Algebraic independence of generalized MMM–classes

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The generalized Miller–Morita–Mumford classes (MMM classes) of a smooth oriented manifold bundle are defined as the image of the characteristic classes of the vertical tangent bundle under the Gysin homomorphism. We show that if the dimension of the manifold is even, then all MMM–classes in rational cohomology are nonzero for some bundle. In odd dimensions, this is also true with one exception: the MMM–class associated with the Hirzebruch $\mathcal{L}$–class is always zero. Moreover, we show that polynomials in the MMM–classes are also nonzero. We also show a similar result for holomorphic fibre bundles and for unoriented bundles.

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1 Introduction and statement of results

Let M be a closed oriented n–dimensional smooth manifold and let $\mathrm{Diff}^+(M)$ be the topological group of all orientation-preserving diffeomorphisms of M, endowed with the Whitney C^∞–topology. A *smooth oriented M–bundle* is a fibre bundle $f\colon E \to B$ with structure group $\mathrm{Diff}^+(M)$ and fibre M. We do not assume that B is a manifold. Let $Q \to B$ be a $\mathrm{Diff}^+(M)$–principal bundle. The *vertical tangent bundle* of the smooth oriented M–bundle $f\colon E := Q \times_{\mathrm{Diff}^+(M)} M \to B$ is the oriented n–dimensional vector bundle $T^f = T_v E := Q \times_{\mathrm{Diff}^+(M)} TM \to E$. An *oriented closed manifold bundle of dimension n* is a map $f\colon E \to B$ such that for any component $C \subset B$, $f\colon f^{-1}(C) \to C$ is a smooth oriented M–bundle for some closed oriented n–manifold M. We sometimes abbreviate this term to *manifold bundle*, because all manifold bundles we consider are oriented (except in Section 8) and have closed fibres.

If $f\colon E \to B$ is an oriented manifold bundle of dimension n, then the *Gysin homomorphism* $f_!\colon H^*(E) \to H^{*-n}(B)$ is defined (the Gysin homomorphism exists with arbitrary coefficients; however all cohomology groups in this paper are with rational coefficients, unless the contrary is explicitly stated). Define a linear map

$$\kappa_E\colon H^*(B\mathrm{SO}(n)) \to H^{*-n}(B)$$

by

$$\kappa_E(c) := f_!(c(T_v E)) \in H^{k-n}(B), \quad c \in H^k(B\mathrm{SO}(n)).$$

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The universal smooth M–bundle $E_M \to B\mathrm{Diff}^+(M)$ gives a map

$$\kappa_{E_M}\colon H^*(B\mathrm{SO}(n)) \to H^{*-n}(B\mathrm{Diff}^+(M)).$$

The homomorphism κ_E is natural in the sense that $h^* \circ \kappa_E = \kappa_{h^*E}$ for any manifold bundle $E \to B$ and any map $h\colon A \to B$ and so the images of κ_E can be viewed as characteristic classes of manifold bundles, which we call *generalized Miller–Morita–Mumford classes* or MMM–classes. Miller [26], Morita [28] and Mumford [30] first studied these classes in the 2–dimensional case.

For a graded vector space V and $n \in \mathbb{Z}$, we denote by $\sigma^{-n}V$ the new graded vector space with $(\sigma^{-n}V)_m = 0$ if $m < 0$ and $(\sigma^{-n}V)_m = V_{m+n}$ for $m \geq 0$. Moreover, $\tilde{\sigma}^{-n}V \subset \sigma^{-n}V$ is the subspace generated by elements in positive degree. Note that κ_E becomes a map $\sigma^{-n}H^*(B\mathrm{SO}(n)) \to H^*(B)$ of graded vector spaces.

Let $\mathcal{R}_n$ be a set of representatives for the oriented diffeomorphism classes of oriented closed n–manifolds (connected or nonconnected) and let $\mathcal{R}_n^0 \subset \mathcal{R}_n$ be the set of connected n–manifolds. Put

$$\mathcal{B}_n := \coprod_{M \in \mathcal{R}_n} B\mathrm{Diff}^+(M), \quad \mathcal{B}_n^0 := \coprod_{M \in \mathcal{R}_n^0} B\mathrm{Diff}^+(M) \subset \mathcal{B}_n.$$

There are tautological manifold bundles on these spaces and therefore we get maps of graded vector spaces

$$\kappa^n\colon \sigma^{-n}H^*(B\mathrm{SO}(n)) \to H^*(\mathcal{B}_n), \quad \kappa^{n,0}\colon \sigma^{-n}H^*(B\mathrm{SO}(n)) \to H^*(\mathcal{B}_n^0),$$

where the map $\kappa^{n,0}$ is the composition of κ^n with the restriction map $H^*(\mathcal{B}_n) \to H^*(\mathcal{B}_n^0)$. Here is our first main result. Recall that we work with rational cohomology throughout the paper.

Theorem A (1) *For all even $n \geq 0$, $\kappa^{n,0}\colon \sigma^{-n}H^*(B\mathrm{SO}(n)) \to H^*(\mathcal{B}_n^0)$ is injective.*

(2) *For all odd $n \geq 1$, the kernel of $\kappa^{n,0}\colon \sigma^{-n}H^*(B\mathrm{SO}(n)) \to H^*(\mathcal{B}_n^0)$ is the linear subspace that is generated by the components $\mathcal{L}_{4d} \in H^{4d}(B\mathrm{SO}(n))$ of the Hirzebruch $\mathcal{L}$–class (for $4d > n$).*

Equivalently, Theorem A says (for even n) that for each $0 \neq c \in \sigma^{-n}H^*(B\mathrm{SO}(n))$, there is a connected n–manifold M and an oriented M–bundle $f\colon E \to B$ such that $\kappa_E(c) \neq 0 \in H^*(B)$. For odd n, there is an analogous reformulation.

Generalized MMM–classes of degree 0 are just the characteristic numbers of the fibre. The linear independence of those is a well-known classical result by Thom [34] and therefore we only need to prove something new in positive degrees (we need to consider the degree 0 case in the proof, though).

For an arbitrary graded $\mathbb{Q}$–vector space V (concentrated in positive degrees), we let ΛV be the free graded-commutative unital $\mathbb{Q}$–algebra generated by V. If A is a graded-commutative $\mathbb{Q}$–algebra, then any graded vector space homomorphism $\phi\colon V \to A$ extends uniquely to a homomorphism $\Lambda'\phi\colon \Lambda V \to A$ of graded algebras such that $\Lambda'\phi \circ s = \phi$ where $s\colon V \to \Lambda V$ is the natural inclusion. Therefore, the map κ^n induces an algebra homomorphism

$$\Lambda'\kappa^n\colon \Lambda\tilde{\sigma}^{-n} H^*(B\mathrm{SO}(n)) \to H^*(\mathcal{B}_n).$$

Our second main result is about $\Lambda'\kappa^n$. Again rational coefficients are understood.

Theorem B (1) *For all even n, the map $\Lambda'\kappa^n$ is injective.*

(2) *For all odd n, the kernel of $\Lambda'\kappa^n$ is the ideal generated by the components $\mathcal{L}_{4d} \in H^{4d}(B\mathrm{SO}(n))$ of the Hirzebruch $\mathcal{L}$–class (for $4d > n$).*

The case $n = 2$ of Theorem A and Theorem B is a well-known result that was first established by Miller [26] and Morita [28] and we make essential use of this in our proof.

We show similar results in the complex case. A *holomorphic fibre bundle* of (complex) dimension m is a proper holomorphic submersion $f\colon E \to B$ between complex manifolds of (complex) codimension $-m$. By Ehresmann's fibration theorem, f is a smooth oriented fibre bundle (but the biholomorphic equivalence class of the fibres is not locally constant). The vertical tangent bundle $T_v E := \ker \mathrm{d}f$ is a complex vector bundle of rank n and for any $c \in H^*(BU(m))$, we can define

$$\kappa_E^{\mathbb{C}}(c) := f_!(c(T_v E)) \in H^{*-2m}(B).$$

Theorem C (1) *For each $0 \neq c \in \sigma^{-2m} H^*(BU(m))$, there exists a holomorphic fibre bundle $f\colon E \to B$ of dimension m on a smooth projective variety B such that $\kappa_E^{\mathbb{C}}(c) \neq 0$.*

(2) *For any $0 \neq c \in \Lambda\tilde{\sigma}^{-2m} H^*(BU(m))$, there exists a holomorphic fibre bundle with m–dimensional fibres on an open complex manifold such that $\Lambda'\kappa_E^{\mathbb{C}}(c) \neq 0$.*

Note that it is far from obvious to say what the universal holomorphic bundle should be. Therefore we do not formulate Theorem C in the language of universal bundles.

The results of this paper can be interpreted in the language of the Madsen–Tillmann–Weiss spectra $\mathrm{MTSO}(n)$ [15], as we will briefly explain. By definition, $\mathrm{MTSO}(n)$ is the Thom spectrum of the inverse of the universal vector bundle $L_n \to B\mathrm{SO}(n)$. If

$f\colon E \to B$ is an oriented manifold bundle of fibre dimension n, then the Pontrjagin–Thom construction yields a spectrum map $\alpha^\flat\colon \Sigma^\infty B_+ \to \mathrm{MTSO}(n)$. The spectrum cohomology of $\mathrm{MTSO}(n)$ is, by the Thom isomorphism, isomorphic to $H^{*+n}(B\mathrm{SO}(n))$. Therefore, $\alpha^\flat$ induces a map of graded vector spaces $\widetilde{\sigma}^{-n} H^*(B\mathrm{SO}(n)) \to H^*(B)$, which is the same as the map κ_E.

If B is connected, then all fibres of f are diffeomorphic. Call one of the fibres M. The oriented manifold M defines an element $[M] \in \pi_0(\Omega^\infty \mathrm{MTSO}(n))$ and this component is denoted by $\Omega^\infty_{[M]} \mathrm{MTSO}(n)$. The adjoint of $\alpha^\flat$ is a map $\alpha\colon B \to \Omega^\infty_{[M]} \mathrm{MTSO}(n)$. Loop sum with any point in $\Omega^\infty_{[M]} \mathrm{MTSO}(n)$ defines a homotopy equivalence $\Omega^\infty_0 \mathrm{MTSO}(n) \to \Omega^\infty_{[M]} \mathrm{MTSO}(n)$, with whose inverse we compose α to obtain a map $B \to \Omega^\infty_0 \mathrm{MTSO}(n)$ still denoted by α. This map induces an algebra map $H^*(\Omega^\infty_0 \mathrm{MTSO}(n)) \to H^*(B)$. Under the classical isomorphism $H^*(\Omega^\infty_0 \mathrm{MTSO}(n)) \cong \Lambda H^{*>0}(\mathrm{MTSO}(n))$ (rational coefficients), this map corresponds to $\Lambda' \kappa^n$. Therefore we can restate Theorems A and B as:

Theorem D *The universal map $\mathcal{B}_n \to \Omega^\infty_0 \mathrm{MTSO}(n)$ induces an injection on rational cohomology if n is even. If n is odd, then the kernel is the ideal generated by the components of the Hirzebruch $\mathcal{L}$–class (under the above isomorphisms).*

Apart from the breakthrough works of Madsen and Tillmann [23], Madsen and Weiss [24] and Galatius, Madsen, Tillmann and Weiss [15], the characteristic classes of manifold bundles related to $\mathrm{MTSO}(n)$ have been studied for example by Galatius and Randal-Williams [14]. Their methods, however, do not suffice to show Theorems A and B, because they do not deal with *closed* manifolds.

One can ask whether an appropriate version of Theorem D holds for unoriented manifold bundles. Any unoriented manifold bundle $f\colon E \to B$ of dimension n induces a map $\beta_E\colon B \to \Omega^\infty \mathrm{MTO}(n)$ into the nonoriented version of the Madsen–Weiss–Tillmann spectrum. Let $\mathcal{A}_n = \coprod_{M \in \mathcal{S}_n} B\mathrm{Diff}(M)$, where $\mathcal{S}_n$ is a set of representatives for the diffeomorphism classes of nonoriented closed n–manifolds. There is a universal (nonoriented) manifold bundle on $\mathcal{A}_n$ and hence a universal map

$$\beta^n\colon \mathcal{A}_n \to \Omega^\infty \mathrm{MTO}(n).$$

The following result is much easier than Theorems A and B (we still assume rational coefficients).

Theorem E *For any n, the map $(\beta^n)^*\colon H^*(\Omega^\infty \mathrm{MTO}(n)) \to H^*(\mathcal{A}_n)$ is injective.*

In general, the construction of manifold bundles and the computation of generalized MMM–classes are rather difficult problems, but in this paper, we can circumvent this:

the only difficult constructions which we need are in the 2–dimensional case, and for that we rely entirely on Miller [26] and Morita [28]. There are some other computations of MMM–classes which we want to mention though we do not need them.

The MMM–classes of bundles with compact connected Lie groups as structure groups are relatively easy to compute due to the "localization formula" of Atiyah and Bott [3]. A special case is the case of homogeneous space bundles of the form $BH \to BG$ where $H \subset G$ are compact Lie groups. In that case, the MMM–classes can be expressed entirely in terms of Lie-theoretic data. Akita, Kawazumi and Uemura [1] apply a similar localization principle to cyclic structure groups.

The MMM–classes associated with *multiplicative sequences* are rather well understood because of the close relationship with genera, ie, ring homomorphisms from the oriented bordism ring to $\mathbb{Q}$; see eg Hirzebruch, Berger and Jung [18]. The theory of elliptic genera shows that many of these MMM–classes are nontrivial. Unfortunately, this is not enough to establish Theorem A.

