Uniqueness of A_∞–structures and Hochschild cohomology

CONSTANZE ROITZHEIM
SARAH WHITEHOUSE

Working over a commutative ground ring, we establish a Hochschild cohomology criterion for uniqueness of derived A_∞–algebra structures in the sense of Sagave. We deduce a Hochschild cohomology criterion for intrinsic formality of a differential graded algebra. This generalizes a classical result of Kadeishvili for the case of a graded algebra over a field.

16E45; 16E40, 55S30

Introduction

A_∞–Structures were introduced by Stasheff [17] in the early 1960s in the study of topological spaces with products. They are now known to arise widely in algebra, geometry and mathematical physics, as well as topology.

We are interested in questions of formality and intrinsic formality for differential graded algebras. Thus we would like to establish conditions under which two differential graded algebras with the same homology are quasi-isomorphic. This has been studied by Keller and others in the case where the ground ring k is a field. It is related to the existence of different A_∞–structures on a minimal model of the differential graded algebra.

An important structural result of Kadeishvili [8] proves the existence of minimal models of differential graded algebras over a field while another classical theorem by Kadeishvili [9] gives a criterion for uniqueness of certain minimal models using Hochschild cohomology.

For the applications we have in mind, which are related to rigidity of the model category structures arising in stable homotopy theory, we will be interested in working over local rings rather than fields. When working with a commutative ground ring rather than a field, one has to work with derived A_∞–algebras as in the world of "classical" A_∞–algebras, a differential graded algebra might not have a minimal model if its homology is not projective. The theory of derived A_∞–algebras was developed by Sagave in [16]. He describes the notion of a minimal model for a differential graded

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algebra A over a commutative ground ring by giving a projective resolution of the homology of A that is compatible with the existing A_∞–structure on A.

Our main result is Theorem 3.7 which extends Kadeishvili's uniqueness theorem to derived A_∞–algebras. For this we develop a new notion of Hochschild cohomology. After further work, in Theorem 4.4 we obtain a Hochschild cohomology criterion for intrinsic formality of a differential graded algebra over a commutative ring rather than a field.

In the subsequent sections we return to classical A_∞–algebras and derive some further generalizations of Kadeishvili's uniqueness criterion. The first of these is Theorem 5.3 which studies uniqueness of an A_∞–structure on a fixed differential graded algebra. The other, Theorem 6.3, discusses differential graded algebras with fixed Massey products on their homology.

An alternative approach is developed by Dugger and Shipley. In [3, Section 3] they consider the classification of quasi-isomorphism types of differential graded algebras with given homology. They do this by building differential graded algebras up degree-wise via a theory of Postnikov sections and k–invariants. To do so requires working with bounded below differential graded algebras, a restriction which does not apply to our methods. The k–invariants live in derived Hochschild cohomology groups of the Postnikov sections with coefficients in the next homology group of the differential graded algebra being built. Their work does not consider A_∞–structures and although also formulated in terms of Hochschild cohomology, there does not seem to be a very direct relationship between their methods and ours. However, we are going to put some of their examples in context throughout our paper.

This paper is organized as follows. In Section 1 we recall basic definitions relating to A_∞–algebras and Hochschild cohomology. In Section 2 we recall Sagave's construction of derived A_∞–algebras and his results about minimal models. This section also introduces the Lie algebra structure which leads to the definition of Hochschild cohomology of a certain class of derived A_∞–algebras in Section 3. At the end of Section 3 we show that the vanishing of certain Hochschild cohomology groups gives a sufficient condition for the existence of a unique derived A_∞–structure on a fixed underlying object. In Section 4 we deduce the criterion for intrinsic formality of differential graded algebras over a commutative ground ring. Finally, in Section 5 and Section 6 we discuss the previously mentioned analogues of these results for classical A_∞–structures. A short appendix is devoted to sign issues.

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1 A quick review of A_∞–algebras

We assume that the reader is familiar with the basic definitions regarding A_∞–algebras and Hochschild cohomology, but we are going to recall some of them in this section to establish notation and assumptions. We are going to be very brief with this; the explicit formulas and definitions regarding derived A_∞–algebras given in the later Sections 2 and 3 specialize to the case of "classical" A_∞–algebras. For greater detail we refer to Keller's introductory paper [10].

The notion of an A_∞–algebra arose with the study of loop spaces in topology and has since become an increasingly important and powerful subject in algebraic topology and homological algebra. Roughly speaking, A_∞–algebras are not necessarily associative algebras with given maps for "multiplying" n elements for each n, unlike in the case of associative algebras where one knows how to multiply n elements from knowing how to multiply two elements.

1.1 Basic definitions

In Section 1 and Section 6 of this paper, k will denote a field of characteristic not equal to 2. In Sections 2 to 5 we will allow k to be a commutative ring rather than a field. Note that in fact Section 1 and Section 6 do not require a ground field as long as all k–modules in question are projective.

All unadorned tensor products are over k. All graded objects will be $\mathbb{Z}$–graded unless stated otherwise. Our convention for the degree of a map f is as follows: a map of graded k–vector spaces $f\colon A \to B$ of degree i consists of a sequence of maps $f^n\colon A^n \to B^{n+i}$. (Later this will be called the internal degree and there will also be a notion of cohomological or external degree.) We often abbreviate "differential graded algebra" to dga.

Definition 1.1 Let $A = \bigoplus_{n\in\mathbb{Z}} A^n$ be a graded k–vector space. An *A_∞–structure* on A is a sequence of k–linear maps

$$m_j\colon A^{\otimes j} \longrightarrow A \quad \text{for } j \geq 1$$

of degree $2-j$ satisfying the equation

$$\sum_{n=r+s+t} (-1)^{rs+t} m_{1+r+t}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0$$

for each $n \geq 1$. An *A_∞–algebra* is a graded k–vector space A together with an A_∞–structure on A.

Further all A_∞–algebras are assumed to be strictly unital; cf Definition 2.1. We are using the sign convention of Sagave [16, (2.6)] and of Lefèvre-Hasegawa [11, 1.2.1.2] rather than of Keller [10].

Note that we are applying the Koszul sign rule when applying such formulas to elements:

$$(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y).$$

In particular, this definition gives us

$$m_1 m_1 = 0,$$

ie m_1 is a differential on A. It also yields the following special cases: if $m_k = 0$ for all $k \neq 2$, then A is simply a graded associative algebra. If $m_k = 0$ for $k \geq 3$, then A is a differential graded algebra.

There are also notions of morphism and quasi-isomorphism of A_∞–algebras; these are special cases of Definitions 2.3 and 2.4.

Notation We sometimes write an A_∞–structure as a formal infinite sum, ie

$$m = m_1 + m_2 + \cdots .$$

Note that all infinite sums in this paper are finite in every degree.

1.2 Hochschild cohomology and Lie structure

Hochschild cohomology is a very powerful tool in many areas around algebra and topology, from relations to the geometry of loop spaces to deformation theory of algebras and realizability questions in topology. The definition of Hochschild cohomology of associative graded algebras can be extended to a definition of Hochschild cohomology of A_∞–algebras. A convenient way of doing this is using a Lie algebra structure on the bigraded k–vector space

$$C^{n,m}(A,A) = \mathrm{Hom}_k^m(A^{\otimes n}, A) = \prod_i \mathrm{Hom}_k((A^{\otimes n})^i, A^{i+m}),$$

where $n \in \mathbb{N}, m \in \mathbb{Z}$ and A is a graded k–vector space.

Explicitly, for $f \in C^{n,k}(A,A)$ and $g \in C^{m,l}(A,A)$ the Lie bracket is given by

$$[f,g] = \sum_{i=0}^{n-1} (-1)^{(n-1)(m-1)+(n-1)l+i(m-1)} f(1^{\otimes i} \otimes g \otimes 1^{\otimes n-i-1})$$
$$-(-1)^{(n+k-1)(m+l-1)} \sum_{i=0}^{m-1} (-1)^{(m-1)(n-1)+(m-1)k+i(n-1)} g(1^{\otimes i} \otimes f \otimes 1^{\otimes m-i-1})$$

which lies in $C^{n+m-1,l+k}(A,A)$. This gives $C^{*,*}(A,A)$ the structure of a graded Lie algebra, where the grading is by total degree shifted by 1; see eg Fialowski and Penkava [5, Section 2], Getzler [7, Section 1], Gerstenhaber [6] or Penkava and Schwarz [13]. Note that the formula given in some of the references has signs arising from the Koszul rule because it is given evaluated on elements rather than as a formula of morphisms. For details on how this formula arises, see Section 2.2 and the Appendix.

Lemma 1.2 *Let $m \in C^{*,*}(A,A)$ of total degree 2. Then m is an A_∞–structure on A if and only if $[m,m]=0$. Further, for such m,*

$$D := [m,-]\colon C^{*,*}(A,A) \longrightarrow C^{*,*}(A,A)$$

is a differential on $C^{*,*}(A,A)$, ie D raises total degree by 1 and satisfies $D\circ D=0$.

Proof The first claim follows immediately from the bracket formula and the fact that 2 is invertible. The fact that $D\circ D=0$ is an immediate consequence of the graded Jacobi identity, while the total degree of D can be computed directly. □

Definition 1.3 Let A be an A_∞–algebra with A_∞–structure m. Then the *Hochschild cohomology of the A_∞–algebra A* is defined as

$$\mathrm{HH}^*(A,A) = H^*\Big(\prod_i C^{i,*-i}(A,A), [m,-]\Big).$$

For this, see, for example, Penkava and Schwarz [13, Section 5]. If A is an associative algebra (ie $m=m_2$), a direct computation using the above definitions shows this recovers the usual definition of the Hochschild cohomology of associative algebras, ie for $f \in C^{n,k}(A,A)$,

$$[m_2,f] = (-1)^k\Big(m_2(1\otimes f) + \sum_{i=0}^{n-1}(-1)^{i+1} f(1^{\otimes i}\otimes m_2\otimes 1^{\otimes n-1-i}) + (-1)^{n+1}m_2(f\otimes 1)\Big).$$

The grading in Definition 1.3 refers to the total degree. In the case of an associative algebra the differential

$$[m_2,-]\colon C^{*,*}(A,A) \longrightarrow C^{*+1,*}(A,A)$$

preserves internal degree so we can split the total degree of the Hochschild cohomology into the cohomological degree and the internal degree. We denote the bigraded

Hochschild cohomology in this special case by $\mathrm{HH}^{*,*}_{\mathrm{alg}}(A,A)$. For a general A_∞–algebra, we do not have a bigrading, but we can introduce a filtration; see Definition 5.2.

For A a dga, the definition can be interpreted in terms of bicomplexes. The dga A has differential m_1 and multiplication m_2. The bigraded module $C^{*,*}(A,A)$ becomes a bicomplex by taking

$$d^v := [m_1, -]\colon C^{*,*} \longrightarrow C^{*,*+1}$$

to be the vertical differential and

$$d^h := [m_2, -]\colon C^{*,*} \longrightarrow C^{*+1,*}$$

to be the horizontal differential. The condition

$$[m_1 + m_2, m_1 + m_2] = 0$$

translates into $(d^v)^2 = 0$, $(d^h)^2 = 0$ and $d^v d^h + d^h d^v = 0$, which are exactly the conditions for $C^{*,*}(A,A)$ to be a bicomplex [18, 1.2.4].

1.3 Minimal models and uniqueness

We now recall a definition and theorem about minimal models of A_∞–algebras. It relates differential graded algebras to A_∞–structures on their homology.

Definition 1.4 An A_∞–algebra is called *minimal* if $m_1 = 0$.

Over a field, one can replace any A_∞–algebra by a quasi-isomorphic minimal one which gives a very convenient way to describe a quasi-isomorphism class of an A_∞–algebra. We are particularly interested in the special case of differential graded algebras.

