only in the relative sense, corresponding to our choice of \mathfrak{s} .

Note that in the case where $g = 0$, this recaptures (as a relatively \mathbb{Z} -graded group) the description of HF^+ for rational homology Seifert fibered spaces given by Némethi in terms of his “computational sequences”. We prove Theorem 10.1 in Section 10.2 below, after giving some sample calculations.

Although we have described here the Floer homology only as a relatively graded group, the absolute grading can be obtained by comparing the summand corresponding to $t = 0$ with the calculation of genus zero Seifert fibered spaces from [17]; see also [13].

10.1 Sample calculations

We begin with some generalities. Suppose that $h: \mathbb{Z} \rightarrow \mathbb{Z}$ is a function with $\lim_{s \rightarrow \pm\infty} h(s) = +\infty$. Let $\delta(s) = h(s) - h(s - 1)$. Clearly, the rank of $\text{Ker } U \subset$

$\mathbb{H}F^+(h)$ agrees with the number of pairs of integers (i, j) with $i < j$, $\delta(i) < 0$, $\delta(j) > 0$ and $\delta(k) = 0$ for all $i < k < j$ (ie these are the local minima of h).

In the case of [Theorem 10.1](#), explicitly finding these local minima is a straightforward matter: in the statement of the theorem, $\delta_t(s)$ differs from a linear function of s by a periodic function whose period is the least common multiple of 2 (when $t \neq 0$) and the integers $\{q_i\}_{i=1}^n$.

We use [Theorem 10.1](#) to calculate the Seifert fibered space over a genus one base, and Seifert invariants $(-1, 1/2, 2/3)$. Note that $H_1(Y; \mathbb{Z}) \cong \mathbb{Z}^2$, and there is a unique Spin^c structure with nontrivial $\mathbb{H}F^+$.

To calculate it, we proceed as follows. Observe that

$$\delta_t(s + 6) = 1 + \delta_t(s).$$

Moreover, the sequence $\{\delta_0(s)\}_{s=0}^5$ is

$$\{0, -1, 0, 0, 0, 0\}.$$

Set $t = 0$. The sequence of integers $\{h_0(i)\}$ clearly has a unique local minimum. It follows that the corresponding summand of $\mathbb{H}F^+(Y)$ is isomorphic to

$$H_1(T^2) \otimes_{\mathbb{Z}} \mathcal{T}^+ \cong (\mathcal{T}^+)^2.$$

In our subsequent descriptions, we will fix an absolute grading lifting the relative grading, with the additional convention that this corresponding summand of $\mathbb{H}F^+(Y)$ is isomorphic to $(\mathcal{T}_{(0)}^+)^2$. Indeed, this convention corresponds to the naturally induced absolute grading on $\mathbb{H}F^+(Y)$, as can be seen by comparing against the case where $g = 0$, and observing that the Seifert fibered space with corresponding invariants is S^3 .

When $t = -1$, all of the minima of the sequence $\{h_{-1}(i)\}_{i \in \mathbb{Z}}$ clearly occur for $-5 \leq i \leq 12$, where $\{\delta_{-1}(s)\}_{s=-5}^{12}$ takes the form

$$\{-3, 0, -2, 0, -2, 1, -2, 1, -1, 1, -1, 2, -1, 2, 0, 2, 0, 3\}.$$

We plot the corresponding function $\{h_{-1}(s)\}_{s=-5}^{12}$ in [Figure 4](#). Thus, the corresponding summand of $\mathbb{H}F^+(Y)$ takes the form

$$\mathcal{T}_{(1)}^+ \oplus \mathbb{Z}_{(1)}^2 \oplus \mathbb{Z}_{(3)}^2,$$

under the normalization convention established in the previous paragraph. Perhaps a few words are in order regarding the normalization. The minimum value of the sequence $\{h_0(s)\}_{s \in \mathbb{Z}}$ is -1 ; ie it has grading -2 . To obtain then the renormalized absolute grading of a well at a local minimum of $h_t(s)$, we must add 2 to this minimal value.

In the case where $t = 1$, we have that $\{\delta_1(i)\}_{i=-5}^{12}$ takes the form

$$\{-1, -2, 0, -2, 0, -1, 0, -1, 1, -1, 1, 0, 1, 0, 2, 0, 2, 1\},$$

which clearly contains all the minima of $\{h_1(i)\}$ for $i \in \mathbb{Z}$. It follows that the corresponding summand of $HF^+(Y)$ is of the form $\mathcal{T}_{(-1)}^+ \oplus \mathbb{F}_{(-1)}$.