Another source of manifold bundles with nontrivial MMM–classes is the following result. If M is an oriented manifold with signature 0, then there exists an oriented manifold bundle $E \to \mathbb{S}^1$ such that E is oriented cobordant to M. Away from the prime 2, this was established by Burdick [7, Theorem 1.2] and Conner [8, Corollary 6.3]. Another proof was given by Neumann [32] based on a result of Jänich [20]. Let $0 \neq x \in H^{4k}(B\mathrm{SO}(4k))$ be a class that is not a multiple of the Hirzebruch class. Then there is a $4k$–manifold M with signature 0 and $\langle x(TM); [M]\rangle \neq 0$ and a manifold bundle $f\colon M \to \mathbb{S}^1$ by the above results. Then $f_!(x(T_vM)) \neq 0 \in H^1(\mathbb{S}^1)$. Therefore, in all dimensions of the form $4k-1$, the statement of Theorem 2.3 is true for classes of degree 1.

In Section 2, we give a detailed overview of the proof of Theorem A. The details of this proof occupies the main bulk of this paper, ie Sections 3, 4 and 5. In Section 6, we show how to derive Theorem B from Theorem A, using Nakaoka stability and the Barratt–Priddy–Quillen Theorem on the infinite symmetric group. Section 7 contains the proof of Theorem C, which is a simple variation of the proofs of Theorem A and Theorem B. In Section 8, we discuss Theorem E. In Appendix A, we recapitulate the definitions and the relevant properties of the Gysin homomorphism and the related transfer.

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1.1 Notation and conventions

All cohomology groups in this paper have rational coefficients, unless an exception is explicitly made. When G is a topological group which acts on the space X, we denote the Borel construction by $E(G;X) := EG \times_G X$. If X_1, X_2 are spaces, then the projections are denoted by $\mathrm{pr}_i\colon X_1 \times X_2 \to X_i$; the index is sometimes omitted when it is clear. Let $V_i \to X_i$, $i = 1,2$ be two vector bundles. Then $V_1 \boxtimes V_2 \to X_1 \times X_2$ denotes the exterior tensor product, ie $V_1 \boxtimes V_2 := \mathrm{pr}_1^* V_1 \otimes \mathrm{pr}_2^* V_2$.

The symbol $\underline{n}$ denotes the set $\{1,\dots,n\}$. If x is an element of a graded vector space, we denote its degree by $|x|$, implicitly assuming that x is homogeneous. Moreover, all sub–vector spaces $W \subset V$ of a graded vector space are assumed to be graded, in other words $W = \bigoplus_n W \cap V_n$. The dual space of a vector space V is always denoted by $V^\vee$. If V is any vector space, then $\mathrm{Sym}^k V$ denotes the k–th symmetric power.

Our notation of standard characteristic classes differs from the customary one. We give them the actual cohomological degree they have as an index. For an example, $p_4(V) \in H^4(X)$ will denote what is commonly known as the first Pontrjagin class of the real vector bundle $V \to X$. We hope that this does not lead to confusion.

2 Outline of the proof of Theorem A

The proof of Theorem A is an eclectic combination of several computations. In this section, we give an outline. The cohomology of $B\mathrm{SO}(n)$ is well known:

$$H^*(B\mathrm{SO}(2m+1)) \cong \mathbb{Q}[p_4,\dots,p_{4m}],$$
$$H^*(B\mathrm{SO}(2m)) \cong \mathbb{Q}[p_4,\dots,p_{4m},\chi]/(\chi^2 - p_{4m}).$$

The cases $n = 0,1$ of Theorem A are empty. The following result shows that κ^n cannot be injective if n is odd.

Theorem 2.1 *For odd n, the kernel of κ^n contains the subspace that is generated by the components $\mathcal{L}_{4d} \in H^{4d}(B\mathrm{SO}(n))$ of the Hirzebruch $\mathcal{L}$–class (for $4d > n$).*

In [10, Section 4.4], the author shows how to derive Theorem 2.1 from the Hirzebruch signature theorem and the multiplicativity of the signature in manifold bundles of odd-dimension: If $f\colon E \to B$ is an oriented manifold bundle with odd-dimensional fibres and B is a closed oriented manifold, then $\mathrm{sign}(E) = 0$. This was first mentioned by Atiyah [2] (without proof), proven later by Meyer [25] and Lück and Ranicki [22]. In [10], we also give an index-theoretic proof of Theorem 2.1 that does not depend on [25] or [22].

Because the components of $\mathcal{L}$ form an additive basis of $H^*(B\mathrm{SO}(3))$, Theorem 2.1 forces κ^3 to be the zero map. Thus Theorem A is also empty in the 3–dimensional case. As mentioned in the introduction, the case $n = 2$ is a classical result. It is the main ingredient for the proof of Theorem A.

Theorem 2.2 *The map $\kappa^{2,0}$ is injective.*

This was first established by Miller [26] and Morita [28]. Today, there are other proofs by Akita, Kawazumi and Uemura [1] and Madsen and Tillmann [23]. Of course, the affirmative solution of the Mumford conjecture by Madsen and Weiss [24] also implies Theorem 2.2.

We denote by $\mathrm{Pont}^*(n) \subset H^*(B\mathrm{SO}(n))$ the subring generated by the Pontrjagin classes. If $V \to X$ is a real vector bundle, then $\mathrm{Pont}^*(V) \subset H^*(X)$ is the subring generated by the Pontrjagin classes of X. The main bulk of work to prove Theorem A is:

Theorem 2.3 (1) *For even n, $\kappa^{n,0}\colon \sigma^{-n}\,\mathrm{Pont}^*(n) \to H^*(\mathcal{B}_n^0)$ is injective.*

(2) *For odd n, the kernel of $\kappa^{n,0}\colon \sigma^{-n}\,\mathrm{Pont}^*(n) \to H^*(\mathcal{B}_n^0)$ is the linear subspace that is generated by $\mathcal{L}_{4d}$ (for $4d \geq n$).*

Theorem 2.3 implies Theorem A: for odd n, $\mathrm{Pont}^*(n) = H^*(B\mathrm{SO}(n))$ and for $n = 2m$, the argument is so short and easy that we give it here. The total space of the unit sphere bundle of the $(2m+1)$–dimensional universal vector bundle on $B\mathrm{SO}(2m+1)$ is homotopy equivalent to $B\mathrm{SO}(2m)$ and the bundle projection corresponds to the inclusion map $f\colon B\mathrm{SO}(2m) \to B\mathrm{SO}(2m+1)$; the induced map $\mathrm{Pont}^*(2m+1) \to \mathrm{Pont}^*(2m)$ is an isomorphism. Any element $x \in H^*(B\mathrm{SO}(2m))$ can be written in a unique way as $x = f^*x_1\chi + f^*x_2$ with $x_i \in H^*(B\mathrm{SO}(2m+1))$. Lemma 2.4 below and Theorem 2.3 immediately imply Theorem A.

Lemma 2.4 *Let $f\colon B\mathrm{SO}(2m) \to B\mathrm{SO}(2m+1)$ be the universal $\mathbb{S}^{2m}$–bundle and let $x = f^*x_1\chi + f^*x_2$ be as above. Then $f_!(x(T_v B\mathrm{SO}(2m))) = 2x_1$.*

Proof The vertical tangent bundle $T_v BSOy(2m)$ is isomorphic to the universal $2m$–dimensional vector bundle; therefore $f_!(x(T_v B\mathrm{SO}(2m))) = f_!(x)$. By Proposition A.3, using the identities $f_!(\chi) = \chi(\mathbb{S}^{2m}) = 2$ and $f_!(1) = 0$, we conclude

$$f_!(x) = f_!(f^*x_1\chi + f^*x_2) = x_1 f_!(\chi) + f_!(1)x_2 = 2x_1. \qquad \square$$

The proof of Theorem 2.3 has two parts. The first part is an induction argument, using Theorem 2.2 as induction beginning and the second part deals with the classes that are missed by the inductive argument (actually both parts of the argument are interwoven;

see below). The idea of the induction is simple. Let n be given. Let $f_i\colon E_i \to B_i$ be manifold bundles of fibre dimension n_i, $i=1,2$, $n_1+n_2=n$. The idea is to consider the product bundle $f=f_1\times f_2\colon E_1\times E_2 \to B_1\times B_2$, which has fibre dimension n. The MMM–classes of the product can be expressed by the MMM–classes of the two factors. It turns out that we can detect "most", but not all MMM–classes on products of lower-dimensional manifold bundles. Proposition 2.5 describes the classes that are missed by this inductive argument.

The *Pontrjagin character* of a real vector bundle $V \to X$ is $\mathrm{ph}(V) := \mathrm{ch}(V\otimes\mathbb{C})$. Since $V\otimes\mathbb{C} \cong \overline{V\otimes\mathbb{C}}$ (it is self-conjugate), it follows that $\mathrm{ph}_{4d+2}(V)=0$, so ph is concentrated in degrees that are divisible by 4. In fact, $\mathrm{ph}_{4d} \in \mathrm{Pont}^{4d}(n)$, $n=\mathrm{rank}(V)$. Note that if V is itself complex, then $\mathrm{ph}(V)=\mathrm{ch}(V\otimes_{\mathbb{R}}\mathbb{C}) = \mathrm{ch}(V\oplus\bar{V})$.

Proposition 2.5 (1) *Let $n=2m$ be even and assume that Theorem 2.3 has been proven for all even dimensions $2l<n$. Then the kernel of $\kappa^{n,0}\colon \sigma^{-n}\,\mathrm{Pont}^*(n) \to H^*(\mathcal{B}_n^0)$ is contained in the span of the components ph_{4d}, $4d\geq n$.*

(2) *Let $n=2m+1\geq 7$ be odd and assume that Theorem 2.3 has been proven for all dimensions less than n. Then the kernel of $\kappa^{n,0}\colon \sigma^{-n}\,\mathrm{Pont}^*(n) \to H^*(\mathcal{B}_n^0)$ is contained in the span of the components ph_{4d} and $\mathcal{L}_{4d}$, $4d\geq 2m+1$.*

The proof is purely algebraic and elementary. We give it in Section 3. By Proposition 2.5 and Theorem 2.2, two steps remain to be done for the proof of Theorem 2.3 and hence Theorem A. We have to show that $\kappa^{n,0}(\mathrm{ph}_{4d})\neq 0$ for all $4d\geq n\geq 4$. Furthermore, we have to show the case $n=5$ of Theorem 2.3 from scratch.

There are two ideas involved: we do explicit computations for bundles of complex projective spaces and then we use what we call "loop space construction" to increase the dimension of the manifolds that are involved.

Let the group $\mathrm{SU}(m+1)$ act on $\mathbb{CP}^m$ in the usual way. Consider the Borel-construction $q\colon E(\mathrm{SU}(m+1);\mathbb{CP}^m) \to B\mathrm{SU}(m+1)$. In Section 5, we will show the following result.

Proposition 2.6 *The class $\kappa_{E(\mathrm{SU}(2k+1);\mathbb{CP}^{2k})}(\mathrm{ph}_{4d}) \in H^{4d-4k}(B\mathrm{SU}(2k+1))$ is nonzero for all $d\geq k$.*

To finish the proof of Theorem A in the even-dimensional case it remains to prove that $\kappa^{4k+2,0}(\mathrm{ph}_{4d})\neq 0$ if $4d\geq 4k+2\geq 6$. To do this, we employ the loop space construction that we describe now.

Let $f\colon E\to X$ be a manifold bundle of dimension n. Let LX be the free loop space of X, ie $LX := \mathrm{map}(\mathbb{S}^1;X)$ and let $\mathrm{ev}\colon \mathbb{S}^1\times LX \to X$ be the evaluation map $\mathrm{ev}(t,\gamma) := \gamma(t)$.

Consider the following diagram (pr is the obvious projection):

$$\begin{array}{ccc} \mathfrak{L}E := \mathbb{S}^1 \times LX \times_X E & \xrightarrow{h} & E \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathbb{S}^1 \times LX & \xrightarrow{\mathrm{ev}} & X \\ \downarrow{\scriptstyle \mathrm{pr}} & & \\ LX & & \end{array}$$

The composition on the left-hand side is denoted $\mathfrak{L}p := \mathrm{pr} \circ f' \colon \mathfrak{L}E \to LX$; this is a topological fibre bundle which we call the *loop space construction* on the bundle E. The fibre over a loop $\gamma \in LX$ is the total space of the bundle $\gamma^* E \to \mathbb{S}^1$. In particular, if the fibre of f is M and if X is simply connected, then $\mathfrak{L}p$ is an $\mathbb{S}^1 \times M$–bundle.

In Section 4, we will prove that there is a structure of a smooth manifold bundle on the loop space construction and that this structure is unique up to isomorphism.

The generalized MMM–classes of $\mathfrak{L}E \to LX$ can be expressed in terms of those of $E \to X$. The result is that the following diagram is commutative (Proposition 4.5):

(2.7)
$$\begin{array}{ccc} \mathrm{Pont}^*(n+1) & \longrightarrow & \mathrm{Pont}^*(n) \\ \downarrow{\scriptstyle \kappa_{\mathfrak{L}E}} & & \downarrow{\scriptstyle \kappa_E} \\ H^{*-n-1}(LX) & \xleftarrow{\mathrm{trg}} & H^{*-n}(X) \end{array}$$

The bottom map is the *transgression*; see Definition 4.4. We can iterate the loop space construction. From an M–bundle $E \to X$ on an r–connected space X, we obtain an $(\mathbb{S}^1)^r \times M$–bundle $\mathfrak{L}^r p \colon \mathfrak{L}^r E \to L^r(X) = \mathrm{map}((\mathbb{S}^1)^r; X)$. Also, the transgression can be iterated and gives $\mathrm{trg}^r \colon H^*(X) \to H^{*-r}(L^r X)$. Now let $f \colon E \to X$ be an M^{4k}–bundle and let $4d \geq 4k + r$. Assume that $f_!(\mathrm{ph}_{4d}(T_v E)) \in H^{4d-4k}(X)$ is nonzero. Since ph_{4d} does not lie in the kernel of the restriction $\mathrm{Pont}^*(4k+r) \to \mathrm{Pont}^*(4k)$, the class $\kappa_{\mathfrak{L}^r E}(\mathrm{ph}_{4d}) \in H^{4d-4k-r}(L^r X)$ is nontrivial provided that trg^r is injective.