Theorem 1.5 (Kadeishvili) *Let A be a differential graded algebra over a field k, and let $H^*(A)$ be its homology module. Then $H^*(A)$ has an A_∞–structure such that*

- *$m_1 = 0$ and the multiplication m_2 is induced by the multiplication on A,*
- *there is a morphism of A_∞–algebras $f\colon H^*(A) \longrightarrow A$ such that f_1 is a quasi-isomorphism.*

This A_∞–algebra $H^*(A)$ is called the minimal model of A.

For more details, see Kadeishvili [8]. Note that the theorem states in particular that the minimal model $H^*(A)$ is quasi-isomorphic to A as an A_∞–algebra.

This is useful in combination with a uniqueness result in Kadeishvili [9].

Definition 1.6 We say that an A_∞–structure m is *trivial* if $m_n = 0$ for $n \geq 3$.

Theorem 1.7 (Kadeishvili) *Let C be a graded k–algebra with multiplication μ. If*

$$\mathrm{HH}^{n,2-n}_{\mathrm{alg}}(C, C) = 0 \quad \textit{for } n \geq 3,$$

then every A_∞–structure on C with $m_1 = 0$ and $m_2 = \mu$ is quasi-isomorphic to the trivial one.

We can reformulate this in terms of formality of dgas. We recall the following standard definitions.

Definition 1.8 (1) A dga A is *formal* if it is quasi-isomorphic to its homology $H^*(A)$ regarded as a dga with trivial differential.

(2) A dga A is *intrinsically formal* if any other dga A' such that $H^*(A) \cong H^*(A')$ as associative algebras is quasi-isomorphic to A.

If a dga is intrinsically formal then it is formal, but the converse need not hold. For example, in [3, Example 3.15], it is shown that there are two quasi-isomorphism types of dgas with homology an exterior algebra over $\mathbb{F}_p$ on an even degree generator. The trivial one is therefore formal but not intrinsically formal.

Using Theorem 1.7 for the case $C = H^*(A)$ yields the following.

Corollary 1.9 *Let A be a dga and $H^*(A)$ its homology algebra. Suppose that*

$$\mathrm{HH}^{n,2-n}_{\mathrm{alg}}(H^*(A), H^*(A)) = 0 \quad \textit{for } n \geq 3.$$

Then A is intrinsically formal.

In Section 5, we will recover these results as special cases of our derived versions.

2 Derived A_∞–algebras

To work with Kadeishvili's minimal models and to establish the uniqueness theorems, one has to assume all dgas as well as their homology algebras to be degreewise projective, hence the assumption of a ground field. However, there are important examples arising from homotopy theory where projectivity cannot be guaranteed. In 2008, Sagave introduced the notion of derived A_∞–algebras, providing a framework for not necessarily projective modules over an arbitrary commutative ground ring [16].

First of all, we recall some definitions and results about derived A_∞–algebras; we refer to Sagave's paper for the finer technical details.

The basic idea is to introduce degreewise projective resolutions for an A_∞–algebra that are compatible with the A_∞–structure. This will introduce another internal grading.

2.1 Definitions, conventions and known results

All definitions and results in this subsection have been developed by Sagave in [16] and we refer to his paper for technical details.

Let k be a commutative ring and let A be an $(\mathbb{N},\mathbb{Z})$*–bigraded* k–module, ie

$$A = \bigoplus_{i\in\mathbb{N}, j\in\mathbb{Z}} A_i^j .$$

A morphism of bigraded k–modules $f\colon A \longrightarrow B$ of bidegree (s,t) is a sequence of maps of k–modules $f\colon A_i^j \longrightarrow B_{i-s}^{j+t}$ for all $i \in \mathbb{N}$ and $j \in \mathbb{Z}$. Again, we follow the Koszul sign convention: for g a morphism of bidegree (s,t) and x an element of bidegree (i,j), we have

$$(f \otimes g)(x \otimes y) = (-1)^{is+jt} f(x) \otimes g(y).$$

The homological (subscript) bidegree is called the *horizontal bidegree* and the cohomological (superscript) bidegree is called the *vertical bidegree*.

Throughout the rest of the paper we also assume that all bigraded modules have no 2–torsion.

Definition 2.1 [16, Definition 2.1] A *derived* A_∞*–structure* (or dA_∞*–structure* for short) on an $(\mathbb{N},\mathbb{Z})$–bigraded k–module A consists of k–linear maps

$$m_{ij}\colon A^{\otimes j} \longrightarrow A$$

of bidegree $(i, 2-(i+j))$ for each $j \geq 1$, $i \geq 0$, satisfying the equation

$$\sum_{\substack{u=i+p, v=j+q-1\\ j=1+r+t}} (-1)^{rq+t+pj} m_{ij}(1^{\otimes r} \otimes m_{pq} \otimes 1^{\otimes t}) = 0 \tag{1}$$

for all $u \geq 0$ and $v \geq 1$. A dA_∞*–algebra* is a bigraded k–module together with a dA_∞–structure.

A dA_∞–algebra A is called *strictly unital* if there is a unit map $\eta\colon k \longrightarrow A$ such that

- $m_{01}(\eta) = 0$,
- $m_{02}(\eta \otimes 1) = 1 = m_{02}(1 \otimes \eta)$,
- $m_{ij}(1^{\otimes r-1} \otimes \eta \otimes 1^{\otimes j-r}) = 0$ for $i + j \geq 3$, $1 \leq r \leq j$.

From now on, all dA_∞–algebras are assumed to be strictly unital.

Remark A dA_∞–algebra concentrated in horizontal degree 0 (and hence with $m_{ij}=0$ for all $i\neq 0$) is the same as an A_∞–algebra.

A dA_∞–algebra with $m_{ij}=0$ except m_{01} and m_{11} is just a bicomplex (with a different sign convention to that encountered earlier) with horizontal differential m_{11} and vertical differential m_{01} as the definition in this case forces $m_{11}m_{11}=0$, $m_{01}m_{01}=0$ and $m_{01}m_{11}-m_{11}m_{01}=0$.

Definition 2.2 A *bidga* is a monoid in the category of bicomplexes; equivalently, a bidga is a dA_∞–algebra with $m_{ij}=0$ for $i+j\geq 3$. (See [16, Definition 2.10 and Remark 2.11].)

Definition 2.3 [16, Definition 2.5] Let A and B be dA_∞–algebras with dA_∞–structures m and $\bar{m}$, respectively. A *morphism of dA_∞–algebras* $f\colon A\longrightarrow B$ consists of a family of k–module maps

$$f_{st}\colon A^{\otimes t}\longrightarrow B$$

of bidegree $(s,1-(s+t))$ satisfying

$$\sum_{\substack{u=i+p,v=j+q-1\\ j=1+r+t}} (-1)^{rq+t+pj} f_{ij}(1^{\otimes r}\otimes m_{pq}\otimes 1^{\otimes t}) = \sum_{\substack{u=i+p_1+\cdots+p_j\\ v=q_1+\cdots+q_j}} (-1)^{\epsilon}\,\bar{m}_{ij}(f_{p_1q_1}\otimes\cdots\otimes f_{p_jq_j}) \tag{2}$$

for all $u\geq 0$ and $v\geq 1$. Here,

$$\epsilon = u+\sum_{w=1}^{j-1}\left(jp_w+w(q_{j-w}-p_w)+q_{j-w}\left(\sum_{s=j-w+1}^{j} p_s+q_s\right)\right).$$

For strictly unital dA_∞–algebras, morphisms are required to satisfy the unit conditions $f_{01}\eta=\bar{\eta}$ and $f_{ij}(1^{\otimes r-1}\otimes\eta\otimes 1^{\otimes j-r})=0$ for $i+j\geq 2$ and $1\leq r\leq j$.

Recall that a quasi-isomorphism of A_∞–algebras is a morphism of A_∞–algebras that induces a quasi-isomorphism of complexes with respect to m_1. In the case of dA_∞–algebras, the role of the quasi-isomorphisms is played by the so-called E_2–equivalences. These are the morphisms that induce an isomorphism of E_2–terms of the spectral sequence computing the homology of the total complex of a bicomplex; see McCleary [12, 2.12].

Notation The equations defining a dA_∞–structure include $m_{01}m_{01}=0$. For a dA_∞–algebra A let H^*_{ver} denote its homology with respect to the *vertical differential* m_{01}. The map m_{01} is called the vertical differential because it raises the vertical degree.

Since the equations defining a dA_∞–structure also include

$$m_{21}m_{01}-m_{11}m_{11}+m_{01}m_{21}=0,$$

it follows that the map m_{11} becomes a differential in horizontal direction on the bigraded module $H^*_{\mathrm{ver}}(A)$, so we can form $H^*_{\mathrm{hor}}(H^*_{\mathrm{ver}}(A)) = H^*(H^*_{\mathrm{ver}}(A), m_{11})$.

Definition 2.4 A morphism $f\colon A \longrightarrow B$ of dA_∞–algebras is called an E_2*–equivalence* if

$$H^*_{\mathrm{hor}}(H^*_{\mathrm{ver}}(f_{01}))$$

is an isomorphism of k–modules; cf [16, Definition 2.19].

We would like to extend some applications of A_∞–algebras to differential graded algebras that are not necessarily projective over the ground ring k or whose homology is not projective. The problem we encounter is that not all differential graded algebras possess a minimal model as an A_∞–algebra. However, Sagave showed that dgas have reasonable minimal models in the world of dA_∞–algebras. For this, one has to apply a special projective resolution.

Definition 2.5 [16, Definition 3.1] Let A be a graded algebra. A *termwise k–projective resolution* of A is a termwise k–projective bidga P with $m_{01}=0$ together with an E_2–equivalence $P \longrightarrow A$.

Definition 2.6 [16, Definition 3.2] Let A be a dga. A *k–projective E_1–resolution of A* is a bidga B together with an E_2–equivalence $B \longrightarrow A$ such that $H^{st}_{\mathrm{ver}}(B)$ is projective for each bidegree. Further, the map $k \longrightarrow H^{00}_{\mathrm{ver}}(B)$ induced by the unit $k \longrightarrow B$ is required to split as a k–module map.

Thus a k–projective E_1–resolution of a dga A induces a termwise k–projective resolution of the graded homology algebra of A.

Sagave then proceeds to show that a k–projective E_1–resolution is unique up to E_2–equivalence.

Theorem 2.7 [16, Theorem 3.4] *Every dga A over k admits a k–projective E_1–resolution. Two such resolutions can be related by a zigzag of E_2–equivalences between k–projective E_1–resolutions.*

Definition 2.8 A dA_∞–algebra is called *minimal* if $m_{01}=0$.

Theorem 2.9 [16, Theorem 1.1] *Let A be a dga over k. Then there is a degreewise k–projective dA_∞–algebra E together with an E_2–equivalence $E \longrightarrow A$ such that*

- *E is minimal,*
- *E is well-defined up to E_2–equivalence,*
- *together with the differential m_{11} and the multiplication m_{02}, E is a termwise k–projective resolution of the graded algebra $H^*(A)$.*

To prove this, Sagave starts with a k–projective E_1–resolution $E \longrightarrow A$. He then shows that the vertical homology $H^*_{\mathrm{ver}}(E)$ admits a dA_∞–structure satisfying the claims of the theorem.

However, not every termwise projective resolution of $H^*(A)$ admits such a structure [16, Remark 4.14.]. For example, consider the dga over $\mathbb{Z}$

$$A = \mathbb{Z}[e]\big/(e^4), \quad \partial(e) = p, \quad |e| = -1,$$

also examined by Dugger and Shipley in [3, Example 3.13]. The bidga

$$C = \mathbb{Z}\left\langle a, b\right\rangle \big/(a^2, b^2, ab - ba), \quad |a| = (1,0), \quad |b| = (0,-2), \quad m_{11}(b) = p$$

is a termwise projective resolution of $H^*(A) = \Lambda_{\mathbb{Z}/p}([e^2])$, but there is no dA_∞–structure on C admitting an E_2–equivalence $C \longrightarrow A$. (For example, Equation (2) for $(u,v) = (2,2)$ forces $m_{22}(b \otimes b) \equiv \pm 1 \bmod p$ whereas Equation (1) for $(u,v) = (2,3)$ forces $m_{22}(b \otimes b) \equiv 0 \bmod p$.)