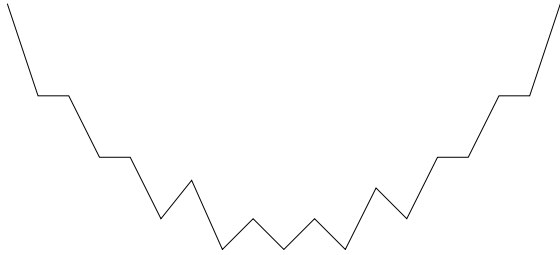


Figure 4: Height function for the Seifert fibered space with invariants $(-1, 1/2, 2/3)$ at $t = -1$

Putting this together, if Y denotes the Seifert fibered space over a genus one base with Seifert invariants $(-1, 1/2, 2/3)$, then its Heegaard Floer homology can be described as a graded $\mathbb{F}[U]$ module by

$$HF^+(Y) \cong \mathbb{F}_{(-1)} \oplus \mathbb{F}_{(1)}^2 \oplus \mathbb{F}_{(3)}^2 \oplus \mathcal{T}_{(-1)}^+ \oplus (\mathcal{T}_{(0)}^+)^2 \oplus \mathcal{T}_{(1)}^+.$$

10.2 Proof of Theorem 10.1

It is useful to have a mild generalization of the rational surgeries formula.

Definition 10.2 Let $K \subset Y$ be a null-homologous knot in a three-manifold Y . Fix an integer a and an n -tuple of rational numbers $\{q_i/r_i\}_{i=1}^n$. Consider an n -tuple of unknotted circles O_i each of which links K once, and which are pairwise mutually unlinked. Let $Y(K, a, \{q_i/r_i\}_{i=1}^n)$ denote the three-manifold obtained as a -surgery on K , followed by $-q_i/r_i$ surgery on each O_i . This three-manifold $Y(K, a, \{q_i/r_i\}_{i=1}^n)$ is said to be obtained as a *generalized rational surgery* on $K \subset Y$, with Seifert invariants $(a, \{q_i/r_i\}_{i=1}^n)$.

For fixed $K \subset Y$, let

$$Y' = Y \# (\#_{i=1}^n L(q_i, r_i)) \quad \text{and} \quad K' = K \# (\#_{i=1}^n O_{q_i/r_i})$$

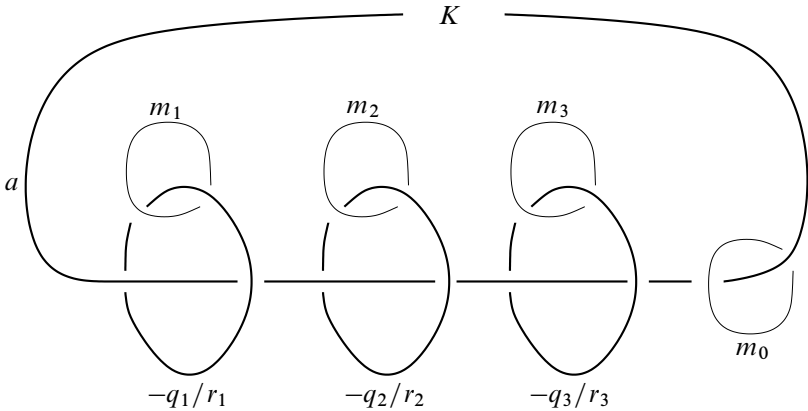


Figure 5: A schematic illustration of generalized rational surgery. We take here $n = 3$. K represents some initial knot, and $\{q_i/r_i\}_{i=1}^3$ represent surgery instructions on the unknots, while K is framed with framing a . The lightly drawn circles represent generators of the homology of the complement of the dark link (they are meridians).

in the notation of Section 7. Of course, $Y(K, a, \{q_i/r_i\}_{i=1}^n)$ can be thought of as the three-manifold gotten by a -surgery on $K' \subset Y'$.

As an example, if we start with the unknot $K \subset S^3$ and form the three-manifold $S^3(K, a, \{q_i/r_i\}_{i=1}^n)$, we obtain the Seifert fibered space whose base has genus zero, n singular fibers and Seifert invariants $(a, \{q_i/r_i\}_{i=1}^n)$, with the standard conventions. More generally, if we start with the Borromean knot $B_g \subset \#^{2g}(S^1 \times S^2)$ (this is the knot obtained by taking zero surgery on two of the components of the Borromean rings to obtain a knot B_1 in $\#(S^1 \times S^2)$, and then taking the connected sum of g copies of B_1), the resulting three-manifold $\#^{2g}(S^1 \times S^2)(B_g, a, \{q_i/r_i\}_{i=1}^n)$ is the Seifert space over a genus g base orbifold with Seifert invariants $(a, \{q_i/r_i\}_{i=1}^n)$.