For a general space X, the transgression is far from being injective, but it is injective in positive degrees if X is simply connected and the rational cohomology of X is a free graded-commutative algebra, compare Proposition 4.7. If X is an addition r–connected, then trg^r is injective (again, in positive degrees).

The base space $B\mathrm{SU}(2k+1)$ of the universal $\mathbb{CP}^{2k}$–bundle in Proposition 2.6 is 3–connected and its rational cohomology is a polynomial algebra and so trg^r is injective for $r = 1, 2, 3$. Therefore, Proposition 2.6 implies that $\kappa^n(\mathrm{ph}_{4d}) \neq 0$ if $n = 4k + r$

for $0 \leq r \leq 3$. This concludes, by Proposition 2.5, the proof of Theorem A in the even-dimensional case.

For the odd-dimensional case, the only thing that is left is the induction beginning (ie the case $n = 5$). This is accomplished by the same method.

Proposition 2.8 *Let* $q\colon E(\mathrm{SU}(3);\mathbb{CP}^2) \to B\mathrm{SU}(3)$ *be the Borel construction. Then the kernel of* $\kappa_{E(\mathrm{SU}(3);\mathbb{CP}^2)}\colon \mathrm{Pont}^{4d+4}(4) \to H^{4d}(B\mathrm{SU}(3))$ *is* 1*–dimensional and spanned by* $\mathcal{L}_{4d+4}$ *(if* $d > 0$*).*

Corollary 2.9 *Let* $\mathfrak{L}E := \mathfrak{L}E(\mathrm{SU}(3);\mathbb{CP}^2) \stackrel{\mathfrak{L}q}{\to} LB\mathrm{SU}(3)$ *(it is an* $\mathbb{S}^1 \times \mathbb{CP}^2$*–bundle). Then the kernel of* $(\mathfrak{L}q)_!\colon \mathrm{Pont}^{*+5}(T_v\mathfrak{L}E) \to H^*(LB\mathrm{SU}(3))$ *is spanned by the components of the Hirzebruch class.*

The corollary follows immediately from Proposition 2.8, diagram (2.7) and Proposition 4.7. This gives the induction beginning and finishes the proof of Theorem A. Actually, it is quite surprising that a single 5–manifold, namely $\mathbb{S}^1 \times \mathbb{CP}^2$, suffices to detect all MMM–classes.

3 The induction step

In this section, we prove Proposition 2.5. First we recall that the Hirzebruch $\mathcal{L}$–class is the multiplicative sequence in the Pontrjagin classes that is associated with the formal power series

$$\sqrt{x}\coth(\sqrt{x}) = \sum_{k=0}^{\infty} \frac{2^{2k} B_{2k}}{(2k)!} x^k,$$

where B_{2k} denotes the Bernoulli number. It is crucial for the proofs of Propositions 2.5 and 2.8 that $B_{2k} \neq 0$.

The main part of the proof is pure linear algebra. Let $n_1 + n_2 = n$; the Whitney sum map $B\mathrm{SO}(n_1) \times B\mathrm{SO}(n_2) \to B\mathrm{SO}(n)$ induces

$$r_{n_1,n_2}\colon \sigma^{-n}\,\mathrm{Pont}^*(n) \to \sigma^{-n_1}\,\mathrm{Pont}^*(n_1) \otimes \sigma^{-n_2}\,\mathrm{Pont}^*(n_2).$$

Furthermore, we let $L(n) \subset \sigma^{-n}\,\mathrm{Pont}^*(n)$ be the subspace spanned by the components of the Hirzebruch $\mathcal{L}$–class. For $n_1 + n_2 = n$, let

$$\tilde{r}_{n_1,n_2}\colon \sigma^{-n}\,\mathrm{Pont}^*(n) \to \sigma^{-n_1}\,\mathrm{Pont}^*(n_1) \otimes (\sigma^{-n_2}\,\mathrm{Pont}^*(n_2)/L(n_2))$$

be the composition of r_{n_1,n_2} with the quotient map.

Lemma 3.1 (1) *Let $n = 2m$. Then the elements* ph_{4d}, $4d \geq n$, *span the intersection* $\bigcap_{m_1+m_2=m, 0<m_1<m} \ker(r_{2m_1,2m_2}) \subset \sigma^{-n}\,\mathrm{Pont}^*(n)$.

(2) *Let $n = 2m+1$. Then the elements* ph_{4d}, $4d \geq n$, *span the intersection* $\bigcap_{m_1+m_2=m, 0<m_1<m} \ker(r_{2m_1,2m_2+1}) \subset \sigma^{-n}\,\mathrm{Pont}^*(n)$.

(3) *Let $n = 2m+1 \geq 7$. Then the elements* ph_{4d} *and* $\mathcal{L}_{4d}$, $4d \geq n$, *span the intersection* $\bigcap_{m_1+m_2=m, 0<m_1<m} \ker(\widetilde{r}_{2m_1,2m_2+1}) \subset \sigma^{-n}\,\mathrm{Pont}^*(2m+1)$.

Remark The first part can also be deduced from Lemma 16.2 in [27].

Proof of Lemma 3.1 (1)–(2) We identify

$$\mathrm{Pont}^*(2m) = \mathrm{Pont}^*(2m+1) = \mathbb{Q}[x_1, \dots, x_m]^{\Sigma_m},$$

where $x_1, \dots, x_m$ are indeterminates of degree 4, the Pontrjagin classes correspond to elementary symmetric functions and ph_{4d} to $x_1^d + \ldots + x_m^d$. Let us introduce some abbreviations. If $S = \{i_1, \dots, i_s\} \subset \underline{m}$, then $V_S := \mathbb{Q}[x_{i_1}, \dots, x_{i_s}]$. Slightly abusing notation, we identify $V_S \subset V_T$ if $S \subset T$ and $V_S \otimes V_T = V_{S \cup T}$ if $S, T \subset \underline{m}$ with $S \cap T = \varnothing$. Moreover, $V_S^{<d}$ denotes the subspace of elements of degree less than d (as usual, all degrees are total degrees). Let $\pi\colon V_{\{1,\dots,m\}} \to V_{\{1,\dots,m\}} / V_{\{1,\dots,m\}}^{<2m}$.

The kernel of $r_{2k,2m-2k} \circ \pi$ agrees, up to a degree shift, with the space of symmetric polynomials in the kernel of the quotient map

$$V_{\{1,\dots,m\}} \to \frac{V_{\{1,\dots,k\}} \otimes V_{\{k+1,\dots,m\}}}{V_{\{1,\dots,k\}}^{<2k} \otimes V_{\{k+1,\dots,m\}} \oplus V_{\{1,\dots,k\}} \otimes V_{\{k+1,\dots,m\}}^{<2m-2k}}$$

and the intersection of the kernels is

$$\mathcal{I} := \bigcap_{k=1}^{m-1} V_{\{1,\dots,k\}}^{<2k} \otimes V_{\{k+1,\dots,m\}} \oplus V_{\{1,\dots,k\}} \otimes V_{\{k+1,\dots,m\}}^{<2m-2k}.$$

Let $4d \geq 2m$. We have to show the following: if a homogeneous symmetric polynomial $p(x_1 \dots, x_m)$ of degree $4d$ lies in $\mathcal{I}$, then p must be a power sum, ie a multiple of ph_{4d}.

Let $\mathcal{P}$ be the set of all partitions of the set $\underline{m}$ into two parts, ie

$$\mathcal{P} = \{(S_1, S_2) | S_i \subset \underline{m}; S_i \neq \varnothing, S_1 \cup S_1 = \underline{m}, S_1 \cap S_2 = \varnothing\}.$$

A *symmetric* polynomial p lies in $\mathcal{I}$ if and only if it lies in

$$U = \bigcap_{(S_1,S_2)\in\mathcal{P}} V_{S_1}^{<2|S_1|} \otimes V_{S_2} \oplus V_{S_1} \otimes V_{S_2}^{<2|S_2|}.$$

Clearly, $V_{S_1}^{<2|S_1|} \otimes V_{S_2} \oplus V_{S_1} \otimes V_{S_2}^{<2|S_2|}$ is spanned by monomials. Therefore U is spanned by monomials, too. Therefore, the space U has the following property: if $p \in U$ is written as a linear combination of monomials $p = \sum_i a_i p_i$ with pairwise distinct monomials p_i and $0 \neq a_i \in \mathbb{Q}$, then $p_i \in U$.

We call the monomials of the form x_i^j *pure* and all the other ones *impure*. We will show that any monomial in U of degree $\geq n$ is pure (clearly, any pure monomial lies in U). This finishes the proof because then any symmetric $p \in U$ must be a linear combination of pure monomials and the symmetry forces p to be a power sum.

Let now p be an impure monomial of degree $4d > 2m$. We want to show that p does not lie in U. Without loss of generality (symmetry!), we can assume that $p = x_1^{d_1} \dots x_{k-1}^{d_{k-1}} x_m^{d_m}$ and $0 < d_m \leq d_j$ for all $j = 1, \dots, k-1$. Note that $k \geq 2$ since p is impure. Let $r := \min\{m-1, 2d_1 + \dots + 2d_{k-1} - 1\}$. Then

$$k-1 \leq r \leq m-1, \quad 2(d_1 + \dots + d_{k-1}) \geq r, \quad 2d_m \geq m-r.$$

These relations mean that p does not lie in

$$V_{\{1,\dots,r-1\}}^{<2(r-1)} \otimes V_{\{r,\dots,m\}} \oplus V_{\{1,\dots,r-1\}} \otimes V_{\{r,\dots,m\}}^{<2(m-r)}$$

and hence not in U. This finishes the proof in degrees $4d > 2m$. The case $4d = 2m$ is similar, but easier and left to the reader. The same argument applies verbatim to part (2) by replacing $2m$ by $2m+1$. □

To show the third part of Lemma 3.1, we need another lemma on multiplicative sequences. Our notation deviates from the standard notation, so we introduce it here. Let $f = \sum_{k=0}^{\infty} f_k x^k \in 1 + x\mathbb{Q}[[x]]$ be a power series and $F(x_1, \dots, x_r) = f(x_1) \dots f(x_r) = \sum_{i \geq 0} F_i(x_1, \dots, x_r)$ be the associate multiplicative sequence, where F_i is homogeneous of degree i (r is an arbitrary number; a priori we should have denoted F_i by the symbol $F_i^{(r)}$. However, $F_i^{(r+1)}(x_1, \dots, x_r, 0) = F_i^{(r)}(x_1, \dots, x_r)$ and so we may suppress r in the notation).

Lemma 3.2 *Let $f = \sum_{k=0}^{\infty} f_k x^k \in 1 + x\mathbb{Q}[[x]]$ be a power series such that $f_k \neq 0$ for all k. Let $F = \sum_{i \geq 0} F_i$ be the corresponding multiplicative sequence. Let $m \geq 3$ and let $h(x_1, \dots, x_m)$ be a symmetric homogeneous polynomial of degree d. Assume that*

$$h(x_1, \dots, x_m) = \sum_{i=0}^{d} a_i x_m^i F_{d-i}(x_1, \dots, x_{m-1}), \quad a_i \in \mathbb{Q}.$$

Then $h(x_1, \dots, x_m) = a_0 F_d(x_1, \dots, x_m)$.

Proof By the definition of multiplicative sequences, we can write

$$h(x_1,\dots,x_m)=\sum_{k=0}^{d}\sum_{i+j=k} a_i f_j x_m^i x_{m-1}^j F_{d-k}(x_1,\dots,x_{m-2})$$

and by symmetry we also have

$$h(x_1,\dots,x_m)=\sum_{k=0}^{d}\sum_{i+j=k} a_i f_j x_{m-1}^i x_m^j F_{d-k}(x_1,\dots,x_{m-2}).$$

The assumption that $f_k\neq 0$ implies that F_i is nonzero (for all i and an arbitrary positive number of variables). Therefore $a_i f_j=a_j f_i$ for all $0\le i+j\le d$ (here the assumption that $m\geq 2$ is essential). Thus $a_j=a_j f_0=f_j a_0$ and hence

$$h(x_1,\dots,x_m)=a_0\sum_{i=0}^{d} f_i x_m^i F_{d-i}(x_1,\dots,x_{m-1})=a_0F_d(x_1,\dots,x_m). \qquad \square$$

Proof of Lemma 3.1 (3) The space $\ker(\tilde{r}_{2m_1,2m_2+1})\subset\sigma^{-n}\operatorname{Pont}^*(2m+1)$ is the sum of the space $\ker(r_{2m_1,2m_2+1})$ and the space of symmetric elements of degree $\geq n$ in $\mathbb{Q}[x_1,\dots,x_{m_1}]\otimes L(2m_2+1)$. There is an inclusion relation $\mathbb{Q}[x_1]\otimes L(2m-1)\subset\mathbb{Q}[x_1,\dots,x_{m_1}]\otimes L(2m_2+1)$ for all $m_1\geq 1$. Thus the intersection agrees with the smallest space, ie $\mathbb{Q}[x_1]\otimes L(2m-1)$.