Definition 2.10 Let A and E be as in Theorem 2.9. Such an E is called a *minimal model* of A.

Remark Note that in the context of Theorem 2.9, the underlying k–module of the minimal model E together with the differentials m_{01} and m_{11} and the multiplication m_{02} form a bidga.

2.2 Lie algebra structure on $C^{*,*}_*(A, A)$

We would like to establish a reasonable notion of Hochschild cohomology for dA_∞–algebras. In order to give a simple description, it is our goal to describe the Hochschild cohomology in terms of a graded Lie algebra structure.

Let A be a $(\mathbb{N}, \mathbb{Z})$–bigraded module without 2–torsion over a commutative ring. Define

$$C^{n,i}_k(A, A) = \prod_{u,v} \mathrm{Hom}((A^{\otimes n})^v_u, A^{v+i}_{u-k}).$$

We are going to define a Lie algebra structure on $C^{*,*}_*(A,A)$ generalizing Section 1.2. First of all, we define a bracket operation that is not a Lie bracket. Then we are going to introduce a shift operation on elements of $C^{*,*}_*(A,A)$ and then define the actual Lie bracket using this shift and the previously defined bracket operation.

For $f\in C^{n,i}_k(A,A)$ and $g\in C^{m,j}_l(A,A)$ we now define

$$\llbracket f,g\rrbracket=\sum_{v=0}^{n-1} f(1^{\otimes v}\otimes g\otimes 1^{\otimes n-v-1})-(-1)^{ij+kl}\sum_{v=0}^{m-1} g(1^{\otimes v}\otimes f\otimes 1^{\otimes m-v-1}) \in C^{n+m-1,i+j}_{k+l}(A,A).$$

This is not the actual Lie bracket but the first step in our construction. For degree and sign reasons we have to introduce a shift map.

Let $S(A)$ be the bigraded module with $S(A)^v_u=A^{v+1}_u$, and so the suspension map $S\colon A\to S(A)$ given by the identity map in each bidegree has internal bidegree $(0,-1)$. Given $f\in C^{n,i}_k(A,A)$, then

$$\sigma(f)=(-1)^{n+i+k-1}S\circ f\circ (S^{-1})^{\otimes n}\in C^{n,i+n-1}_k(S(A),S(A)).$$

Conversely, for $F\in C^{m,j}_l(S(A),S(A))$, we define

$$\sigma^{-1}(F)=(-1)^{j+l+\binom{m}{2}}S^{-1}\circ F\circ S^{\otimes m}\in C^{m,j+1-m}_l(A,A),$$

so $\sigma^{-1}(\sigma(f))=f$.

Particularly, for $m_{ij}\in C^{j,2-(i+j)}_i(A,A)$, we have $\sigma(m_{ij})\in C^{j,1-i}_i(S(A),S(A))$. Note that the notation $\sigma(f)$ does not mean applying a shift functor to f.

We now define

$$\begin{aligned}[f,g]&:=\sigma^{-1}\llbracket\sigma(f),\sigma(g)\rrbracket\\ &=\sum_{v=0}^{n-1}(-1)^{(n-1)(m-1)+v(m-1)+j(n-1)}f(1^{\otimes v}\otimes g\otimes 1^{\otimes n-v-1})\\ &\qquad -(-1)^{\langle f,g\rangle}\sum_{v=0}^{m-1}(-1)^{(m-1)(n-1)+v(n-1)+i(m-1)}g(1^{\otimes v}\otimes f\otimes 1^{\otimes m-v-1})\\ &\qquad\qquad \in C^{n+m-1,i+j}_{k+l}(A,A)\end{aligned}$$

for $f\in C^{n,i}_k(A,A)$ and $g\in C^{m,j}_l(A,A)$. Here, $\langle f,g\rangle:=(n+i-1)(m+j-1)+kl$. (See the Appendix for this computation.) It is easy to see that in the case of bigraded modules concentrated in horizontal degree 0 this specializes to the Lie algebra structure given in Section 1.2.

As earlier, we use formal infinite sums of morphisms. These are now bigraded and any such sum is actually finite in any given bidegree.

Remark It is also possible to work with a different definition of the shift σ on morphisms. Instead of our convention

$$\sigma(f) = (-1)^{k+n+i-1} S \circ f \circ (S^{-1})^{\otimes n},$$

it is also possible to work with

$$\overline{\sigma}(f) = (-1)^{k+n+i-1} S \circ f \circ (S^{\otimes n})^{-1}$$

as in [10, 3.6] which differs from the above σ by the sign $(-1)^p$, where $p = \binom{n}{2}$. Working with $\overline{\sigma}$ would recover Keller's sign convention in the definition of A_∞–algebras and their morphisms, whereas our choice of σ recovers the signs of Lefèvre-Hasegawa and Sagave.

It is convenient to describe the above bracket in terms of a composition product as in [6].

Definition 2.11 For $f \in C_k^{n,i}(A, A)$ and $g \in C_l^{m,j}(A, A)$ we define the composition product $\circ$ by

$$\begin{aligned} f \circ g &= \sum_{v=0}^{n-1} \sigma^{-1}\Big((\sigma(f)(1^{\otimes v} \otimes \sigma(g) \otimes 1^{\otimes n-v-1})\Big) \\ &= \sum_{v=0}^{n-1} (-1)^{(m-1)(n-1)+v(m-1)+j(n-1)} f(1^{\otimes v} \otimes g \otimes 1^{\otimes n-v-1}) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \in C_{k+l}^{n+m-1,i+j}(A, A). \end{aligned}$$

Hence, we have that

$$[f, g] = f \circ g - (-1)^{\langle f,g \rangle} g \circ f.$$

We will show that with this bracket $C_*^{*,*}(A, A)$ can be regarded as a bigraded Lie algebra in the sense of the following definition.

Definition 2.12 A bigraded k–module $X = \bigoplus X_i^j$ is a bigraded Lie algebra if there is a bracket operation $[-,-]\colon X \otimes X \to X$ satisfying

- $[g, f] = -(-1)^{ab+kl}[f, g]$,
- $(-1)^{ac+km}[[f, g], h] + (-1)^{ab+kl}[[g, h], f] + (-1)^{bc+lm}[[h, f], g] = 0$,

for $f \in X_k^a$, $g \in X_l^b$, $h \in X_m^c$.

Proposition 2.13 *The above bracket gives $C_*^{*,*}(A, A)$ the structure of a bigraded Lie algebra for the bigrading where $f \in C_k^{n,i}$ is given bidegree $(k, n+i-1)$; ie for all $f, g, h \in C_*^{*,*}(A, A)$,*

- $[g, f] = -(-1)^{\langle f,g\rangle}[f, g]$,
- $(-1)^{\langle f,h\rangle}[[f, g], h] + (-1)^{\langle g,f\rangle}[[g, h], f] + (-1)^{\langle h,g\rangle}[[h, f], g] = 0$.

Proof The first point is immediate. For the graded Jacobi identity we will show that the composition product $\circ$ makes $C_*^{*,*}(A, A)$ a bigraded pre-Lie ring in the sense that for $f \in C_k^{n,i}(A, A)$, $g \in C_l^{m,j}(A, A)$ and $h \in C_w^{u,v}(A, A)$, we have

$$(h \circ f) \circ g - (-1)^{\langle f,g\rangle}(h \circ g) \circ f = h \circ (f \circ g) - (-1)^{\langle f,g\rangle} h \circ (g \circ f). \tag{3}$$

We can then apply a direct computation analogous to the proof of Theorem 1 of [6] which proves the claim. (For this, we note that $\langle f \circ g, h\rangle = \langle f, h\rangle + \langle g, h\rangle$.)

To prove the Equation (3), we note that

$$f \circ g = \sigma^{-1}(\sigma(f) \odot \sigma(g))$$

with
$$F \odot G := \sum_{r=1}^{n-1} F(1^{\otimes r} \otimes G \otimes 1^{\otimes n-r-1}).$$

This is going to simplify the signs in (3) considerably since this equation is equivalent to

$$(H \odot F) \odot G - (-1)^{\langle f,g\rangle}(H \odot G) \odot F = H \odot (F \odot G) - (-1)^{\langle f,g\rangle} H \odot (G \odot F) \tag{4}$$

for $F = \sigma(f)$, $G = \sigma(g)$ and $H = \sigma(h)$. We have

$$(H \odot F) \odot G = \left(\sum_{r=0}^{u-1} H(1^{\otimes r} \otimes F \otimes 1^{\otimes u-r-1}) \right) \odot G$$

$$
\begin{aligned}
&= (-1)^{\langle f,g\rangle} \sum_{r=0}^{u-1} \sum_{a+b=r-1} H(1^{\otimes a} \otimes G \otimes 1^{\otimes b} \otimes F \otimes 1^{\otimes u-r-1}) \\
&\quad + \sum_{r=0}^{u-1} \sum_{s=0}^{n-1} H\big(1^{\otimes r} \otimes F(1^{\otimes s} \otimes G \otimes 1^{\otimes n-s-1}) \otimes 1^{\otimes u-r-1}\big) \\
&\quad + \sum_{r=0}^{u-1} \sum_{a+b=u-r-2} H(1^{\otimes r} \otimes F \otimes 1^{\otimes a} \otimes G \otimes 1^{\otimes b}).
\end{aligned}
$$

Note that the sign $(-1)^{\langle f,g\rangle}$ in the first summand arises from the Koszul sign rule for interchanging F and G. Using this, we can read off the Equation (4), from which (3) follows. □

Now we would like to describe derived A_∞–structures in terms of this Lie algebra structure, but first we have to introduce another operation which alters signs.

Definition 2.14 For $f \in C_k^{n,i}(A,A)$ define $f^{\#} = (-1)^k f \in C_k^{n,i}(A,A)$.

This operation satisfies

- $(f^{\#})^{\#} = f$,
- $(f \circ g)^{\#} = f^{\#} \circ g^{\#}$,
- $[f,g]^{\#} = [f^{\#}, g^{\#}]$.

Proposition 2.15 *Let A be a bigraded k–module without 2–torsion with given map $\eta\colon k \to A$. Let*

$$m = \sum_{i \geq 0,\, j \geq 1} m_{ij}$$

with $m_{ij} \in C_i^{j,2-(i+j)}(A,A)$ satisfying the unit conditions of Definition 2.1.

Then the following are equivalent:

- *m is a derived A_∞–structure on A.*
- *$m \circ m^{\#} = 0$.*
- *$[m, m^{\#}] = 0$.*

Proof The equivalence of the first two points follows immediately from the definitions. For the equivalence of the last two points let us consider the part $[m,m^{\#}]_u$ of $[m,m^{\#}]$

that lies in horizontal degree u. We have

$$\begin{aligned}[m,m^{\#}]_u &= \sum_{u=i+p} \big(m_{ij}\circ m^{\#}_{pq} - (-1)^{ip+(i-1)(p-1)} m^{\#}_{ij}\circ m_{pq}\big)\\ &= \sum_{u=i+p} \big((-1)^p m_{ij}\circ m_{pq} - (-1)^{u+1+i} m_{ij}\circ m_{pq}\big).\end{aligned}$$

We are going to distinguish between the cases u even and u odd. For even $u = i+p$, the sum splits into the cases where either both i and p are even or both i and p are odd. In either case, we can read off that

$$[m,m^{\#}]_u = 2(m\circ m^{\#})_u.$$

The case of u odd follows similarly. □

3 Hochschild cohomology and uniqueness of derived A_∞–algebras

3.1 Hochschild cohomology of dA_∞–algebras

We would like to define a notion of Hochschild cohomology for dA_∞–algebras that extends the classical, nonderived case. However, this is not as straightforward as before. In the classical case of an A_∞–algebra A with A_∞–structure m, we could define a differential on $C^{*,*}(A,A)$ via $D=[m,-]$. This satisfies $D\circ D = [m,[m,-]] = 0$ since $[m,m]=0$. But in the derived case the signs are slightly more complicated which means we can only guarantee $[m,m^{\#}]=0$. We can still define Hochschild cohomology for a certain class of dA_∞–algebra which includes the cases we are interested in.

