The Meyer functions for projective varieties and their application to local signatures for fibered 4–manifolds

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We study a secondary invariant, called the Meyer function, on the fundamental group of the complement of the dual variety of a smooth projective variety. This invariant has played an important role when studying the local signatures of fibered 4–manifolds from topological point of view. As an application of our study, we define a local signature for generic nonhyperelliptic fibrations of genus 4 and 5 and compute some examples.

1 Introduction

Let Σ_g be a closed oriented C^∞–surface of genus $g \geq 0$. The *mapping class group* of Σ_g, which we denote by Γ_g, is the group of orientation preserving diffeomorphisms of Σ_g modulo isotopy. The group cohomology of Γ_g attracts attentions because (1) its elements give rise to a characteristic class of oriented Σ_g–bundles, (2) over the rational coefficients, it is isomorphic to the cohomology of the moduli space of compact Riemann surfaces of genus g.

As for the degree two part, the cohomology group itself has been determined. Harer [12] proved that $H^2(\Gamma_g; \mathbb{Z}) \cong \mathbb{Z}$ for $g \geq 3$, and the groups $H^2(\Gamma_1; \mathbb{Z}) \cong \mathbb{Z}/12\mathbb{Z}$ and $H^2(\Gamma_2; \mathbb{Z}) \cong \mathbb{Z}/10\mathbb{Z}$ are classically known. However, as a reflection of the fact that Γ_g is related to various mathematical objects, various 2–cocycles of Γ_g are known, arising in different contexts.

One of these is *Meyer's signature cocycle* τ_g, introduced by W Meyer [20] and rediscovered later by Turaev [27]. The definition involves the signature of 4–manifolds and will be recalled in this section.

The main object we study here is the *Meyer function*, a secondary invariant associated to τ_g, originating in the work of Meyer [20]. He showed that for $g = 1$ or 2, there exists a unique $\mathbb{Q}$–valued 1–cochain $\phi_g \colon \Gamma_g \to \mathbb{Q}$ whose coboundary equals τ_g. He also gave an explicit formula for ϕ_1. Note that Γ_1 is isomorphic to $\mathrm{SL}(2; \mathbb{Z})$. Atiyah [4] showed interesting aspects of ϕ_1 as a function: he showed that the value of ϕ_1 for a

hyperbolic element $\alpha \in \mathrm{SL}(2; \mathbb{Z})$ coincides with various values associated to α such as the special value of a Shimizu L–function determined by α, an arithmetic invariant, or the η–invariant of the mapping torus of α, a differential-geometric invariant.

Several recent works give higher genera or higher dimensional analogues of ϕ_1 or ϕ_2. Endo [8] and Morifuji [22] study the Meyer function on the hyperelliptic mapping class group. Endo [8] gives an application to the local signature for hyperelliptic fibrations and Morifuji [22] studies a relation to the η–invariant of mapping tori. Iida [14] studies, as a higher dimensional generalization of ϕ_2, the Meyer function for the family of smooth theta divisors.

The Meyer function for a projective variety   In this paper we give other analogues of Meyer's ϕ_1 or ϕ_2 and discuss their applications. We consider the family of Riemann surfaces constructed as follows.

Let $X \subset \mathbb{P}_N$ be a smooth complex projective variety of dimension n, embedded in the complex projective space of dimension N. Throughout the paper, we assume that

$$N > n \geq 2.$$

Let $k := N - n + 1$. We denote by $G_k(\mathbb{P}_N)$ the set of all k–planes of $\mathbb{P}_N$. Let D_X be the set of k–planes meeting X not transversally. When $n = 2$, D_X is the classical dual variety of X. In Gel'fand, Kapranov and Zelevinsky [9], D_X is called the k–th associated subvariety of X. Let $U^X = G_k(\mathbb{P}_N) \setminus D_X$. For $W \in U^X$, W and X meet transversally so their intersection $X \cap W$ has a natural structure of a compact Riemann surface. Thus setting

$$\mathcal{C}^X := \{(x, W) \in \mathbb{P}_N \times U^X : x \in X \cap W\},$$

the second projection

(1.0.1) $p_X := p_2|_{\mathcal{C}^X} \colon \mathcal{C}^X \to U^X$

is a complex analytic family of compact Riemann surfaces. Let g be the genus of the fibers and

$$\rho_X \colon \pi_1(U^X) \to \Gamma_g$$

the topological monodromy (see the conventions below) of (1.0.1). Let $\rho_X^* \tau_g$ be the pullback of τ_g by ρ_X.

Theorem 1.0.2   (Theorem 3.1.1)   *There exists a unique $\mathbb{Q}$–valued 1–cochain $\phi_X \colon \pi_1(U^X) \to \mathbb{Q}$ whose coboundary equals $\rho_X^* \tau_g$. In particular, we have $\rho_X^*[\tau_g] = 0 \in H^2(\pi_1(U^X); \mathbb{Q})$.*

We remark here that if $\rho\colon G \to \Gamma_g$ is a homomorphism from a group G to Γ_g and $\phi\colon G \to \mathbb{Q}$ is a 1–cochain whose coboundary equals $\rho^*\tau_g$, then ϕ is always a class function: $\phi(xyx^{-1}) = \phi(y)$ for $x, y \in G$. This is easily derived from properties of τ_g (see Kuno [16, Appendix]). In particular the above ϕ_X is a class function. We call ϕ_X *the Meyer function associated to* $X \subset \mathbb{P}_N$. In fact this theorem is regarded as a further generalization of [16], where the case of X being the d–th Veronese image of $\mathbb{P}_2$ is studied. Our proof is based on a geometric feature of τ_g, and applications of the Novikov additivity of the signature are essential.

There are several studies on the fundamental group of the complement of the dual variety (or more generally the associated subvariety in a Grassmannian); for example, see Carlson and Toledo [6; 7] and recent work of Shimada [26]. However, it is still a mysterious object and lots of the properties are unknown. The function ϕ_X tells us some information about $\pi_1(U^X)$ under a mild condition; see Proposition 3.6.1.

An application to local signatures One reason to seek a generalization of Meyer's ϕ_1 or ϕ_2 comes from the motivation to treat *localization of the signature* for the case of the genus of the fibers greater than two via Meyer functions.

Let us go back to the case of $g = 1$ or 2 for explanation. The coboundary condition $\delta\phi_g = \tau_g$ leads to an immediate consequence: for an oriented surface bundle of genus ≤ 2 over a closed oriented surface, the signature of the total space is zero. Proceeding further, let M (resp. B) be a closed oriented C^∞–manifold of dimension 4 (resp. 2) and $f\colon M \to B$ a proper surjective C^∞–map having the structure of surface bundle of genus g, outside of finitely many points $b_1, \ldots, b_m \in B$. We call such a triple (M, f, B) a *fibered* 4–*manifold*. The fiber germ $\mathcal{F}_i$ over b_i is called a singular fiber germ. Typical examples are elliptic surfaces or Lefschetz fibrations.

In the above situation, the advantage of ϕ_1 or ϕ_2 is that we can associate each singular fiber germ with its local invariant $\sigma(\mathcal{F}_i) \in \mathbb{Q}$, called the *local signature*. The adjective "local" comes from the equality

$$\mathrm{Sign}(M) = \sum_{i=1}^{m} \sigma(\mathcal{F}_i).$$

The definition of $\sigma(\mathcal{F}_i)$ is given by

$$(1.0.3) \qquad \sigma(\mathcal{F}_i) = \phi_g(x_i) + \mathrm{Sign}(N(f^{-1}(b_i))),$$

where $x_i \in \Gamma_g$ is the local monodromy around b_i and $\mathrm{Sign}(N(f^{-1}(b_i)))$ is the signature of a fiber neighborhood of $f^{-1}(b_i)$. This formulation first appeared in Y Matsumoto's papers [18; 19]. For generalizations of this story to higher genera, there is an obstruction: the class $[\tau_g]$ is a generator of $H^2(\Gamma_g; \mathbb{Q}) \cong \mathbb{Q}$ for $g \geq 3$.

Local signatures are also studied from the complex-geometric or algebro-geometric point of view; see Ashikaga and Konno [2] and Ashikaga and Yoshikawa [3]. In these settings, a local signature is defined another way and can be defined even if $g \geq 3$, by assigning some algebro-geometric conditions on the general fibers. There is an important point to note here: when $g \geq 3$, there is a fiber germ with a nontrivial local signature but which is topologically a trivial Σ_g–bundle. To capture such phenomena, it is insufficient just to look at the shape of $f^{-1}(b_i)$ or the local monodromy x_i, hence we need to modify (1.0.3).

As for higher genera analogues of Matsumoto's approach, Endo [8] studied the local signature for hyperelliptic fibrations. In [16] the author discussed nonhyperelliptic fibrations of genus 3. In this paper using the Meyer functions ϕ_X for particular choices of X, we will discuss nonhyperelliptic fibrations of genus 4 or 5. The modification of (1.0.3) is achieved by introducing a group with some universal property and the Meyer function on that group. The monodromy x_i in (1.0.3) is replaced by the *lifted monodromy*; see Definition 4.1.10. One advantage of our local signature is that we only need the complex structures on the general fibers so it is not necessary f itself should be holomorphic. Although we don't know whether our local signature is the same as the others [2; 3], we will observe the coincidence on some examples of singular fiber germs.

Organization of the paper   Section 2 is a preparation for Section 3. We describe the tangent space of D_X and study the situation when a holomorphic disk intersects D_X transversally. These considerations will be used in Proposition 3.2.3. In Section 3 we prove Theorem 1.0.2 by a purely topological argument. Using the method of Lefschetz pencils, we give a formula for the value of ϕ_X on a special element, called the *lasso*. We also study the second bounded cohomology of $\pi_1(U^X)$. In Section 4 applications to nonhyperelliptic fibrations of genus 4 or 5 are discussed. First we explain our approach to local signatures via Meyer functions (see Proposition 4.1.8), then proceed to the particular cases. When the genus is 4, we assign the general fibers to be "of rank 4". When the genus is 5, we assign the general fibers to be nontrigonal. In the case of genus 4, we compute the value of our local signature for some fiber germs.

In the rest of this introduction we fix conventions and recall Meyer's signature cocycle.

Topological monodromy   We adopt the following: (1) For any two mapping classes f_1 and f_2, the multiplication $f_1 \circ f_2$ means that f_2 is applied first. (2) For any two homotopy classes of based loops ℓ_1 and ℓ_2, their product $\ell_1 \cdot \ell_2$ means that ℓ_1 is traversed first.

Let $p: \mathcal{C} \to B$ be an oriented Σ_g–bundle. Choose a base point $b_0 \in B$ and fix an identification $\phi: \Sigma_g \xrightarrow{\cong} p^{-1}(b_0)$. For each based loop $\ell: [0, 1] \to B$ the pullback $\ell^*\mathcal{C} \to [0, 1]$ by ℓ is a trivial Σ_g–bundle. Hence there exists a trivialization $\Phi: \Sigma_g \times [0, 1] \to \ell^*\mathcal{C}$ such that $\Phi(x, 0) = \phi(x)$, $x \in \Sigma_g$. By assigning the isotopy class of $\Phi(\,\cdot\,, 1)^{-1} \circ \phi$ to the homotopy class of ℓ, we obtain a map ρ, called the *topological monodromy* of $p: \mathcal{C} \to B$, from $\pi_1(B, b_0)$ to Γ_g. Under the conventions above, ρ is a homomorphism.

Meyer's signature cocycle Let P denote the pair of pants, ie, $P = S^2 \setminus \bigcup_{i=1}^{3} \operatorname{Int} D_i$ where D_i, $1 \leq i \leq 3$, are the three disjoint closed disks in the 2–sphere S^2. Choose a base point $p_0 \in \operatorname{Int} P$ and fix based loops ℓ_i, $1 \leq i \leq 3$, such that each ℓ_i is homotopic to the loop traveling once the boundary ∂D_i by counterclockwise manner, and the product $\ell_1 \cdot \ell_2 \cdot \ell_3$ is nullhomotopic. For $(f_1, f_2) \in \Gamma_g \times \Gamma_g$, we can construct an oriented Σ_g–bundle $E(f_1, f_2)$ over P such that the topological monodromy $\pi_1(P, p_0) \to \Gamma_g$ sends $[\ell_i]$ to f_i for $i = 1, 2$. The bundle $E(f_1, f_2)$ is a compact C^∞–manifold of dimension 4 endowed with the natural orientation. Thus the signature of $E(f_1, f_2)$ is defined and we set

$$\tau_g(f_1, f_2) := \operatorname{Sign}(E(f_1, f_2)).$$

By the Novikov additivity of the signature, τ_g turns out to be a 2–cocycle of Γ_g. The class $[\tau_g] \in H^2(\Gamma_g; \mathbb{Z})$ equals $1/3$ times the first MMM class; see Miller [21], Morita [23] and Mumford [25].

There is a linear-algebraic description of τ_g given by Meyer [20]. Let $\Gamma_g \to \operatorname{Sp}(2g; \mathbb{Z})$ be the homomorphism obtained by the action of Γ_g on the first homology of Σ_g, and let A_1 and A_2 be the image of f_1 and f_2 by this homomorphism, respectively. Let

$$J = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix},$$

where I_g is the $g \times g$ identity matrix, and consider the linear space

$$V_{A_1, A_2} := \{(x, y) \in \mathbb{R}^{2g} \oplus \mathbb{R}^{2g} : (A_1^{-1} - I_{2g})x + (A_2 - I_{2g})y = 0\},$$

where I_{2g} is the $2g \times 2g$ identity matrix. Then

$$\langle (x, y), (x', y') \rangle_{A_1, A_2} = {}^t(x + y)J(I_{2g} - A_2)y'$$

turns out to be a symmetric bilinear form on V_{A_1, A_2} hence its signature is defined. As proved in [20], we have

(1.0.4) $$\tau_g(f_1, f_2) = \operatorname{Sign}(V_{A_1, A_2}, \langle \, , \, \rangle_{A_1, A_2}).$$

Here we correct some errors about signs in [16]. In the Appendix of [16], we have adopted the same notation about topological monodromies as this paper and have defined $\tau_g(f_1, f_2) = -\operatorname{Sign}(E(f_1, f_2))$. Then Definition 7.1 on page 943 should be corrected as

$$\operatorname{loc.sig}^{\mathcal{Q}}(\mathcal{F}) := -\phi^4(\theta(\mathcal{F}^0)_*(\gamma)) + \operatorname{Sign}(E).$$

The equation $\operatorname{Sign}(\pi^{-1}(B_0)) = \sum_{i=1}^n \phi^4(\theta(\mathcal{F}_i^0)_*(\gamma))$ in the proof of Theorem 7.2 on page 944 should be corrected by multiplying the right hand side by -1. The proof of Proposition 5.1 on page 936 should be corrected similarly and all the values of the Meyer function in [16] should be multiplied by -1.

Notation For a 1–cochain $\phi\colon G \to A$ of a group G with coefficient in an abelian group A, its coboundary means the map $\delta\phi\colon G \times G \to A$ defined by

$$\delta\phi(x, y) = \phi(x) - \phi(xy) + \phi(y).$$

For a complex manifold M, we denote by K_M the canonical divisor of M. More generally, for a possibly singular variety Y, we denote by ω_Y the dualizing sheaf of Y. We use this notion only when Y is given as a hypersurface in a complex manifold M. In this case ω_Y is an invertible sheaf on Y given by the adjunction formula:

$$\omega_Y = (K_M + Y)|_Y.$$

For integers p, q with $0 < p < q$, we denote by $G_{p,q}$ the Grassmannian of all p–planes of $\mathbb{C}^q$. Note that $G_k(\mathbb{P}_N)$ is naturally isomorphic to $G_{k+1,N+1}$.

2   Preliminaries from complex algebraic geometry

In this section we describe some properties of D_X. When $n = 2$, D_X is an irreducible variety in $G_{N-1}(\mathbb{P}_N) = \mathbb{P}_N^\vee$, the dual projective space of $\mathbb{P}_N$, and D_X is classically known as *the dual variety of X*. In fact, the treatment here is a generalization of the treatments in Sections 1 and 2 of K Lamotke's paper [17] to the case of general n. Corollary 2.2.3, Proposition 2.3.8 and Theorem 2.3.12 will be used in later sections. Let

$$\mathcal{W} := \{(x, W) \in \mathbb{P}_N \times G_k(\mathbb{P}_N) : x \in X \cap W\}.$$

Then there are two projections $p_1\colon \mathcal{W} \to X$, and

$$p_2\colon \mathcal{W} \to G_k(\mathbb{P}_N).$$

2.1 Coordinate description of p_2

In the following we give an explicit coordinate description of p_2.

Let $(x^0, W_0) \in \mathcal{W}$. By choosing appropriate homogeneous coordinates $[x_0 : x_1 : \cdots : x_N]$ of $\mathbb{P}_N$, we may assume that $x^0 = [1 : 0 : \cdots : 0]$ and W_0 is given by $x_{k+1} = \cdots = x_N = 0$.

We first introduce local coordinates of $\mathcal{W}$ near (x^0, W_0). For $x \in X$, $p_1^{-1}(x)$ is the set of k–planes of $\mathbb{P}_N$ through x, which is isomorphic to $G_{k,N}$. The open set $\{x_0 \neq 0\}$ of $\mathbb{P}_N$ is identified with $\mathbb{C}^N$ by

$$(2.1.1) \qquad [x_0 : x_1 : \cdots : x_N] \mapsto (x_1/x_0, \ldots, x_N/x_0).$$

Thus for $(x, \mathring{W}) \in (X \cap \{x_0 \neq 0\}) \times G_{k,N}$, considering the affine subspace $x + \mathring{W} \subset \mathbb{C}^N \cong \{x_0 \neq 0\}$ and taking its closure in $\mathbb{P}_N$, we have a trivialization of p_1 over $X \cap \{x_0 \neq 0\}$:

$$(2.1.2) \qquad (X \cap \{x_0 \neq 0\}) \times G_{k,N} \xrightarrow{\cong} p_1^{-1}(X \cap \{x_0 \neq 0\}).$$

Let
$$e_i := (0, \ldots, \overset{i}{1}, \ldots, 0)^t \in \mathbb{C}^N$$

for $1 \leq i \leq N$. In view of (2.1.1), $\mathring{W}_0 := W_0 \cap \{x_0 \neq 0\}$ is the k–plane of $\mathbb{C}^N$ spanned by $e_1, \ldots, e_k$. Let W_1 be the subspace of $\mathbb{C}^N$ spanned by $e_{k+1}, \ldots, e_N$. For $F \in \mathrm{Hom}(\mathring{W}_0, W_1)$ let $\mathring{W}(F) := \mathrm{Span}(f_1, \ldots, f_k) \in G_{k,N}$, where $f_i := e_i + F(e_i)$, $1 \leq i \leq k$. By this mapping $F \mapsto \mathring{W}(F)$, $\mathcal{U} := \{\mathring{W} \in G_{k,N} : \mathring{W} \cap W_1 = 0\}$ is identified with $\mathrm{Hom}(\mathring{W}_0, W_1)$. Introducing $\{F_i^j\}$ by

$$F(e_i) = \sum_{j=k+1}^{N} F_i^j e_j, \quad 1 \leq i \leq k,$$

for $F \in \mathrm{Hom}(\mathring{W}_0, W_1)$, the set of functions $\{F_i^j\}$ serves as local coordinates of $\mathcal{U}$.