Any homogeneous symmetric polynomial $f\in\mathbb{Q}[x_1]\otimes L(2m-1)$ of degree $4d$ can be written as $f(x_1,\dots,x_m)=\sum_{i=0}^{d} a_i x_1^i\mathcal{L}_{4i}(x_2,\dots,x_m)$ with $a_i\in\mathbb{Q}$. By Lemma 3.2, f is a multiple of the Hirzebruch class (here we use that the coefficients of the power series $\sqrt{x}\coth(\sqrt{x})$ are nonzero). $\square$

Proof of Proposition 2.5 Let $f_i\colon E_i\to B_i$, $i=1,2$, be two oriented manifold bundles of fibre dimension $n_i>0$ with $n_1+n_2=n$. Consider $f_1\times f_2\colon E_1\times E_2\to B_1\times B_2$, which is an oriented manifold bundle of fibre dimension n. The Gysin homomorphism is compatible with products in the sense that

$$(f_1\times f_2)_!(x_1\times x_2)=(f_1)_!(x_1)\times(f_2)_!(x_2)$$

for all $x_i\in\operatorname{Pont}^*(T_vE_i)$ by Proposition A.4 (the signs are all $+1$ since x_i has even degree). Therefore the diagram

(3.3)
$$\begin{array}{ccc}
\sigma^{-n}\operatorname{Pont}^*(n) & \xrightarrow{\kappa^{n,0}} & H^*(\mathcal{B}_n^0)\\
\big\downarrow{\scriptstyle r_{n_1,n_2}} & & \big\downarrow\\
\sigma^{-n_1}\operatorname{Pont}^*(n_1)\otimes\sigma^{-n_2}\operatorname{Pont}(n_2) & \xrightarrow{\kappa^{n_1,0}\otimes\kappa^{n_2,0}} & H^*(\mathcal{B}_{n_1}^0)\otimes H^*(\mathcal{B}_{n_2}^0)
\end{array}$$

is commutative; the left-hand side vertical map is induced by the Whitney sum and the right hand side vertical map is induced by taking product bundles. If n_1, n_2 are both even, then by induction hypothesis, $\kappa^{n_1,0} \otimes \kappa^{n_2,0}$ is injective. A straightforward application of Lemma 3.1 (1) completes the proof for even dimensions. In the odd-dimensional case, let $n = n_1 + n_2$, n_1 even and n_2 odd, replace diagram (3.3) by

$$\begin{array}{ccc} \sigma^{-n}\,\mathrm{Pont}^*(n) & \xrightarrow{\kappa^{n,0}} & H^*(\mathcal{B}_n^0) \\ \downarrow{\scriptstyle \tilde{r}_{n_1,n_2}} & & \downarrow \\ \sigma^{-n_1}\,\mathrm{Pont}^*(n_1) \otimes \sigma^{-n_2}\,\mathrm{Pont}(n_2)/L(n_2) & \xrightarrow{\kappa^{n_1,0}\otimes\kappa^{n_2,0}} & H^*(\mathcal{B}_{n_1}^0) \otimes H^*(\mathcal{B}_{n_2}^0) \end{array}$$

and complete the proof by applying Lemma 3.1 (3) (the bottom map is injective by induction hypothesis again). □

4 The loop space construction

The loop space construction

Let M be an oriented closed n–manifold and $f\colon E \to X$ a smooth oriented M–bundle. Let LX be the free loop space of X and let $\mathrm{ev}\colon \mathbb{S}^1 \times LX \to X$ be the evaluation map $\mathrm{ev}(t,\gamma) := \gamma(t)$. Recall that the loop space construction $\mathfrak{L}p = \mathrm{pr} \circ f'$ is defined by the diagram:

$$\begin{array}{ccc} \mathfrak{L}E := \mathbb{S}^1 \times LX \times_X E & \xrightarrow{h} & E \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathbb{S}^1 \times LX & \xrightarrow{\mathrm{ev}} & X \\ \downarrow{\scriptstyle \mathrm{pr}} & & \\ LX & & \end{array}$$

We wish to prove that $\mathfrak{L}E \to LX$ has the structure of a smooth manifold bundle. This requires a detour.

Let G be a topological group, let $P \to \mathbb{S}^1$ be a G–principal bundle and $EG \to BG$ be a universal principal bundle. Let A_P be the space of all G–equivariant maps $P \to EG$. Mapping a G–map $P \to EG$ to the underlying map $\mathbb{S}^1 \to BG$ defines a map $A_P \to L_P BG$, where $L_P BG \subset LBG$ is the connected component of maps $\mathbb{S}^1 \to BG$ that pull back the universal G–bundle to a bundle isomorphic to P. It is straightforward to check that $A_P \to L_P BG$ is a principal bundle with group $\mathrm{Aut}(P)$, the group of all bundle automorphisms of P. Because A_P is contractible (see [19, Chapter 7, Theorem 3.4]), it follows that $B\,\mathrm{Aut}(P) \simeq L_P BG$.

The projection $P \to \mathbb{S}^1$ induces a map $A_P \times_{\mathrm{Aut}(P)} P \to \mathbb{S}^1 \times L_P BG$, which is a G–principal bundle. It is not hard to see that the two G–principal bundles $A_P \times_{\mathrm{Aut}(P)} P$ and $\mathrm{ev}^*(EG)$ on $\mathbb{S}^1 \times L_P BG$ are isomorphic.

Let M be a G–manifold such that G acts by diffeomorphisms on M and let $P \times_G M \to \mathbb{S}^1$ be the M–bundle associated with P. This is a left $\mathrm{Aut}(P)$–space. We obtain a fibre bundle with fibre $Q := P \times_G M$

$$A_P \times_{\mathrm{Aut}(P)} Q \to L_P BG,$$

and because the projection $Q \to \mathbb{S}^1$ is equivariant with respect to the group $\mathrm{Aut}(P)$, it factors as a composition

$$A_P \times_{\mathrm{Aut}(P)} Q \to \mathbb{S}^1 \times L_P BG \to L_P BG.$$

The first map is the M–bundle associated with the G–principal bundle $A_P \times_{\mathrm{Aut}(P)} P \to \mathbb{S}^1 \times L_P BG$ and the second is the projection. These arguments show the following result:

Proposition 4.1 *Let M be a G–manifold. The fibre bundle $A_P \times_{\mathrm{Aut}(P)} Q \to L_P BG$ is isomorphic to the loop construction applied to the bundle $E(G;M) \to BG$ (or rather the restriction of it to the component $L_P BG$).*

This shows a relation of the loop space construction to loop groups which might be illuminating on its own. We use it on a more technical level to endow the loop space construction with the structure of a smooth manifold bundle. We have seen above that the fibre of $\mathfrak{L}E \to LX$ is $Q = P \times_G M$ for some G–bundle on $\mathbb{S}^1$; which is a smooth manifold. But the structure group is $\mathrm{Aut}(P)$, which does not act by diffeomorphisms on Q. We now can show that the structure group can be reduced to a subgroup of the diffeomorphism group.

Proposition 4.2 *Let $E \to X$ be a smooth manifold bundle. Then the loop space construction $\mathfrak{L}E \to LX$ has the structure of a smooth manifold bundle and this structure is unique up to isomorphism.*

Proof Without loss of generality, X is connected. We confine ourselves to the component $L_0X \subset LX$ containing the constant loop. The cautious reader can check that this is the only case we need in the sequel. The general case is similar, but the notation is more involved.

It is clearly sufficient to consider the universal bundle $E_M \to B\mathrm{Diff}(M)$. By Proposition 4.1, the structure group of $\mathfrak{L}E_M|_{L_0 B\mathrm{Diff}(M)} \to L_0 B\mathrm{Diff}(M)$ is the automorphism group of the trivial $\mathrm{Diff}(M)$–bundle on $\mathbb{S}^1$, which is the same as $\mathrm{map}(S^1; \mathrm{Diff}(M))$. Let $C^\infty(\mathbb{S}^1; \mathrm{Diff}(M))$ be the group of all maps $\mathbb{S}^1 \to \mathrm{Diff}(M)$ such that the adjoint map $\mathbb{S}^1 \times M \to M$ is smooth, endowed with the C^∞–topology. To complete the proof, we have to show that $C^\infty(\mathbb{S}^1; \mathrm{Diff}(M)) \to \mathrm{map}(\mathbb{S}^1; \mathrm{Diff}(M))$ is a weak homotopy equivalence.

Let a map $\varphi\colon (\mathbb{D}^r; \mathbb{S}^{r-1}) \to (\mathrm{map}(\mathbb{S}^1; \mathrm{Diff}(M)); C^\infty(\mathbb{S}^1; \mathrm{Diff}(M)))$ be given. The adjoint is a map

$$\mathbb{D}^r \times \mathbb{S}^1 \times M \to M$$

and it is well-known that it can be approximated arbitrarily close by smooth maps; see eg [17, Chapter 2.2]. Such an approximation is the same as a relative null-homotopy of φ, which completes the argument. □

Characteristic classes of the loop space construction

Let us compute the generalized MMM–classes of the bundle $\mathfrak{L}f\colon \mathfrak{L}E \to LX$ in terms of those of the original bundle $f\colon E \to X$. Let $x \in \mathrm{Pont}^*(n+1)$. The vertical tangent bundle of $\mathfrak{L}f = \mathrm{pr} \circ f'$ is isomorphic to $(f')^* T^{\mathrm{pr}} \oplus T^{f'} \cong \mathbb{R} \oplus h^* T^f$. Therefore

(4.3)
$$(\mathfrak{L}f)_!(x(T^{\mathfrak{L}f})) = \mathrm{pr}_!\, f'_!(x(h^*T^f)) = \mathrm{pr}_!\, f'_! h^*(x(T^f)) = \mathrm{pr}_!\, \mathrm{ev}^*\, f_!(x(T^f)),$$

using the naturality and transitivity of the Gysin homomorphism (see Proposition A.3 (1) and (4)). The first equation is true because $x \in \mathrm{Pont}^*(n+1)$. We can rephrase this formula using the notion of the transgression homomorphism.

Definition 4.4 Let X be a space, LX its free loop space, ev: $\mathbb{S}^1 \times LX \to X$ the evaluation map and pr: $\mathbb{S}^1 \times LX \to LX$ be the projection onto the first factor. The *transgression* is the homomorphism

$$\mathrm{trg} := \mathrm{pr}_! \circ \mathrm{ev}^*\colon H^*(X) \to H^{*-1}(LX).$$

Formula (4.3) becomes:

Proposition 4.5 *The diagram*

$$\begin{array}{ccc} \mathrm{Pont}^k(n+1) & \longrightarrow & \mathrm{Pont}^k(n) \\ \downarrow{\scriptstyle \kappa_{\mathfrak{L}E}} & & \downarrow{\scriptstyle \kappa_E} \\ H^{k-n-1}(LX) & \xleftarrow{\mathrm{trg}} & H^{k-n}(X) \end{array}$$

is commutative.

The usefulness of the above construction stems from the fact that the transgression is injective in some cases, which we will explain now. But first we need a slightly different description of the transgression or rather the composition

$$H^p(X) \stackrel{\mathrm{trg}}{\to} H^{p-1}(LX) \stackrel{\mathrm{inc}^*}{\to} H^{p-1}(\Omega X),$$

where inc: $\Omega X \to LX$ denotes the inclusion of the based loop space into the free one. Let X be a simply connected pointed space (the only case that we will need). Let $K(p) = K(\mathbb{Q}; p)$ be the Eilenberg–Mac Lane space and recall that the set of homotopy classes $[X; K(p)]$ is isomorphic to $H^p(X)$. Furthermore $\Omega K(p) = K(p-1)$. Consider the composition

$$\text{(4.6)} \quad H^p(X) = [X, K(p)] \stackrel{\Omega}{\to} [\Omega X; \Omega K(p)] = [\Omega X; K(p-1)] = H^{p-1}(\Omega X).$$

We claim that this coincides with $\mathrm{inc}^* \circ \mathrm{trg}$. The evaluation map $\mathbb{S}^1 \times \Omega X \to X$ can be factored as

$$\mathbb{S}^1 \times \Omega X \stackrel{e}{\to} \mathbb{S}^1 \wedge \Omega X \stackrel{u}{\to} X.$$

The map $\mathrm{pr}_! \circ e^*\colon H^*(\mathbb{S}^1 \wedge \Omega X) \to H^{*-1}(\Omega X)$ is equal to the suspension isomorphism σ and so $\mathrm{inc}^* \circ \mathrm{trg} = \sigma \circ u^*$. That $\sigma \circ u^*$ and (4.6) agree follows from formal consideration ($\mathbb{S}^1 \wedge -$ and Ω are adjoint functors).