Definition 3.1 Let $m = \sum_{i\geq 0, j\geq 1} m_{ij}$ be a dA_∞–structure. Then we denote the horizontal even degree part by m_{even} and the horizontal odd degree part by m_{odd}, ie,

$$m_{\mathrm{even}} = \sum_{i \text{ even}} m_{ij} \quad\text{and}\quad m_{\mathrm{odd}} = \sum_{i \text{ odd}} m_{ij}.$$

Remark Since m is a dA_∞–structure, by Proposition 2.15 we have $(m_{\mathrm{even}}+m_{\mathrm{odd}})\circ(m_{\mathrm{even}}-m_{\mathrm{odd}}) = 0$, which splits as

$$m_{\mathrm{even}}\circ m_{\mathrm{even}} = m_{\mathrm{odd}}\circ m_{\mathrm{odd}} \quad\text{and}\quad m_{\mathrm{even}}\circ m_{\mathrm{odd}} = m_{\mathrm{odd}}\circ m_{\mathrm{even}}.$$

Definition 3.2 We call a derived A_∞–structure m *orthogonal* if

$$m_{\mathrm{even}}\circ m_{\mathrm{even}} = 0 \quad\text{or, equivalently,}\quad m_{\mathrm{odd}}\circ m_{\mathrm{odd}} = 0.$$

Example Bidgas are orthogonal since they have $m_{\mathrm{odd}} = m_{11}$ and $m_{11}\circ m_{11} = 0$.

Lemma 3.3 *Let A be a bigraded k–module without 2–torsion and let $m = \sum_{i,j} m_{ij}$ be an orthogonal derived A_∞–structure on A. Define*

$$D\colon C_*^{*,*}(A,A) \longrightarrow C_*^{*,*}(A,A)$$

via

$$D(f) = [m_{\mathrm{even}}, f^{\#}] + [m_{\mathrm{odd}}, f] = (-1)^k [m_{\mathrm{even}}, f] + [m_{\mathrm{odd}}, f] \quad \text{for } f \in C_k^{n,i}(A,A).$$

Then D satisfies $D \circ D = 0$. Also, D raises the total degree by 1, so D is a differential on $C_*^{*,*}(A,A)$.

Proof The map D raises degree by 1 since m has total degree 2. Let us look at $D(D(f))$. Assume that f has horizontal internal degree k. Then for even p the horizontal degree of $[m_{pq}, f]$ has the same parity as k whereas for odd p the horizontal degree of $[m_{pq}, f]$ has the parity of $k+1$. This means that

$$[m_{\mathrm{even}}, f]^{\#} = (-1)^k [m_{\mathrm{even}}, f] \quad \text{and} \quad [m_{\mathrm{odd}}, f]^{\#} = (-1)^{k+1} [m_{\mathrm{odd}}, f].$$

Thus, we obtain

$$D((-1)^k [m_{\mathrm{even}}, f]) = (-1)^k \big((-1)^k [m_{\mathrm{even}}, [m_{\mathrm{even}}, f]] + [m_{\mathrm{odd}}, [m_{\mathrm{even}}, f]] \big)$$

and
$$D([m_{\mathrm{odd}}, f]) = (-1)^{k+1} [m_{\mathrm{even}}, [m_{\mathrm{odd}}, f]] + [m_{\mathrm{odd}}, [m_{\mathrm{odd}}, f]]$$

which together give us

$$\begin{aligned} D(D(f)) = {} & [m_{\mathrm{even}}, [m_{\mathrm{even}}, f]] + (-1)^k [m_{\mathrm{odd}}, [m_{\mathrm{even}}, f]] \\ & + (-1)^{k+1} [m_{\mathrm{even}}, [m_{\mathrm{odd}}, f]] + [m_{\mathrm{odd}}, [m_{\mathrm{odd}}, f]]. \end{aligned} \tag{5}$$

Since m is assumed to be orthogonal, we can directly compute that

$$[m_{\mathrm{even}}, [m_{\mathrm{even}}, f]] = 0 = [m_{\mathrm{odd}}, [m_{\mathrm{odd}}, f]].$$

From the graded Jacobi identity established in Proposition 2.13 we conclude that

$$[m_{\mathrm{odd}}, [m_{\mathrm{even}}, f]] = [m_{\mathrm{even}}, [m_{\mathrm{odd}}, f]].$$

Putting this together, we can read off the desired equation $D \circ D = 0$. □

Definition 3.4 Let A be an orthogonal dA_∞–algebra with orthogonal dA_∞–structure m. Then the Hochschild cohomology of A as a dA_∞–algebra is defined as

$$\mathrm{HH}^*(A,A) := H^*\Big(\prod_{i,j} C_j^{i,*-i-j}(A,A), D \Big).$$

The grading in the above definition of Hochschild cohomology denotes the total degree.

Remark If A has dA_∞–structure $m = m_{11} + m_{02}$ (ie A is a bidga with trivial vertical differential), then this definition specializes to Sagave's definition [16, Section 5] of Hochschild cohomology of bidgas with trivial vertical differential.

In this very special case of a bidga with trivial vertical differential, one grading is preserved by both m_{11} and m_{02} so that we have bigraded Hochschild cohomology groups:

$$\mathrm{HH}^t(A, A) = \prod_{s \geq 0} \mathrm{HH}^{s,t-s}(A, A),$$

where $\mathrm{HH}^{s,r}(A, A) = H^s(\prod_n C^{n,r}_{*-n}(A, A), D)$. We denote the Hochschild cohomology in this special case by $\mathrm{HH}^{*,*}_{\mathrm{bidga}}(A, A)$.

3.2 Uniqueness of derived A_∞–algebras

The overall goal of this section is to establish a uniqueness result analogous to Kadeishvili's (Theorem 1.7) for the possibility of extending an existing dA_∞–structure on a minimal model. A minimal model of a differential graded algebra has an underlying bidga with zero vertical differential. Let $\mu = m_{02}$ denote the multiplication of this bidga and $\partial = m_{11}$ the horizontal differential.

The first step is to look into how to perturb an existing dA_∞–structure by certain elements b of total degree 1.

Definition 3.5 Let A be a bidga with multiplication $m_{02} = \mu$, horizontal differential $m_{11} = \partial$ and vertical differential $m_{01} = 0$. Then

$$a = \sum_{i \geq 0,\, j \geq 1} a_{ij}, \quad a_{ij} \in C_i^{j,2-(i+j)}(A, A), \quad i + j \geq 3,$$

is a *twisting cochain* if $\partial + \mu + a$ is a dA_∞–structure.

Remark Note that by Proposition 2.15 a is a twisting cochain if and only if we have

$$[\partial + \mu + a, \partial^\# + \mu^\# + a^\#] = 0.$$

Letting D be the differential corresponding to the orthogonal dA_∞–structure $m = \partial + \mu$, this is equivalent to the *derived Maurer–Cartan formula*

$$2D(a) = -[a, a^\#] + 4[\partial, a_{\mathrm{odd}}], \tag{6}$$

as can be verified quickly by splitting a into even and odd horizontal degree parts and using that $[\partial+\mu,\partial+\mu]=0$. Hence, an element

$$a=\sum_{i,j} a_{ij},\quad a_{ij}\in C_i^{j,2-(i+j)}(A,A),\quad i+j\geq 3,$$

is a twisting cochain if and only if a satisfies the above derived Maurer–Cartan formula.

Lemma 3.6 *Let A be a bidga with multiplication $m_{02}=\mu$, horizontal differential $m_{11}=\partial$ and vertical differential $m_{01}=0$. Let*

$$a=\sum_{i,j} a_{ij},\quad a_{ij}\in C_i^{j,2-(i+j)}(A,A),\quad i+j\geq 3,$$

be a twisting cochain. Let either

(A) $b\in C_k^{n-1,2-(n+k)}(A,A)$*, for $k+n\geq 3$, with $[\partial,b]=0$, or*

(B) $b\in C_{k-1}^{n,2-(n+k)}(A,A)$*, for $k+n\geq 3$, with $[\mu,b]=0$.*

Then there is a twisting cochain $\overline{a}$ satisfying

- *the dA_∞–structures $\partial+\mu+a$ and $\overline{m}=\partial+\mu+\overline{a}$ are E_2–equivalent,*
- *$\overline{a}_{uv}=a_{uv}$ for $u<k$ or $v<n-1$ or $(u,v)=(k,n-1)$ in case* (A) *and for $u<k-1$ or $v<n$ or $(u,v)=(k-1,n)$ in case* (B)*,*
- *$\overline{a}_{kn}=a_{kn}-[\mu,b]$ in case* (A)*,*
- *$\overline{a}_{kn}=a_{kn}-[\partial,b]$ in case* (B)*.*

Proof This is a lengthy but direct computation using the definition of a morphism of dA_∞–algebras. The twisting cochain $\overline{a}$ is going to be determined by $\partial+\mu+a$ being E_2–equivalent to $\partial+\mu+\overline{a}$ via the equivalence $\mathrm{id}+b$. We will only do case (A) explicitly since the other case can be read off the proof of this one.

Let $f:=\mathrm{id}+b$. We consider what it means for there to be a dA_∞–structure $\overline{m}=\partial+\mu+\overline{a}$ on A such that $f\colon (A,m)\longrightarrow(A,\overline{m})$ is a morphism of dA_∞–structures, ie the Equation (2) in Definition 2.3 is satisfied. Using $f_{01}=\mathrm{id}$, $f_{k,n-1}=b$ and $f_{ij}=0$ in all other degrees as well as $m=\mu+a$ and $\overline{m}=\mu+\overline{a}$, we write down (2). The left-hand side of (2) is only nonzero for $(i,j)=(0,1)$ and $(i,j)=(k,n-1)$. Thus, we obtain

$$(-1)^u m_{uv}+\sum_{r=0}^{n-2}(-1)^{r(v-n)+(n-r)+(u-k)(n-1)}b(1^{\otimes r}\otimes m_{u-k,v+2-n}\otimes 1^{\otimes n-2-r}).$$

The sum can only be nonzero if $u \geq k$ and $v \geq n-1$ and $(u,v) \neq (k,n-1)$. In the special case $(u,v) = (k,n)$ we get

$$(-1)^k a_{kn} + \sum_{r=0}^{n-2} (-1)^{n-r} b(1^{\otimes r} \otimes \mu \otimes 1^{\otimes n-2-r}).$$

For $(u,v) = (k+1, n-1)$, the result is

$$(-1)^{k+1} a_{k+1,n-1} - \sum_{r=0}^{n-2} b(1^{\otimes r} \otimes \partial \otimes 1^{\otimes n-2-r}).$$

On the right-hand side of (2) we have

$$(-1)^u \overline{m}_{uv} + \sum_{\substack{u=i+p_1+\cdots+p_j \\ v=q_1+\cdots+q_j}} (-1)^{\epsilon} \overline{m}_{ij}(f_{p_1q_1} \otimes \cdots \otimes f_{p_jq_j}) \tag{7}$$

where at least one of the $f_{p_rq_r}$ in the sum has to be $f_{k,n-1} = b$ and ϵ is as in Definition 2.3. The following four special cases are to be considered. First, we note that, since we have $\overline{m}_{01} = 0$, the sum is zero for $(u,v) = (k,n-1)$. For $(u,v) = (k,n)$, we obtain

$$(-1)^k \overline{a}_{kn} + (-1)^n \mu(1 \otimes b) + \mu(b \otimes 1),$$

for $(u,v) = (k+1, n-1)$ we have

$$(-1)^{k+1} \overline{a}_{k+1,n-1} + (-1)^{k+1} \partial(b)$$

and for $(u,v) = (2k, 2n-2)$ the result is

$$\overline{a}_{2k,2n-2} + (-1)^{nk} \mu(b \otimes b) + \sum_{r=0}^{n-1} (-1)^{\epsilon} \overline{a}_{k,n}(1^r \otimes b \otimes 1^{n-1-r}).$$

In all other cases each summand appearing in the sum in (7) has $i + j \geq 3$. Further, the sum in (7) can only be nonzero for $u \geq i + k$ and $v \geq (n-1) + (j-1)$.