Choose a sufficiently small local coordinate neighborhood $(U; t_1, \ldots, t_n)$ of X centered at x^0 (ie, x^0 corresponds to the origin $(0, \ldots, 0)$) so that $U \subset X \cap \{x_0 \neq 0\} \subset \mathbb{C}^N$. Then points in U can be expressed as $x(t_1, \ldots, t_n) = (x_1, \ldots, x_N)$ where $x_i = x_i(t_1, \ldots, t_n)$, $1 \leq i \leq N$ are holomorphic in $t_1, \ldots, t_n$.

In view of (2.1.2),

$$(2.1.3) \qquad \left(U \times \mathcal{U}; t_1, \ldots, t_n, \{F_i^j\}\right)$$

can be used as a local coordinate neighborhood of $\mathcal{W}$ centered at (x^0, W_0).

Next we introduce local coordinates of $G_k(\mathbb{P}_N)$ near W_0. Let

$$\widehat{e}_i = (0, \ldots, \overset{i+1}{1}, \ldots, 0)^t \in \mathbb{C}^{N+1},$$

$0 \leq i \leq N$ and $\widehat{W}_0 = \mathrm{Span}(\widehat{e}_0, \widehat{e}_1, \ldots, \widehat{e}_k)$, $\widehat{W}_1 = \mathrm{Span}(\widehat{e}_{k+1}, \ldots, \widehat{e}_N)$. Then by the natural isomorphism $G_k(\mathbb{P}_N) \cong G_{k+1,N+1}$, W_0 corresponds to $\widehat{W}_0$ (recall that W_0 is given by $x_{k+1} = \cdots x_N = 0$). For $G \in \mathrm{Hom}(\widehat{W}_0, \widehat{W}_1)$ let $W(G) :=$ $\mathrm{Span}(g_0, g_1, \ldots, g_k)$, where $g_i := \widehat{e}_i + G(\widehat{e}_i)$, $0 \leq i \leq k$. By this mapping $G \mapsto W(G)$, $\mathcal{V} := \{W \in G_{k+1,N+1} : W \cap \widehat{W}_1 = 0\}$ is identified with $\mathrm{Hom}(\widehat{W}_0, \widehat{W}_1)$. Introducing $\{G_i^j\}$ by

$$G(\widehat{e}_i) = \sum_{j=k+1}^{N} G_i^j \widehat{e}_j, \quad 0 \leq i \leq k,$$

for $G \in \mathrm{Hom}(\widehat{W}_0, \widehat{W}_1)$, then

(2.1.4) $(\mathcal{V}, \{G_i^j\})$

is a local coordinate neighborhood of $G_{k+1,N+1} \cong G_k(\mathbb{P}_N)$ near W_0.

Now for $(t_1, \ldots, t_n, \{F_i^j\}) \in U \times \mathcal{U}$, using the local coordinates (2.1.3) and (2.1.4), write $p_2(t_1, \ldots, t_n, \{F_i^j\}) = G(t_1, \ldots, t_n, \{F_i^j\}) \in \mathrm{Hom}(\widehat{W}_0, \widehat{W}_1)$. The closure of the affine space $x(t_1, \ldots, t_n) + \overset{\circ}{W}(F) \subset \mathbb{C}^N$ in $\mathbb{P}_N$ must be equal to $W(G(t_1, \ldots, t_n, \{F_i^j\}))$. On the other hand, at the level of $G_{k+1,N+1}$, the closure is a $(k+1)$–plane spanned by $(1, x_1, \ldots, x_n)^t$ and $\widehat{f}_1, \ldots, \widehat{f}_k$, where for $1 \leq i \leq k$, $\widehat{f}_i \in \mathbb{C}^{N+1}$ is the image of f_i under the inclusion $\mathbb{C}^N \hookrightarrow \mathbb{C}^{N+1}$, $(y_1, \ldots, y_N) \mapsto (0, y_1, \ldots, y_N)$. From these we establish the following:

Lemma 2.1.5 (Local description of p_2) *Let $(U \times \mathcal{U}; t_1, \ldots, t_n, \{F_i^j\})$ and $(\mathcal{V}, \{G_i^j\})$ be the local coordinate neighborhoods of $\mathcal{W}$ and $G_k(\mathbb{P}_N)$, respectively, as above, and let $G_i^j = G_i^j(t_1, \ldots, t_n, \{F_i^j\})$ be the local coordinates of the point $W(G(t_1, \ldots, t_n, \{F_i^j\}))$. Then we have*

$$G_i^j = F_i^j \qquad\qquad \text{for } 1 \leq i \leq k,\ k+1 \leq j \leq N,$$

(2.1.6) $$G_0^j = x_j - \sum_{i=1}^{k} x_i F_i^j \qquad\qquad \text{for } k+1 \leq j \leq N.$$

2.2 Irreducibility of D_X

The k–th associated subvariety D_X is irreducible. Although this might be well known, here we include the proof of it together with the irreducibility of some loci related to D_X.

Definition 2.2.1 Define the subsets of $\mathcal{W}$ as follows:

$$\mathcal{D} := \{(x, W) \in \mathcal{W} : T_x X + T_x W \neq T_x \mathbb{P}_N\},$$

and for integers $i \geq 1$,

$$\mathcal{Y}_i := \{(x, W) \in \mathcal{W} : \dim(T_x X + T_x W) = N + 1 - i\}$$
$$= \{(x, W) \in \mathcal{W} : \dim(T_x X \cap T_x W) = i\}.$$

Finally, for $i \geq 1$, $\mathcal{D}_i := \bigcup_{j \geq i} \mathcal{Y}_j$.

Note that $\mathcal{Y}_i$ is empty for $i > \max(n, k)$ and $\mathcal{Y}_i = \mathcal{D}_i \setminus \mathcal{D}_{i+1}$. Also we have $p_2(\mathcal{D}) = D_X$. All $\mathcal{D}_i$ are closed analytic subsets of $\mathcal{W}$ and we have a filtration

$$\mathcal{D}_1 = \mathcal{W} \supset \mathcal{D}_2 = \mathcal{D} \supset \mathcal{D}_3 \supset \mathcal{D}_4 \supset \cdots \supset \mathcal{D}_{\max(n,k)}.$$

Using [Lemma 2.1.5](#), we can verify that $\mathcal{D}$ is the set of critical points of $p_2 : \mathcal{W} \to G_k(\mathbb{P}_N)$ and $\mathcal{Y}_i$ (resp. $\mathcal{D}_i$) is the set of points $(x, W) \in \mathcal{W}$ such that the differential $(p_2)_* : T_{(x,W)} \mathcal{W} \to T_W G_k(\mathbb{P}_N)$ has corank $i - 1$ (resp. corank $\geq i - 1$).

Theorem 2.2.2   (1) *For each i, $\mathcal{Y}_i$ is a connected submanifold of $\mathcal{W}$ with codimension $i^2 - i$ and is open and dense in $\mathcal{D}_i$.*

   (2) *For each i, $\mathcal{D}_i$ is an irreducible analytic subset of $\mathcal{W}$ with codimension $i^2 - i$.*

Since $\dim \mathcal{W} = \dim G_k(\mathbb{P}_N) + 1$ and p_2 is a proper holomorphic map, we get the following which we will use later.

Corollary 2.2.3 *The set D_X is an irreducible analytic subset of $G_k(\mathbb{P}_N)$ with codimension ≥ 1. If the codimension of D_X is 1, $p_2(\mathcal{D}_3)$ is a proper analytic subset of D_X.*

Proof of [Theorem 2.2.2](#)   Let us introduce some notation. Let V be a fixed n–dimensional subspace of $\mathbb{C}^N$ and for $i = 1, 2, \ldots$, let

$$Y_i := \{W \in G_{k,N} : \dim(V + W) = N + 1 - i\},$$

and $D_i = \bigcup_{j \geq i} Y_j$. By trivializations of the pair of holomorphic vector bundles $(T\mathbb{P}_N|_X, TX)$ on X, we see that $p_1|_{\mathcal{D}_i}$ and $p_1|_{\mathcal{Y}_i}$ are holomorphic fiber bundle with fiber isomorphic to D_i and Y_i, respectively. Since Y_i is open and dense in D_i, $\mathcal{Y}_i$ is also open and dense in $\mathcal{D}_i$. We see that Y_i has a structure of a connected complex manifold of dimension $kn - k + i - i^2$. This can be seen by considering the projection $Y_i \to G_i(V)$, $W \mapsto W \cap V$ ($G_i(V)$ is the Grassmannian of all i–planes of V). This shows $\mathcal{Y}_i$ is a connected complex manifold with the desired codimension.

We next prove the second part. Since $\mathcal{Y}_i$ is contained in $\mathcal{D}_i$, the first part shows that the set of smooth points of $\mathcal{D}_i$ is connected, hence $\mathcal{D}_i$ is irreducible. Also we have $\dim \mathcal{D}_i = \dim \mathcal{Y}_i$. This completes the proof. $\square$

2.3 The tangent space of D_X for the case D_X is a hypersurface

In this section we describe the tangent space of D_X at a generic point of D_X under the assumption that the codimension of D_X is 1, ie, D_X is a hypersurface of $G_k(\mathbb{P}_N)$. Then $p_2|_{\mathcal{D}} \colon \mathcal{D} \to D_X$ is a dominant regular map between projective varieties of the same dimension. By Sard's lemma for varieties (see Chapter 3 of Mumford [24] for instance), there exists a proper analytic subset $E' \subset D_X$ such that

(1) E' contains $S(D_X)$, the set of singular points of D_X,

(2) the differential $(p_2|_{\mathcal{D}})_* \colon T_{(x,W)}\mathcal{D} \to T_W D_X$ is an isomorphism for $(x, W) \in (p_2|_{\mathcal{D}})^{-1}(D_X \setminus E') \setminus S(\mathcal{D})$, where $S(\mathcal{D})$ denotes the set of singular points of $\mathcal{D}$.

By Theorem 2.2.2, $\mathcal{Y}_2$ is contained in $\mathcal{D} \setminus S(\mathcal{D})$ so $S(\mathcal{D}) \subset \mathcal{D}_3$. Setting $E := E' \cup p_2(\mathcal{D}_3)$ then this is a proper analytic subset of D_X by Corollary 2.2.3. Now we have:

Lemma 2.3.1 *Suppose that the codimension of D_X is 1. Then there exists a proper analytic subset E of D_X such that*

(1) *E contains $S(D_X)$,*

(2) *$(p_2|_{\mathcal{D}})^{-1}(D_X \setminus E) \subset \mathcal{Y}_2$, in particular $(p_2|_{\mathcal{D}})^{-1}(D_X \setminus E)$ is contained in $\mathcal{D} \setminus S(\mathcal{D})$,*

(3) *the differential $(p_2|_{\mathcal{D}})_*$ is an isomorphism at $(x, W) \in (p_2|_{\mathcal{D}})^{-1}(D_X \setminus E)$.*

Suppose that $(x^0, W_0) \in \mathcal{D}$ and $W_0 = p_2(x^0, W_0) \in D_X \setminus E$. By Lemma 2.3.1, $(x^0, W_0) \in \mathcal{Y}_2$. Then the following relations among the subspaces of $T_{W_0} G_k(\mathbb{P}_N)$ hold:

$$(p_2)_*(T_{(x^0, W_0)}\mathcal{W}) \supset (p_2)_*(T_{(x^0, W_0)}\mathcal{Y}_2) = (p_2|_{\mathcal{D}})_*(T_{(x^0, W_0)}\mathcal{D}) = T_{W_0} D_X.$$

Since $(x^0, W_0) \in \mathcal{Y}_2$, $(p_2)_*(T_{(x^0, W_0)}\mathcal{W})$ has codimension 1 in $T_{W_0} G_k(\mathbb{P}_N)$ (see Section 2.2). Therefore, we have

(2.3.2) $T_{W_0} D_X = (p_2)_*(T_{(x^0, W_0)}\mathcal{W}).$

Recall the local coordinates of $\mathcal{W}$ and $G_k(\mathbb{P}_N)$ in Section 2.1. By Lemma 2.1.5, we see that $(p_2)_*(T_{(x^0,W_0)}\mathcal{W})$ is generated by the n vectors

$$
\textbf{(2.3.3)} \qquad \frac{\partial x_{k+1}}{\partial t_i}(0)\frac{\partial}{\partial G_0^{k+1}} + \ldots + \frac{\partial x_N}{\partial t_i}(0)\frac{\partial}{\partial G_0^N},
$$

$1 \le i \le n$, and the $k(n-1)$ vectors

$$
\textbf{(2.3.4)} \qquad \frac{\partial}{\partial G_i^j},
$$

$1 \le i \le k$, $k+1 \le j \le N$. Let

$$
\frac{\partial \boldsymbol{x}}{\partial t_i} = \left(\frac{\partial x_{k+1}}{\partial t_i}, \ldots, \frac{\partial x_N}{\partial t_i} \right)^t \in \mathbb{C}^{n-1}, \quad 1 \le i \le n.
$$

Since $(x^0, W_0) \in \mathcal{Y}_2$ the rank of the matrix

$$
\left(\frac{\partial \boldsymbol{x}}{\partial t_1}(0), \ldots, \frac{\partial \boldsymbol{x}}{\partial t_n}(0) \right)
$$

is $(n-2)$. By an arrangement of indices we may assume that

$$
\textbf{(2.3.5)} \qquad \frac{\partial \boldsymbol{x}}{\partial t_i}(0), \ 1 \le i \le n-2, \quad \text{are linearly independent,}
$$

then the other two column vectors are in the linear span of these.

Proposition 2.3.6 *Suppose that the codimension of D_X is 1 and let E be as in Lemma 2.3.1. Let $(x^0, W_0) \in \mathcal{D}$ and suppose $W_0 \in D_X \setminus E$. Then under the assumption (2.3.5), the tangent space $T_{W_0} D_X$ is given by*

$$
\textbf{(2.3.7)} \qquad T_{W_0} D_X = \left\{ \sum_{i,j} u_i^j \frac{\partial}{\partial G_i^j} : \det\left(\boldsymbol{u}, \frac{\partial \boldsymbol{x}}{\partial t_1}(0), \ldots, \frac{\partial \boldsymbol{x}}{\partial t_{n-2}}(0) \right) = 0 \right\}.
$$

Here, $\boldsymbol{u} = (u_0^{k+1}, \ldots, u_0^N)^t \in \mathbb{C}^{n-1}$.

Proof By the assumption (2.3.5), the right hand side of (2.3.7) is a hyperplane of $T_{W_0} G_k(\mathbb{P}_N)$. Also, the vectors (2.3.3) and (2.3.4) are clearly contained in the right hand side of (2.3.7). This completes the proof.   □

Regarding $p_2 \colon \mathcal{W} \to G_k(\mathbb{P}_N)$ as a family of algebraic curves, we investigate its pullback by a mapping into $G_k(\mathbb{P}_N)$ which does not meet E and is transverse to D_X. We first show that the total space of the pullback has the structure of a manifold.

Proposition 2.3.8 *Suppose that the codimension of D_X is 1 and let E be as in Lemma 2.3.1. Let B be a C^∞–manifold of dimension ≥ 2 and let $\iota\colon B \to G_k(\mathbb{P}_N)$ be a C^∞–map satisfying $\iota^{-1}(E) = \varnothing$ and transverse to D_X. Then, the pullback*

$$\iota^*\mathcal{W} := \{(b, (x, W)) \in B \times \mathcal{W} : \iota(b) = W\}$$
$$\cong \{(b, x) \in B \times X : x \in \iota(b)\}$$

of p_2 by ι has the natural structure of a C^∞–manifold as a C^∞–submanifold of $B \times X$. Moreover, if B is a complex manifold and ι is a holomorphic map, $\iota^*\mathcal{W}$ has the natural structure of a complex manifold as a complex submanifold of $B \times X$.

Proof We only treat the case B is the small disk $\Delta := \{z \in \mathbb{C} : |z| < \varepsilon\}$, $\varepsilon > 0$ and ι is a holomorphic map such that $\iota^{-1}(D_X) = \{0\}$. A similar argument proves the general case (see also Lemma 2.4 in [16]).

By the assumption, we have $\iota(0) \in D_X \setminus E$ and the transversality

$$(2.3.9) \qquad\qquad \iota_*(T_0\Delta) + T_{\iota(0)}D_X = T_{\iota(0)}G_k(\mathbb{P}_N).$$

Let $(z_0, x^0) \in \iota^*\mathcal{W}$ and write $W_0 := \iota(z_0)$. Choosing the local coordinates of $\mathcal{W}$ and $G_k(\mathbb{P}_N)$ as in Section 2.1 we denote by ι_i^j the coordinate expression of ι with respect to the local coordinates $\{G_i^j\}$. In particular, we have $\iota_i^j(z_0) = 0$. By Lemma 2.1.5, the local equation of $\iota^*\mathcal{W}$ near (z_0, x^0) is given by

$$-\iota_0^j(z) + x_j(t_1, \ldots, t_n) - \sum_{i=1}^{k} x_i(t_1, \ldots, t_n)\iota_i^j(z) = 0$$

for $k+1 \leq j \leq N$. Let $\psi_j(z, t_1, \ldots, t_n)$ be the left hand side of the above equation. The Jacobian matrix of $(\psi_{k+1}, \ldots, \psi_N)$ at $(z_0, 0, \ldots, 0)$ is the $(n-1) \times (n+1)$ matrix

$$(2.3.10) \qquad\qquad \left(-\iota'(z_0), \frac{\partial \boldsymbol{x}}{\partial t_1}(0), \ldots, \frac{\partial \boldsymbol{x}}{\partial t_n}(0)\right),$$

where $\iota' = (d\iota_0^{k+1}/dz, \ldots, d\iota_0^N/dz)$. We claim that this matrix is of full rank. Suppose $(x^0, W_0) \notin \mathcal{D}$, namely $(x^0, W_0) \in \mathcal{Y}_1$. Then the $(n-1) \times n$ matrix obtained by deleting the first column of (2.3.10) is already of full rank so is (2.3.10).