Of course,

$$H_1(Y'; \mathbb{Z}) \cong \frac{H_1(Y - K; \mathbb{Z}) \oplus \mathbb{Z}m_1 \oplus \cdots \oplus \mathbb{Z}m_n}{\left(\begin{array}{l} a \cdot m_0 + \sum_{i=1}^n m_i = 0, \\ r_i \cdot m_0 - q_i \cdot m_i = 0, \quad i = 1, \dots, n \end{array} \right)},$$

where here $m_0 \in H_1(Y - K; \mathbb{Z})$ denotes the homology class of the meridian of K . We can change basis, letting $g_i = a_i \cdot m_0 + b_i \cdot m_i$, where here $a_i \cdot q_i + r_i \cdot b_i = 1$, to obtain a presentation

$$H_1(Y'; \mathbb{Z}) \cong \frac{H_1(Y - K; \mathbb{Z}) \oplus \mathbb{Z}g_1 \oplus \cdots \oplus \mathbb{Z}g_n}{\left(\begin{array}{l} a \cdot m_0 + \sum_{i=1}^n r_i \cdot g_i = 0, \\ m_0 - q_i \cdot g_i = 0, \quad i = 1, \dots, n \end{array} \right)}.$$

Consider the map

$$\beta: H_1(Y' - K'; \mathbb{Z}) \longrightarrow H_1(Y - K; \mathbb{Z})$$

defined by
$$\beta(\xi_0 + \xi_1 \cdot g_1 + \cdots + \xi_n \cdot g_n) = \xi_0 + \left(\sum_{i=1}^n \left\lfloor \frac{\xi_i}{q_i} \right\rfloor \right).$$

Letting $N = a \cdot m_0 + \sum_{i=1}^n r_i \cdot g_i$ be the push-off of K , we have that

$$H_1(Y(K, a, \{q_i/r_i\}_{i=1}^n); \mathbb{Z}) \cong H_1(Y' - K')/\mathbb{Z} \cdot N.$$

Given $E \in H_1(Y - K)$, define

$$\mathbb{A}_{[E]}^+ = \bigoplus_{s \in \mathbb{Z}} (s, A_{\beta(E+s \cdot N)}^+(Y, K)) \quad \text{and} \quad \mathbb{B}_{[E]}^+ = \bigoplus_{s \in \mathbb{Z}} (s, B_{\beta(E+s \cdot N)}^+(Y)),$$

and
$$D_{[E]}^+: \mathbb{A}_{[E]}^+ \longrightarrow \mathbb{B}_{[E]}^+$$

with
$$D_{[E]}^+ \{a_s\}_{s \in \mathbb{Z}} = \{b_s\}_{s \in \mathbb{Z}}$$

where $a_s \in A_{\deg(E+s \cdot N)}^+$ and $b_s \in B_{\deg(E+s \cdot N)}^+$ by

$$b_s = h_{\beta(E+(s-1) \cdot N)}^+(a_{s-1}) + v_{\beta(E+s \cdot N)}^+(a_s).$$

Proposition 10.3 *Let $K \subset Y$ be a null-homologous knot, and fix $E \in \text{Spin}^c(Y' - K')$. Suppose moreover that $b_1(Y(K, a, \{q_i/r_i\})) = b_1(Y)$. There is a map*

$$f: H_1(Y' - K') \longrightarrow \text{Spin}^c(Y(K, a, \{q_i/r_i\}))$$

with the property that for each $E \in Y' - K'$ for which $c_1(f(E))$ is a torsion class, we have that $HF^+(Y(K, a, \{q_i/r_i\}_{i=1}^n), f(E))$ is identified with the homology of the mapping cone $D_{[E]}^+: \mathbb{A}_{[E]}^+ \longrightarrow \mathbb{B}_{[E]}^+$.

Proof Like [Theorem 1.1](#), this follows from a direct application of [Theorems 6.1, 5.1](#), and the calculation of the invariant for $O_{q/r} \subset L(q, r)$ of [Lemma 7.1](#). □

[Theorem 10.1](#) follows quickly from [Proposition 10.3](#), together with the calculation of the knot Floer homology of the Borromean knot (cf [Section 9](#) of [\[19\]](#)), which we summarize here:

Lemma 10.4 *Let $K \subset \#^{2g}(S^2 \times S^1)$ be the Borromean knot. The Spin^c structure over $\#^{2g}(S^2 \times S^1)$ with trivial first Chern class is the only one whose induced knot filtration consists of nonzero groups. For that chain complex C , we have a splitting*

$$(51) \quad C \cong \bigoplus_t \Lambda^{t+g} H_1(\Sigma; \mathbb{F}) \otimes \mathbb{F}[U, U^{-1}],$$

where this splitting corresponds to the various summands $C\{i - j = t\}$. In particular, there are identifications

$$A_s^+(\#^{2g}(S^2 \times S^1), K) \cong \bigoplus_t \Lambda^{t+g} H_1(\Sigma; \mathbb{F}) \otimes \mathcal{T}^+,$$

$$C\{i \geq 0\}(\#^{2g}(S^2 \times S^1), K) \cong \bigoplus_t \Lambda^{t+g} H_1(\Sigma; \mathbb{Z}) \otimes \mathcal{T}_+^+$$