Now recall that

$$[\partial, b] = \partial(b) - (-1)^k \sum_{r=0}^{n-2} b(1^{\otimes r} \otimes \partial \otimes 1^{\otimes n-2-r}),$$

$$[\mu, b] = (-1)^{n+k} \left(\mu(1 \otimes b) + (-1)^n \mu(b \otimes 1) + \sum_{r=0}^{n-2} (-1)^{r+1} b(1^{\otimes r} \otimes \mu \otimes 1^{\otimes n-2-r}) \right).$$

Further, note that we have assumed that $[\partial, b] = 0$.

Putting all this together, we can read off that for (u,v) with either $u<k$ or $v<n-1$ and for $(u,v)=(k,n-1)$, we have

$$\overline{a}_{uv}=a_{uv}.$$

For (u,v)=(k,n), we get

$$\begin{aligned}\overline{a}_{kn}=a_{kn}-(-1)^k\Big(&\mu(b\otimes 1)+(-1)^n\mu(1\otimes b)\\&+(-1)^{n-1}\sum_{r=0}^{n-2}(-1)^r b(1^{\otimes r}\otimes\mu\otimes 1^{\otimes n-2-r})\Big)\\=a_{kn}-[\mu,b].&\end{aligned}$$

For $(u,v)=(k+1,n-1)$ we have

$$\begin{aligned}\overline{a}_{k+1,n-1}&=a_{k+1,n-1}+(-1)^k\sum_{r=0}^{n-2}b(1^{\otimes r}\otimes\partial\otimes 1^{\otimes n-2-r})-\partial(b)\\&=a_{k+1,n-1}-[\partial,b]=a_{k+1,n-1}.\end{aligned}$$

For $(u,v)=(2k,2n-2)$ we have

$$\begin{aligned}\overline{a}_{2k,2n-2}=a_{2k,2n-2}&\\&+\sum_{r=0}^{n-2}(-1)^{m+n-r+k(n-1)}b(1^{\otimes r}\otimes a_kn\otimes 1^{\otimes n-2-r})-(-1)^{nk}\mu(b\otimes b)\\&+\sum_{r=0}^{n-1}(-1)^{\epsilon}\overline{a}_{kn}(1^{\otimes r}\otimes b\otimes 1^{\otimes n-1-r}).\end{aligned}$$

Finally for $(u,v)\neq(k,n),(k+1,n-1)$ or $(2k,2n-2)$ with $u\geq k$ and $v\geq n-1$, we have

$$\begin{aligned}\overline{a}_{uv}=a_{uv}+(-1)^u\sum_{r=0}^{n-2}&(-1)^{r(v-n)+(n-r)+(u-k)(n-1)}b(1^{\otimes r}\otimes m_{u-k,v+2-n}\otimes 1^{\otimes n-2-r})\\-(-1)^u&\sum_{\substack{u=i+p_1+\cdots+p_j\\ v=q_1+\cdots+q_j\\ \text{at least one } q_j\neq 1}}(-1)^{\epsilon}\overline{a}_{ij}(f_{p_1q_1}\otimes\cdots\otimes f_{p_jq_j}).\end{aligned}$$

Note that the second sum in the last equation can only be nonzero if $i+j\geq 3$, $u\geq k+i$ and $v\geq(n-1)+(j-1)$. Also, for fixed (u,v), the right-hand side of the last equation only uses $\overline{a}_{pq}$ with $p<u$ and $q<v$. The same thing happens in the

case $(u,v)=(2k,2n-2)$. This proves that the $\overline{a}$ in the statement of our lemma can be constructed inductively.

One can then check degreewise that $\overline{m}=\partial+\mu+\overline{a}$ defines a dA_∞–structure by showing that $[\overline{m},\overline{m}^\#]=0$. The morphism f is an E_2–equivalence since $f_{01}=\mathrm{id}$. □

Remark Note that in the situation of the above lemma, in both cases we have in particular that $\overline{a}_{uv}=a_{uv}$ whenever $u+v<k+n$.

We can now formulate a derived version of Kadeishvili's uniqueness theorem.

Theorem 3.7 *Let A be a bidga with multiplication $m_{02}=\mu$, horizontal differential $m_{11}=\partial$ and vertical differential $m_{01}=0$. If $\mathrm{HH}^{r,2-r}_{\text{bidga}}(A,A)=0$ for $r\geq 3$, then every dA_∞–structure on A with $m_{02}=\mu$, $m_{11}=\partial$ and $m_{01}=0$ is E_2–equivalent to the trivial one, ie the one with $m_{02}=\mu$, $m_{11}=\partial$ and $m_{ij}=0$ for $(i,j)\neq(0,2)$ or $(1,1)$.*

Proof Let $m=\partial+\mu+a$ be a dA_∞–structure on A with

$$a=\sum_{k+n\geq 3}a_{kn},\quad a_{kn}\in C_k^{n,2-(k+n)}(A,A).$$

We want to show that m is E_2–equivalent to the dA_∞–structure $\partial+\mu$.

We now fix $t\geq 3$ and show that m is equivalent to a dA_∞–structure with $a_{kn}=0$ for $k+n=t$. We show this by induction on k. Assuming that $a_{ij}=0$ for $i+j=t$ and $i<k$, we will show that m is equivalent to a dA_∞–structure with $\overline{m}=\partial+\mu+\overline{a}$ with $\overline{a}_{kn}=0$ and $\overline{a}_{ij}=a_{ij}=0$ for $i+j=t$, $i<k$ and $i+j<t$.

Because m is a dA_∞–structure, by Lemma 3.3 we have $[\partial+\mu+a,\partial^\#+\mu^\#+a^\#]$=0. Since A is also a bidga, we have $[\partial+\mu,\partial^\#+\mu^\#]=0$. Hence, a is a twisting cochain satisfying the Maurer–Cartan formula

$$2D(a)=-[a,a^\#]+4[\partial,a_{\text{odd}}]$$

as explained in (6). Further, we have

$$D(-)=[\mu,(-)^\#]+[\partial,-]$$

with $\quad [\mu,(-)^\#]\colon C_*^{*,*}(A,A)\longrightarrow C_*^{*+1,*}(A,A)$

and $\quad [\partial,-]\colon C_*^{*,*}(A,A)\longrightarrow C_{*+1}^{*,*}(A,A),$

so $[\mu,a_{kn}^\#]$ lives in the tridegree $(n+1,k,2-(k+n))$–part of $D(a)$ and $[\partial,a_{kn}]$ lives in tridegree $(n,k+1,2-(k+n))$. However, on the other side of (6) the tridegree

$(n+1, k, 2-(k+n))$–part as well as the $(n, k+1, 2-(k+n))$–part of $[a, a^\#]$ is zero since $[a, a^\#]$ can only be nonzero in degrees (u, v, w) with $u+v \geq 5$ whereas $n+1+k = 4$. Here we are adopting the convention for tridegrees that an element in $C_k^{n,i}(A, A)$ has tridegree (n, k, i).

Thus according to the Maurer–Cartan formula, $D(a_{kn})$ lives in $2[\partial, a_{\mathrm{odd}}]$. This information splits into the equations

$$[\mu, a^\#_{kn}] = \epsilon_1 2[\partial, a_{k-1,n+1}], \qquad \epsilon_1 \in \{0, 1\}$$

and

$$[\partial, a_{kn}] = \epsilon_2 2[\partial, a_{kn}], \qquad \epsilon_2 \in \{0, 1\}$$

where $\epsilon_2 = 0$ for k even by definition since the right hand side is supposed to be a summand of $2[\partial, a_{\mathrm{odd}}]$. Thus, we can also conclude that $[\mu, a^\#_{kn}] = 0$ since our induction assumption gives $a_{k-1,n+1} = 0$.

For k odd, we are left with $[\partial, a_{kn}] = \epsilon_2 2[\partial, a_{kn}]$, $\epsilon_2 \in \{0, 1\}$, from which we can immediately read off that $[\partial, a_{kn}] = 0$.

Hence, in any case $D(a_{kn}) = 0$ and a_{kn} is a cocycle in $C_k^{n,2-(n+k)}(A, A)$, so

$$[a_{kn}] \in \mathrm{HH}^{k+n,2-k-n}_{\mathrm{bidga}}(A, A).$$

However, $\mathrm{HH}^{k+n,2-k-n}_{\mathrm{bidga}}(A, A)$ is zero by assumption, so there must be a b in total degree 1 with $D(b) = a_{kn}$.

So, analogously to the proof of Theorem 5.3, there is a $b_1 \in C_k^{n-1,2-(k+n)}(A, A)$ with $[\partial, b_1] = 0$ and $[\mu, b_1] = a_{kn}$ and $b_2 \in C_{k-1}^{n,2-(k+n)}(A, A)$ with $[\mu, b_2] = 0$ and $[\partial, b_2] = a_{kn}$.

Applying Lemma 3.6 to b_1, there is a dA_∞–structure $\bar{m} = \partial + \mu + \bar{a}_{ij}$ with $\bar{a}_{ij} \in C_i^{j,2-(i+j)}(A, A)$, $i+j \geq 3$ such that $\bar{m}$ is E_2–equivalent to m, $\bar{a}_{kn} = a_{kn} - [\mu, b_1] = 0$ and $\bar{a}_{ij} = a_{ij}$ for $i+j < t$ and $i+j = t$, $i < k$, which proves our claim. $\square$

Example In [4, Proposition 4.2], Dugger and Shipley consider the dga

$$A = \mathbb{Z}\langle e, x, y\rangle \big/ (e^2 = 0,\ ex + xe = x^2,\ xy = yx = 1),$$
$$\partial(e) = p,\ \partial(x) = 0,\ \partial(y) = 0, \qquad |e| = |x| = 1,\ |y| = -1.$$

This is a dga over $\mathbb{Z}$ which has homology $H_n(A) = \mathbb{Z}/p$ in every degree n. (Note that Dugger and Shipley use homological grading.) They then prove in Theorem 4.5 that A is not formal.

In [15] Sagave gives a projective E_1–resolution B of A. He then constructs the first degrees of a minimal model structure on the induced termwise projective resolution $P = H^*_{\mathrm{ver}}(B)$ and shows that this gives a nontrivial class in $\mathrm{HH}^{3,-1}_{\mathrm{bidga}}(P, P)$.

Theorem 3.7 will be used in the next section to give a sufficient criterion for the existence of a unique dga realising a fixed homology algebra over a ground ring rather than a ground field. To prove this derived analogue of Corollary 1.9, we first have to investigate the behaviour of Hochschild cohomology of degreewise projective resolutions under E_2–equivalence.

4 Invariance under E_2–equivalence and intrinsic formality

In order to establish our uniqueness criterion we need an invariance result for Hochschild cohomology under E_2–equivalence. To prove this we will need to define Hochschild cohomology with coefficients. We will carry this out here only for the special case we need. In future work we hope to study the general case, but this would take us too far afield here.

Thus we will concentrate on the case of relevance to us, namely bidgas with $m_{01} = 0$. Invariance under E_2–equivalence in this situation is also discussed in [16, Section 5]. We begin by spelling out concretely what a bidga with $m_{01} = 0$ is.

A bidga with $m_{01} = 0$ is a bigraded module A_i^j equipped with maps

$$m_{11}\colon A_i^j \to A_{i-1}^j$$

and

$$m_{02}\colon (A \otimes A)_i^j \to A_i^j$$

with relations which specify that m_{02} is associative, m_{11} is a differential and m_{11} is a derivation with respect to m_{02}. These relations come from the cases $(u = 0, v = 3)$, $(u = 2, v = 1)$ and $(u = 1, v = 2)$ respectively of the defining relations; all other relations are trivial. Notice that this is just a dga with an extra grading.