Suppose $(x^0, W_0) \in \mathcal{D}$. Then by the assumption, $z_0 = 0$ and $W_0 \in D_X \setminus E$. Proposition 2.3.6 and (2.3.9) shows that

$$(2.3.11) \qquad\qquad \det\left(\iota'(0), \frac{\partial \boldsymbol{x}}{\partial t_1}(0), \ldots, \frac{\partial \boldsymbol{x}}{\partial t_{n-2}}(0)\right) \neq 0,$$

therefore (2.3.10) is of full rank also in this case. By the implicit function theorem the assertion follows. $\qquad\square$

Let Δ and ι be as in the proof of Proposition 2.3.8. The pullback $\iota^*\mathcal{W} = \{(z, x) \in \Delta \times X : x \in \iota(z)\}$ has the natural projection $f_\iota \colon \iota^*\mathcal{W} \to \Delta$. Explicitly, f_ι is given by $f_\iota(z, x) = z$.

Theorem 2.3.12 *Notation is as above. Then, $(z_0, x^0) \in \iota^*\mathcal{W}$ is a critical point of f_ι if and only if $z_0 = 0$ and $(x^0, \iota(0)) \in \mathcal{D}$. All the critical points are nondegenerate.*

In fact, we will see in Corollary 3.4.6 that there is only one critical point. By an argument like the Morse lemma, we see that near each critical point f_ι looks like $(z_1, z_2) \mapsto z_1^2 + z_2^2$. The next section is devoted to the proof of this theorem.

2.4 Proof of Theorem 2.3.12

We will use the notation in the proof of Proposition 2.3.8.

Let $(z_0, x^0) \in \iota^*\mathcal{W}$ and write $W_0 := \iota(z_0)$. Suppose $(x^0, W_0) \notin \mathcal{D}$. Then we may assume that in the column vectors of (2.3.10)

$$\frac{\partial \boldsymbol{x}}{\partial t_1}(0), \dots, \frac{\partial \boldsymbol{x}}{\partial t_{n-1}}(0)$$

are linearly independent. By the implicit function theorem, there exist local coordinates (s_1, s_2) of $\iota^*\mathcal{W}$ centered at (z_0, x^0) such that the points near (z_0, x^0) can be expressed as $(z, t_1, \dots, t_{n-1}, t_n)$ where

$$z = s_1, t_1 = t_1(s_1, s_2), \dots, t_{n-1} = t_{n-1}(s_1, s_2), t_n = s_2,$$

and $t_i(s_1, s_2)$ are holomorphic in s_1, s_2. Since $f_\iota(z, x) = z = s_1$, (z_0, x^0) is not a critical point of f_ι.

Suppose $(x^0, W_0) \in \mathcal{D}$. Then as we have seen in the proof of Proposition 2.3.8, $z_0 = 0$, $W_0 \in D_X \setminus E$, and $(x^0, W_0) \in \mathcal{Y}_2$. We may assume (2.3.5). Then we have the inequality (2.3.11) hence there exist local coordinates (s_1, s_2) of $\iota^*\mathcal{W}$ centered at (z_0, x^0) such that

(2.4.1) $z = z(s_1, s_2), t_1 = t_1(s_1, s_2), \dots, t_{n-2} = t_{n-2}(s_1, s_2), t_{n-1} = s_1, t_n = s_2$.

For $k + 1 \leq j \leq N$, differentiating the identity

$$\psi_j(z(s_1, s_2), t_1(s_1, s_2), \dots, t_{n-2}(s_1, s_2), s_1, s_2) = 0$$

with respect to s_1 and setting $(s_1, s_2) = (0, 0)$, we have

$$-\frac{\partial z}{\partial s_1}(0)\iota'(0) + \sum_{i=1}^{n-2} \frac{\partial t_i}{\partial s_1}(0)\frac{\partial \boldsymbol{x}}{\partial t_i}(0) + \frac{\partial \boldsymbol{x}}{\partial t_{n-1}}(0) = 0.$$

But by the transversality, $\iota'(0)$ is not contained in $\mathrm{Span}\,(\partial x/\partial t_i(0))_{1\le i\le n}$. Thus we have $\partial z/\partial s_1(0) = 0$ and the identity

(2.4.2)
$$\sum_{i=1}^{n-2} \frac{\partial t_i}{\partial s_1}(0)\frac{\partial x}{\partial t_i}(0) + \frac{\partial x}{\partial t_{n-1}}(0) = 0.$$

Similarly, we have $\partial z/\partial s_2(0) = 0$ and the identity

(2.4.3)
$$\sum_{i=1}^{n-2} \frac{\partial t_i}{\partial s_2}(0)\frac{\partial x}{\partial t_i}(0) + \frac{\partial x}{\partial t_n}(0) = 0.$$

This shows that $(0, x^0)$ is a critical point of f_ι. We have proved the first part.

To accomplish the proof, we must show that all the critical points are nondegenerate. We need to compute the Hessian of $z(s_1, s_2)$ at $(s_1, s_2) = (0, 0)$ where $z(s_1, s_2)$ is as in (2.4.1). For this purpose we give a system of local equations for the submanifold $\mathcal{Y}_2$, and we rephrase the fact that the differential $(p_2|_{\mathcal{D}})_*\colon T_{(x^0, W_0)}\mathcal{D} = T_{(x^0, W_0)}\mathcal{Y}_2 \to T_{W_0}D_X$ is an isomorphism.

Now take the local coordinates (2.1.3) of $\mathcal{W}$. Then for (x, W) in this coordinate neighborhood, $T_x X = T_x(X \cap \mathcal{U}_0) \subset \mathbb{C}^N$ is spanned by the n vectors

$$\alpha_i = \left(\frac{\partial x_1}{\partial t_i}, \dots, \frac{\partial x_N}{\partial t_i}\right)^t, \quad 1 \le i \le n,$$

and $T_x W = T_x(W \cap \mathcal{U}_0)$ is spanned by the k vectors

$$\beta_i = (0, \dots, \overset{i}{1}, \dots, 0, F_i^{k+1}, \dots, F_i^N)^t, \quad 1 \le i \le k.$$

$(x, W) \in \mathcal{Y}_2$ if and only if the linear span of these $n + k = N + 1$ vectors is $(N-1)$–dimensional. On the other hand, the origin (x^0, W_0) is in $\mathcal{Y}_2$ and by the assumption (2.3.5) $\alpha_1, \dots, \alpha_{n-2}, \beta_1, \dots, \beta_k$ are linearly independent at (x^0, W_0). Therefore, the vanishing of the two determinants $\det(\alpha_1, \dots, \alpha_{n-2}, \alpha_{n-1}, \beta_1, \dots, \beta_k)$ and $\det(\alpha_1, \dots, \alpha_{n-2}, \alpha_n, \beta_1, \dots, \beta_k)$ gives a system of local equations for $\mathcal{Y}_2$ near (x^0, W_0). By elementary transformations of matrices, we see that these determinants are equal up to sign to Φ_{n-1} and Φ_n respectively, where

$$\Phi_\nu = \det\left(\frac{\partial x}{\partial t_1} - \sum_{i=1}^{k} \frac{\partial x_i}{\partial t_1}\boldsymbol{F}_i, \cdots, \frac{\partial x}{\partial t_{n-2}} - \sum_{i=1}^{k} \frac{\partial x_i}{\partial t_{n-2}}\boldsymbol{F}_i, \frac{\partial x}{\partial t_\nu} - \sum_{i=1}^{k} \frac{\partial x_i}{\partial t_\nu}\boldsymbol{F}_i\right).$$

Here, $\boldsymbol{F}_i = (F_i^{k+1}, \dots, F_i^N)^t$. Hence $\mathcal{Y}_2$ is locally given by $\Phi_{n-1} = \Phi_n = 0$.

Now the fact that $(p_2|_{\mathcal{D}})_*\colon T_{(x^0, W_0)}\mathcal{Y}_2 \to T_{W_0}D_X$ is an isomorphism can be rephrased as follows: the rank of the Jacobian matrix of $(\Phi_{n-1}, \Phi_n, p_2)$ at (x^0, W_0) is equal

to dim $D_X + 2 = (k+1)(n-1) + 1$. Again by elementary transformations, this is equivalent to the following:

Lemma 2.4.4 *Let $(x^0, \mathcal{W}_0)$ and let Φ_{n-1} and Φ_n be as in the above. Then the rank of the $(n+1) \times n$ matrix*

$$(2.4.5) \qquad \begin{pmatrix} \dfrac{\partial \Phi_{n-1}}{\partial t_1}(0) & \cdots & \dfrac{\partial \Phi_{n-1}}{\partial t_n}(0) \\ \dfrac{\partial \Phi_n}{\partial t_1}(0) & \cdots & \dfrac{\partial \Phi_n}{\partial t_n}(0) \\ \dfrac{\partial x}{\partial t_1}(0) & \cdots & \dfrac{\partial x}{\partial t_n}(0) \end{pmatrix}$$

is equal to $(k+1)(n-1) + 1 - k(n-1) = n$.

We perform the following two elementary transformation to (2.4.5): let C_i be the i–th column of (2.4.5), then (1) add $\sum_{i=1}^{n-2} \partial t_i / \partial s_1(0) C_i$ to the $(n-1)$–th column, and (2) add $\sum_{i=1}^{n-2} \partial t_i / \partial s_2(0) C_i$ to the n–th column. Then by (2.4.2) and (2.4.3), (2.4.5) is transformed into

$$\begin{pmatrix} \dfrac{\partial \Phi_{n-1}}{\partial t_1}(0) & \cdots & \dfrac{\partial \Phi_{n-1}}{\partial t_{n-2}}(0) & A_{11} & A_{12} \\ \dfrac{\partial \Phi_n}{\partial t_1}(0) & \cdots & \dfrac{\partial \Phi_n}{\partial t_{n-2}}(0) & A_{21} & A_{22} \\ \dfrac{\partial x}{\partial t_1}(0) & \cdots & \dfrac{\partial x}{\partial t_{n-2}}(0) & 0 & 0 \end{pmatrix},$$

where

$$(2.4.6) \qquad A_{\lambda\mu} = \sum_{i=1}^{n-2} \frac{\partial t_i}{\partial s_\mu}(0) \frac{\partial \Phi_{n+\lambda-2}}{\partial t_i}(0) + \frac{\partial \Phi_{n+\lambda-2}}{\partial t_{n+\mu-2}}(0).$$

Now combining Lemma 2.4.4 and (2.3.5), we see that

$$(2.4.7) \qquad \det \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \neq 0.$$

Lemma 2.4.8 *We have the equality*

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = A_0 \begin{pmatrix} \dfrac{\partial^2 z}{\partial s_1^2}(0) & \dfrac{\partial^2 z}{\partial s_2 \partial s_1}(0) \\ \dfrac{\partial^2 z}{\partial s_1 \partial s_2}(0) & \dfrac{\partial^2 z}{\partial s_2^2}(0) \end{pmatrix},$$

where $A_0 = \det (\partial x / \partial t_1(0), \ldots, \partial x / \partial t_{n-2}(0), \iota'(0))$.

By (2.3.11), (2.4.7) and Lemma 2.4.8, it follows that the Hessian of $z(s_1, s_2)$ at $(s_1, s_2) = (0, 0)$ is nonzero. Thus, $(0, x^0)$ is a nondegenerate critical point of f_ι. This completes the proof of Theorem 2.3.12, modulo Lemma 2.4.8.

Lemma 2.4.8 can be proved by a straightforward computation. We only give an outline of the proof of $A_{11} = A_0 \partial^2 z / \partial s_1{}^2(0)$. The first claim is that

$$(2.4.9) \quad \frac{\partial \Phi_{n-1}}{\partial t_i}(0)$$
$$= \det\left(\frac{\partial x}{\partial t_1}(0), \ldots, \frac{\partial x}{\partial t_{n-2}}(0), \sum_{\ell=1}^{n-2} \frac{\partial t_\ell}{\partial s_1}(0) \frac{\partial^2 x}{\partial t_i \partial t_\ell}(0) + \frac{\partial^2 x}{\partial t_i \partial t_{n-1}}(0) \right),$$

which can be proved by using (2.4.2).

For $k + 1 \le j \le N$, differentiating twice the identity

$$\psi_j(z(s_1, s_2), t_1(s_1, s_2), \ldots, t_{n-2}(s_1, s_2), s_1, s_2) = 0$$

with respect to s_1 and setting $(s_1, s_2) = (0, 0)$, we have

$$(2.4.10) \quad -\frac{d\iota_0^j}{dz}(0) \frac{\partial^2 z}{\partial s_1{}^2}(0)$$
$$+ \sum_{i=1}^{n}\left(\left(\sum_{h=1}^{n} \frac{\partial^2 x_j}{\partial t_h \partial t_i}(0) \frac{\partial t_h}{\partial s_1}(0) \right) \frac{\partial t_i}{\partial s_1}(0) + \frac{\partial x_j}{\partial t_i}(0) \frac{\partial^2 t_i}{\partial s_1{}^2}(0) \right) = 0$$

(note that $\partial z / \partial s_1(0) = \partial z / \partial s_2(0) = 0$, $\partial t_{n-1} / \partial s_1(0) = 1$, and $\partial t_n / \partial s_1(0) = 0$).

Using (2.4.6), (2.4.9) and (2.4.10), we can get the desired formula.

3 The Meyer function for a projective variety

3.1 Main theorem

Recall the situation arising from $X \subset \mathbb{P}_N$ as described in Section 1. We focus on the topological monodromy

$$\rho_X \colon \pi_1(U^X) \to \Gamma_g$$

of (1.0.1).

Theorem 3.1.1 *There exists a uniquely determined $\mathbb{Q}$–valued 1–cochain $\phi_X \colon \pi_1(U^X) \to \mathbb{Q}$ whose coboundary equals $\rho_X^* \tau_g$. In particular, we have $\rho_X^*[\tau_g] = 0 \in H^2(\pi_1(U^X); \mathbb{Q})$.*

Here we comment about the group $\pi_1(U^X)$. If the codimension of D_X is ≥ 2, $\pi_1(U^X)$ is trivial since $G_k(\mathbb{P}_N)$ is simply connected. Suppose the codimension of D_X is 1. Then $\pi_1(U^X)$ is finitely presentable since U^X is an affine algebraic variety, and the first Betti number $b_1(\pi_1(U^X))$ is zero (see Lemma 3.3.1). Moreover, $\pi_1(U^X)$ is normally generated by a single element, called a *lasso*. Roughly speaking, a lasso is an element of $\pi_1(U^X)$ going once around D_X. The precise definition is as follows. We fix some base point in U^X. Let W_0 be a smooth point of D_X and $(z_1, \ldots, z_m)$ be local coordinates of $G_k(\mathbb{P}_N)$ centered at W_0, such that D_X is locally given by $z_1 = 0$. For a sufficiently small $\varepsilon > 0$, consider the loop

$$[0,1] \to U^X, \ t \mapsto (\varepsilon e^{2\pi\sqrt{-1}t}, 0, \ldots, 0)$$

defined in this coordinate neighborhood. Joining this loop with a path in U^X from the base point of U^X to $(\varepsilon, 0, \ldots, 0)$, we get an element of $\pi_1(U^X)$, which is called a lasso around D_X. The irreducibility of D_X implies that all lassos are conjugate to each other and $\pi_1(G_k(\mathbb{P}_N)) = 1$ implies that $\pi_1(U^X)$ is normally generated by a lasso. Since ϕ_X is a class function (see Section 1), the value of ϕ_X on any lasso is constant. This value can be computed from various invariants of X. For details, see Section 3.5. Nevertheless, the values of ϕ_X on an element other than lasso seems difficult to know.

The proof of Theorem 3.1.1 will be given in the next two sections. The following argument is a generalization of Sections 3 and 4 of [16].

3.2 Proof of the existence

It suffices to consider the case when the codimension of D_X is 1. The existence of ϕ_X is equivalent to $\rho_X^*[\tau_g] = 0 \in H^2(\pi_1(U^X); \mathbb{Q})$.

We first embed $H^2(\pi_1(U^X); \mathbb{Q})$ into another space. Let $V_{k+1,N+1}$ be the (complex) Stiefel manifold of all $(k+1)$–frames of $\mathbb{C}^{N+1}$. Regarding $\mathbb{P}_N$ as the projectivization of $\mathbb{C}^{N+1}$ we have the natural projection $q: V_{k+1,N+1} \to G_k(\mathbb{P}_N)$, which is a principal $\mathrm{GL}(k+1; \mathbb{C})$ bundle. Let $\tilde{D}_X = q^{-1}(D_X)$, $\tilde{U}^X = V_{k+1,N+1} \setminus \tilde{D}_X$, and $\tilde{E} = q^{-1}(E)$. For simplicity we use the same letter q for the restriction $q|_{\tilde{U}^X}: \tilde{U}^X \to U^X$.

We have the following short exact sequence with rational coefficients:

$$H^0(U^X) \xrightarrow{\cup c_1} H^2(U^X) \xrightarrow{q^*} H^2(\tilde{U}^X).$$

This is derived from the 5–term exact sequence of the principal bundle $q: \tilde{U}^X \to U^X$. Here, c_1 is the first Chern class, which is the restriction of a generator of $H^2(G_k(\mathbb{P}_N))$ to $H^2(U^X)$. But since D_X is of codimension 1, the first Chern class $c_1([D_X]) \in H^2(G_k(\mathbb{P}_N))$ is defined and is also a generator of $H^2(G_k(\mathbb{P}_N))$. The point here is that

$H^2(G_k(\mathbb{P}_N))$ is of rank 1. Clearly the restriction of $c_1([D_X])$ to $U^X = G_k(\mathbb{P}_N) \setminus D_X$ is zero, therefore c_1 is also zero. Thus, we have the injective homomorphism

$$(3.2.1) \qquad q^*\colon H^2(U^X;\mathbb{Q}) \hookrightarrow H^2(\widetilde{U}^X;\mathbb{Q}).$$

Let $\chi\colon \pi_1(\widetilde{U}^X) \to \pi_1(U^X)$ be the homomorphism between fundamental groups induced by q. Since for any space X there is the natural injection $H^2(\pi_1(X)) \to H^2(X)$ of the second cohomology with arbitrary coefficients, (3.2.1) implies that we have the injective homomorphism

$$(3.2.2) \qquad \chi^*\colon H^2(\pi_1(U^X);\mathbb{Q}) \hookrightarrow H^2(\pi_1(\widetilde{U}^X);\mathbb{Q})$$

induced by χ.