Proposition 4.7 *Let X be a simply connected space such that the rational cohomology ring $H^*(X) \cong \Lambda V$ is a free graded-commutative algebra on a finite-dimensional graded vector space V. Then $H^*(LX)$ is a free-graded commutative algebra on a finite-dimensional vector space as well and the transgression $\widetilde{H}^*(X) \to H^{*-1}(LX)$ is injective.*

Proof Of course, the transgression is not a ring homomorphism. Instead, the following product formula holds ($\eta\colon LX \to X$ is the evaluation at the base-point; $\eta(\gamma) := \gamma(1)$):

$$\text{(4.8)} \qquad \mathrm{trg}(x_1 x_2) = (-1)^{|x_1|} \eta^* x_1 \,\mathrm{trg}(x_2) + \mathrm{trg}(x_1) \eta^* x_2.$$

This is shown as follows. Write $\mathrm{ev}^* x_i = 1 \times a_i + u \times b_i \in H^*(\mathbb{S}^1 \times LX)$ for some $a_i, b_i \in H^*(LX)$; $u \in H^1(\mathbb{S}^1)$ is the standard generator. Then $\mathrm{trg}(x_i) = \mathrm{pr}_! \,\mathrm{ev}^* x_i = b_i$ and $\eta^* x_i = a_i$. Formula (4.8) follows immediately from

$$\mathrm{ev}^*(x_1 x_2) = \mathrm{ev}^*(x_1)\,\mathrm{ev}^*(x_2) = 1 \times a_1 a_2 + u \times (b_1 a_2 + (-1)^{|x_1|} a_1 b_2).$$

Let $K(V^\vee) := \prod_k K(V_k^\vee; k)$ be the graded Eilenberg–Mac Lane space of the graded vector space $V^\vee$. There is a natural isomorphism $H^*(K(V^\vee)) \cong \Lambda V$ and the tautological map $s\colon X \to K(V^\vee)$ induces the identity in the sense that

$$\begin{array}{ccc} H^*(K(V^\vee)) & \xrightarrow{s^*} & H^*(X) \\ \downarrow \cong & & \downarrow \cong \\ \Lambda V & = & \Lambda V \end{array}$$

commutes. In particular, s is a rational homotopy equivalence. Because $\pi_1(X) = 0$, LX is nilpotent and there is a rational homotopy equivalence $(LX)_{\mathbb{Q}} \simeq L(X_{\mathbb{Q}}) \simeq L(K(V^\vee))$. The diagram

$$\begin{array}{ccc} H^*(X) & \xrightarrow{\operatorname{trg}} & H^{*-1}(LX) \\ s^* \uparrow & & \uparrow Ls^* \\ H^*(K(V^\vee)) & \xrightarrow{\operatorname{trg}} & H^{*-1}(LK(V^\vee)) \end{array}$$

is commutative and the vertical arrows are isomorphisms. Thus we can assume that $X = K(V^\vee)$.

The algebra map $\tau\colon \Lambda(\eta^* V \oplus \operatorname{trg}(V)) \to H^*(LX)$ induced by $\eta^* \oplus \operatorname{trg}\colon V \oplus \sigma^{-1} V \to H^*(LX)$ is an isomorphism by the following argument. Since a product of Eilenberg–Mac Lane spaces is an abelian topological group, the fibration

$$\Omega X \overset{\operatorname{inc}}{\to} LX \overset{\eta}{\to} X$$

is a product and thus it has a retraction $r\colon LX \to \Omega X$. The maps η^* and r^* induce an isomorphism $H^*(LX) \cong H^*(\Omega X) \otimes H^*(X)$.

Because the composition

$$H^*(X) \overset{\operatorname{trg}}{\to} H^{*-1}(LX) \overset{\operatorname{inc}^*}{\to} H^{*-1}(\Omega X)$$

is the same as (4.6), $\operatorname{inc}^* \circ \operatorname{trg}$ maps V to a generating subspace of the target. It follows that τ is an epimorphism which has to be an isomorphism by a dimension count.

Let $x_1, \dots, x_n$ be a basis of V consisting of homogeneous elements, $a_i := \eta^* x_i$. Let d_i be the degree of x_i. From formula (4.8), one derives the identity

$$\operatorname{trg}(x_1^{m_1} \cdots x_n^{m_n}) = \sum_{i=1}^{n} \epsilon_i m_i a_1^{m_1} \cdots a_{i-1}^{m_{i-1}} \operatorname{trg}(x_i) a_i^{m_i - 1} a_{i+1}^{m_{i+1}} \cdots a_n^{m_n}$$

with $\epsilon_i = (-1)^{d_1 m_1 + \ldots + d_{i-1} m_{i-1}}$, which implies that trg is injective because the terms on the right hand side are all linearly independent. □

Remark An alternative proof of Proposition 4.7 uses a bit rational homotopy theory, more precisely, the formula for the minimal model of LX in terms of the minimal model of X; see Sullivan and Vigué-Poirrier [35]. The details are left to the interested reader.

Lemma 4.9 *Let G be a simply connected compact Lie group. Then the spaces BG, LBG, L^2BG satisfy the assumptions of Proposition 4.7.*

Proof The case of BG is a well-known result generally attributed to Borel. Since BG is 3–connected, the spaces LBG and L^2BG are simply connected and therefore the first half of the statement of Proposition 4.7 can be applied. □

5 Computations for $\mathbb{CP}^m$–bundles

Let $V \to X$ be an $(m+1)$–dimensional hermitian complex vector bundle. Let $q\colon \mathbb{P}(V) \to X$ be the projective bundle of V (its fibre is $\mathbb{CP}^m$ and its structure group is $\mathbb{P}U(m+1)$. The finite isogeny $\mathrm{SU}(m+1) \to \mathbb{P}U(m+1)$ induces a rational homotopy equivalence $B\mathrm{SU}(m+1) \to B\mathbb{P}U(m+1)$ of simply connected spaces. Therefore we conclude:

Lemma 5.1 *If $x \in H^*(BU(m))$ and $p(x_2,\dots,x_{m+1})$ is a polynomial such that $q_!(x(T_v\mathbb{P}(V))) = p(c_4(V),\dots,c_{2m+2})$ for any vector bundle $V \to X$ with structure group $\mathrm{SU}(m+1)$, then the same identity is true for all $(m+1)$–dimensional vector bundles (in other words, the first Chern class c_2 does not occur).*

We will use this Lemma in Section 7. From now on, we restrict our attention to hermitian vector bundles with trivialized determinant and Lemma 5.1 tells us that we do not loose anything. There is a tautological complex line bundle $L_V \to \mathbb{P}(V)$ and the first Chern class of $L_V^\vee$ is denoted by $z_V \in H^2(\mathbb{P}(V))$. There are natural isomorphisms

$$\textbf{(5.2)} \qquad T_v\mathbb{P}(V) \cong L_V^\vee \otimes (L_V)^\perp \quad \text{and} \quad T_v\mathbb{P}(V) \oplus \mathbb{C} \cong q^*V \otimes L_V^\vee$$

where $(L_V)^\perp \subset q^*V$ is the orthogonal complement of L_V. Because

$$\mathbb{CP}^m = \mathrm{SU}(m+1)/S(U(1)\times U(m)), \quad S(U(1)\times U(m)) := \mathrm{SU}(m+1)\cap U(1)\times U(m),$$

we can identify the total space of the universal bundle $E(\mathrm{SU}(m+1),\mathbb{CP}^m)$ with $B(S(U(1)\times U(m)))$. Under this identification, the classifying map of the vertical tangent bundle corresponds to the map induced by the group homomorphism

$$\textbf{(5.3)} \qquad S(U(1)\times U(m)) \to U(m) \subset \mathrm{SO}(2m), \quad \begin{pmatrix} z & 0 \\ 0 & A \end{pmatrix} \mapsto z^{-1}A.$$

The Pontrjagin character for $\mathbb{CP}^m$–bundles

In principle, the MMM–classes of the universal $\mathbb{CP}^m$–bundle $E(\mathrm{SU}(m+1), \mathbb{CP}^m)$ can be computed using the formula on page 51 of Hirzebruch's lecture notes [18]. However, this formula is not appropriate to show Propositions 2.8 and 2.6. Therefore we follow another path. First we turn to the proof of Proposition 2.6, which follows immediately from Lemma 5.4 below.

Our method is to use the Leray–Hirsch Theorem for the computation of generalized MMM–classes. Let $V \to X$ be a complex vector bundle of rank $m+1$, $q\colon \mathbb{P}(V) \to X$, $L_V \to \mathbb{P}(V)$ and $z_V \in H^2(\mathbb{P}(V))$ as above.

As an $H^*(X)$–algebra, $H^*(\mathbb{P}(V))$ is isomorphic to $H^*(X)[z_V]/(\sum_i c_{2i}(V) z_V^{m+1-i})$. The set $\{1, z_V, \dots, z_V^m\}$ is a $H^*(X)$–basis of $H^*(\mathbb{P}(V))$. Moreover, $q_!$ is the $H^*(X)$–linear map determined by $q_!(z_V^i) = 0$ for $0 \le i \le n-1$ and $q_!(z_V^m) = 1$. The higher powers of z_V can be expressed explicitly in terms of this basis. This gives an algorithm to compute $q_!$, which is not very manageable in general. But it is manageable if all but one of the Chern classes of V are zero.

Lemma 5.4 *Let $X = B\mathrm{SU}(2)$ and $V \to X$ the universal 2–dimensional vector bundle. Then $H^*(B\mathrm{SU}(2)) = \mathbb{Q}[u]$ where $u \in H^4(B\mathrm{SU}(2))$ is the second Chern class of V. Consider the projective bundle $q\colon \mathbb{P}(V \oplus \mathbb{C}^{m-1}) \to X$, which is a $\mathbb{CP}^m$–bundle. Then*

$$q_!\big(\mathrm{ch}(T_v\mathbb{P}(V \oplus \mathbb{C}^{m-1}))\big) = \sum_{p=0}^{\infty} a_p u^p,$$

where
$$a_p = (-1)^p \left(\frac{m-1}{(m+2p)!} + \sum_{k+l=p} \frac{2}{(m+2k)!(2l)!} \right) \neq 0.$$

Proof By the isomorphism (5.2), we obtain

$$q_!\big(\mathrm{ch}(T_v\mathbb{P}(V \oplus \mathbb{C}^{m-1}))\big) = (\mathrm{ch}(V) + m - 1) q_!(\mathrm{ch}(L^\vee_{V \oplus \mathbb{C}^{m-1}})). \tag{5.5}$$

When we write the total Chern class of V formally as $c(V) = (1+x_1)(1+x_2)$, then $x := x_1 = -x_2$ and $u = -x^2$. Thus $\mathrm{ch}(V) = \exp(x_1) + \exp(x_2) = 2\cosh(x) = 2\cos(\sqrt{u})$.

Let us compute

$$q_!(\mathrm{ch}(L^\vee_{V \oplus \mathbb{C}^{m-1}})) = \sum_{l=0}^{\infty} \frac{1}{l!} q_!(z^l_{V \oplus \mathbb{C}^{m-1}}).$$

With $z := z_{V \oplus \mathbb{C}^{m-1}}$, we get the algebra isomorphism

$$H^*(\mathbb{P}(V \oplus \mathbb{C}^{m-1})) \cong \mathbb{Q}[q^*u, z]/(z^{m+1} + q^*u z^{m-1}).$$

Therefore, for $l \geq 0$,

$$z^{m+2l+1} = (-1)^{l+1} z^{m-1} q^* u^{l+1}, \quad z^{m+2l} = (-1)^l z^m q^* u^l.$$

Therefore $q_!(z^{m+2l+1}) = 0$ and $q_!(z^{m+2l}) = (-1)^l u^l$ and

$$q_!(\mathrm{ch}(L^{\vee}_{V \oplus \mathbb{C}^{m-1}})) = \sum_{l=0}^{\infty} \frac{(-1)^l}{(m+2l)!} u^l. \tag{5.6}$$

A combination of formulae (5.6) and (5.5) finishes the proof. □

The case of $\mathbb{CP}^2$–bundles

Here we show Proposition 2.8. Consider the $\mathbb{CP}^2$–bundle $q\colon BS(U(1) \times U(2)) \to B\mathrm{SU}(3)$. The proof is in several steps and we start it by showing that $\mathcal{L}_{4d+4}$ indeed lies in the kernel, Lemma 5.9 (we need this twice later on). To proceed, we consider the diagram

$$\begin{array}{ccc} & & BT \\ & & \big\downarrow g \\ B\mathrm{SO}(4) & \xleftarrow{\ h\ } & BS(U(1) \times U(2)) \\ & & \big\downarrow q \\ & & B\mathrm{SU}(3) \end{array} \tag{5.7}$$

where h is the classifying map of the vertical tangent bundle and T is the standard maximal torus of $\mathrm{SU}(3)$ (the group of diagonal matrices of determinant 1). Abbreviate $f := q \circ g$.

The first goal is to show that the image of the composition

$$H^{4d+4}(B\mathrm{SO}(4)) \xrightarrow{h^*} H^{4d+4}(BS(U(1) \times U(2))) \xrightarrow{q_!} H^{4d}(B\mathrm{SU}(3)) \tag{5.8}$$

is at least d–dimensional (Lemma 5.13). To achieve this, we first relate it to the transfer (Lemma 5.11; Lemma 5.10 is preliminary). All spaces in the diagram (5.7) are classifying spaces of compact Lie group whose cohomology is conveniently and naturally expressed in terms of Lie theory by the Chern–Weil isomorphisms. Therefore we use complex coefficients in this section. We give a description of the transfer in terms of the Chern–Weil isomorphism which is probably well-known, Lemma 5.12. This description is used in the proof of Lemma 5.13.

Because $\dim H^{4d+4}(B\mathrm{SO}(4)) = d + 2$, Lemma 5.13 says that the kernel of (5.8) has dimension less or equal than 2. In Lemma 5.16, we show that the kernel has dimension 2 and that its intersection with $\mathrm{Pont}^{4d+4}(4)$ has dimension 1, generated by $\mathcal{L}_{4d+4}$, which shows the theorem. Let us start with the real proof.

Lemma 5.9 *Let $\mathcal{L}$ be the total Hirzebruch $\mathcal{L}$–class. Then*

$$q_!(\mathcal{L}(T^q)) = 1 \in H^*(B\mathrm{SU}(3)).$$

In particular, $\mathcal{L}_{4d}$ lies in the kernel of $\kappa_q\colon \mathrm{Pont}^{4d}(4) \to H^{4d-4}(B\mathrm{SU}(3))$ for all $d \geq 2$.