It is straightforward to see what a module over such a thing should be; it is just a dg module with an extra grading.

Definition 4.1 Let A be a bidga with $m_{01} = 0$. A *left A–module M* is a bigraded module $\{M_i^j\}$ over the ground ring equipped with a horizontal differential

$$\bar{m}_{11}\colon M_i^j \to M_{i-1}^j$$

and an associative action $\bar{m}^l_{02}\colon (A \otimes M)_i^j \to M_i^j$ such that the following diagram commutes:

$$\begin{tikzcd} A \otimes M \arrow[r, "\bar{m}^l_{02}"] \arrow[d, "m_{11}\otimes 1 + 1 \otimes \bar{m}_{11}"'] & M \arrow[d, "\bar{m}_{11}"] \\ A \otimes M \arrow[r, "\bar{m}^l_{02}"] & M \end{tikzcd}$$

A right A–module is defined in the obvious way, with a right action map

$$\bar{m}^r_{02}\colon M \otimes A \to M.$$

And an A–bimodule is simultaneously a left and right A–module with the obvious compatibility condition on the left and right actions.

Notice that a morphism of bidgas $A \to A'$ between bidgas with $m_{01} = 0$ makes A' into an A–bimodule.

Let us also spell out what an E_2–equivalence $f\colon A \to A'$ between bidgas with $m_{01} = 0$ is. This is just a morphism $f\colon A \to A'$ inducing an isomorphism on horizontal homology. (So we can think of such an f as a quasi-isomorphism if we think of A and A' as complexes with respect to horizontal differentials.)

Now let A be a bidga with $m_{01} = 0$ and let M be an A–bimodule. Let

$$C^{n,i}_k(A, M) = \prod_{u,v} \operatorname{Hom}\left((A^{\otimes n})^v_u, M^{v+i}_{u-k}\right)$$

and for $f \in C^{n,i}_k(A, M)$ define

$$\begin{aligned} Df = (-1)^{k+n+i-1}\bar{m}^r_{02}(f \otimes 1) + (-1)^{k+i}\bar{m}^l_{02}(1 \otimes f) \\ + (-1)^{k+n+i} f \circ m_{02} + \bar{m}_{11} \circ f + (-1)^{k+1} f \circ m_{11}. \end{aligned}$$

Then D is a differential, allowing us to make the following definition.

Definition 4.2 For A a bidga with $m_{01} = 0$ and M an A–bimodule the *Hochschild cohomology of A with coefficients in M* is defined by

$$\mathrm{HH}^{s,r}_{\mathrm{bidga}}(A, M) = H^s\left(\prod_n C^{n,r}_{*-n}(A, M), D\right).$$

This is a covariant functor of M and a contravariant functor of A. In the case where $M = A$, regarded as a bimodule over itself, this agrees with the earlier definition of $\mathrm{HH}^{*,*}_{\mathrm{bidga}}(A, A)$. Indeed the formula above for the differential D just becomes

$$Df = (-1)^k[m_{02}, f] + [m_{11}, f].$$

Proposition 4.3 *Let (A, m) and $(A', \bar{m})$ be bidgas with $m_{01} = \bar{m}_{01} = 0$ and which are degreewise projective over k. Let $f\colon A \to A'$ be an E_2–equivalence. Then f induces an isomorphism of Hochschild cohomology groups*

$$\mathrm{HH}^{*,*}_{\mathrm{bidga}}(A, A) \cong \mathrm{HH}^{*,*}_{\mathrm{bidga}}(A', A').$$

Proof For each i, we can interpret the Hochschild cohomology $HH^{*,i}_{\mathrm{bidga}}(A,M)$ as the cohomology of a (right half-plane) bicomplex. This works very similarly to the case of Hochschild cohomology of a dga discussed earlier. One differential, say D_1, is given by the m_{11} part of the formula for D and the other, say D_2, by the m_{02} part.

Now consider A and M as complexes with respect to their differentials m_{11} and $\bar{m}_{11}$ (with an extra grading). The differential D_1 on $\mathrm{Hom}^i_*(A^{\otimes p}, M)$ is the induced differential via the tensor product and Hom functors of complexes. For bounded below and degreewise projective complexes the ordinary Hom and tensor product functors agree with the derived versions and are therefore quasi-isomorphism invariant.

Thus the morphism $f\colon A \to A'$ induces column-wise quasi-isomorphisms of bicomplexes $C^{*,i}_*(A,A) \to C^{*,i}_*(A,A')$ and $C^{*,i}_*(A',A') \to C^{*,i}_*(A,A')$. It follows that the induced maps of total complexes are quasi-isomorphisms and

$$\mathrm{HH}^{*,*}_{\mathrm{bidga}}(A,A) \cong \mathrm{HH}^{*,*}_{\mathrm{bidga}}(A',A'). \qquad \square$$

Now we are in a position to give our criterion for intrinsic formality.

Theorem 4.4 *Let A be a dga and E its minimal model with dA_∞–structure m. By $\widetilde{E}$, we denote the underlying bidga of E, ie $\widetilde{E} = E$ as k–modules together with dA_∞–structure $\widetilde{m} = m_{11} + m_{02}$. If*

$$\mathrm{HH}^{m,2-m}_{\mathrm{bidga}}(\widetilde{E},\widetilde{E}) = 0 \quad \text{for } m \geq 3,$$

then A is intrinsically formal.

Proof Applying Theorem 3.7 to $\widetilde{E}$, we obtain that every dA_∞ structure on $\widetilde{E}$ is E_2–equivalent to the trivial one. By definition of minimal model, A is E_2–equivalent to E. Thus A is E_2–equivalent to $(\widetilde{E}, \mathrm{triv})$. Again by definition of minimal model, $(\widetilde{E}, \mathrm{triv})$ is E_2–equivalent to $(H^*(A), \mathrm{triv})$. Thus we have an E_2–equivalence between A and $(H^*(A), \mathrm{triv})$ and since these are both dgas an E_2–equivalence is a quasi-isomorphism. So A is formal.

Now let A' be a dga with $H^*(A) \cong H^*(A')$ as associative algebras, let E' be a minimal model of A' and let $\widetilde{E}'$ be its underlying bidga. We have E_2–equivalences

$$\widetilde{E}' \simeq (H^*(A'), \mathrm{triv}) \simeq (H^*(A), \mathrm{triv}) \simeq \widetilde{E}.$$

Thus $\widetilde{E}'$ and $\widetilde{E}$ are E_2–equivalent bidgas. By definition of minimal model they are degreewise projective and have $m_{01} = 0$. Applying Proposition 4.3 gives

$$\mathrm{HH}^{m,2-m}_{\mathrm{bidga}}(\widetilde{E}',\widetilde{E}') \cong \mathrm{HH}^{m,2-m}_{\mathrm{bidga}}(\widetilde{E},\widetilde{E}).$$

So the Hochschild cohomology of $\widetilde{E}'$ is zero in the relevant range and the argument of the preceding paragraph shows that A' is also formal.

Since A and A' are both formal, the hypothesis $H^*(A) \cong H^*(A')$ means they are quasi-isomorphic. □

5 Uniqueness of classical A_∞–structures

In this section k is still a commutative ground ring without 2–torsion unless stated otherwise. We use Hochschild cohomology of differential graded algebras to give a uniqueness criterion for extending the differential and multiplication of a fixed dga to an A_∞–structure. In the case of a trivial differential this recovers Kadeishvili's classical Theorem 1.7. We then apply this to an example in homotopy theory.

Fix a differential graded algebra A with differential $m_1 = \partial$ and multiplication $m_2 = \mu$. We would like to consider the set of all A_∞–structures on A (up to quasi-isomorphism) that extend the differential graded algebra structure, ie A_∞–structures of the form $m = \partial + \mu + m_3 + m_4 + \cdots$. Let us write $a = m_3 + m_4 + \cdots$.

Recall that $m = \partial + \mu + a$ is an A_∞–structure if and only if a satisfies the Maurer–Cartan formula and that such a are called *twisting cochains*. In this classical case the Maurer–Cartan formula reads

$$-D(a) = \frac{1}{2}[a, a]$$

if 2 is invertible in k or, equivalently, $-D(a) = a \circ a$ where $\circ$ denotes the composition product; see eg [5, Section 2] and (6).

Lemma 5.1 *Let A be a dga with differential ∂ and multiplication μ, and let a be a twisting cochain. Further, for $n \geq 3$, let either $p \in C^{n,1-n}(A, A)$ with*

$$d^h(p) = [\mu, p] = 0$$

or $p \in C^{n-1,2-n}(A, A)$ with

$$d^v(p) = [\partial, p] = 0.$$

Then there is a twisting cochain $\overline{a}$ such that

- *the A_∞–structures $\partial + \mu + \overline{a}$ and $\partial + \mu + a$ are quasi-isomorphic,*
- *$\overline{a}_i = a_i$ for $i \leq n-1$,*
- *$\overline{a}_n = a_n - D(p)$.* □

We omit the proof since it is very similar to that of Lemma 3.6. For the case where A is a graded algebra rather than a dga, the analogous result is mentioned without proof in [9, Section 4].

With the help of Lemma 5.1, we can now prove the sufficient condition for a unique A_∞–structure on a dga A extending the existing differential and multiplication. This is only a minor generalization of Kadeishvili's classical result [9, Theorem 1] in the zero differential case, but we have not been able to find a reference.

To formulate the uniqueness results of this section and Section 6 we have to look deeper into the grading of the Hochschild cohomology of A_∞–algebras and the internal grading of representing cocycles. An element of $\mathrm{HH}^n(A,A)$ can be nonuniquely expressed as

$$[x] = [x_0 + x_1 + x_2 + \cdots] \quad \text{with } x_i \in C^{i,n-i}(A,A).$$

However, while the sum of the x_i is a cocycle the individual summands are not necessarily cocycles themselves. So generally we do not get a decomposition of $\mathrm{HH}^n(A,A)$ as $\prod_i \mathrm{HH}^{i,n-i}(A,A)$. To keep track of the internal degrees we introduce a decreasing filtration on $\mathrm{HH}^*(A,A)$.

Definition 5.2 For an A_∞–algebra A, let

$$F^k\,\mathrm{HH}^n(A,A) = \left\{ [x] \in \mathrm{HH}^n(A,A) \;\middle|\; x \in \prod_{i \geq k} C^{i,n-i}(A,A) \right\}.$$

This means that $F^k\,\mathrm{HH}^n(A,A)$ consists of all those elements of $\mathrm{HH}^n(A,A)$ whose representing cocycles can be written as a sum of $x_i \in C^{i,n-i}(A,A)$ with $i \geq k$.

Note that in the case of a bidga the filtration F^* given in Definition 5.2 agrees with the usual filtration arising from the column-wise filtration on the bicomplex; see eg [12, 2.2 and 2.4].

Theorem 5.3 *Let A be a dga with differential ∂ and multiplication μ. If*

$$F^3\,\mathrm{HH}^2(A,A) = 0,$$

then any A_∞–structure on A with $m_1 = \partial$ and $m_2 = \mu$ is quasi-isomorphic to $\partial + \mu$.

Proof Let a be a twisting cochain. Assuming that there is a $k \geq 3$ such that $a_i = 0$ for $i < k$, we are going to show that there is a twisting cochain $\overline{a}$ that is equivalent to a and satisfies $\overline{a}_i = 0$ for $i \leq k$, ie we are killing off the bottom summand. By induction, it follows that a is equivalent to zero.