Next we show $\chi^* \rho_X^* [\tau_g] = 0 \in H^2(\pi_1(\widetilde{U}^X); \mathbb{Z})$. Let

$$\widetilde{\mathcal{W}} := \left\{ (x, \widetilde{W}) \in \mathbb{P}_N \times V_{k+1,N+1} : x \in X \cap q(\widetilde{W}) \right\}$$

and $\widetilde{p}_2 \colon \widetilde{\mathcal{W}} \to V_{k+1,N+1}$ be the second projection, and

$$\widetilde{\mathcal{C}^X} := \left\{ (x, \widetilde{W}) \in \mathbb{P}_N \times \widetilde{U}^X : x \in X \cap q(\widetilde{W}) \right\}.$$

The second projection $\widetilde{p}_X \colon \widetilde{\mathcal{C}^X} \to \widetilde{U}^X$ is a family of Riemann surfaces, which is the pullback of $p_X \colon \mathcal{C}^X \to U^X$ by q. The associated topological monodromy is $\widetilde{\rho}_X := \rho_X \circ \chi$.

We construct a 1–cochain $c\colon \pi_1(\widetilde{U}^X) \to \mathbb{Z}$ whose coboundary δc coincides with $\widetilde{\rho}_X^* \tau_g$. The point here is $V_{k+1,N+1} \setminus \widetilde{E}$ is 2–connected. This follows from the two facts: (1) the Stiefel manifold $V_{k+1,N+1}$ is $2(N-k)$–connected and $2(N-k) = 2n - 2 \geq 2$, and (2) the complex codimension of $\widetilde{E} \subset V_{k+1,N+1}$ is ≥ 2 (see Lemma 2.3.1). All of the spaces that we consider in the rest of this section as well as all of the maps are based, otherwise stated.

Construction of c Let $\ell\colon S^1 \to \widetilde{U}^X$ be a C^∞–loop, ie, a C^∞–map from S^1 to $\widetilde{U}^X$. Since $V_{k+1,N+1} \setminus \widetilde{E}$ is simply connected we can extend ℓ to a C^∞–map $\widetilde{\ell}\colon D^2 \to V_{k+1,N+1} \setminus \widetilde{E}$ which is transverse to $\widetilde{D}_X$. Here we make the identifications $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and $D^2 = \{z \in \mathbb{C} : |z| \leq 1\}$, and endow them the usual orientation: the orientation of D^2 is induced by that of $\mathbb{C}$ and S^1 goes around D^2 by counterclockwise manner.

By Proposition 2.3.8 the pullback $\widetilde{\ell}^* \widetilde{\mathcal{W}} := (q \circ \widetilde{\ell})^* \mathcal{W}$ has the natural structure of a compact oriented 4–dimensional C^∞–manifold with boundary. The orientation is

induced by the orientation of D^2 and that of the general fibers of $\widetilde{\ell}^*\widetilde{\mathcal{W}}$, which have the natural orientations as compact Riemann surfaces. Set

$$c([\ell]) := \mathrm{Sign}(\widetilde{\ell}^*\widetilde{\mathcal{W}}) \in \mathbb{Z}.$$

Here $[\ell] \in \pi_1(\widetilde{U}^X)$ is the element represented by ℓ, and the right hand side is the signature of $\widetilde{\ell}^*\widetilde{\mathcal{W}}$.

Proposition 3.2.3 *The above definition of c is well defined. The 1–cochain c is a class function on $\pi_1(\widetilde{U}^X)$ and $c(x^{-1}) = -c(x)$ for $x \in \pi_1(\widetilde{U}^X)$. We have $\delta c = -\widetilde{\rho}_X^*\tau_g$.*

Proof Let ℓ_0 and ℓ_1 be C^∞–loops in $\widetilde{U}^X$. Suppose that the elements of $\pi_1(\widetilde{U}^X)$ represented by them are conjugate to each other. Then there exists a C^∞–homotopy $H\colon S^1 \times [0,1] \to \widetilde{U}^X$ such that $H(\cdot, 0) = \ell_0$ and $H(\cdot, 1) = \ell_1$ (caution: we do not require that $H(\cdot, t)$ is a base preserving map for every $t \in [0,1]$). Identify the 2–sphere S^2 as

$$S^2 \cong (S^1 \times [0,1]) \cup (D^2 \times \{0\}) \cup (D^2 \times \{1\})$$

and take some extensions $\widetilde{\ell}_i\colon D^2 \times \{i\} \to V_{k+1,N+1} \setminus \widetilde{E}$ of ℓ_i for $i = 0, 1$. Then piecing H, $\widetilde{\ell}_0$, and $\widetilde{\ell}_1$ together, we can construct a C^∞–map $\widetilde{H}\colon S^2 \to V_{k+1,N+1} \setminus \widetilde{E}$ which is transverse to $\widetilde{D}_X$. Introduce the orientation of S^2 such that $D^2 \times \{0\} \hookrightarrow S^2$ is orientation preserving. Then $D^2 \times \{1\} \hookrightarrow S^2$ is orientation reversing and the pullback $\widetilde{H}^*\widetilde{\mathcal{W}}$ is a closed oriented 4–dimensional C^∞–manifold. Moreover, since $\pi_2(V_{k+1,N+1} \setminus \widetilde{E}) = 0$, $\widetilde{H}$ extends to a C^∞–map from the 3–ball to $V_{k+1,N+1} \setminus \widetilde{E}$ which is transverse to $\widetilde{D}_X$. Hence $\widetilde{H}^*\widetilde{\mathcal{W}}$ is the boundary of a 5–dimensional manifold and the signature of $\widetilde{H}^*\widetilde{\mathcal{W}}$ is zero. Now by the Novikov additivity of the signature we have

$$0 = \mathrm{Sign}(\widetilde{H}^*\widetilde{\mathcal{W}}) = \mathrm{Sign}(\widetilde{\ell}_0^*\widetilde{\mathcal{W}}) - \mathrm{Sign}(\widetilde{\ell}_1^*\widetilde{\mathcal{W}}).$$

This proves that c is well defined and c is a class function, ie, $c(xyx^{-1}) = c(y)$ for $x, y \in \pi_1(\widetilde{U}^X)$. Since changing the orientation of a manifold changes the sign of its signature, the property $c(x^{-1}) = -c(x)$ is clear.

We next prove that $\delta c = -\widetilde{\rho}_X^*\tau_g$, ie,

(3.2.4) $$c([\ell_0]) + c([\ell_1]) - c([\ell_0][\ell_1]) = -\widetilde{\rho}_X^*\tau_g([\ell_0], [\ell_1])$$

for any based C^∞–loops ℓ_0 and ℓ_1. Let D_i, $0 \leq i \leq 2$, be embedded three disjoint closed 2–disks in S^2 and we denote its boundary circle by S_i^1. Let $P := S^2 \setminus \coprod_{i=0}^2 \mathrm{Int}(D_i)$. Since P has the homotopy type of the bouquet $S^1 \vee S^1$, we can construct a C^∞–map $L\colon P \to \widetilde{U}^X$ such that the restriction of L to $S_i^1 \cong S^1$ is equal to ℓ_i for $i = 0, 1$. Then the restriction of L to S_2^1 is homotopic to the inverse

of the composition loop $\ell_0 \cdot \ell_1$. Notice that $\mathrm{Sign}(L^*\widetilde{\mathcal{W}})$ is equal to $\widetilde{\rho}_X^* \tau_g([\ell_0], [\ell_1])$. Take some extensions $\widetilde{\ell}_0$, $\widetilde{\ell}_1$, and $\widetilde{\ell_0 \cdot \ell_1}$ of ℓ_0, ℓ_1, and $\ell_0 \cdot \ell_1$, respectively. Then by piecing them and L together we have a C^∞–map $\widetilde{L} \colon S^2 \to V_{k+1,N+1} \setminus \widetilde{E}$ which is transverse to D_X. Again, the vanishing of $\pi_2(V_{k+1,N+1} \setminus \widetilde{E})$ implies that the signature of $\widetilde{L}^*\widetilde{\mathcal{W}}$ is zero. Finally, by the Novikov additivity we have

$$0 = \mathrm{Sign}(\widetilde{L}^*\widetilde{\mathcal{W}}) = \mathrm{Sign}(\widetilde{\ell}_0^*\widetilde{\mathcal{W}}) + \mathrm{Sign}(\widetilde{\ell}_0^*\widetilde{\mathcal{W}}) - \mathrm{Sign}(\widetilde{\ell_0 \cdot \ell_1}^*\widetilde{\mathcal{W}}) + \mathrm{Sign}(L^*\widetilde{\mathcal{W}}),$$

but this equation is equivalent to (3.2.4). This completes the proof. □

By Proposition 3.2.3, we have $\chi^* \rho_X^* [\tau_g] = 0 \in H^2(\pi_1(\widetilde{U}^X); \mathbb{Z})$. Combining this with the injection (3.2.2) we have $\rho_X^* [\tau_g] = 0 \in H^2(\pi_1(U^X); \mathbb{Q})$. This completes the proof of the existence of ϕ_X.

3.3 Proof of the uniqueness

The uniqueness of ϕ_X follows from the following:

Lemma 3.3.1 *The first cohomology group of $\pi_1(U^X)$ is trivial over rationals:*

$$H^1(\pi_1(U^X); \mathbb{Q}) = \mathrm{Hom}(\pi_1(U^X), \mathbb{Q}) = 0.$$

Proof It suffices to consider the case when the codimension of D_X is 1. Consider the following commutative diagram among (co)homology groups with integer coefficients:

$$
\begin{array}{ccccccc}
\mathbb{Z} \cong H_2(G_k(\mathbb{P}_N)) & \longrightarrow & H_2(G_k(\mathbb{P}_N), U^X) & \longrightarrow & H_1(U^X) & \longrightarrow & 0 \\
\cong \downarrow & & \downarrow \cong & & & & \\
H^{2\dim G_k(\mathbb{P}_N)-2}(G_k(\mathbb{P}_N)) & \xrightarrow{\ j^* \ } & H^{2\dim G_k(\mathbb{P}_N)-2}(D_X) \cong \mathbb{Z} & & & &
\end{array}
$$

The vertical isomorphisms are Poincaré duality. Note that $H^{2\dim G_k(\mathbb{P}_N)-2}(D_X) \cong \mathbb{Z}$ since D_X is irreducible. The first horizontal sequence is exact and is a part of the homology sequence of the pair $(G_k(\mathbb{P}_N), U^X)$ and j^* is induced by the inclusion $D_X \hookrightarrow G_k(\mathbb{P}_N)$. Then the generator of $H_2(G_k(\mathbb{P}_N))$ is mapped to a positive integer times the generator of $H^{2\dim G_k(\mathbb{P}_N)-2}(D_X)$, the fundamental class of D_X (this positive integer is denoted by $\deg D_X$ and will be studied in the next section). Thus $H_1(U^X)$, which is isomorphic to the abelianization of $\pi_1(U^X)$, is a cyclic group of finite order. This completes the proof. □

Now Theorem 3.1.1 is established.

3.4 Theory of Lefschetz pencils

In this section we recall the definition of the degree of an analytic subset in a Grassmannian and describe a method to compute the degree of $D_X \subset G_k(\mathbb{P}_N)$. This will be used to compute the value of ϕ_X on a lasso.

First we treat the case of classical dual varieties, namely when $n = 2$. Let $X \subset \mathbb{P}_N$ be a smooth projective surface. Then $G_k(\mathbb{P}_N) = G_{N-1}(\mathbb{P}_N)$ is nothing but the dual projective space $\mathbb{P}_N^\vee$ and D_X is the dual variety of X. Let L be a line of $\mathbb{P}_N^\vee$ avoiding the singular points of D_X and meeting D_X transversally. Note that generic lines of $\mathbb{P}_N^\vee$ satisfy this condition. We denote by i_L the inclusion $L \hookrightarrow \mathbb{P}_N^\vee$. Then as explained by Katz [15] or Lamotke [17, (1.6.3)], $i_L^* \mathcal{W} \to L$ is a *holomorphic Lefschetz fibration* with the set of critical values being $L \cap D_X$, in the following sense.

Definition 3.4.1 Let Y be a complex surface and C a compact Riemann surface. A proper surjective holomorphic map $f \colon Y \to C$ is called a holomorphic Lefschetz fibration if the number of critical values of f is finite and over each critical value, there exists only one critical point near which f locally looks like $(z_1, z_2) \mapsto z_1^2 + z_2^2$.

In particular $\deg D_X$, which is equal to the number $\sharp(L \cap D_X)$ by definition, is equal to the number of critical points of $i_L^* \mathcal{W} \to L$.

More generally for a projective variety $X \subset \mathbb{P}_N$, we can take a generic line L of $\mathbb{P}_N^\vee$ and consider the family of hyperplane sections $\{H \cap X\}_{H \in L}$ of X, parametrized by L. This construction is called *Lefschetz pencils* and very useful to study the topology of X.

We slightly generalize the above construction to the case of general n. We shall start by giving the definition of the *degree of an analytic subset in a Grassmannian* following [9, Chapter 3, Section 2-A].

Recall that $G_{p,q}$ is the Grassmannian of p–planes of $\mathbb{C}^q$. By a *line of $G_{p,q}$* is meant a curve embedded in $G_{p,q}$ which can be written as

(3.4.2) $$P_{NM} := \{W \in G_{p,q} : N \subset W \subset M\}$$

for some $(p-1)$–plane N and $(p+1)$–plane M satisfying $N \subset M$.

Let Z be an analytic subset of $G_{p,q}$. Take a line P_{MN} of $G_{p,q}$ such that

(1) $P_{MN} \cap S(Z) = \varnothing$, where $S(Z)$ denotes the set of singular points of Z,

(2) P_{MN} and Z meet transversally.

Then the intersection $P_{MN} \cap Z$ consists of finitely many points. We define $\deg Z \in \mathbb{Z}$ by

$$\deg Z := \sharp(P_{MN} \cap Z)$$

where P_{MN} satisfies the above two conditions. When Z is a hypersurface this number is positive (see [10, page 64]), and has the following topological interpretation. Let $c_1([Z]) \in H^2(G_{p,q}; \mathbb{Z})$ be the first Chern class of the line bundle over $G_{p,q}$ determined by Z and $[P_{NM}] \in H_2(G_{p,q}; \mathbb{Z})$ the homology class represented by the embedded 1–dimensional projective space P_{NM}. Note that $[P_{NM}]$ is a generator of $H_2(G_{p,q}; \mathbb{Z}) \cong \mathbb{Z}$. Then $\deg Z$ is equal to the result of the Kronecker pairing $\langle c_1([Z]), [P_{NM}] \rangle$. If $n = 2$, this degree coincides with the usual definition of the degree of a projective hypersurface in $G_{N-1}(\mathbb{P}_N) = \mathbb{P}_N^\vee$. When the codimension of Z is ≥ 2, $\deg Z = 0$.

We remark that generic lines of $G_{p,q}$ satisfy the above two conditions in the following sense. Let us consider the space parametrizing all lines of $G_{p,q}$; namely, let

$$\mathcal{L}_{p,q} := \left\{ (N, M) \in G_{p-1,q} \times G_{p+1,q} : N \subset M \right\}.$$

Then the set of $(N, M) \in \mathcal{L}_{p,q}$ such that P_{NM} satisfying the above two conditions is nonempty and Zariski open in $\mathcal{L}_{p,q}$. This is proved by an application of Sard's lemma for varieties to the second projection $\{(z, (N, M)) \in Z \times \mathcal{L}_{p,q} : z \in P_{NM}\} \to \mathcal{L}_{p,q}$.

Let us return to our setting: $X \subset \mathbb{P}_N$ is a n–dimensional smooth projective variety, D_X is the k–th associated variety. Let $E_X = E \subset D_X$ be as in Lemma 2.3.1 when the codimension of D_X is 1, $E_X = \varnothing$ when the codimension of D_X is ≥ 2. Here we put the subscript X to E to indicate its dependence on X. Let

$$\mathcal{L}_k(\mathbb{P}_N) := \{ (N, M) \in G_{k-1}(\mathbb{P}_N) \times G_{k+1}(\mathbb{P}_N) : N \subset M \}.$$

This is clearly isomorphic to $\mathcal{L}_{k+1, N+1}$. For $(N, M) \in \mathcal{L}_k(\mathbb{P}_N)$, the corresponding line $P_{NM} \subset G_k(\mathbb{P}_N)$ is defined by the same way as (3.4.2).

The existence of Lefschetz pencils for the case of general n is stated as follows.

Theorem 3.4.3 (Existence of Lefschetz pencils) *Let P_{NM} be a line of $G_k(\mathbb{P}_N)$ not meeting E_X and meeting D_X transversally. Then the projection $i_{NM}^* \mathcal{W} \to P_{NM}$, where $i_{NM} : P_{NM} \hookrightarrow G_k(\mathbb{P}_N)$ denotes the inclusion, is a holomorphic Lefschetz fibration in the sense of Definition 3.4.1. Moreover $\deg D_X$ is equal to the number of critical points of $i_{NM}^* \mathcal{W} \to P_{NM}$.*

By Theorem 2.3.12, the remaining to show is the number of critical points over each critical value is just one. If $n = 2$, there is nothing to prove as remarked before Definition 3.4.1. We only remark that in the proof of (1.6.3) of [17], the biduality theorem plays a key role.

To reduce the case of general n to the case of $n = 2$, we will cut X with a generic $(k+1)$–plane. The result will be a smooth projective surface in the $(k+1)$–plane. We prepare some notation. P_{NM} can be considered as a line of $M^\vee$, the dual projective space of M. Then we write it by L_N. For $M \in G_k(\mathbb{P}_N)$, let $X' := M \cap X \subset M$. If M meets X transversally, X' is a smooth surface in M. Then we can consider $D_{X'}$ and $E_{X'}$ in $M^\vee$.