Proof We offer three methods since they are all interesting. The first and most elementary method is a direct computation that can be found in [18, page 51 ff].

The second method is to use the loop space construction, Proposition 4.5 and then Theorem 2.1 for the resulting $\mathbb{S}^1 \times \mathbb{CP}^2$–bundle.

The third argument is also index-theoretic. By the family index theorem [4], the class $q_!(\mathcal{L}(T^q))$ equals the Chern character of the index bundle of the signature operator on the bundle q. The group $\mathrm{SU}(3)$ acts by *isometries* on $\mathbb{CP}^2$ with respect to the Fubini–Study metric. Therefore the index bundle is the bundle on $B\mathrm{SU}(3)$ associated with the $\mathrm{SU}(3)$–action on the kernel of the signature operator on $\mathbb{CP}^2$ (the cokernel is zero). But the kernel consists of harmonic forms and so, by the Hodge decomposition of differential forms, any isometry that is homotopic to the identity acts trivially on the space of harmonic forms. Since $\mathrm{SU}(3)$ is connected, the action of the whole group is trivial. Thus the index bundle is the trivial bundle of rank 1 whose Chern character vanishes in positive degrees. □

Lemma 5.10 *Let $x \in H^{4d+4}(B\mathrm{SO}(4))$. Then there is a unique $a \in \mathbb{Q}$ and a unique polynomial F such that*

$$x = a\mathcal{L}_{4d+4} + \chi F(\chi, p_4).$$

Proof The uniqueness is clear. Because $p_8 = \chi^2$, we can write

$$x = bp_4^{d+1} + \chi C_1(p_4, \chi)$$

for an appropriate polynomial C_1 and $b \in \mathbb{Q}$. Then write

$$\mathcal{L}_{4d+4} = a_{d+1} p_4^{d+1} + p_8 C_2(p_4, p_8)$$

for $a_{d+1} = 2(2^{2d+2} B_{2d+2}/(2d+2)!) \neq 0 \in \mathbb{Q}$ and a certain polynomial C_2. Put

$$a := \frac{b}{a_{d+1}}, \quad F(\chi, p_4) := C_1(p_4, \chi) - \frac{b}{a_{d+1}} \chi C_2(p_4, p_8). \qquad \square$$

Lemma 5.11 *The image of the composition* (5.8) *agrees with the image of*

$$H^{4d}(B\mathrm{SO}(4)) \xrightarrow{(hg)^*} H^{4d}(BT) \xrightarrow{\mathrm{trf}_f^*} H^{4d}(B\mathrm{SU}(3)).$$

Proof By Lemma 5.10, we can write $q_!(h^*(x))$ as

$$q_!(h^*(x)) = q_!(ah^*\mathcal{L}_{4d+4}) + q_!(h^*\chi h^* F(\chi, p_4)) = 0 + \mathrm{trf}_q^* h^*(F(\chi, p_4)),$$

the second equality holds by Lemma 5.9 and Definition A.5. We can write

$$\mathrm{trf}_q^* h^*(F(\chi, p_4)) = \frac{1}{2}\mathrm{trf}_f^* g^* h^*(F(\chi, p_4)),$$

because $\mathrm{trf}_f^* = \mathrm{trf}_{qg}^* = \mathrm{trf}_q^* \circ \mathrm{trf}_g^*$ and because $\mathrm{trf}_g^* g^*$ is the multiplication with the Euler number of the fibre of g, which is 2 since g is an $\mathbb{S}^2$–bundle. Because $H^*(B\mathrm{SO}(4)) = \mathbb{Q}[p_4, \chi]$, the proof is complete. □

We are going to employ the Chern–Weil isomorphism, which we briefly recall now. More details and complete proofs can be found in Dupont [9, Chapter 8]. Let G be a compact connected Lie group with maximal torus T and Weyl group W. Let $f\colon BT \to BG$ be the universal G/T–bundle. Let $\mathfrak{t}$ be the Lie algebra of T and $\mathfrak{g}$ be the Lie algebra of G. Let $\mathrm{Sym}^*(\mathfrak{g}_{\mathbb{C}}^\vee)^G$ be the $\mathbb{C}$–algebra of all G–invariant polynomials on the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Recall the Chern–Weil isomorphism CW: $\mathrm{Sym}^*(\mathfrak{g}_{\mathbb{C}}^\vee)^G \cong H^*(BG;\mathbb{C})$, which is natural in G. Moreover, the restriction res: $\mathrm{Sym}^*(\mathfrak{g}_{\mathbb{C}}^\vee)^G \to \mathrm{Sym}^*(\mathfrak{t}_{\mathbb{C}}^\vee)^W$ is an isomorphism. In other words, there is a commutative diagram:

$$\begin{array}{ccccc}
H^*(BG;\mathbb{C}) & & \xrightarrow{\quad f^* \quad} & & H^*(BT;\mathbb{C}) \, . \\
\uparrow{\scriptstyle \mathrm{CW}\,\cong} & & & & \uparrow{\scriptstyle \mathrm{CW}\,\cong} \\
\mathrm{Sym}^*(\mathfrak{g}_{\mathbb{C}}^\vee)^G & \xrightarrow[\mathrm{res}]{\cong} & \mathrm{Sym}^*(\mathfrak{t}_{\mathbb{C}}^\vee)^W & \xrightarrow{\subset} & \mathrm{Sym}^*(\mathfrak{t}_{\mathbb{C}}^\vee)
\end{array}$$

Now we express the transfer (see Definition A.5) $\mathrm{trf}_f^*\colon H^*(BT;\mathbb{C}) \to H^*(BG;\mathbb{C})$ as a map $\mathrm{Sym}^*(\mathfrak{t}_{\mathbb{C}}^\vee) \to \mathrm{Sym}^*(\mathfrak{t}_{\mathbb{C}}^\vee)^W$.

Lemma 5.12 *As maps* $\mathrm{Sym}^*(\mathfrak{t}_{\mathbb{C}}^\vee) \to \mathrm{Sym}^*(\mathfrak{t}_{\mathbb{C}}^\vee)^W$, *the transfer* trf_f^* *and the averaging operator* $F \mapsto \sum_{w\in W} w^* F$ *are equal.*

Proof The left G–action on G/T (by multiplication) commutes with the right-action of W (by conjugation). Therefore there is a fibre-preserving right-action of W on the bundle $E(G;G/T) \to BG$. The total space $E(G;G/T)$ is homotopy equivalent to BT and the homotopy equivalence is W–equivariant. In particular, $f \circ w = f$

for all $w \in W$. Therefore $\mathrm{trf}_f^* = \mathrm{trf}_{f\circ w}^* = \mathrm{trf}_f^* \circ w^*$ (use Proposition A.6 (1), (3)). In other words, the transfer is W–equivariant when considered as a map $\mathrm{Sym}^*(\mathfrak{t}_{\mathbb{C}}^\vee) \to \mathrm{Sym}^*(\mathfrak{t}_{\mathbb{C}}^\vee)^W$. The composition

$$\mathrm{Sym}^*(\mathfrak{t}_{\mathbb{C}}^\vee)^W \xrightarrow{f^*} \mathrm{Sym}^*(\mathfrak{t}_{\mathbb{C}}^\vee) \xrightarrow{\mathrm{trf}_f^*} \mathrm{Sym}^*(\mathfrak{t}_{\mathbb{C}}^\vee)^W$$

is the map $\mathrm{trf}_f^* f^*$, which is the multiplication by the Euler number $\chi(G/T)$ of the fibre by Proposition A.6 (2). On the other hand, it is well-known that $\chi(G/T)$ coincides with the order $|W|$ of the Weyl group.

Recall the following elementary fact from representation theory of finite groups. Let H be a finite group and V an H–representation. Let $h\colon V \to V^H$ be an equivariant linear map. If the restriction of h to V^H is a multiple of the identity, then h is a multiple of the averaging operator $v \mapsto \sum_{h\in H} hv$. Apply this to the transfer. □

Lemma 5.13 *The image of the composition*

$$H^{4d}(B\mathrm{SO}(4)) \xrightarrow{(hg)^*} H^{4d}(BT) \xrightarrow{\mathrm{trf}_f^*} H^{4d}(B\mathrm{SU}(3))$$

from Lemma 5.11 has dimension at least d.

Proof We write the complexified Lie algebra of T as

$$\mathfrak{t} = \{(x_1, x_2, x_3) \in \mathbb{C}^3 \mid x_1 + x_2 + x_3 = 0\}.$$

The Weyl group of $\mathrm{SU}(3)$ is Σ_3, acting by the permutation representation. We write x_1, x_2, x_3 for the coordinate functions on $\mathfrak{t}$.

Under the map $h \circ g\colon BT \to B\mathrm{SO}(4)$, the elements χ and p_1 are mapped by

(5.14) $$\chi \mapsto (x_2 - x_1)(x_3 - x_1), \quad p_4 \mapsto (x_2 - x_1)^2 + (x_3 - x_1)^2,$$

using the isomorphism (5.2) or the equivalent expression (5.3). These elements lie in the 3–dimensional space $V := \mathrm{Sym}^2(\mathfrak{t}_{\mathbb{C}}^\vee)$ on which we now introduce the basis

$$z_1 = (x_2 - x_1)(x_3 - x_1), \quad z_2 = (x_1 - x_2)(x_3 - x_2), \quad z_3 = (x_2 - x_3)(x_1 - x_3).$$

The Weyl group acts by permutations on that basis. Rewriting (5.14) yields

(5.15) $$\chi \mapsto z_1, \quad p_4 \mapsto 2z_1 + z_2 + z_3 = z_1 + s_1,$$

where s_i denotes the W–invariant element $s_i := z_1^i + z_2^i + z_3^i$.

In view of Lemma 5.12, we have to show that the image of the $(d+1)$–dimensional subspace $X := \mathrm{span}\{z_1^k(z_1 + s_1)^{d-k} \mid k = 0, \dots, d\}$ of $\mathrm{Sym}^d V$ under the averaging operator $\Phi = \sum_{\sigma \in \Sigma_3} \sigma$ has dimension d. To this end, abbreviate $v_{k,d} = z_1^k (z_1 + s_1)^{d-k}$ and note that

$$\Phi(v_{k,d}) = 2 \sum_j \binom{d-k}{j} s_1^j s_{d-j}.$$

Let C be the $((d+1) \times (d+1))$–matrix with entries $c_{j,k} = \binom{d-k}{j}$ $(0 \le j,k \le d)$; C is nonsingular because its entries below the antidiagonal are zero and the entries on the antidiagonal are 1, therefore $\det(C) = \pm 1$. Therefore the equation

$$\Phi\Big(\sum_k a_k v_{k,d} \Big) = s_1^j s_{d-j}$$

has a solution (a_k) in $\mathbb{C}^{d+1}$ and the image of X under Φ contains the elements

$$s_1^d, s_1^{d-1} s_1, s_1^{d-2} s_2, \dots, s_1 s_{d-1}, s_d.$$

We claim that these polynomials span an d–dimensional vector space and show this claim by induction on d. The case $d = 2$ is trivial.

Because the multiplication by s_1 is injective, it suffices to show that s_d is not a linear combination of $s_1^d, s_1^{d-1} s_1, s_1^{d-2} s_2, \dots, s_1 s_{d-1}$. Assume, to the contrary, that

$$s_d(z_1, z_2, z_3) = \sum_{j=0}^{d-1} c_j s_j(z_1, z_2, z_3) s_1^{d-j}(z_1, z_2, z_3), \quad c_j \in \mathbb{C}.$$

Restricting to the subspace defined by $z_1 + z_2 + z_3 = 0$, we get the equation

$$z_1^d + z_2^d + (-z_1 - z_2)^d = \sum_{j=0}^{d-1} c_j s_j(z_1, z_2, z_3)(z_1 + z_2 + z_3)^{d-j} = 0,$$

which is obviously wrong for all $d \ge 2$. □

Lemma 5.16 *The kernel of the composition* (5.8) *is* 2*–dimensional. The intersection of the kernel with* Pont^{4d+4} *is* 1*–dimensional and* $\mathcal{L}_{4d+4}$ *is a generator.*

Proof By Lemmas 5.13 and 5.11 and since $\dim H^{4d+4}(B\mathrm{SO}(4)) = d+2$, the kernel is at most 2–dimensional. By Lemma 5.9, $\mathcal{L}_{4d+4}$ lies in the kernel. Another element of the kernel is $(p_4 - \chi)^{d+1}$. To see this, look at formula (5.15): $g^* h^*(p_4 - \chi) = s_1 \in V^{\Sigma_3} = \mathrm{Im}\, f^*$. Since g^* is injective, it follows that $h^*(p_4 - \chi) = q^* y$ for a certain y. It follows that

$$q_!(h^*(p_4 - \chi)^{d+1}) = q_!(q^* y^{d+1} 1) = y^{d+1} q_!(1) = 0.$$

This means that any element in the kernel of the map (5.8) can be written as $a_1\mathcal{L}_{4d+4} + a_2(p_4-\chi)^{d+1}$ (in a unique way, since $\mathcal{L}_{4d+4}$ and $(p_4-\chi)^{d+1}$ are linearly independent). But $a_1\mathcal{L}_{4d+4} + a_2(p_4-\chi)^{d+1}$ belongs to $\mathrm{Pont}^{4d+4}(4)$ if and only if $a_2 = 0$. □

Remark It is known that the oriented cobordism ring Ω_*^{SO} is generated by total spaces of $\mathbb{CP}^2$–bundles (this is unfinished work by R Jung, recently written down by S Führing [13]). In other words, there exist $\mathbb{CP}^2$–bundles with arbitrarily prescribed characteristic numbers. This ought to be related to Proposition 2.8. However, I was not able to find an explicit relation.