So let a now be a twisting cochain such that there is a $k \geq 3$ with $a_i = 0$ for $i < k$. Considering the Maurer–Cartan equation

$$-D(a) = a \circ a$$

in bidegrees $(k+1, 2-k)$ and $(k, 3-k)$, we see that $D(a_k) = 0$ for degree reasons, so a_k is a cocycle and $[a_k] \in F^k\,\mathrm{HH}^2(A, A)$. Since $F^k\,\mathrm{HH}^2(A, A) = 0$, a_k also has to be a coboundary, ie there is a cochain p in total degree 1 with $D(p) = a_k$. This p is the sum of two cochains p_1 and p_2 with $p_1 \in C^{k,1-k}(A, A)$ and $p_2 \in C^{k-1,2-k}(A, A)$. We have $d^v(p_1) + d^h(p_2) = a_k$ and $d^h(p_1) = d^v(p_2) = 0$ for degree reasons.

$$\begin{tikzcd}
C^{k-1,3-k}(A,A) & C^{k,3-k}(A,A) & \cdots \\
C^{k-1,2-k}(A,A) \arrow[u, "d^v"] \arrow[r, "d^h"] & C^{k,2-k}(A,A) \arrow[u, "d^v"] \arrow[r, "d^h"] & C^{k+1,2-k}(A,A) \\
\cdots & C^{k,1-k}(A,A) \arrow[u, "d^v"] \arrow[r, "d^h"] & C^{k+1,1-k}(A,A)
\end{tikzcd}$$

Applying Lemma 5.1 for p_1 and p_2, we obtain that there is a twisting cochain $\overline{a}$ quasi-isomorphic to a with $\overline{a}_i = 0$ for $i < k$ and $\overline{a}_k = a_k - D(p) = 0$, which completes our proof. □

Example Consider the dga over the p–local integers

$$A = \mathbb{Z}_{(p)}[x] \otimes \Lambda_{\mathbb{Z}_{(p)}}(e) \big/ (x^m, x^{m-1}e), \quad \partial(x) = pe, \quad |e| = -(2p-3), |x| = -(2p-2)$$

where $m \geq 2$. We can compute its Hochschild cohomology as a dga by applying the spectral sequence for the homology of the total complex of a bicomplex [12, 2.15]. Its E_1–term is the Hochschild cohomology of A as a graded algebra.

To obtain this, we note that for an A–bimodule M

$$\begin{aligned}\mathrm{HH}^{*,*}_{\mathrm{alg}}(A, M) &\cong \mathrm{HH}^{*,*}_{\mathrm{alg}}(\mathbb{Z}_{(p)}[x]/(x^m) \otimes \Lambda_{\mathbb{Z}_{(p)}}(e), M) \\ &\cong \mathrm{HH}^{*,*}_{\mathrm{alg}}(\mathbb{Z}_{(p)}[x]/(x^m), M) \otimes \mathrm{HH}^{*,*}_{\mathrm{alg}}(\Lambda_{\mathbb{Z}_{(p)}}(e), \mathbb{Z}_{(p)}).\end{aligned}$$

(Use [2, XI.1] for the second isomorphism. The first follows from a change-of-rings spectral sequence; see [12].) Computing each factor separately, we obtain

$$\mathrm{HH}^{*,*}_{\mathrm{alg}}(A, A) = \mathbb{Z}_{(p)}[f, \tau] \otimes \Lambda_{\mathbb{Z}_{(p)}}(\sigma) \otimes A$$

with $|f| = (1, -|e|)$, $|\tau| = (2, -m|x|)$ and $|\sigma| = (1, -|x|)$ for A viewed as a graded algebra.

Already at this E_1–stage we can read off that

$$\mathrm{HH}^{n,2-n}_{\mathrm{alg}}(A,A)=0 \quad \text{for } n\geq 3,$$

so $F^3\,\mathrm{HH}^2(A,A)=0$ for A as a dga. Hence $\mu+\partial$ is the only A_∞–structure on A with $m_1=\partial$ and $m_2=\mu$.

Also note that the homology of A coincides with the stable homotopy groups of the $K_{(p)}$–local sphere in a certain range, ie

$$H^{-i}(A)=\pi_i(L_1S^0) \quad \text{for } 0\leq i\leq (m-1)(2p-2)-1.$$

Combining Kadeishvili's result on minimal models with Theorem 5.3, we recover the following result which we already stated earlier as Corollary 1.9.

Corollary 5.4 *Let A be a dga over a ground field and $H^*(A)$ its homology algebra. Suppose that*

$$\mathrm{HH}^{n,2-n}_{\mathrm{alg}}(H^*(A),H^*(A))=0 \quad \textit{for } n\geq 3.$$

Then A is intrinsically formal.

Proof We apply Theorem 5.3 to $H^*(A)$ with the trivial differential to see that any A_∞–structure on this is quasi-isomorphic to the trivial one. So in particular the minimal model is quasi-isomorphic to the trivial structure. But the minimal model is quasi-isomorphic to A, so A is formal.

Now given a dga A' with $H^*(A')\cong H^*(A)$, the same argument shows that A' is also formal and thus that A' is quasi-isomorphic to A. □

We note that the corollary follows from the special case of Theorem 5.3 where the dga has trivial differential.

6 Massey products

Massey products provide some very useful additional structure when studying differential graded algebras and their homology. They are closely related to Toda brackets in triangulated categories which have strong applications in homotopy theory. Here we explain the relationship between Massey products and the m_3 part of A_∞–structures; see also [1, Lemma 5.14].

In this section, k denotes a field of characteristic not 2.

Let A be a differential graded algebra and $\alpha_1, \alpha_2, \alpha_3$ elements in the homology $H^*(A)$ such that $\alpha_1\alpha_2 = 0$ and $\alpha_2\alpha_3 = 0$. That means that for chosen representing cocycles a_i of α_i there is an element u_i such that $d(u_i) = (-1)^{1+|a_i|}a_ia_{i+1}$. With those elements, one can now define the Massey product of α_1, α_2 and α_3 as follows.

Definition 6.1 Let α_1, α_2 and α_3 be as above. Then the *Massey product*

$$\langle\alpha_1, \alpha_2, \alpha_3\rangle \subset H^{|a_1|+|a_2|+|a_3|-1}(A)$$

is defined as the set of homology classes of the elements

$$(-1)^{1+|a_1|}a_1u_2 + (-1)^{1+|u_1|}u_1a_3$$

ranging over all possible choices of representing cocycles a_i of the α_i and u_i such that $d(u_i) = (-1)^{1+|a_i|}a_ia_{i+1}$.

Note that the Massey product $\langle\alpha_1, \alpha_2, \alpha_3\rangle$ is a set rather than an element as the choices one makes can be altered by appropriate cocycles. Hence, if one fixes any x in the Massey product, for any other x' in the Massey product there is a

$$y \in \alpha_1 H^{|\alpha_3|+|\alpha_2|-1}(A) \oplus H^{|\alpha_2|+|\alpha_1|-1}(A)\alpha_3$$

such that $x' = x + y$. The group

$$\alpha_1 H^{|\alpha_3|+|\alpha_2|-1}(A) \oplus H^{|\alpha_2|+|\alpha_1|-1}(A)\alpha_3$$

is called the *indeterminacy* of $\langle\alpha_1, \alpha_2, \alpha_3\rangle$. So a Massey product consists of only one element if and only if its indeterminacy is zero. For more details on Massey products, see eg Ravenel [14, A.1.4].

Example Let k be a field of characteristic different from 2. Consider the following noncommutative differential graded algebra

$$A = k\,\langle x, y\rangle \big/ (x^3, y^2, xy = -yx), \quad \partial(x) = 0, \partial(y) = x^2, \quad |x| = 2, |y| = 3.$$

Its homology has a copy of k in degrees $0, 2, 5$ and 7 and zero elsewhere. Let $[x]$ and $[xy]$ denote the homology classes of x and xy respectively. Then

$$2[xy] = \langle [x], [x], [x]\rangle \in H^5(A),$$

the indeterminacy being zero for degree reasons.

Example The dga

$$A = \mathbb{Z}_{(p)}[x] \otimes \Lambda_{\mathbb{Z}_{(p)}}(e) \big/ (x^m, x^{m-1}e), \;\; \partial(x) = pe, \;\; |e| = -(2p-3), \;\; |x| = -(2p-2)$$

considered in the previous section has nontrivial Massey products. Take a_k to be an order p element in $H^{-(2p-2)k+1}(A)$. Then

$$\langle a_i, p, a_j \rangle = a_{i+j}.$$

This is related to the Toda bracket relation $\langle \alpha_i, p, \alpha_j \rangle = \alpha_{i+j}$ in the homotopy groups of the $K_{(p)}$–local sphere $\pi_* L_1 S^0$.

In the context of A_∞–algebras, Massey products can be reformulated using minimal models which were introduced in the previous section. We quote the following well-known result (see also [1, Lemma 5.14]).

Lemma 6.2 *Let A be a dga and $H^*(A)$ its minimal model with A_∞–structure m. Let $\alpha_1, \alpha_2, \alpha_3 \in H^*(A)$. If the Massey product $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is defined in $H^*(A)$, then*

$$(-1)^{|\alpha_1|+|\alpha_2|+1} m_3(\alpha_1 \otimes \alpha_2 \otimes \alpha_3) \in \langle \alpha_1, \alpha_2, \alpha_3 \rangle. \qquad \square$$

Hence, if A and B are differential graded algebras with isomorphic homology algebras $H^*(A)$ and $H^*(B)$, then they have the same Massey products if the A_∞–structures of the minimal models have identical m_3. (The converse is not necessarily true – see the discussion at the end of this section.)

Theorem 6.3 *Let A be a dga whose minimal model $H^*(A)$ satisfies $m_i = 0$ for $i \neq 2, 3$ and let $\bar{m}$ be an A_∞–structure on $H^*(A)$ with $\bar{m}_2 = m_2$ and $\bar{m}_3 = m_3$. If $F^4\,\mathrm{HH}^2(H^*(A), H^*(A)) = 0$, then $\bar{m}$ and m are quasi-isomorphic.*

Proof The proof is extremely similar to the proof of Theorem 5.3. The differential in the Hochschild complex for $H^*(A)$ is $D = D_2 + D_3$ with

$$D_2 = [m_2, -]\colon C^{n,k}(H^*(A), H^*(A)) \longrightarrow C^{n+1,k}(H^*(A), H^*(A))$$

and $$D_3 = [m_3, -]\colon C^{n,k}(H^*(A), H^*(A)) \longrightarrow C^{n+2,k-1}(H^*(A), H^*(A)).$$

Assume there is an A_∞–structure $\bar{m}$ on $H^*(A)$ with

$$\bar{m} = m_2 + m_3 + a_4 + a_5 + \cdots.$$

Let $$a = a_4 + a_5 + \cdots.$$

Because $m = m_2 + m_3$ is an A_∞–structure on the minimal model by assumption, we know that a is a twisting cochain, ie a satisfies the Maurer–Cartan equation. Again, for degree reasons $D(a_4) = 0$ and so there is

$$
\begin{array}{llll}
 & p_2 \in C^{3,-2}(H^*(A), H^*(A)) & \text{and} & p_3 \in C^{2,-1}(H^*(A), H^*(A)) \\
\text{with} & D_2(p_2) + D_3(p_3) = a_4 & \text{and} & D_3(p_2) = D_2(p_3) = 0.
\end{array}
$$

The analogue of Lemma 5.1 also holds in this case: for any

$$
\begin{array}{lll}
 & p \in C^{n,1-n}(H^*(A), H^*(A)) & \text{with } D_3(p) = 0 \\
\text{or} & p \in C^{n+1,-n}(H^*(A), H^*(A)) & \text{with } D_2(p) = 0,
\end{array}
$$

there is a twisting cochain $\overline{a} = \overline{a}_4 + \overline{a}_5 + \cdots$ such that

- $\overline{a}$ is equivalent to a,
- $\overline{a}_k = a_k$ for $k \leq n$,
- $\overline{a}_{n+1} = a_{n+1} - D(p)$.

The rest of the proof follows the same steps as the proof of Theorem 5.3. □

Of course one would like to apply this theorem to a minimal model

$$(H^*(A), m = m_2 + m_3)$$

of a dga A to obtain a uniqueness result analogous to Corollary 1.9 and conclude that the vanishing of the right Hochschild cohomology groups implies that A is the only dga up to quasi-isomorphism with the given homology and Massey products.