Lemma 3.4.4 *There exists a point* $(N, M) \in \mathcal{L}_k(\mathbb{P}_N)$ *such that*

(1) *the* $(k+1)$–*plane* M *meets* X *transversally (hence* X' *is a smooth projective surface),*

(2) *the line* P_{NM} *does not meet* E_X *and meets* D_X *transversally,*

(3) *the line* L_N *does not meet* $E_{X'}$ *and meets* $D_{X'}$ *transversally.*

Proof The set of points in $\mathcal{L}_k(\mathbb{P}_N)$ satisfying the conditions 1 and 2 is nonempty and Zariski open in $\mathcal{L}_k(\mathbb{P}_N)$. Let (N', M) be a point in this set. Since the set of lines of $M^\vee$ not meeting $E_{X'}$ and meeting $D_{X'}$ transversally is nonempty and Zariski open in the space of all lines of $M^\vee$, there exists a line L_N near $L_{N'}$ such that (N, M) satisfies all the three conditions. $\square$

Implications of the Lemma Let (N, M) be as in [Lemma 3.4.4]. We have the natural inclusion $\iota_M : M^\vee \hookrightarrow G_k(\mathbb{P}_N)$. Since M meets X transversally, for $H \in M^\vee$ the conditions $H \in D_{X'}$ and $\iota_M(H) \in D_X$ are equivalent. Therefore we have the injection

$$(3.4.5) \qquad \iota_M|_{U^{X'}} : U^{X'} \hookrightarrow U^X,$$

where $U^{X'} = M^\vee \setminus D^{X'}$, and the bijection $\iota_M|_{L_N \cap D_{X'}} : L_N \cap D_{X'} \overset{\cong}{\to} P_{NM} \cap D_X$. In particular, D_X is a hypersurface if and only if $D_{X'}$ is a hypersurface and we have $\deg D_X = \deg D_{X'}$.

For simplicity we identify $L_N \cap D_{X'}$ with $P_{NM} \cap D_X$ and write it by D_{NM}. Let $U_{NM} = P_{NM} \setminus D_{NM}$. By the inclusion $U_{NM} \hookrightarrow U^X$ (resp. $U_{NM} \hookrightarrow U^{X'}$), any loop in U_{NM} going once around a point of D_{NM} is mapped to a lasso around D_X (resp. $D_{X'}$), hence a lasso around $D_{X'}$ is mapped to a lasso around D_X by the map [(3.4.5)]. Consider the group homomorphism $j_M : \pi_1(U^{X'}) \to \pi_1(U^X)$ induced by [(3.4.5)]. The uniqueness of $\phi_{X'}$ shows that $j_M^* \phi_X$ coincides with $\phi_{X'}$. Thus the value of ϕ_X on a lasso around D_X coincides with the value of $\phi_{X'}$ on a lasso around $D_{X'}$.

In this way we can reduce the computation of $\deg D_X$ or the value of ϕ_X on a lasso around D_X to the case of $n = 2$.

Proof of [Theorem 3.4.3] Let (N, M) be as in [Lemma 3.4.4]. Let $i'_N : L_N \to M^\vee$ be the inclusion and $\mathcal{W}' := \{(x, W) \in M \times M^\vee : x \in X' \cap W\}$. We can consider the

pullback $(i'_N)^* \mathcal{W}'$. Since $\dim X' = 2$ the remark right after the statement of Theorem 3.4.3 applies, so $(i'_N)^* \mathcal{W}' \to L_N$ is a holomorphic Lefschetz fibration. Therefore, $i^*_{NM} \mathcal{W} \to P_{NM}$ is also a holomorphic Lefschetz fibration because ι_M induces the isomorphism

$$(i'_N)^* \mathcal{W}' \xrightarrow{\cong} i^*_{NM} \mathcal{W}$$

between the families of algebraic curves over $L_N = P_{NM}$. Thus we have proved that there exists a line $P_0 = P_{NM}$ of $G_k(\mathbb{P}_N)$ not meeting E_X and meeting D_X transversally such that the projection $i^*_{NM} \mathcal{W} \to P_{NM}$ is a holomorphic Lefschetz fibration.

Let P_1 be a line not meeting E_X and meeting D_X transversally. For $j = 0, 1$, we denote the inclusion map $P_j \hookrightarrow G_k(\mathbb{P}_N)$ by i_j. Let $\mathcal{V}$ be the space of lines of $G_k(\mathbb{P}_N)$ not meeting E_X and meeting D_X transversally. This is nonempty and Zariski open, hence connected. Thus there exists a differentiable path in $\mathcal{V}$ joining P_0 and P_1, inducing a deformation equivalence of class C^∞ between $i_0^* \mathcal{W}$ and $i_1^* \mathcal{W}$ as a family of algebraic curves over 1–dimensional projective space. We already know $i_0^* \mathcal{W} \to P_0$ is a holomorphic Lefschetz fibration, so $i_1^* \mathcal{W} \to P_1$ is also a holomorphic Lefschetz fibration. $\square$

Taking into account that $D_X \setminus E_X$ is connected, similar argument shows the following:

Corollary 3.4.6 *In Theorem 2.3.12, the number of critical points of $f_\iota \colon \iota^* \mathcal{W} \to \Delta$ is 1. The singular fiber $f_\iota^{-1}(0)$ has just one nodal singularity and its topological type does not depend on the choice of ι.*

3.5 Computations

In this section we will give a formula for the value of ϕ_X on a lasso around D_X from the data of various invariants of X. In view of the discussion of the implications of Lemma 3.4.4 in Section 3.4, we may focus on the case $n = 2$.

First we review theory of Lefschetz pencils following [17]. Let $X \subset \mathbb{P}_N$ be a smooth projective surface and L a generic line of $\mathbb{P}_N^\vee$, as in the beginning of Section 3.4. Let $D_L = L \cap D_X$, $U_L = L \setminus D_L$ and $X_H = H \cap X$ for $H \in L$. We have a family of Riemann surfaces over U_L by restricting $i_L^* \mathcal{W} \to L$. Choosing a base point $H_0 \in U_L$, let ρ_X be the associated topological monodromy. The source of ρ_X is the fundamental group $\pi_1(U_L) = \pi_1(U_L, H_0)$.

Picard–Lefschetz formula For $H' \in D_L$ choose a path ℓ from H_0 to H' and let σ be the element of $\pi_1(U_L)$ represented by a loop going to a point nearby H' along ℓ, then going once around H' by counterclockwise manner and then coming back along ℓ.

Then the famous Picard–Lefschetz formula says that $\rho_X(\sigma)$ is the *inverse* (recall our conventions about monodromies) of the right hand Dehn twist along some simple closed curve C_σ, called the *vanishing cycle*, on X_{H_0}. The adjective "vanishing" comes in because looking at the fiber $H \cap X$ when H moves along ℓ, $X_{H'}$ looks like obtained from X_{H_0} by pinching C_σ into a point.

Let V be the submodule of $H_1(X_{H_0})$ generated by all the vanishing cycles. Homology with coefficients in some principal ideal domain is considered. Then the equality

(3.5.1) $$V = \mathrm{Ker}(i_* \colon H_1(X_{H_0}) \to H_1(X)),$$

where i_* is induced by the inclusion $X_{H_0} \hookrightarrow X$, holds. See [17, (3.8.2)]. In particular, if $H_1(X) = 0$, the vanishing cycles generate the homology of the reference fiber X_{H_0}.

Lemma 3.5.2 *Suppose the genus of X_{H_0} is positive and for some principal ideal domain R, the rank of $H_1(X) = H_1(X; R)$ is less than twice the genus of X_{H_0}. Then every singular fiber of $i_L^* \mathcal{W} \to L$, ie, the inverse image of a point of D_L, is irreducible.*

Proof First remark that for any choice of H' and ℓ, $\sigma \in \pi_1(U_L)$ is mapped to a lasso around D_X by the homomorphism $\pi_1(U_L) \to \pi_1(U^X)$ induced by the inclusion. Thus for any two vanishing cycles the Dehn twists along them are conjugate to each other in the mapping class group of X_{H_0}.

Suppose there exists a reducible fiber. This means that there exists a vanishing cycle which is a separating simple closed curve. Then all the vanishing cycles are separating by the remark above. Since any separating simple closed curve is zero as a homology class, this implies $V = 0$. But by the assumption and (3.5.1) we also have $V = H_1(X_{H_0}) \neq 0$, a contradiction. This completes the proof. $\square$

Proposition 3.5.3 *Let $X \subset \mathbb{P}_N$ be a smooth projective surface and g the genus of a generic hyperplane section $H \cap X$, $H \in U^X$. Assume that $g > 0$ and the rank of $H_1(X; R)$ is less than $2g$ for some principal ideal domain R. Suppose D_X is a hypersurface and let $\sigma_X \in \pi_1(U^X)$ be a lasso around D_X. Then we have*

$$\phi_X(\sigma_X) = \frac{\mathrm{Sign}\, X - \deg X}{\chi(X) + \deg X - 2(2 - 2g)}.$$

Here, $\mathrm{Sign}\, X$ is the signature of X as a closed oriented 4–manifold and $\chi(X)$ is the Euler–Poincaré characteristic of X, and $\deg X$ is the usual degree of X (ie, the number of intersecting points with a generic complementary dimensional plane to X).

Proof Let L be a generic line of $\mathbb{P}_N^\vee$ as in the beginning of this section. As in [17, (1.6.1)] the axis $A = \bigcap_{H \in L} H$ of the pencil meets X transversally, and $i_L^* \mathcal{W}$ is the

blow up of X at the $\deg X$ points $A \cap X$ hence diffeomorphic to the connected sum $X \# (\deg X)\overline{\mathbb{P}}_2$. Therefore we have

$$(3.5.4) \qquad \operatorname{Sign}(i_L^* \mathcal{W}) = \operatorname{Sign} X - \deg X$$

and $\chi(i_L^* \mathcal{W}) = \chi(X) + \deg X$.

Let $D_L = \{H_1, \ldots, H_d\}$, where $d = \deg D_X$ and for $1 \le i \le d$, let $\sigma_i \in \pi_1(U_L)$ be the element obtained by substituting H_i for H' in the definition of σ; see the beginning of this section. As elements of $\pi_1(U^X)$, all σ_i are lassos around D_X.

Let $D_i \subset L$ be a small closed 2–disk around H_i. We write by f_L the projection $i_L^* \mathcal{W} \to L$ and write $X_i = f_L^{-1}(D_i)$. Let $X_0 = X \setminus \coprod_i \operatorname{Int} X_i$. By Lemma 3.5.2, $f_L^{-1}(H_i)$ is irreducible hence the signature of X_i is zero. Using the Novikov additivity, we have $\operatorname{Sign}(i_L^* \mathcal{W}) = \operatorname{Sign} X_0$. By Meyer's signature formula [20, Satz 1] and $\rho_X^* \tau_g = \delta\phi_X$, we have

$$(3.5.5) \qquad \operatorname{Sign}(i_L^* \mathcal{W}) = \operatorname{Sign} X_0 = \sum_{i=1}^{d} \phi_X(\sigma_i) = d\phi_X(\sigma_X).$$

On the other hand since all the singular fibers have one nodal singularity, there are d singular fibers with Euler contribution $+1$ (see [5, (11.4) Proposition]), thus

$$(3.5.6) \qquad d = \chi(i_L^* \mathcal{W}) - 2(2 - 2g) = \chi(X) + \deg X - 2(2 - 2g).$$

The proposition follows from (3.5.4), (3.5.5) and (3.5.6). $\qquad\qquad\qquad\square$

Note that by (3.5.6) we can express $\deg D_X$ in terms of $\chi(X)$, $\deg X$, and g. The genus g is expressed as follows. For $H \in U^X$, let $C = H \cap X$. By the adjunction formula we have $c_1(C) = c_1(X)|_C - h$, where h is the hyperplane class, thus

$$2 - 2g = \chi(C) = \langle c_1(C), [C] \rangle = \langle c_1(X)h - h^2, [X] \rangle = \langle c_1(X)h, [X] \rangle - \deg X.$$

Example 3.5.7 Let $m \ge 1$ and $n_1, \ldots, n_m \ge 2$ be integers and let $X \subset \mathbb{P}_{m+2}$ be a smooth complete intersection of type $(n_1, \ldots, n_m)$. Namely X is given as the zero set of some homogeneous polynomials $f_1, \ldots, f_m$ where f_i is of degree n_i.

Proposition 3.5.8 Let X be as above and assume that $(m, n_1, \ldots, n_m) \ne (1, 2)$. Then D_X is a hypersurface. Let $\sigma_X \in \pi_1(U^X)$ be a lasso around D_X. We have

$$\phi_X(\sigma_X) = \frac{m - \sum_{i=1}^{m} n_i^2}{3 \left(\dfrac{m^2 + m}{2} + \sum_{i=1}^{m} n_i^2 - (m+1)\sum_{i=1}^{m} n_i + \sum_{i<j} n_i n_j \right)}.$$

Proof We have $\deg X = n_1 \cdots n_m$ and using the adjunction formula we can compute

$$\chi(X) = c_2(X)$$

(3.5.9)
$$= n_1 \cdots n_m \left(\binom{m+3}{2} + \sum_{i=1}^{m} n_i^2 - (m+3) \sum_{i=1}^{m} n_i + \sum_{i<j} n_i n_j \right),$$

(3.5.10)
$$\text{Sign } X = \frac{n_1 \cdots n_m}{3} \left(m + 3 - \sum_{i=1}^{m} n_i^2 \right).$$

For $H \in U^X$, $C = H \cap X$ is a smooth complete intersection of type $(n_1, \ldots, n_m, 1)$. Using the adjunction formula we have

$$2 - 2g = \chi(C) = n_1 \cdots n_m \left(m + 2 - \sum_{i=1}^{m} n_i \right),$$

and following the argument in the proof of [Proposition 3.5.3](), $\deg D_X$ is given by

$$\deg D_X = n_1 \cdots n_m \left(\frac{m^2 + m}{2} + \sum_{i=1}^{m} n_i^2 - (m+1) \sum_{i=1}^{m} n_i + \sum_{i<j} n_i n_j \right).$$

We claim that $\deg D_X$ is positive. If $m = 1$, $\deg D_X = n_1(n_1 - 1)^2 > 0$. If $m \geq 2$, Using the inequality

(3.5.11)
$$\sum_{i=1}^{m} n_i^2 \geq \frac{2}{m-1} \sum_{i<j} n_i n_j$$

for $n_i \geq 0$ (this is easily derived from the geometric-arithmetic mean inequality), we have

$$\deg D_X \geq n_1 \cdots n_m \left(\frac{m^2 + m}{2} + \frac{m+1}{m-1} \sum_{i<j} n_i n_j - (m+1) \sum_{i=1}^{m} n_i \right)$$

$$= n_1 \cdots n_m \frac{m+1}{m-1} \sum_{i<j} (n_i - 1)(n_j - 1).$$

Thus in any case $\deg D_X > 0$, ie, D_X is a hypersurface. Also we can show $\chi(C) \leq 0$ hence $g > 0$ except for the case $m = 1$ and $n_1 = 2$. Finally, X is simply connected. This follows from the Zariski theorem of Lefschetz type; see [17, (8.1.1)]. Now [Proposition 3.5.3]() can be applied, and combining the above computations all together we have the result. □

The next example is a generalization of the above, but it will illustrate that for a fixed variety, how the value of the Meyer function on a lasso depends on a choice of its projective embedding.

Example 3.5.12 Let $m \geq 0$, $n_1, \ldots, n_m \geq 2$, $n \geq 2$, and $d \geq 1$ be integers. When $m = 0$, we assume that $d \geq 2$. Let $v_d \colon \mathbb{P}_{m+n} \hookrightarrow \mathbb{P}_N$ be the Veronese embedding of degree d. Here,

$$N = \binom{n+m+d}{d} - 1.$$

Let X be the v_d–image of a smooth complete intersection in $\mathbb{P}_{m+n}$ of type $(n_1, \ldots, n_m)$. When $m = 0$, X is by definition the v_d–image of $\mathbb{P}_n$.

Proposition 3.5.13 Let X be as above and assume that $(d, m, n_1, \ldots, n_m) \neq (1, 1, 2)$ and $(n, d, m) \neq (2, 2, 0)$. Then the k–th associated variety D_X is a hypersurface. Let $\sigma_X \in \pi_1(U^X)$ be a lasso around D_X. We have

$$\phi_X(\sigma_X) = \frac{\alpha_X}{\beta_X},$$

where $\alpha_X = \dfrac{m+n+1 - \sum_{i=1}^m n_i^2 - (n+1)d^2}{3}$

and $\beta_X = \dbinom{m+n+1}{2} + \sum_{i=1}^m n_i^2 + \sum_{i<j} n_i n_j - (m+n+1)\left(\sum_{i=1}^m n_i + nd\right)$
$$+ nd \sum_{i=1}^m n_i + \frac{(n^2+n)d^2}{2}.$$

Proof Let $(N, M) \in \mathcal{L}_k(\mathbb{P}_N)$ be as in Lemma 3.4.4 and $X' = M \cap X$. We may focus on $X' \subset M$. We will show that $D_{X'}$ is a hypersurface and compute the value $\phi_{X'}(\sigma_{X'})$, which must coincide with $\phi_X(\sigma_X)$, where $\sigma_{X'} \in \pi_1(U^{X'})$ is a lasso around $D_{X'}$.

First of all, the pullback $v_d^{-1}(X')$ is a smooth complete intersection in $\mathbb{P}_{m+n}$ of type

$$(n_1, \ldots, n_m, \underbrace{d, \ldots, d}_{n-2}).$$

Thus X' is simply connected, and the invariants $\chi(X')$ and $\mathrm{Sign}\, X'$ can be computed from (3.5.9), (3.5.10). Also, we have $\deg X' = \deg X = n_1 \cdots n_m d^n$. From these we can see that $\mathrm{Sign}\, X' - \deg X' = n_1 \cdots n_m d^{n-2} \alpha_X$. For $W \in U^{X'} \subset U^X$, $C := W \cap X = W \cap X'$ is a smooth complete intersection in $\mathbb{P}_{m+n}$ of type

$$(n_1, \ldots, n_m, \underbrace{d, \ldots, d}_{n-1}).$$

Thus the genus g of C is seen by

$$2 - 2g = \chi(C) = n_1 \cdots n_m d^{n-1} \left(m + n + 1 - \sum_{i=1}^{m} n_i - (n-1)d \right).$$

It is easy to see that under the assumption, we have $\chi(C) \leq 0$ hence $g > 0$. Using (3.5.6) and our knowledge of $\chi(X')$, $\deg X'$, and $\chi(C)$ gives

$$\deg D_{X'} = n_1 \cdots n_m d^{n-2} \beta_X.$$

We claim that $\deg D_{X'}$ is positive. Now we have the inequality

$$nd^2 + \sum_{i=1}^{m} n_i^2 \geq \frac{2}{m+n-1} \left(\sum_{i<j} n_i n_j + nd \sum_{i=1}^{m} n_i + \binom{n}{2} d^2 \right),$$

the same kind as (3.5.11). Using this, we have

$$\deg D_{X'} \geq n_1 \cdots n_m d^{n-2} \frac{m+n+1}{m+n-1} \left(\sum_{i<j} (n_i - 1)(n_j - 1) + n \sum_{i=1}^{m} (n_i - 1)(d - 1) \right.$$
$$\left. + \binom{n}{2} (d-1)^2 \right).$$

This shows $\deg D_{X'} > 0$ except for the case $d = 1$ and $m = 1$. In this case, we have $\deg D_{X'} = n_1(n_1 - 1)^2 > 0$. Thus $D_{X'}$ is a hypersurface, so is D_X. Applying Proposition 3.5.3, we have $\phi_X(\sigma_X) = \phi_{X'}(\sigma_{X'}) = (\text{Sign } X' - \deg X')/\deg D_{X'} = \alpha_X/\beta_X$, as desired. $\square$

3.6 Bounded cohomology of $\pi_1(U^X)$

For a group G, we denote by $H_b^*(G; \mathbb{R})$ the bounded cohomology group of G. Namely, $H_b^*(G; \mathbb{R})$ is the cohomology of the cochain complex of $\mathbb{R}$–valued *bounded* cochains of G. In this section we show that the second bounded cohomology of $\pi_1(U^X)$ is nontrivial under a certain mild condition.