6 From linear to algebraic independence

In this section, we show Theorem B, based on Theorem A whose proof we just completed. It is this step where we have to sacrifice the connectedness of the manifolds. Recall the spaces $\mathcal{B}_n$ and $\mathcal{B}_n^0$ defined in the introduction. The main step is:

Proposition 6.1 *Let $W \subset \tilde{\sigma}^{-n} H^*(B\mathrm{SO}(n))$ be a linear subspace such that $\kappa^n\colon W \to H^*(\mathcal{B}_n^0)$ is injective. Then the extension $\Lambda'\kappa^{n,0}\colon \Lambda W \to H^*(\mathcal{B}_n)$ is injective.*

Assuming Proposition 6.1 for the moment, we can show Theorem B.

Proof of Theorem B If n is even, then Theorem B is an immediate consequence of Theorem A and Proposition 6.1.

If n is odd, we need a little argument. If $W \subset V$ are graded vector spaces, then $\Lambda(V/W) \cong \Lambda(V)/(W)$, where (W) is the 2–sided ideal generated by W. Let $V := \tilde{\sigma}^{-n} H^*(B\mathrm{SO}(n))$ and let W be the span of the Hirzebruch $\mathcal{L}$–classes. Choose a complement $U \subset V$ of W. By Theorem A, $\kappa^{n,0}\colon U \to H^*(\coprod_{M\in\mathcal{R}_n} B\mathrm{Diff}^+(M))$ is injective; whence $\Lambda(U) \to H^*(\coprod_{M\in\mathcal{R}_n} B\mathrm{Diff}^+(M))$ is injective by Proposition 6.1. But $\Lambda(U) \cong \Lambda(V)/(W)$ and therefore the statement follows. □

Proof of Proposition 6.1 Without loss of generality, we can assume that W is finite-dimensional.

There exist connected n–manifolds $M_1, \dots, M_r$ such that $\kappa^n\colon W \to H^*(\mathcal{B}_n^0) \to H^*(\coprod_{i=1}^r B\mathrm{Diff}^+(M_i))$ is injective. Put $G_i := \mathrm{Diff}^+(M_i)$ and $M := \coprod_{i=1}^r M_i$. The group $\prod_{i=1}^r G_i$ acts on M, separately on each factor. Thus there is a smooth M–bundle $E \to \prod_{i=1}^r BG_i$. A choice of base-points in BG_i defines a map $\coprod_{i=1}^r BG_i \to \prod_{i=1}^r BG_i$ and it is easy to see that the diagram

$$\begin{array}{ccc} W & \xrightarrow{\kappa^{n,0}} & H^*(\mathcal{B}_n^0) \\ \downarrow{\scriptstyle \kappa_E} & & \downarrow \\ H^*(\prod_{i=1}^r BG_i) & \longrightarrow & H^*(\coprod_{i=1}^r BG_i) \end{array}$$

commutes. Therefore $\kappa_E \colon W \to H^*(\prod_{i=1}^r BG_i)$ is injective. The purpose of this argument is to show that we can find a single (possibly disconnected) manifold M and a smooth M–bundle $f \colon E \to B$ on a connected base space such that $\kappa_E \colon W \to H^*(B)$ is injective.

Let $m \in \mathbb{N}$ and let Σ_m be the symmetric group. Now we consider the diagram

$$\begin{array}{ccc} E' & \xrightarrow{p'} & E \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ E(\Sigma_m; \underline{m} \times B^m) & \xrightarrow{p} & B \\ \downarrow{\scriptstyle q} & & \\ E(\Sigma_m; B^m) & & \end{array} \tag{6.2}$$

where the map p is given by the Σ_m–equivariant map $\underline{m} \times B^m \to B$, $(i, x_1, \ldots, x_m) \mapsto x_i$; the square is a pullback and the composition $q \circ f'$ is a smooth $\underline{m} \times M$–bundle (note the similarity to the loop space construction).

In the same way as in the proof of Proposition 4.5 (formula (4.3)), one sees that the diagram

$$\begin{array}{ccc} W & \xrightarrow{\kappa_E} & H^*(B) \\ & \searrow{\scriptstyle \kappa_{E'}} & \downarrow{\scriptstyle q_! \circ p^*} \\ & & H^*(E(\Sigma_m; B^m)) \end{array} \tag{6.3}$$

commutes. Hence the induced diagram

$$\begin{array}{ccc} \Lambda W & \xrightarrow{\Lambda' \kappa_E} & \Lambda H^*(B) \\ & \searrow{\scriptstyle \Lambda' \kappa_{E'}} & \downarrow{\scriptstyle \Lambda(q_! \circ p^*)} \\ & & H^*(E(\Sigma_m; B^m)) \end{array} \tag{6.4}$$

commutes as well. The top horizontal map is injective by assumption.

The map $q_! \circ p^*\colon H^*(B) \to H^*(E(\Sigma_m; B^m))$ induces an algebra map $\Lambda \widetilde{H}^*(B) \to H^*(E(\Sigma_m; B^m))$ which is an isomorphism in degrees $* \leq (m-1)/2$. This is a combination of the Barratt–Priddy–Quillen theorem and homological stability for symmetric groups (Nakaoka [31] et alii). See Ebert and Giansiracusa [11, Section 5.3] for details and references. □

Remark It is obvious that it is necessary to consider disconnected manifolds in the above proof of Theorem B. We do not know whether Theorem B remains true if $\Lambda' \kappa^n$ is replaced by $\Lambda' \kappa^{n,0}$. In the 2–dimensional case, the situation is different. All published proofs of Theorem 2.2 show that $\Lambda' \kappa^{n,0}$ is injective. For the passage from $\kappa^{2,0}$ to $\Lambda' \kappa^{2,0}$, the use of Harer's homological stability theorem for the mapping class groups is essential, while the stability result is not necessary to show that $\kappa^{2,0}$ is injective (this point is most obvious in Miller's proof [26]). Since a large portion of the proof of Theorem A relies on the 2–dimensional case, there are partial results for $\Lambda' \kappa^{n,0}$; see eg Giansiracusa [16] for a result in the 4–dimensional case.

7 The holomorphic case

In this section, we prove Theorem C, which is parallel to the proofs of Theorems A and B. So we sketch only the differences.

The proofs of Theorem 2.2 given by Miller and Morita show that Theorem C holds if $m = 1$. The inductive procedure works in the same way; the proof of Proposition 2.5 is easily adjusted and shows that only the classes of the form $\kappa_E(\mathrm{ch}_{2d})$, $2d \geq 2m$ cannot be detected on products.

If $q\colon E \to BU(m+1)$ is the universal $\mathbb{CP}^m$–bundle, then the class $q_!(\mathrm{ch}_{2d}(T_v E))$ is nonzero if $2d \geq 2m$ and $d - m \equiv 0 \pmod 2$ by Proposition 2.8. Of course, $BU(m)$ is not a complex manifold; nevertheless it can be approximated by the Grassmann manifolds $\mathrm{Gr}_m(\mathbb{C}^r)$ of m–dimensional quotients of $\mathbb{C}^r$ for $r \gg m$, which is a projective variety. The tautological vector bundle on $\mathrm{Gr}_m(\mathbb{C}^r)$ is a holomorphic vector bundle and hence its projectivization is a holomorphic fibre bundle.

Thus we are left with showing that $\kappa^m_{\mathbb{C}}(\mathrm{ch}_{2d}) \neq 0$ if $2m \leq 2d$ and $d - m \equiv 1 \pmod 2$. The loop space construction as in Section 4 does not make sense in the holomorphic realm. One could replace $\mathbb{S}^1$ by $\mathbb{CP}^1$ in the loop space construction and the space $\mathrm{map}(\mathbb{S}^1, BU(m+1))$ by the space of holomorphic maps $\mathrm{hol}_k(\mathbb{CP}^1; \mathrm{Gr}_{m+1}(\mathbb{C}^r)$ and then use the fact (proven by Segal [33] and Kirwan [21]) that the space of holomorphic maps into a Grassmannian is a good homotopical approximation to the space of all maps, but we prefer a more direct route. Let $T \to \mathbb{CP}^1$ and $L \to \mathbb{CP}^r$ be the tautological line bundles. Consider the 2–dimensional vector bundle $V=(\mathbb{C}\oplus T)\boxtimes L\to\mathbb{CP}^1\times\mathbb{CP}^r$. Its total Chern class is $c(V)=(1+1\times x)(1+z\times 1+1\times x)$, where $x\in H^2(\mathbb{CP}^r)$ and $z\in H^2(\mathbb{CP}^1)$ are the usual generators. Therefore the second Chern class is $u=1\times x^2+z\times x$ and $u^l=1\times x^{2l}+lz\times x^{2l-1}\neq 0$ for $r\gg 2l$. Consider the composite bundle

$$\mathbb{P}(V\oplus\mathbb{C}^{m-2})\xrightarrow{q}\mathbb{CP}^1\times\mathbb{CP}^r\xrightarrow{\mathrm{pr}}\mathbb{CP}^r$$

with fibre $\mathbb{CP}^1\times\mathbb{CP}^{m-1}$. A computation similar to the one in (4.3) (and using Proposition A.4, (4)) shows that

$$\mathrm{pr}_!\,q_!(\mathrm{ch}(T^{\mathrm{pr}\circ q}))=\mathrm{pr}_!\,q_!(\mathrm{ch}(T^q))+\mathrm{pr}_!(q_!q^*\,\mathrm{ch}(T^{\mathrm{pr}}))=\mathrm{pr}_!\,q_!(\mathrm{ch}(T^q)).$$

By Lemma 5.4 and Lemma 5.1,

$$\mathrm{pr}_!\,q_!(\mathrm{ch}(T^q))=\mathrm{pr}_!\sum_{l=0}^{\infty}b_lu^l,$$

where $b_l=(-1)^l\big((m-2)/(m-1+2p)!+\sum_{k+p=l}(2/(m-1+2k)!(2p)!)\big)\neq 0$; see Lemma 5.4. But $\mathrm{pr}_!(u^l)=lx^{2l-1}$. This finishes the proof that $\kappa_E^{\mathbb{C}}(\mathrm{ch}_{2d})\neq 0$ for a certain m–dimensional manifold bundle with $m-d$ odd.

To show the second half of Theorem C, we replace the space $E\Sigma_m$ by the configuration space $C^m(\mathbb{C}^r)$ of m numbered points in $\mathbb{C}^r$ for sufficiently large r.

8 The unoriented case

In this section, we consider unoriented manifold bundles and prove Theorem E. We continue to work in rational cohomology. For the notation of twisted cohomology and the Gysin homomorphism for nonoriented bundles, we refer to Appendix A.

Let $f\colon E\to B$ be an unoriented manifold bundle of dimension n. Let ω be the orientation twist on $BO(n)$. If $\phi\colon E\to BO(n)$ is a classifying map for the vertical tangent bundle of E, then $\phi^*\omega$ is the orientation twist for T_vE. Therefore denoting the orientation twist on E by ω will not lead to confusion. Let $c\in H^{*+\omega}(BO(n))$. Then $c(T_vE):=\phi^*c\in H^{*+\omega}(E)$. We can define characteristic classes of unoriented manifold bundles by

$$\lambda_E(c):=f_!(c(T_vE))\in H^{*-n}(B).$$

Let $\mathcal{A}_n:=\coprod_{M\in\mathcal{S}_n}B\mathrm{Diff}(M)$; here $\mathcal{S}_n$ denotes a set of representatives for the diffeomorphism classes of unoriented closed n–manifolds. As in the oriented case, $\mathcal{A}_n$ is the base space of an unoriented manifold bundle. Thus we have maps

$$\lambda^n\colon \sigma^{-n} H^{*+\omega}(BO(n)) \to H^*(\mathcal{A}_n), \tag{8.1}$$

$$\Lambda'\lambda^n\colon \Lambda\sigma^{-n} H^{*+\omega}(BO(n)) \to H^*(\mathcal{A}_n). \tag{8.2}$$

We will first show that λ^n and $\Lambda'\lambda^n$ are injective (Proposition 8.5). Then we will interpret the result in terms of Madsen–Weiss–Tillmann spectra. But our first job is to compute $H^{*+\omega}(BO(n))$.

Proposition 8.3 (1) *If n is odd, then $H^{*+\omega}(BO(n)) = 0$.*

(2) *$H^{*+\omega}(BO(2m))$ is a free graded module of rank 1 over the untwisted cohomology $H^*(BO(2m)) \cong \mathrm{Pont}(2n)$, generated by the single element $\chi \in H^{2m+\omega}(BO(2m))$.*

Proof Look at the Leray–Serre spectral sequence of the fibration $\mathbb{Z}/2 \to B\mathrm{SO}(n) \to BO(n)$, which for degree reasons collapses at the E_2–stage. It follows that there is an isomorphism $H^*(B\mathrm{SO}(n)) \cong H^*(BO(n); V)$ where V is the coefficient system obtained by the regular representation of $\pi_1(BO(n)) = \mathbb{Z}/2$. But V is the direct sum of the trivial coefficient system ω and the trivial 1–dimensional coefficient system $\mathbb{Q}$. Thus

$$H^*(BO(n)) \oplus H^{*+\omega}(BO(n)) \cong H^*(B\mathrm{SO}(n)), \tag{8.4}$$

where the isomorphism is given by the map $B\mathrm{SO}(n) \to BO(n)$. The group $\mathbb{Z}/2$ acts by conjugation on $H^*(B\mathrm{SO}(n))$. It acts trivially on the summand $H^*(BO(n))$ and by the sign representation on the summand $H^{*+\omega}(BO(n))$. This implies that $H^{*+\omega}(BO(n)) \to H^*(B\mathrm{SO}(n))$ maps onto the isotypical summand corresponding to the sign representation.