Moreover, the following squares commute:

$$\begin{array}{ccccc} \Lambda^{t+g} H_1(\Sigma; \mathbb{F}) \otimes \mathcal{T}^+ & \xrightarrow{U^{\max(0, t-s)}} & \Lambda^{t+g} H_1(\Sigma; \mathbb{F}) \otimes \mathcal{T}^+ & & \\ \downarrow & & \downarrow & & \\ A_s^+ & \xrightarrow{v^+} & B^+ & & \\ \Lambda^{t+g} H_1(\Sigma; \mathbb{F}) \otimes \mathcal{T}^+ \xrightarrow{U^{\max(0, s-t)}} \Lambda^{t+g} H_1(\Sigma; \mathbb{F}) \otimes \mathcal{T}^+ & \xrightarrow{\cong} & \Lambda^{-t+g} H_1(\Sigma; \mathbb{F}) \otimes \mathcal{T}^+ & & \\ \downarrow & & \downarrow & & \\ A_s^+ & \xrightarrow{h^+} & B^+ & \xrightarrow{=} & B^+, \end{array}$$

where here all the vertical maps are induced by inclusions in the identification of Equation (51).

Proof Most of the above statements are a direct result of Proposition 9.2 of [19], which, together with the Künneth principle, gives that

$$C\{i, j\} = U^{-i} \otimes \Lambda^{g-i+j} H^1(\Sigma; \mathbb{F}),$$

with no differentials. The second square involves the identification between $C\{i \geq 0\}$ and $C\{j \geq 0\}$, which sends the subset $C\{i - j = t\}$ to $C\{i - j = -t\}$, as explained in Proposition 5.2 of [27]. (Note that this is where we use most significantly the fact that we are using coefficients over \mathbb{F} ; over coefficients in \mathbb{Z} the involution has a more complicated form, described in [9].) \square

Lemma 10.5 Given a map $\delta: \mathbb{Z} \rightarrow \mathbb{Z}$, consider the chain map

$$D_\delta: \bigoplus_s A_s \longrightarrow \bigoplus_s B_s,$$

where all $A_s \cong \mathcal{T}^+ \cong B_s$, defined by

$$D_\delta(\{a_s\}_{s \in \mathbb{Z}}) = \{b_s\}_{s \in \mathbb{Z}}$$

where $b_s = U^{\max(-\delta(s), 0)} a_s + U^{\max(\delta(s+1), 0)} a_{s+1}$.

Letting

$$h(s) = \sum_{i=0}^s \delta(i),$$

we have that the homology of the mapping cone of D_δ is $\mathbb{H}\mathbb{F}^+(h)$, provided that

$$(52) \quad \lim_{s \rightarrow \pm\infty} h(s) = \pm\infty.$$

Proof The map D_δ is surjective. It remains to identify its kernel. Given $(i, j) \in W_n(h)$, consider the element $\{a_s\}_{s \in \mathbb{Z}}$ defined by the property that

$$a_s = \begin{cases} (-1)^k U^{h(k)-n} & \text{if } i < k < j, \\ 0 & \text{otherwise.} \end{cases}$$

By linearity, we can extend this to a homomorphism $W_n(h) \rightarrow \text{Ker } D_\delta$. It is straightforward to verify that this extends to an isomorphism $\mathbb{H}\mathbb{F}^+(h) \rightarrow \text{Ker } D_\delta$. \square

Proof of Theorem 10.1 According to the adjunction inequality (cf Theorem 8.1 of [20]), since the Thurston norm of Y is trivial, it follows that $HF^+(Y, \mathfrak{s})$ is trivial for all Spin^c structures with nontorsion $c_1(\mathfrak{s})$. According to the combination of Proposition 10.3 and Lemma 10.4, given E , the Floer homology $HF^+(Y, [E])$ splits as

$$HF^+(Y, [E]) \cong \bigoplus_{t \in \mathbb{Z}} \Lambda^{g+t} H^1(\Sigma; \mathbb{F}) \otimes X(t),$$

where here $X(t)$ is the homology of a chain complex satisfying the hypotheses of Lemma 10.5, for the function $\delta_t: \mathbb{Z} \rightarrow \mathbb{Z}$ as in the statement of the theorem. The theorem now follows from the calculation in Lemma 10.5. Note that Equation (52) holds, since the orbifold has positive degree. \square

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