Proposition 3.6.1 *Let $X \subset \mathbb{P}_N$ be a smooth projective variety of dimension ≥ 2 such that D_X is a hypersurface. Suppose the value of ϕ_X on a lasso around D_X is neither equal to 0 nor -1. Then the bounded cohomology $H_b^2(\pi_1(U^X); \mathbb{R})$ is nontrivial and the natural comparison map $H_b^2(\pi_1(U^X); \mathbb{R}) \to H^2(\pi_1(U^X); \mathbb{R})$ is not injective.*

We need a lemma.

Lemma 3.6.2 *Let $T \in \Gamma_g$ be the right hand Dehn twist along a nonseparating simple closed curve on Σ_g. Then for any integer $n \geq 1$, we have*

$$\tau_g(T^{-1}, T^{-n}) = -1.$$

Proof We use the description (1.0.4) of τ_g. By the formulas (12) and (13) of [20], it suffices to prove $\mathrm{Sign}(V_{A,A^n}, \langle \ , \ \rangle_{A,A^n}) = -1$ where

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

corresponding to the inverse of the right Dehn twist along a nonseparating simple closed curve on the torus. We have

$$V_{A,A^n} = \left\{ (x, y) \in \mathbb{R}^2 \oplus \mathbb{R}^2 : \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & -n \\ 0 & 0 \end{pmatrix} y = 0 \right\},$$

thus the vectors $((1,0)^t, (0,0)^t)$, $((0,0)^t, (1,0)^t)$, and $((0,n)^t, (0,1)^t)$ form a basis for V_{A,A^n}. The presentation matrix of $\langle \ , \ \rangle_{A,A^n}$ with respect to this basis is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -n(n+1) \end{pmatrix}.$$

This completes the proof.   □

Proof of Proposition 3.6.1 Note that τ_g is a bounded 2–cocycle of Γ_g. More precisely, for $f_1, f_2 \in \Gamma_g$ we have $|\tau_g(f_1, f_2)| \leq 4g$. Thus $\rho_X^* \tau_g$ is also a bounded 2–cocycle. Since ϕ_X is a unique 1–cochain cobounding $\rho_X^* \tau_g$, it suffices to show that ϕ_X is unbounded.

Let σ_X be a lasso around D_X. By the Picard–Lefschetz formula, $\rho_X(\sigma_X)$ is the inverse of the right hand Dehn twist along a simple closed curve. We claim that this curve is nonseparating. For, if this is separating, $\rho_X(\sigma_X) \in \Gamma_g$ does act trivially on the homology of Σ_g. Combining this with the fact that $\pi_1(U^X)$ is normally generated by σ_X, we deduce that the image $\rho_X(\pi_1(U^X))$ acts trivially on the homology of Σ_g. Hence $\rho_X^* \tau_g$ is zero as a cocycle. Since $\delta\phi_X = \rho_X^* \tau_g = 0$ and $\phi_X(\sigma_X) \neq 0$ it follows that ϕ_X is a nontrivial homomorphism from $\pi_1(U^X)$ to $\mathbb{Q}$, contradicting to Lemma 3.3.1.

Now by $\delta\phi_X = \rho_X^* \tau_g$ and Lemma 3.6.2, we have

$$\phi_X(\sigma_X^n) = n\phi_X(\sigma_X) - \sum_{i=1}^{n-1} \tau_g(\rho_X(\sigma_X), \rho_X(\sigma_X^i)) = n\phi_X(\sigma_X) + n - 1$$

for $n \geq 1$. Since $\phi_X(\sigma_X) \neq -1$, this shows the unboundedness of ϕ_X. $\qquad\square$

It is known that if a discrete group is amenable, then its bounded cohomology vanishes in positive degrees (see Gromov [11]). Thus:

Corollary 3.6.3 *Let $X \subset \mathbb{P}_N$ be a smooth projective variety satisfying the hypothesis of Proposition 3.6.1. Then the fundamental group $\pi_1(U^X)$ is not amenable.*

As an example, when X is the one of those in Proposition 3.5.13, we can check that $\alpha_X < 0$ and $\alpha_X + \beta_X > 0$. Therefore Proposition 3.6.1 can be applied to this situation.

4 Applications to local signatures

4.1 An approach to local signatures via Meyer functions

Let $\mathbb{M}_g$ be the moduli space of compact Riemann surfaces of genus g and $\mathcal{A}$ a subset of $\mathbb{M}_g$. We introduce the notion of an $\mathcal{A}$–fibration and a local signature with respect to $\mathcal{A}$.

Definition 4.1.1 Let B be a topological space.

(1) A triple $\xi = (\mathcal{C}, p, B)$ is called an $\mathcal{A}$–*family* (on B) if $p\colon \mathcal{C} \to B$ is a continuous family of compact Riemann surfaces with each fiber being an element of $\mathcal{A}$.

(2) Let ξ_0, ξ_1 be $\mathcal{A}$–families on B. They are called *isotopic* if there exists an $\mathcal{A}$–family ξ on $B \times [0, 1]$ such that for $i = 0, 1$, the restriction of ξ to $B \times \{i\}$ is isomorphic to ξ_i as continuous family of Riemann surfaces on B, where $B \times \{i\}$ is identified with B by $(b, i) \mapsto b$.

(3) We denote by $\mathcal{A}(B)$ the set of isotopy classes of $\mathcal{A}$–families over B.

Let $\xi = (\mathcal{C}, p, B)$ be an $\mathcal{A}$–family and $\psi\colon B' \to B$ a continuous map. By taking the fiber product of ψ and p, the pullback $\psi^*\xi$ by ψ is naturally defined as an $\mathcal{A}$–family on B'. In this way we get a category of $\mathcal{A}$–families, which we denote by $\mathcal{A}'$. Moreover this association induces the map $\eta_{\xi, B'}\colon [B', B] \to \mathcal{A}(B')$, where $[B', B]$ is the set of homotopy classes of continuous maps from B' to B.

Definition 4.1.2 An $\mathcal{A}$–family $\xi_u = (\mathcal{C}_u, p_u, B_u)$ over a path connected space B_u having the homotopy type of a CW–complex is called *universal* if $\eta_{\xi_u, B}$ is bijective for any topological space B having the homotopy type of a CW–complex. We denote by $\rho_u \colon \pi_1(B_u) \to \Gamma_g$ the topological monodromy of $p_u \colon \mathcal{C}_u \to B_u$.

A universal $\mathcal{A}$–family is uniquely determined up to isotopy if it exists: if $\xi'_u = (\mathcal{C}'_u, p'_u, B'_u)$ is another universal $\mathcal{A}$–family then there exist continuous maps $\psi \colon B_u \to B'_u$ and $\psi' \colon B'_u \to B_u$ such that $\psi^*\xi'_u$ and ξ_u (resp. $\psi'^*\xi_u$ and ξ'_u) are isotopic. In particular B_u and B'_u are homotopy equivalent.

In some situations as we will see, we can construct a universal $\mathcal{A}$–family from a certain $\mathcal{A}$–family with group action. The following proposition is used to verify the universality of such a family. To state the proposition we prepare a terminology. Let $\xi^0 = (\mathcal{C}^0, p^0, B^0)$ be a $\mathcal{A}$–family on a connected C^∞–manifold B^0, and let $\mathcal{G}$ be a Lie group acting on $\mathcal{C}^0$ and B^0 from the left such that p^0 is $\mathcal{G}$–equivariant. Let $\mathcal{G}'$ be the category defined as follows: the objects consist of (P, π, B, E) such that $\pi \colon P \to B$ is a principal $\mathcal{G}$–bundle on B (the $\mathcal{G}$–action on P being from the left) and $E \colon P \to B^0$ is a $\mathcal{G}$–equivariant map, and the morphisms from (P, π, B, E) and (P', π', B', E') are the bundle maps from P to P' compatible with E and E'.

Proposition 4.1.3 (A criterion for universality) *Let $\mathcal{A}$ be a subset of $\mathbb{M}_g$ and $\xi^0 = (\mathcal{C}^0, p^0, B^0)$ an $\mathcal{A}$–family as above. Suppose there is a covariant functor from $\mathcal{A}'$ to $\mathcal{G}'$ associating an $\mathcal{A}$–family $\xi = (\mathcal{C}, p, B)$ with $(P(\xi), \pi, B, E_\xi)$, and satisfying the following conditions:*

(1) *$E_\xi^*\xi^0$ and $\pi^*\xi$ are isomorphic as continuous families of Riemann surfaces on $P(\xi)$.*

(2) *As for the object associated to ξ^0, we can take a trivial $\mathcal{G}$–bundle $P(\xi^0) = \mathcal{G} \times B^0$ and a $\mathcal{G}$–equivariant map E_{ξ^0} such that $E_{\xi^0}(g, b) = g \cdot b$. Moreover, for any $g \in \mathcal{G}$ the bundle map $\bar{g} \colon P(\xi^0) \to P(\xi^0)$ induced by the maps $B^0 \to B^0$, $b \mapsto g \cdot b$ and $\mathcal{C}^0 \to \mathcal{C}^0$, $c \mapsto g \cdot c$ is given by $\bar{g}(g', b) = (g'g^{-1}, g \cdot b)$.*

Let $E\mathcal{G} \to B\mathcal{G}$ be a universal principal $\mathcal{G}$–bundle (the $\mathcal{G}$–action on $E\mathcal{G}$ being from the right). Taking the Borel constructions $B_\mathcal{G}^0 = E\mathcal{G} \times_\mathcal{G} B^0$ and $\mathcal{C}_\mathcal{G}^0$ we obtain an $\mathcal{A}$–family $\xi_\mathcal{G}^0 = (\mathcal{C}_\mathcal{G}^0, p_\mathcal{G}^0, B_\mathcal{G}^0)$. Then, $\xi_\mathcal{G}^0$ is a universal $\mathcal{A}$–family.

Proof Let B be a space having the homotopy type of a CW–complex. For simplicity we write $\eta = \eta_{\xi_\mathcal{G}^0, B}$. We construct a candidate for the inverse of η. Let $\xi = (\mathcal{C}, p, B)$ be an $\mathcal{A}$–family on B. Take a principal $\mathcal{G}$–bundle $P = P(\xi)$ and a $\mathcal{G}$–equivariant map $E = E_\xi$ associated to ξ. Considering the Borel construction $P_\mathcal{G} = E\mathcal{G} \times_\mathcal{G} P$, let

$T\colon P_{\mathcal{G}} \to B$ be the map induced from the projection $\pi\colon P \to B$. This is an $E\mathcal{G}$–bundle, thus by Dold's theorem it has a section: a map $\zeta\colon B \to P_{\mathcal{G}}$ such that $T \circ \zeta = \mathrm{id}_B$. Let $E_{\mathcal{G}}\colon P_{\mathcal{G}} \to B_{\mathcal{G}}^0$ be the map induced from E. Now the isomorphism $E^*\xi^0 \cong \pi^*\xi$ induces the isomorphism $E_{\mathcal{G}}^*\xi_{\mathcal{G}}^0 \cong T^*\xi$ and $E_{\mathcal{G}} \circ \zeta$ is a continuous map from B to $B_{\mathcal{G}}^0$ such that

$$(E_{\mathcal{G}} \circ \zeta)^*\xi_{\mathcal{G}}^0 = \zeta^* E_{\mathcal{G}}^*\xi_{\mathcal{G}}^0 \cong \zeta^* T^*\xi = (T \circ \zeta)^*\xi = \xi.$$

This shows η is surjective. In fact, using the functoriality we can show that the homotopy class of $E_{\mathcal{G}} \circ \zeta$ depends only on the isotopy class of ξ. In this way we have the map $\theta\colon \mathcal{A}(B) \to [B, B_{\mathcal{G}}^0]$ satisfying $\eta \circ \theta = id_{\mathcal{A}(B)}$.

Here we consider the above construction applied to $\xi_{\mathcal{G}}^0$. We have $P(\xi^0) = \mathcal{G} \times B^0$ with the projection $\pi^0\colon P(\xi^0) \to B^0$, $(g, b) \mapsto b$ and the $\mathcal{G}$–equivariant map $E_{\xi^0}\colon P(\xi^0) \to B^0$, $(g, b) \mapsto g \cdot b$. By the functoriality, the $\mathcal{G}$–bundle $\pi_u\colon P^u \to B_{\mathcal{G}}^0$ and the $\mathcal{G}$–equivariant map $E^u\colon P^u \to B^0$ associated to $\xi_{\mathcal{G}}^0$ is described as follows.

Take the Borel construction $P^u := E\mathcal{G} \times_{\mathcal{G}} P(\xi^0)$ where the $\mathcal{G}$–action on $P(\xi^0)$ is given by $g \cdot (g', b) = (g'g^{-1}, g \cdot b)$. Define $\pi_u\colon P^u \to B_{\mathcal{G}}^0$ and $E^u\colon P^u \to B^0$ by

$$\pi_u([e, (g, b)]) = [e, b], \text{ and } E^u([e, (g, b)]) = g \cdot b,$$

where $[e, b]$ denotes the element of $B_{\mathcal{G}}^0$ represented by $(e, b) \in E\mathcal{G} \times B^0$, etc. Given the $\mathcal{G}$–action on P^u by $g \cdot [e, (g', b)] = [e, (gg', b)]$, π_u is a principal $\mathcal{G}$–bundle and E^u is $\mathcal{G}$–equivariant. Also the isomorphism $E_{\xi^0}^*\xi_0 \cong \pi^{0*}\xi_0$ induces $E^{u*}\xi_0 \cong \pi_u^*\xi_{\mathcal{G}}^0$. Notice that $T_u\colon P_{\mathcal{G}}^u \to B_{\mathcal{G}}^0$ has a section ζ_u given by $\zeta_u([e, b]) = [e, [e, (\mathrm{id}_{\mathcal{G}}, b)]]$.

Now we show $\theta \circ \eta = \mathrm{id}_{[B, B_{\mathcal{G}}^0]}$, which will complete the proof. Let $\psi\colon B \to B_{\mathcal{G}}^0$ be a continuous map. By the functoriality, we can use the fiber product $\psi^* P^u$ as the $\mathcal{G}$–bundle associated to $\psi^*\xi_{\mathcal{G}}^0$. Pulling back ζ_u, we have a section $\psi^*\zeta_u$ of $T\colon (\psi^* P^u)_{\mathcal{G}} \to B$ which makes the following diagram commutative.

$$
\begin{array}{ccccc}
(\psi^* P^u)_{\mathcal{G}} & \xrightarrow{\bar{\psi}_{\mathcal{G}}} & P_{\mathcal{G}}^u & \xrightarrow{E_{\mathcal{G}}^u} & B_{\mathcal{G}}^0 \\
\Big\uparrow{\scriptstyle \psi^*\zeta_u} & & \Big\uparrow{\scriptstyle \zeta_u} & & \\
B & \xrightarrow{\psi} & B_{\mathcal{G}}^0 & &
\end{array}
$$

Notice that $E_{\mathcal{G}}^u \circ \zeta_u = \mathrm{id}_{B_{\mathcal{G}}^0}$. Following the construction of θ we have

$$\theta([\psi^*\xi_{\mathcal{G}}^0]) = [E_{\mathcal{G}}^u \circ \bar{\psi}_{\mathcal{G}} \circ \psi^*\zeta_u] = [E_{\mathcal{G}}^u \circ \zeta_u \circ \psi] = [\psi],$$

which shows $\theta \circ \eta = \mathrm{id}_{[B, B_{\mathcal{G}}^0]}$. $\qquad\square$

Definition 4.1.4 (1) Let Δ be a closed oriented 2–disk with the center b. A 4·-tuple $\mathcal{F} = (S, f, \Delta, b)$ is called an $\mathcal{A}$–*degeneration* if S is a C^∞–manifold of dimension 4 and $f: S \to \Delta$ is a proper surjective C^∞–map, and the restriction of f to $\Delta \setminus \{b\}$ is given a structure of $\mathcal{A}$–family. We denote by $\xi_{\mathcal{F}}$ this $\mathcal{A}$–family.

(2) Let $\mathcal{F} = (S, f, \Delta, b)$ and $\mathcal{F}' = (S', f', \Delta', b')$ be $\mathcal{A}$–degenerations. We say $\mathcal{F}$ and $\mathcal{F}'$ are *equivalent* if taking suitably smaller disks $\Delta_0 \subset \Delta$ with $b \in \Delta_0$ and $\Delta_0' \subset \Delta'$ with $b' \in \Delta_0'$, there exist an orientation preserving homeomorphism $\psi: (\Delta_0, b) \to (\Delta_0', b')$ such that $\psi^* \xi_{\mathcal{F}'}$ is isotopic to the restriction of $\xi_{\mathcal{F}}$ to $\Delta_0 \setminus \{b\}$.

(3) We denote by $\mathcal{A}_{\mathrm{loc}}$ the set of all equivalence classes of $\mathcal{A}$–degenerations. We often identify an element of $\mathcal{A}_{\mathrm{loc}}$ with its representative. Each element of $\mathcal{A}_{\mathrm{loc}}$ is called a fiber germ. A *smooth fiber germ* is an element of $\mathcal{A}_{\mathrm{loc}}$ obtained by an $\mathcal{A}$–family $\xi = (\mathcal{C}, p, \Delta)$.

Definition 4.1.5 Let M be a closed oriented 4–manifold and B a closed oriented 2–manifold. A proper surjective C^∞–map $f: M \to B$ is called an $\mathcal{A}$–*fibration* if there exist finitely many points $b_1, \ldots, b_m \in B$ such that the restriction of f to $B \setminus \{b_1, \ldots, b_m\}$ is given a structure of $\mathcal{A}$–family.