On the other hand, it is easy to see that $\mathrm{Pont}^*(n) \subset H^*(B\mathrm{SO}(n))$ is fixed by the $\mathbb{Z}/2$–action. Since $\mathrm{Pont}^*(2m+1) = H^*(B\mathrm{SO}(2m+1))$, this finishes the proof for odd n.

If $n = 2m$, then $\mathbb{Z}/2$ acts trivially on $\mathrm{Pont}^*(2m)$ and $H^*(B\mathrm{SO}(2m))$ decomposes under the $\mathbb{Z}/2$–action as

$$H^*(B\mathrm{SO}(2m)) \cong \mathrm{Pont}^*(2m) \oplus \chi\,\mathrm{Pont}^*(2m). \qquad \square$$

Proposition 8.5 *The maps λ^n and $\Lambda'\lambda^n$ are injective.*

Proof There is nothing to show if n is odd, so let $n = 2m$. Let $f\colon E \to B$ be an unoriented manifold bundle with orientation twist ω. There exists an oriented manifold bundle $g\colon \widetilde{E} \to B$ and a two-sheeted covering $p\colon \widetilde{E} \to E$ which restricts to the orientation covering on each fibre; see [12] (the point is that the orientation cover is a truly natural construction). If $c \in H^{*+\omega}(BO(n))$, then

$$f_!(c(T_v E)) = \frac{1}{2} f_! p_! p^* c(T_v E) = g_! c(T_v \widetilde{E})$$

by Propositions A.3 (4) and A.6 (2). So in order to show that $f_!(c(T_v E))$ is nonzero for some manifold bundle, it suffices to find an oriented manifold bundle $g\colon \widetilde{E} \to B$ that is the fibrewise orientation cover of an unoriented bundle $f\colon E \to B$ such that $\kappa_E(j^* c) \neq 0$, where $j\colon B\mathrm{SO}(n) \to BO(n)$ is the natural inclusion.

Consider the action of $\mathrm{SO}(2m+1)$ on $\mathbb{S}^{2m}$. Clearly, the antipodal map on $\mathbb{S}^{2m}$ is $\mathrm{SO}(2m+1)$–equivariant; this gives an $\mathrm{SO}(2m+1)$–equivariant map $\mathbb{S}^{2m} \to \mathbb{RP}^{2m}$. So there is a two-sheeted covering of manifold bundles over $B\mathrm{SO}(2m+1)$:

$$p\colon E(\mathrm{SO}(2m+1), \mathbb{S}^{2m}) \to E(\mathrm{SO}(2m+1), \mathbb{RP}^{2m}).$$

which restricts to the orientation cover on each fibre. Let $c \in H^{*+\omega}(BO(n))$. By Proposition 8.3, we can write $j^* c = \chi x$ for $x \in \mathrm{Pont}^*(n)$. In Lemma 2.4, we have shown that $\kappa_{E(\mathrm{SO}(2m+1);\mathbb{S}^{2m})}(\chi x) = 2x$, which is nonzero if $c \neq 0$. This shows that λ^{2m} is injective.

The argument of Section 6 apply verbatim to show that the injectivity of λ^n implies the injectivity of $\Lambda' \lambda^n$. □

To finish the proof of Theorem E, we have to relate the maps $\Lambda' \lambda^n$ and $(\beta^n)^*$. By the Thom isomorphism, the rational cohomology of the infinite loop space is

$$H^*(\Omega_0^\infty \mathrm{MTO}(n)) \cong \Lambda \widetilde{\sigma}^{-n} H^{*+\omega}(BO(n)).$$

Moreover the diagram

$$\begin{array}{ccc} \Lambda \widetilde{\sigma}^{-n} H^{*+\omega}(BO(n)) & \xrightarrow{\Lambda' \lambda^n} & H^*(\mathcal{A}_n) \\ \downarrow & \nearrow & \\ H^*(\Omega_0^\infty \mathrm{MTO}(n)) & & \end{array}$$

is commutative. This follows as in the oriented case; see [10, Section 2.4].

Appendix A Gysin homomorphisms and the transfer

Here we give a brief recapitulation of Gysin homomorphisms for manifold bundles. In order to include the case of nonoriented manifold bundles, we generalize the well-known construction a bit. The important point is to track down the choice of orientations in the classical construction.

We continue to work with rational coefficients, though everything can be done over $\mathbb{Z}$ as well. First we need to contemplate a moment about local coefficient systems. A special class of coefficient system are the *twists*. A twist ω on X is a group bundle on X with fibre $\mathbb{Q}$ and structure group $\{\pm 1\} \subset \mathbb{Q}^{\times}$. We denote the cohomology of a space X in this coefficient system by $H^{*+\omega}(X)$ and use a similar convention for homology. More general, if U is another coefficient system on X, not necessarily a twist, we denote by $H^{*+\omega}(X;U)$ the cohomology with coefficients in $U \otimes \omega$. The tensor product of two twists ω and τ is denoted by $\omega+\tau$. Note that the twist $\omega+\omega$ admits a *canonical* trivialization. The cup product in twisted cohomology is a pairing

$$H^{p+\omega}(X) \otimes H^{q+\tau}(X) \to H^{p+q+\omega+\tau}(X),$$

in particular, $H^{*+\omega}(X)$ is a module over $H^*(X)$.

Let $V \to X$ be a real vector bundle of rank n. The vector bundle gives rise to a twist $\omega(V)$, the fibre at $x \in X$ being $H^n(V_x; V_x \setminus 0)$. The Thom isomorphism reads (for an arbitrary twist τ)

$$H^{*+\tau}(X) \cong H^{*+n+\tau+\omega(V)}(V, V \setminus 0). \qquad \textbf{(A.1)}$$

If V is orientable, then any orientation defines a trivialization of the twist $\omega(V)$, giving an isomorphism $H^{*+n+\omega}(V, V\setminus 0) \cong H^{*+n}(V, V \setminus 0)$. This isomorphism depends of course on the choice of the orientation, but the isomorphism (A.1) is a perfectly canonical gadget.

Similarly, let M be a closed n–manifold and let $\omega = \omega(TM)$. There is a *canonical* fundamental class $[M] \in H_{n+\omega}(M)$. An orientation of M, if it exists, yields an isomorphism $H_{n+\omega}(M) \cong H_n(M)$ which again depends on the choice of the orientation. There is a cap product pairing

$$H^{s+\omega}(M) \otimes H_{t+\omega}(M) \to H_{t-s}(M).$$

The twist vanishes since $\omega+\omega$ has a canonical trivialization.

A consequence of these remarks is that for any closed manifold M, there is a canonical integration homomorphism

$$\cap [M]\colon H^{n+\omega}(M) \to H_0(M) \to \mathbb{Z}$$

which does not depend on any choice and which is $\mathrm{Diff}(M)$–invariant.

Let $f\colon E \to B$ be a manifold bundle of dimension n and $\omega := \omega(T_v E)$. Let τ be a twist on B. The Gysin homomorphism $f_!\colon H^{k+n+f^*\tau+\omega}(E) \to H^{k+\tau}(B)$ of a smooth closed oriented manifold bundle is defined by means of the Leray–Serre spectral sequence; see eg [29, page 147 ff] for the oriented case. Let $E_r^{p,q}$ be the Leray–Serre spectral sequence of the fibration $E \to B$. The Gysin homomorphism $f_!$ is defined as the composition

$$\textbf{(A.2)} \quad f_!\colon H^{k+n+f^*\tau+\omega}(E) \longrightarrow E_\infty^{k,n} \subset E_2^{k,n} = H^{k+\tau}(B; \underline{H^{n+\omega}(M)}) \xrightarrow{\cap[M]} H^{k+\tau}(B).$$

If E is orientable and τ trivial, then any choice of an orientation defines an isomorphism $H^{k+n+\omega}(E) \cong H^{k+n}(E)$ and by composition with that isomorphism we obtain the Gysin homomorphism used in the main part of this paper.

Below there is a list of the main properties of the Gysin homomorphism. The proof can be found in [6, Section 8]. The generalization to nontrivial twists is straightforward.

Proposition A.3 *Let M be a closed n–manifold and $f\colon E \to B$ be smooth M–bundle with orientation twist $\omega = \omega(T_v E)$ on E.*

(1) Naturality: *If*

$$\begin{array}{ccc} E' & \xrightarrow{\hat{g}} & E \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ B' & \xrightarrow{g} & B \end{array}$$

is a pullback-square, then $f'_! \circ \hat{g}^* = g^* \circ f_!$.

(2) $H^*(B)$–linearity: *If $x \in H^{*+\omega}(E)$ and $y \in H^*(B)$, then $f_!((f^*y)x) = y f_!(x)$.*

(3) Normalization: *If M is an n–manifold whose fundamental class is $[M] \in H_{n+\omega(TM)}(M)$ and $f\colon M \to *$ the constant map, then $f_!(x) = \langle x; [M]\rangle 1$ for all $x \in H^{*+\omega(TM)}(M)$.*

(4) Transitivity: *If N is another closed manifold and $g\colon X \to E$ be a smooth N–bundle, then $(f \circ g)_! = f_! \circ g_!$.*

All computations in the present paper that involve Gysin homomorphisms only use Proposition A.3. In fact, one can show that the formulae in Proposition A.3 characterize the Gysin homomorphism uniquely. Hence one can view these formulae as a list of axioms for the Gysin homomorphism. The following properties are straightforward consequences of Proposition A.3.

Proposition A.4 *Let $f\colon E \to B$ be an oriented smooth n–manifold bundle.*

(1) *For all $x \in H^*(E)$ and $y \in H^*(B)$,*

$$f_!(xf^*(y)) = (-1)^{n|y|} f_!(f^*(y)x) = (-1)^{(|x|-n)|y|} f_!(x)y.$$

(2) *Let $f_i\colon E_i \to B_i$, $i = 1,2$, be two oriented manifold bundles of fibre dimension n_i. Consider the oriented manifold bundle $f = f_1 \times f_2\colon E = E_1 \times E_2 \to B = B_1 \times B_2$ of fibre dimension $n = n_1 + n_2$. Then for all $x_i \in H^*(E_i)$,*

$$(f_1 \times f_2)_!(x_1 \times x_2) = (-1)^{n_2|x_1|}(f_1)_!(x_1) \times (f_2)_!(x_2).$$

(3) *If f is a homeomorphism (the fibre is a point), then $f_! = (f^{-1})^*$.*

(4) *If the fibres of f have positive dimension, then $f_! \circ f^* = 0$.*

Another construction of Gysin homomorphisms is homotopy-theoretic in nature and uses the Pontrjagin–Thom construction; see eg Becker and Gottlieb [5]. If $f\colon E \to B$ is a smooth manifold bundle with closed fibres, then the Pontrjagin–Thom map is a map $\mathrm{PT}_f\colon \Sigma^\infty B_+ \to \mathrm{Th}(-T_v E)$ of spectra ($\mathrm{Th}(-T_v E)$ is the Thom spectrum of the stable vector bundle $-T_v E$. The Thom isomorphism of $T_v E$ is an isomorphism th: $H^{*+\omega(T_v E)}(\Sigma^\infty E_+) \cong H^{*-n}(\mathrm{Th}(-T_v E))$ and the Gysin homomorphism is the composition ($\omega := \omega(T_v E)$)

$$f_! = \mathrm{PT}_f^* \circ \mathrm{th}\colon H^{k+\omega}(E) \xrightarrow{\mathrm{th}} H^{k-n}(\mathrm{Th}(-T_v E)) \xrightarrow{\mathrm{PT}_f^*} H^{k-n}(\Sigma^\infty B_+) = H^{n-k}(B).$$

Closely related to the Gysin homomorphism is the *transfer*. Any real vector bundle $V \to X$ of rank n has an Euler class $\chi \in H^{n+\omega(V)}(X)$ in twisted cohomology; it is the image of 1 under the composition

$$H^0(X) \xrightarrow{(A.1)} H^{n+\omega(V)}(V, V \setminus 0) \xrightarrow{\zeta} H^{n+\omega(V)}(X),$$

where ζ is the zero section.

Definition A.5 Let $f\colon E \to B$ be a manifold bundle. Then the *transfer* is the map $\mathrm{trf}_f^*\colon H^*(E) \to H^*(B)$ given by $\mathrm{trf}_f^*(x) := f_!(\chi(T_v E)x)$.

Note that on both, source and target, of the transfer, there is no twist. Thus orientation ought not play any role here. In fact, trf_f^* is induced by a stable homotopy class $\mathrm{trf}_f\colon \Sigma^\infty B_+ \to \Sigma^\infty E_+$ which only depends on the bundle and not on the orientation;

moreover this stable homotopy class can be defined for more general bundles than we consider here. In this paper, we use this homotopy-theoretic only implicitly in the proof of Proposition 6.1. What we need to know are the following properties which are straightforward consequences of Proposition A.3.

Proposition A.6 *Let $f\colon E \to B$ be an oriented smooth n–manifold bundle.*

(1) *If $g\colon F \to E$ is another smooth oriented manifold bundle, then $\mathrm{trf}^*_{f\circ g} = \mathrm{trf}^*_f \circ \mathrm{trf}^*_g$.*
(2) *The composition $\mathrm{trf}^*_f\, f^*\colon H^*(B) \to H^*(B)$ is multiplication by the Euler number $\chi(M)$ of the fibre.*
(3) *If $f\colon E \to B$ is a homeomorphism (the fibre is a point), then $\mathrm{trf}^*_f = (f^{-1})^*$.*

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