This does not quite work- to give the same Massey products on minimal models of dgas with the same homology algebras, m_3 only needs to agree on triples (a, b, c) with $ab = 0 = bc$. For example, in [1, Example 5.15 and Proposition 5.16] Benson, Krause and Schwede constructed an example of a dga with trivial Massey products but nontrivial m_3.

It would also be interesting to study the implication of Massey products regarding uniqueness criteria in the derived case.

Appendix A Signs in the Lie bracket

In this appendix we verify the signs appearing in the Lie bracket of Section 2.2. The special case where $k = l = 0$ recovers the signs in Section 1.2.

Lemma A.1 *In the context of Section 2.2,*

$$\begin{aligned}[f,g] &:= \sigma^{-1}[\![\sigma(f),\sigma(g)]\!] \\ &= \sum_{v=0}^{n-1}(-1)^{(n-1)(m-1)+v(m-1)+j(n-1)} f(1^{\otimes v}\otimes g\otimes 1^{\otimes n-v-1}) \\ &\qquad -(-1)^{\langle f,g\rangle}\sum_{v=0}^{m-1}(-1)^{(m-1)(n-1)+v(n-1)+i(m-1)} g(1^{\otimes v}\otimes f\otimes 1^{\otimes m-v-1}) \\ &\qquad\qquad \in C_{k+l}^{n+m-1,i+j}(A,A)\end{aligned}$$

for $f\in C_k^{n,i}(A,A)$ and $g\in C_l^{m,j}(A,A)$. Here, $\langle f,g\rangle := (n+i-1)(m+j-1)+kl$.

Proof Throughout this proof, by $\circ$, we mean the actual composition of morphisms rather than the previously used composition product.

The signs arise from the Koszul sign rule for interchanging morphisms. For morphisms f,g,h and u, we have

$$(f\otimes g)\circ(h\otimes u) = (-1)^{is+jt}(f\circ h)\otimes(g\circ u)$$

with g having internal bidegree (i,j) and h having internal bidegree (s,t).

We then obtain

$$\begin{aligned}\sigma^{-1}[\![\sigma(f),\sigma(g)]\!] &= \sigma^{-1}\Bigg(\sum_{v=0}^{n-1}\sigma(f)(1^{\otimes v}\otimes\sigma(g)\otimes 1^{\otimes n-v-1})\Bigg) \\ &\qquad -(-1)^{\langle f,g\rangle}\sigma^{-1}\Bigg(\sum_{v=0}^{m-1}\sigma(g)(1^{\otimes v}\otimes\sigma(f)\otimes 1^{\otimes m-v-1})\Bigg).\end{aligned}$$

For reasons of symmetry and linearity we are only going to explicitly compute

$$\sigma^{-1}\big(\sigma(f)(1^{\otimes v}\otimes\sigma(g)\otimes 1^{\otimes n-v-1})\big).$$

Up to sign, this is $f(1^{\otimes v}\otimes g\otimes 1^{\otimes n-v-1})$ and we now calculate the sign.

The term $\sigma(f)(1^{\otimes v}\otimes\sigma(g)\otimes 1^{\otimes n-v-1})$ lies in $C^{n+m-1,i+j+n+m-2}_{k+l}(S(A),S(A))$, so

$$\begin{aligned}
&\sigma^{-1}\big(\sigma(f)(1^{\otimes v}\otimes\sigma(g)\otimes 1^{\otimes n-v-1})\big)\\
&=(-1)^{i+j+n+m+k+l+\binom{n+m-1}{2}}S^{-1}\circ\big(\sigma(f)(1^{\otimes v}\otimes\sigma(g)\otimes 1^{\otimes n-v-1})\big)\circ S^{\otimes n+m-1}\\
&=(-1)^{\binom{n+m-1}{2}}S^{-1}\circ S\circ f\circ(S^{-1})^{\otimes n}\\
&\qquad\qquad\circ\big(1^{\otimes v}\otimes(S\circ g\circ(S^{-1})^{\otimes m})\otimes 1^{\otimes n-v-1}\big)\circ S^{\otimes n+m-1}\\
&=(-1)^{\binom{n+m-1}{2}}f\circ\big((S^{-1})^{\otimes v}\otimes S^{-1}\otimes(S^{-1})^{\otimes n-v-1}\big)\\
&\qquad\qquad\circ\big(1^{\otimes v}\otimes(S\circ g\circ(S^{-1})^{\otimes m})\otimes 1^{\otimes n-v-1}\big)\circ S^{\otimes n+m-1}.
\end{aligned}$$

In the next step we are obtaining a new sign $(-1)^{(n-v-1)(j+m-1)}$ by interchanging $(S^{-1})^{\otimes n-v-1}$ with $1^{\otimes v}\otimes(S\circ g\circ(S^{-1})^{\otimes m})$. Interchanging S^{-1} and $1^{\otimes v}$ does not introduce any new signs, so we continue with

$$\begin{aligned}
&(-1)^{\binom{n+m-1}{2}+(n-v-1)(j+m-1)}\\
&\qquad f\circ\big((S^{-1})^{\otimes v}\otimes(S^{-1}\circ S\circ g\circ(S^{-1})^{\otimes m})\otimes(S^{-1})^{\otimes n-v-1}\big)\circ S^{\otimes n+m-1}\\
&=(-1)^{\binom{n+m-1}{2}+(n-v-1)(j+m-1)}\\
&\qquad f\circ\big((S^{-1})^{\otimes v}\otimes g\circ(S^{-1})^{\otimes m}\otimes(S^{-1})^{\otimes n-v-1}\big)\circ\big(S^{\otimes v}\otimes S^{\otimes m}\otimes S^{\otimes n-v-1}\big).
\end{aligned}$$

Since $(S^{-1})^{\otimes a}=(-1)^{\binom{a}{2}}(S^{\otimes a})^{-1}$, we continue with

$$\begin{aligned}
&(-1)^{\binom{n+m-1}{2}+\binom{n-v-1}{2}+\binom{m}{2}+\binom{v}{2}+(n-v-1)(j+m-1)}\\
&\qquad f\circ\big((S^{\otimes v})^{-1}\otimes g\circ(S^{\otimes m})^{-1}\otimes(S^{\otimes n-v-1})^{-1}\big)\circ\big(S^{\otimes v}\otimes S^{\otimes m}\otimes S^{\otimes n-v-1}\big).
\end{aligned}$$

We have that

$$\binom{n+m-1}{2}+\binom{n-v-1}{2}+\binom{m}{2}+\binom{v}{2}\equiv(n-1)(v+m)+v\pmod 2$$

so we can simplify the sign in the above expression to give

$$\begin{aligned}
&(-1)^{(n-1)(v+m)+v+(n-v-1)(j+m-1)}\\
&\qquad f\circ\big((S^{\otimes v})^{-1}\otimes g\circ(S^{\otimes m})^{-1}\otimes(S^{\otimes n-v-1})^{-1}\big)\circ\big(S^{\otimes v}\otimes S^{\otimes m}\otimes S^{\otimes n-v-1}\big).
\end{aligned}$$

We then interchange $S^{\otimes v}$ with $g \circ (S^{\otimes m})^{-1} \otimes (S^{\otimes n-v-1})^{-1}$ which in addition gives us the new sign $(-1)^{(j+m+n-v-1)v}$, so we have

$$(-1)^{(n-1)(v+m)+v+(n-v-1)(j+m-1)+(j+m+n-v-1)v}$$
$$f \circ \big(1^{\otimes v} \otimes g \circ (S^{\otimes m})^{-1} \otimes (S^{\otimes n-v-1})^{-1}\big) \circ \big(1^{\otimes v} \otimes S^{\otimes m} \otimes S^{\otimes n-v-1}\big).$$

Finally, we add to the sign by interchanging $S^{\otimes m}$ with $(S^{\otimes n-v-1})^{-1}$, so we end up with

$$(-1)^{(n-1)(v+m)+v+(n-v-1)(j+m-1)+(j+m+n-v-1)v+m(n-v-1)}$$
$$f \circ (1^{\otimes v} \otimes g \otimes 1^{\otimes n-v-1}).$$

We can then simplify the above sign to

$$(-1)^{(n-1)(m-1)+v(m-1)+(n-1)j}$$

which proves our claim. □

References

[1] **D Benson**, **H Krause**, **S Schwede**, *Introduction to realizability of modules over Tate cohomology*, from: "Representations of algebras and related topics", (R-O Buchweitz, H Lenzing, editors), Fields Inst. Commun. 45, Amer. Math. Soc. (2005) 81–97 MR2146241

[2] **H Cartan**, **S Eilenberg**, *Homological algebra*, Princeton Univ. Press (1956) MR0077480

[3] **D Dugger**, **B Shipley**, *Topological equivalences for differential graded algebras*, Adv. Math. 212 (2007) 37–61 MR2319762

[4] **D Dugger**, **B Shipley**, *A curious example of triangulated-equivalent model categories which are not Quillen equivalent*, Algebr. Geom. Topol. 9 (2009) 135–166 MR2482071

[5] **A Fialowski**, **M Penkava**, *Deformation theory of infinity algebras*, J. Algebra 255 (2002) 59–88 MR1935035

[6] **M Gerstenhaber**, *The cohomology structure of an associative ring*, Ann. of Math. (2) 78 (1963) 267–288 MR0161898

[7] **E Getzler**, *Batalin–Vilkovisky algebras and two-dimensional topological field theories*, Comm. Math. Phys. 159 (1994) 265–285 MR1256989

[8] **T V Kadeishvili**, *On the theory of homology of fiber spaces*, Uspekhi Mat. Nauk 35 (1980) 183–188 MR580645 International Topology Conference (Moscow State Univ., Moscow, 1979)

[9] **T V Kadeishvili**, *The structure of the $A(\infty)$–algebra, and the Hochschild and Harrison cohomologies*, Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk Gruzin. SSR 91 (1988) 19–27 MR1029003

[10] **B Keller**, *Introduction to A–infinity algebras and modules*, Homology Homotopy Appl. 3 (2001) 1–35 MR1854636

[11] **K Lefèvre-Hasegawa**, *Sur les A_∞–catégories*, PhD thesis, Université de Paris 7 – Denis Diderot (2003) Available at http://people.math.jussieu.fr/~keller/lefevre/publ.html

[12] **J McCleary**, *A user's guide to spectral sequences*, second edition, Cambridge Studies in Advanced Math. 58, Cambridge Univ. Press (2001) MR1793722

[13] **M Penkava**, **A Schwarz**, *A_∞ algebras and the cohomology of moduli spaces*, from: "Lie groups and Lie algebras: E B Dynkin's Seminar", (S G Gindikin, E B Vinberg, editors), Amer. Math. Soc. Transl. Ser. 2 169, Amer. Math. Soc. (1995) 91–107 MR1364455

[14] **D C Ravenel**, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Math. 121, Academic Press, Orlando, FL (1986) MR860042

[15] **S Sagave**, *A derived A_∞–algebra model for a dga studied by Dugger and Shipley*, Unpublished note (2009)

[16] **S Sagave**, *DG–algebras and derived A_∞–algebras*, J. Reine Angew. Math. 639 (2010) 73–105 MR2608191

[17] **J D Stasheff**, *Homotopy associativity of H–spaces. I, II*, Trans. Amer. Math. Soc. 108 (1963), 275-292; ibid. 108 (1963) 293–312 MR0158400

[18] **C A Weibel**, *An introduction to homological algebra*, Cambridge Studies in Advanced Math. 38, Cambridge Univ. Press (1994) MR1269324

Department of Mathematics, University of Glasgow
University Gardens, Glasgow, G12 8QW, UK

School of Mathematics and Statistics, University of Sheffield
Hicks Building, Hounsfield Road, Sheffield S3 7RH, UK

constanze.roitzheim@glasgow.ac.uk, s.whitehouse@sheffield.ac.uk

http://www.maths.gla.ac.uk/~croitzheim/,
http://www.sarah-whitehouse.staff.shef.ac.uk/

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