The triple (M, f, B) is a fibered 4–manifold in the sense of Section 1. For an $\mathcal{A}$–fibration $f: M \to B$, let $\mathcal{F}_i$ be the element of $\mathcal{A}_{\mathrm{loc}}$ obtained by restricting f to a small closed disk neighborhood Δ_i of b_i. We formulate the notion of a local signature in our setting.

Definition 4.1.6 Let $\mathcal{A}$ be a subset of $\mathbb{M}_g$. A function $\sigma_{\mathcal{A}}: \mathcal{A}_{\mathrm{loc}} \to \mathbb{Q}$ is called a local signature with respect to $\mathcal{A}$ if

(1) for a smooth fiber germ $\mathcal{F}$, $\sigma_{\mathcal{A}}(\mathcal{F}) = 0$, and

(2) for any $\mathcal{A}$–fibration $f: M \to B$, we have *the global signature formula*:

$$(\textbf{4.1.7}) \qquad \mathrm{Sign}(M) = \sum_{i=1}^{m} \sigma_{\mathcal{A}}(\mathcal{F}_i).$$

Proposition 4.1.8 *Let $\mathcal{A}$ be a subset of $\mathbb{M}_g$. Suppose there exist a universal $\mathcal{A}$–family $\xi_u = (\mathcal{C}_u, p_u, B_u)$ and a $\mathbb{Q}$–valued 1–cochain $\phi_{\mathcal{A}}: \pi_1(B_u) \to \mathbb{Q}$ such that $\delta \phi_{\mathcal{A}} = \rho_u^* \tau_g$. Then there exists a local signature with respect to $\mathcal{A}$.*

Proof Let $\mathcal{F} = (S, f, \Delta, b) \in \mathcal{A}_{\mathrm{loc}}$. Since $\eta = \eta_{\xi_u, \Delta \setminus \{b\}}$ is bijective, there exists uniquely up to homotopy a continuous map $g_{\mathcal{F}}: \Delta \setminus \{b\} \to B_u$ such that $\eta([g_{\mathcal{F}}]) = [\xi_{\mathcal{F}}]$.

We denote by $\partial\Delta$ the element of $\pi_1(\Delta \setminus \{b\})$ represented by the loop going once around the boundary of Δ by counterclockwise manner. Then we obtain an element $x_{\mathcal{F}} = g_{\mathcal{F}*}(\partial\Delta) \in \pi_1(B_u)$, which is uniquely determined up to conjugacy. Since the equality $\delta\phi_{\mathcal{A}} = \rho_u^*\tau_g$ implies that $\phi_{\mathcal{A}}$ is a class function (see Section 1), the value $\phi_{\mathcal{A}}(x_{\mathcal{F}})$ is well defined.

Now define $\sigma_{\mathcal{A}} \colon \mathcal{A}_{\text{loc}} \to \mathbb{Q}$ by

(4.1.9) $$\sigma_{\mathcal{A}}(\mathcal{F}) = \phi_{\mathcal{A}}(x_{\mathcal{F}}) + \text{Sign}(S).$$

If $\mathcal{F}$ is a smooth fiber germ, $g_{\mathcal{F}}$ extends to a continuous map from Δ. So $x_{\mathcal{F}} \in \pi_1(B_u)$ is trivial, hence $\phi_{\mathcal{A}}(x_{\mathcal{F}}) = 0$. Also we have $\text{Sign}(S) = 0$ since topologically $f \colon S \to \Delta$ is just a trivial Σ_g–bundle. The first condition in Definition 4.1.6 is verified. The second condition is verified by an argument similar to the proof of Proposition 3.5.3, so we omit the detail (see also [16, Theorem 7.2]). $\qquad\square$

Definition 4.1.10 For $\mathcal{F} \in \mathcal{A}_{\text{loc}}$, we call $x_{\mathcal{F}} \in \pi_1(B_u)$ appeared in the proof of Proposition 4.1.8 the *lifted monodromy*. This is uniquely determined up to conjugacy.

4.2 Fibrations of rank 4 nonhyperelliptic curves of genus 4

Let C be a nonhyperelliptic Riemann surface of genus 4. Its canonical image is a $(2, 3)$ complete intersection in $\mathbb{P}_3$ hence is contained in a uniquely determined quadric. We say C is *of rank 4* if this quadric is of rank 4. Let $\mathcal{R}^4 \subset \mathbb{M}_4$ be the set of rank 4 nonhyperelliptic Riemann surfaces of genus 4. $\mathcal{R}^4$ is Zariski open in $\mathbb{M}_4$.

Let $s \colon \mathbb{P}_1 \times \mathbb{P}_1 \to \mathbb{P}_3$ be the Segre embedding. Explicitly, s is given by

$$s([a_0 : a_1], [b_0 : b_1]) = [a_0 b_0 : a_0 b_1 : a_1 b_0 : a_1 b_1],$$

using the homogeneous coordinates. Let $V_{3,3} = \mathbb{C}[a_0, a_1]^3 \otimes \mathbb{C}[b_0, b_1]^3$ be the space of $(3, 3)$ homogeneous polynomials, and let

$$s_{3,3} \colon \mathbb{P}_1 \times \mathbb{P}_1 \to \mathbb{P}(V_{3,3}{}^\vee) \cong \mathbb{P}(V_{3,3})^\vee \cong \mathbb{P}_{15}$$

be the embedding induced from the evaluation map $\mathbb{C}^2 \times \mathbb{C}^2 \to V_{3,3}{}^\vee = \text{Hom}(V_{3,3}, \mathbb{C})$. Set

$$X = \text{Im}(s_{3,3}).$$

Consider the group $\mathcal{G} = \text{Aut}(\mathbb{P}_1 \times \mathbb{P}_1)$. Of course $\mathcal{G}$ acts on $\mathbb{P}_1 \times \mathbb{P}_1$ (from the left), inducing an action of $\mathcal{G}$ on $\mathbb{P}_{15}$ so that $s_{3,3}$ is $\mathcal{G}$–equivariant. Moreover $\mathcal{G}$ naturally acts on $\mathbb{P}(V_{3,3})$ from the left.

Let $D_X \subset \mathbb{P}_{15}^\vee = \mathbb{P}(V_{3,3})$ be the dual variety of X and $U^X = \mathbb{P}(V_{3,3}) \setminus D_X$. D_X is preserved by the $\mathcal{G}$–action. Also $\mathcal{G}$ acts on $\mathcal{C}^X \subset \mathbb{P}_{15} \times U^X$ diagonally, and the

projection $p_X\colon \mathcal{C}^X \to U^X$ is $\mathcal{G}$–equivariant. Note that for $W \in U^X$, the fiber $p_X^{-1}(W)$ is isomorphic to the smooth curve in $\mathbb{P}_1 \times \mathbb{P}_1$ determined by a $(3,3)$ homogeneous polynomial, which is an element of $\mathcal{R}^4$ since the restriction of s to the curve gives its canonical embedding and the image is contained in $s(\mathbb{P}_1 \times \mathbb{P}_1)$, which is a smooth quadric $x_0 x_3 - x_1 x_2 = 0$. Thus, $\xi^X = (\mathcal{C}^X, p_X, U^X)$ is a $\mathcal{R}^4$–family.

Now we will show that ξ^X and the $\mathcal{G}$–action on it satisfies the conditions in Proposition 4.1.3. We need to construct a principal $\mathcal{G}$–bundle from a $\mathcal{R}^4$–family.

First we consider the case of a single element $C \in \mathcal{R}^4$. We denote by $\Omega^1(C)$ the space of holomorphic 1–forms on C. The unique quadric containing the canonical image of C corresponds to the 1–dimensional kernel of the natural map $t_2\colon \mathrm{Sym}^2 \Omega^1(C) \to H^0(C; K_C^{\otimes 2})$. Here, K_C is the canonical bundle of C. Note that t_2 is surjective by Max Noether's theorem (see [10, page 253]).

If we take a basis $\omega = (\omega_0, \omega_1, \omega_2, \omega_3)$ of $\Omega^1(C)$, an explicit form of $\mathrm{Ker}(t_2)$ is obtained as follows. Let $\varphi_0, \varphi_1, \varphi_2, \varphi_3 \in \Omega^1(C)^*$ be the dual basis of ω. Then $\mathrm{Sym}^2 \Omega^1(C)$ is identified with the space S_4 of 4×4 symmetric matrices by assigning $B \in S_4$ with the quadratic function

$$\Omega^1(C)^* \to \mathbb{C}, \ x_0\varphi_0 + x_1\varphi_1 + x_2\varphi_2 + x_3\varphi_3 \mapsto (x_0, x_1, x_2, x_3)B(x_0, x_1, x_2, x_3)^t.$$

Hence a choice of a basis ω of $\Omega^1(C)$ determines the element $B(\omega) \in \mathbb{P}(S_4)$ corresponding to $\mathrm{Ker}(t_2)$, and the image of the canonical map $\iota_\omega\colon C \hookrightarrow \mathbb{P}_3$, $c \mapsto [\omega_0(c) : \omega_1(c) : \omega_2(c) : \omega_3(c)]$ is contained in the quadric determined by $B(\omega)$.

Let $P(C)$ be the set of ω modulo $\mathbb{C}^*$ such that the quadric determined by $B(\omega)$ is equal to $\{x_0 x_3 - x_1 x_2 = 0\}$. Namely,

$$P(C) = \{\omega \bmod \mathbb{C}^* : \omega \text{ is a basis of } \Omega^1(C) \text{ and } B(\omega) = H\},$$

where $H \in \mathbb{P}(S_4)$ is represented by

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Now consider the group $\mathrm{PO}_4^H(\mathbb{C}) = \{A \in \mathrm{PGL}(4) : A^t H A = H\}$, which acts on $P(C)$ from the left freely and transitively by $A \cdot (\omega_0, \omega_1, \omega_2, \omega_3) \bmod \mathbb{C}^* = (\omega_0, \omega_1, \omega_2, \omega_3)A^t \bmod \mathbb{C}^*$. In fact, this group is isomorphic to $\mathcal{G}$ and the isomorphism is induced by the action of $\mathrm{PO}_4^H(\mathbb{C})$ on $\mathbb{P}_3$ (as a subgroup of $\mathrm{PGL}(4)$) preserving $s(\mathbb{P}_1 \times \mathbb{P}_1)$. Therefore, $\mathcal{G}$ acts on $P(C)$ freely and transitively. Finally, define the map

$$E_C\colon P(C) \to U^X$$

as follows. Again by Max Noether's theorem, the natural map $t_3: \mathrm{Sym}^3 \, \Omega^1(C) \to H^0(C; K_C^{\otimes 3})$ is surjective. Choose $h \in \mathrm{Ker}(t_3)$ which is not divided by elements of $\mathrm{Ker}(t_2)$. Let $\omega \in P(C)$. Then h is identified with a homogeneous polynomial of degree 3 in determinates x_0, x_1, x_2, x_3. We denote it by h^ω. The canonical image $\iota_\omega(C)$ is given by $x_0 x_3 - x_1 x_2 = h^\omega(x_0, x_1, x_2, x_3) = 0$. Set

$$E_C(\omega) = h^\omega(a_0 b_0, a_0 b_1, a_1 b_0, a_1 b_1) \in \mathbb{P}(V_{3,3}).$$

Since the zero set of $h^\omega(a_0 b_0, a_0 b_1, a_1 b_0, a_1 b_1)$ is isomorphic to C, we have $E_C(\omega) \in U^X$. $E_C(\omega)$ does not depend on the choice of h, and we can verify that E_C is $\mathcal{G}$–equivariant.

Now let $\xi = (\mathcal{C}, p, B)$ be a $\mathcal{R}^4$–family. Applying the above construction to all the fibers, we get a principal $\mathcal{G}$–bundle

$$P(\xi) = \bigcup_{b \in B} P(p^{-1}(b))$$

and by piecing together $E_{p^{-1}(b)}$, $b \in B$, we get a $\mathcal{G}$–equivariant map

$$E_\xi: P(\xi) \to U^X.$$

The first condition in Proposition 4.1.3 is clear from the construction. So far we have only used the objects arising from holomorphic 1–forms on Riemann surfaces, which behave naturally under pullback by biholomorphic maps. Thus the functoriality is also true. As to the second condition, we can describe $P(\xi^X)$ as follows. For $W \in U^X$, $W \cap X$ is isomorphic to $v_3^{-1}(W \cap X) \subset \mathbb{P}_3$. Let ω_W be the basis of $\Omega^1(W \cap X)$ corresponding to the homogeneous coordinates $[x_0 : x_1 : x_2 : x_3]$ of $\mathbb{P}_3$. Then the isomorphism

$$P(\xi_X) \cong \mathcal{G} \times U^X$$

is given by assigning $(A, W) \in \mathcal{G} \times U^X$ with $A \cdot \omega_W$. We can check that for $A \in \mathcal{G}$ and $W \in U^X$, ω_W corresponds to $A^{-1} \cdot \omega_{A \cdot W}$ by the isomorphism $W \cap X \to (A \cdot W) \cap X$ induced by A. The second condition follows from this. Applying Proposition 4.1.3, we have a universal $\mathcal{R}^4$–family $\xi_{\mathcal{G}}^X = (\mathcal{C}_{\mathcal{G}}^X, p_u, U_{\mathcal{G}}^X)$. Here, $p_u = (p_X)_{\mathcal{G}}$.

Theorem 4.2.1 *Let $\mathcal{R}^4$ be the set of rank 4 nonhyperelliptic Riemann surfaces of genus 4 and X, $\mathcal{G}$ as above. Then $\xi_{\mathcal{G}}^X = (\mathcal{C}_{\mathcal{G}}^X, p_u, U_{\mathcal{G}}^X)$ is a universal $\mathcal{R}^4$–family. We denote by $\rho_u: \pi_1(U_{\mathcal{G}}^X) \to \Gamma_4$ the topological monodromy of $p_u: \mathcal{C}_{\mathcal{G}}^X \to U_{\mathcal{G}}^X$. Then there exists a unique $\mathbb{Q}$–valued 1–cochain $\phi_{\mathcal{R}^4}: \pi_1(U_{\mathcal{G}}^X) \to \mathbb{Q}$ whose coboundary equals $\rho_u^* \tau_4$.*

Proof We only have to prove the latter part. Consider the map $U_{\mathcal{G}}^X \to B\mathcal{G}$ induced from the projection $E\mathcal{G} \to B\mathcal{G}$. This is a U^X–bundle. By the homotopy exact sequence, we have the exact sequence

$$(4.2.2) \qquad\qquad \pi_1(\mathcal{G}) \to \pi_1(U^X) \xrightarrow{i} \pi_1(U_{\mathcal{G}}^X) \to \pi_0(\mathcal{G}) \to *,$$

where i is induced from the inclusion. But $\pi_1(\mathcal{G}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and $\pi_0(\mathcal{G})$ consists of two points, since $\mathcal{G}$ is isomorphic to the semidirect product $(\mathrm{PGL}(2) \times \mathrm{PGL}(2)) \ltimes \mathbb{Z}/2\mathbb{Z}$. So (4.2.2) shows that $i^*\colon H^2(\pi_1(U_{\mathcal{G}}^X);\mathbb{Q}) \to H^2(\pi_1(U^X);\mathbb{Q})$ is injective. By Theorem 3.1.1, we have $i^*\rho_u^*[\tau_4] = 0 \in H^2(\pi_1(U^X);\mathbb{Q})$. Therefore we also have $\rho_u^*[\tau_4] = 0 \in H^2(\pi_1(U_{\mathcal{G}}^X);\mathbb{Q})$. This shows the existence of $\phi_{\mathcal{R}^4}$.

On the other hand, in the proof of Lemma 3.3.1 we have seen that the abelianization of $\pi_1(U^X)$ is a cyclic group of finite order. Combining this fact with (4.2.2), we see that the abelianization of $\pi_1(U_{\mathcal{G}}^X)$ is a finite abelian group. This shows the uniqueness of $\phi_{\mathcal{R}^4}$. $\square$

Combining this with Proposition 4.1.8, we have:

Corollary 4.2.3 Let $\mathcal{R}^4$ be the set of rank 4 nonhyperelliptic Riemann surfaces of genus 4. Then the formula

$$(4.2.4) \qquad\qquad \sigma_{\mathcal{R}^4}(\mathcal{F}) = \phi_{\mathcal{R}^4}(x_{\mathcal{F}}) + \mathrm{Sign}(S)$$

for $\mathcal{F} = (S,\pi,\Delta,b) \in \mathcal{R}_{\mathrm{loc}}^4$ (see (4.1.9)) gives a local signature with respect to $\mathcal{R}^4$.

4.3 Some computations of $\sigma_{\mathcal{R}^4}$ and $\phi_{\mathcal{R}^4}$

In this section we compute the value of our $\sigma_{\mathcal{R}^4}$ or $\phi_{\mathcal{R}^4}$ for some examples.

Lemma 4.3.1 Let $X = s_{3,3}(\mathbb{P}_1 \times \mathbb{P}_1)$ be as defined in Section 4.2. Then D_X is a hypersurface and $\deg D_X = 34$. For a lasso σ_X around D_X, we have $\phi_X(\sigma_X) = -9/17$.

Proof Since $X \cong \mathbb{P}_1 \times \mathbb{P}_1$, X is simply connected and we have $\mathrm{Sign}\, X = 0$, $\chi(X) = 4$. Also we have $\deg X = 18$. By Proposition 3.5.3 and (3.5.6), the assertion follows. $\square$

Let $\iota\colon \Delta \to \mathbb{P}(V_{3,3})$ be as in Proposition 2.3.8. Then we get a $\mathcal{R}^4$–degeneration $\iota^*\mathcal{W} \to \Delta$ (see Theorem 2.3.12), which we denote by $\mathcal{F}_I$ and call a *singular fiber germ of type I*. In this case we can choose $x_{\mathcal{F}_I}$ in (4.2.4) to be the image of a lasso around D_X. By Lemma 3.5.2 the signature of the fiber neighborhood is 0. Thus we have:

Proposition 4.3.2 $\sigma_{\mathcal{R}^4}(\mathcal{F}_I) = \phi_{\mathcal{R}^4}(x_{\mathcal{F}_I}) = -9/17$.

In the following let $\Delta = \{z \in \mathbb{C} : |z| \le \varepsilon\}$ for a sufficiently small real number $\varepsilon > 0$.

Example 4.3.3 Let

$$\Phi(z, a_0, a_1, b_0, b_1) = \varphi^0(a_0, a_1){b_0}^3 + ({a_0}^3 + z^6 {a_1}^3){b_0}{b_1}^2 + z^9 \varphi^3(a_0, a_1){b_1}^3$$

and $S_\Phi \subset \Delta \times \mathbb{P}_1 \times \mathbb{P}_1$ be the zero locus of Φ. Here φ^0, φ^3 are generic homogeneous polynomials of degree 3. Let $f' \colon S_\Phi \to \Delta$ be the first projection.

The zero locus S_Φ has an isolated singularity at $(0, [0:1], [0:1])$. Applying the resolution process given by Ashikaga [1], we will obtain a resolution $\varpi \colon \widetilde{S}_\Phi \to S_\Phi$ of the singularity. By successive blow down of (-1)–curves contained in the fiber at 0, we finally get the fiber germ what we want.

In the below we describe the resolution process. We shall introduce the inhomogeneous coordinates $a = a_0/a_1$ and $b = b_0/b_1$.

First we recall a terminology from [1]. Let W be a complex manifold of dimension two and L a holomorphic line bundle on W. Let $\overline{L} = \mathbb{P}(\mathcal{O}_W \oplus \mathcal{O}_W(L))$ the associated $\mathbb{P}_1$ bundle. This is the $\mathbb{P}_1$ bundle on W whose fiber at $w \in W$ is the projectivization of the dual space of $\mathbb{C} \oplus L_w$. Here L_w is the fiber of $L \to W$ at w. Let $T = \mathcal{O}_{\overline{L}}(1)$. This is the line bundle on $\overline{L}$ whose fiber at $\overline{\ell}$ (which is a line of $(\mathbb{C} \oplus L_w)^*$ for some $w \in W$) is the dual space of $\overline{\ell}$. Let S be an irreducible reduced divisor on $\overline{L}$ which is linearly equivalent to $3T$. In [1], the triple (S, W, L) is called a *triple section surface*.

Let $W = \Delta \times \mathbb{P}_1$ with $(z, [a_0 : a_1])$ the global coordinates, and L a trivial line bundle on W. We can introduce the homogeneous fiber coordinates $[b_0 : b_1]$ for $\overline{L}$ by assigning the linear functional on $\mathbb{C} \oplus L_w = \mathbb{C} \oplus \mathbb{C}$, given by $(c_0, c_1) \mapsto b_0 c_0 + b_1 c_1$, to (b_0, b_1). Our S_Φ, which is the zero locus of Φ, is naturally identified with an irreducible reduced divisor on $\overline{L}$. Then (S_Φ, W, L) is a triple section surface in the above sense.

Let $\tau_1 \colon W_1 \to W$ be the blow up at the origin $p_1 = (0, [0:1]) \in W$ and let $\widehat{\tau}_1 = (\overline{\tau}_1, \tau_1) \colon (S_1, W_1, L_1) \to (S_\Phi, W, L)$ be the triplet blow-up at p_1 with $\ell_1 = 1$, in the sense of [1, page 181], W_1 is covered by the two coordinate neighborhoods $U_z = \{(z, \widetilde{a})\}$ and $U_a = \{(\widetilde{z}, a)\}$, and τ_1 is given by $\tau_1(z, \widetilde{a}) = (z, \widetilde{a}z)$ on U_z, and $\tau_1(\widetilde{z}, a) = (a\widetilde{z}, a)$ on U_a. Note that $L_1 = \tau_1^* L - E_1$ where E_1 is the exceptional curve of τ_1.

Next, let $p_2 = (0, 0) \in U_z$ and let $\tau_2 \colon W_2 \to W_1$ be the blow up at p_2. Let $\widehat{\tau}_2 = (\overline{\tau}_2, \tau_2) \colon (S_2, W_2, L_2) \to (S_1, W_1, L_1)$ be the triplet blow-up at p_2 with $\ell_2 = 2$. (S_2, W_2, L_2) is also a triple section surface. Note that $L_2 = \tau_2^* L_1 - 2E_2$ where E_2 is the exceptional curve of τ_2.
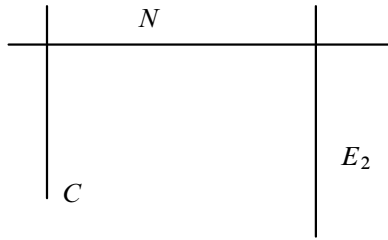
Figure 1

There is a natural map $W_2 \to W \to \Delta$, whose fiber at $0 \in \Delta$ looks like Figure 1. Here, N is the proper transform of $E_1 \subset W_1$, and C is the proper transform of $\{z = 0\} \subset W$. All the irreducible components are curves of genus 0. We denote by $\hat{\pi}_2$ the natural projection $S_2 \to W_2$ and let $N^* = \hat{\pi}_2^{-1}(N)$. $N^* \subset \bar{L}_2$ is a curve of genus 0.

Let $\bar{\sigma} \colon M \to \bar{L}_2$ be the blow up with center N^* and let $\widetilde{S}_\Phi$ be the proper transform of S_2 by $\bar{\sigma}$ (see [1, page 187]). Then $\widetilde{S}_\Phi$ turns out to be nonsingular. Setting ϖ to be the natural map from $\widetilde{S}_\Phi$ to S_Φ, we get a resolution $\varpi \colon \widetilde{S}_\Phi \to S_\Phi$.

The fiber $(f' \circ \varpi)^{-1}(0) \subset \widetilde{S}_\Phi$ looks like Figure 2. Here C_1, N_1, N_2 are curves of
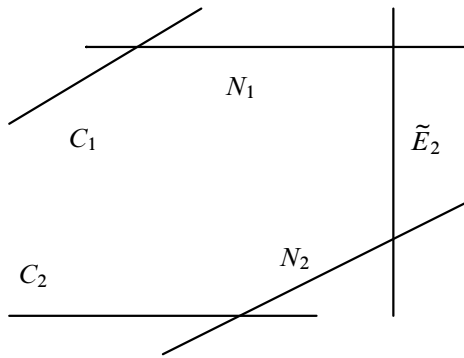

Figure 2

genus 0, C_2 is a curve of genus 1, and $\widetilde{E}_2$ is a curve of genus 3. The self intersection numbers are: $C_1^2 = N_2^2 = -1$, $N_1^2 = C_2^2 = -2$, and $\widetilde{E}_2^2 = -3$. The inverse images of C, N, and E_2 by the natural map $\widetilde{S}_\Phi \to W_2$ are $C_1 \amalg C_2$, $N_1 \amalg N_2$, and $\widetilde{E}_2$, respectively.

Note that we have a triple covering $\widetilde{S}_\Phi \to W_2$. The restrictions of this map to C_2 or $\widetilde{E}_2$ gives a double covering $C_2 \to C$ with 4 simple branch points or a triple

covering $\widetilde{E}_2 \to E_2$ with 10 simple branch points. As a divisor, $(f' \circ \varpi)^{-1}(0) = C_1 + N_1 + \widetilde{E}_2 + 2N_2 + C_2$.

Finally let $\widetilde{S}_\Phi \to \overline{S}_\Phi$ be the contraction obtained by repeating blow down of (-1) curves in the fiber at $0 \in \Delta$ until the resulting surface contains no more such curves (we need to blow down three times). Let $\overline{C}_2$ (resp. $\overline{E}_2$) be the image of C_2 (resp. $\widetilde{E}_2$) by this contraction. They are curves of genus 1 and 3 respectively, and $\overline{C}_2 \cdot \overline{E}_2 = 1$, $\overline{C}_2^2 = \overline{E}_2^2 = -1$.

Let $f_\Phi \colon \overline{S}_\Phi \to \Delta$ be the map induced from $f' \circ \varpi$. Then $\mathcal{F}_{3,1} := (\overline{S}_\Phi, f_\Phi, \Delta, 0)$ is a $\mathcal{R}^4$–degeneration with $f_\Phi^{-1}(0)$ being homeomorphic to the one point union of a surface of genus 1 and a surface of genus 3.

Proposition 4.3.4 $\sigma_{\mathcal{R}^4}(\mathcal{F}_{3,1}) = 11/17$ and $\phi_{\mathcal{R}^4}(x_{\mathcal{F}_{3,1}}) = 28/17$.

Proof The idea is to globalize the fiber germ $\mathcal{F}_{3,1}$ by a small perturbation. As a result we will obtain a $\mathcal{R}^4$–fibration such that the set of singular fiber germs consists of one $\mathcal{F}_{3,1}$ and $\mathcal{F}_I$'s. Then we can compute the number of singular fiber germs, and get the value of $\sigma_{\mathcal{R}^4}(\mathcal{F}_{3,1})$ by the global signature formula (4.1.7).

Let $\alpha \gg 0$ be a sufficiently large natural number and let $\varphi = \varphi(a_0, a_1, b_0, b_1)$ be a generic $(3, 3)$ homogeneous polynomial. Set $\Phi' = \Phi + z^\alpha \varphi$. Regarding z as an affine coordinate of $\mathbb{P}_1$, let $S_{\Phi'} \subset \mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$ be the zero locus of Φ'.

The zero locus $S_{\Phi'}$ has an isolated singularity at $(0, [0 : 1], [0 : 1])$, and the same resolution process as that of S_Φ can be applied. Let $\widetilde{S}_{\Phi'}$ and $\overline{S}_{\Phi'}$ be the result of the process corresponding to $\widetilde{S}_\Phi$ and $\overline{S}_\Phi$, respectively. Then $\widetilde{S}_{\Phi'}$ is a nonsingular compact complex surface, and the induced projection $f_{\Phi'} \colon \overline{S}_{\Phi'} \to \mathbb{P}_1$ is a $\mathcal{R}^4$–fibration. The singular fiber germ at 0 is $\mathcal{F}_{3,1}$, and since ψ is chosen to be generic, the other singular fiber germs is of type I.

We compute the various invariants. First of all, the holomorphic Euler characteristic of $S_{\Phi'}$ is computed as $\chi(\mathcal{O}_{S_{\Phi'}}) = 4\alpha - 3$, and the self intersection number of the dualizing sheaf (see Introduction) of $S_{\Phi'}$ is computed as $\omega_{S_{\Phi'}}{}^2 = 14\alpha - 24$. By using Lemma 1.10 and the formula in [1, page 187], we have $\chi(\mathcal{O}_{\widetilde{S}_{\Phi'}}) = 4\alpha - 10$ and $\omega_{\widetilde{S}_{\Phi'}}{}^2 = 14\alpha - 49$. Thus, we have $\chi(\mathcal{O}_{\overline{S}_{\Phi'}}) = 4\alpha - 10$ and $\omega_{\overline{S}_{\Phi'}}{}^2 = 14\alpha - 46$. By the Noether formula and the Hirzebruch signature formula, we have $\chi(\overline{S}_{\Phi'}) = 34\alpha - 74$ and $\mathrm{Sign}(\overline{S}_{\Phi'}) = -18\alpha + 34$.

As (3.5.6), the number of singular fiber germs of $f_{\Phi'} \colon \overline{S}_{\Phi'} \to \mathbb{P}_1$ is computed as

$$\chi(\overline{S}_{\Phi'}) - 2 \cdot (2 - 2 \cdot 4) = 34\alpha - 62.$$

In particular, the number of fiber germs of type I is $34\alpha - 63$. By the global signature formula, we have

$$\text{Sign}(\bar{S}_{\Phi'}) = -18\alpha + 34 = (34\alpha - 63) \cdot (-9/17) + \sigma_{\mathcal{R}^4}(\mathcal{F}_{3,1}).$$

Finally, since the signature of $\bar{S}_\Phi$ is -1, $\phi_{\mathcal{R}^4}(\mathcal{F}_{3,1}) = \sigma_{\mathcal{R}^4}(\mathcal{F}_{3,1}) - 1$. This completes the proof. □

Example 4.3.5 Let

$$\Phi(z, a_0, a_1, b_0, b_1) = \varphi^0(a_0, a_1)b_0{}^3 + \varphi^1(a_0, a_1)b_0{}^2 b_1 + a_0{}^3 b_0 b_1{}^2 + z^6 \varphi^3(a_0, a_1)b_1{}^3$$

and $S_\Phi \subset \Delta \times \mathbb{P}_1 \times \mathbb{P}_1$ be the zero locus of Φ. Here φ^0, φ^1 and φ^3 are generic homogeneous polynomials of degree 3.

S_Φ has an isolated singularity at $(0, [0:1], [0:1])$. If we write $\varphi^i(a)$ or $\Phi(z, a, b)$ instead of $\varphi^i(a, 1)$ or $\Phi(z, a, 1, b, 1)$ respectively, then

$$\Phi(z, a, b) = (\varphi^0(a)b + \varphi^1(a))(b^2 + A^3 b + Z^6),$$

where $A = a(\varphi^0(a)b + \varphi^1(a))^{-1/3}$ and $Z = z\{\varphi^3(a)/(\varphi^0(a)b + \varphi^1(a))\}^{1/6}$. Note that

$$b^2 + a^3 b + z^6 = \left(b + \frac{a^3}{2}\right)^2 - \frac{a^6}{4} + Z^6.$$

Set $h(b, z, a) = b^2 + z^6 - a^6$ and let $S_h \subset \mathbb{C}^3$ be the zero locus of h. Define $f': S_h \to \Delta$ by $f'(b, z, a) = z$. Then the singularity of S_Φ is analytically equivalent to the hypersurface singularity $(S_h, 0)$. In this case Horikawa's canonical resolution for double coverings (see [13, Section 2]) can be used for a resolution of $(S_h, 0)$.

The process is as follows. Let W be an open neighborhood of 0 in $\mathbb{C}^2$ with global coordinates (z, a) and let $\tau: \widetilde{W} \to W$ be the blow up at the origin. $\widetilde{W}$ is covered by the two coordinate neighborhoods U_z and U_a as in Example 4.3.3, and τ is given by the same formula as τ_1. The picture of $\widetilde{W}$ is like Figure 3. Here E is the exceptional
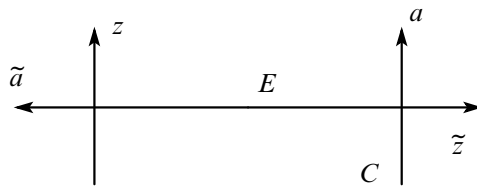

Figure 3

curve of τ and C is the proper transform of $\{z = 0\} \subset W$.

Let $\widetilde{S}_h$ be the double covering on $\widetilde{W}$ defined by the following: on U_z, let $\widetilde{S}_h^z$ be the hypersurface in $\mathbb{C}^3$ given by $x^2 + 1 - \widetilde{a}^6 = 0$ and on U_a, let $\widetilde{S}_h^a$ be given by $x^2 + \widetilde{z}^6 - 1 = 0$. Identifying $\widetilde{S}_h^z$ with $\widetilde{S}_h^a$ over $U_z \cap U_a$ by the map $(x, z, \widetilde{a}) \mapsto (x/\widetilde{a}^3, \widetilde{z}, a)$, we get the resulting nonsingular surface $\widetilde{S}_h$. Define $\varpi \colon \widetilde{S}_h \to S_h$ by $\varpi(x, z, \widetilde{a}) = (xz^3, z, \widetilde{a})$ on $\widetilde{S}_h^z$, and $\varpi(x, \widetilde{z}, a) = (xa^3, a\widetilde{z}, a)$ on $\widetilde{S}_h^a$. Then this is a resolution of singularity. $(f' \circ \varpi)^{-1}(0)$ looks like in Figure 4.
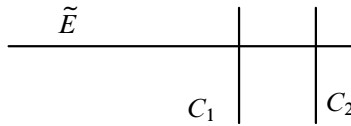

Figure 4

Here $\widetilde{E}$ (resp. $C_1 \amalg C_2$) is the proper transform of E (resp. C). $\widetilde{E}$ is a curve of genus 2. Note that the restriction of $\widetilde{S}_h \to \widetilde{W}$ to $\widetilde{E}$ gives a double covering $\widetilde{E} \to E \cong \mathbb{P}^1$ with 6 simple branch points.

Now applying the above process, we can resolve the singularity of S_Φ. Let $\varpi \colon \widetilde{S}_\Phi \to S_\Phi$ be such a resolution. Then $(f' \circ \varpi)^{-1}(0)$ consists of the three components, $\widetilde{E}$, C_1 and C_2, like Figure 4 (using the same letter). Note that

$$\Phi(0, a, b) = b(\varphi^0(a)b^2 + \varphi^1(a)b + a^3).$$

We assume that C_1 (resp. C_2) is the component corresponding to $b = 0$ (resp. $\varphi^0(a)b^2 + \varphi^1(a)b + a^3 = 0$). C_1 is a curve of genus 0 and C_2 is a curve of genus 2. The self intersection numbers are: $\widetilde{E}^2 = -2$, $C_1{}^2 = -1$, $C_2{}^2 = -1$.

Let $\widetilde{S}_\Phi \to \overline{S}_\Phi$ be the blow down of $\overline{C}_1$ and $f_\Phi \colon \overline{S}_\Phi \to \Delta$ the map induced from $f' \circ \varpi$. Then $\mathcal{F}_{2,2} := (\overline{S}_\Phi, f_\Phi, \Delta, 0)$ is a $\mathcal{R}^4$–degeneration with $f_\Phi^{-1}(0)$ being homeomorphic to the one points union of two curves of genus 2.

Proposition 4.3.6 $\sigma_{\mathcal{R}^4}(\mathcal{F}_{2,2}) = 19/17$ and $\phi_{\mathcal{R}^4}(x_{\mathcal{F}_{2,2}}) = 36/17$.

Proof The idea of proof is the same as Proposition 4.3.4. Let α, φ, Φ', and $S_{\Phi'}$ be the same as in the proof of Proposition 4.3.4. We construct $\widetilde{S}_{\Phi'}$ and $f_{\Phi'} \colon \overline{S}_{\Phi'} \to \mathbb{P}_1$ by a similar manner to Proposition 4.3.4 except for using the resolution of $(S_h, 0)$ described as above.

We have $\chi(\mathcal{O}_{S_{\Phi'}}) = 4\alpha - 3$, and $\omega_{S_{\Phi'}}{}^2 = 14\alpha - 24$. Using Lemma 6 of [13], we have $\chi(\mathcal{O}_{\widetilde{S}_{\Phi'}}) = 4\alpha - 6$, and $\omega_{\widetilde{S}_{\Phi'}}{}^2 = 14\alpha - 32$. Thus we have $\chi(\mathcal{O}_{\overline{S}_{\Phi'}}) = 4\alpha - 6$

and $\omega_{\bar{S}_{\Phi'}}{}^2 = 14\alpha - 31$, therefore $\chi(\bar{S}_{\Phi'}) = 34\alpha - 41$ and $\mathrm{Sign}(\bar{S}_{\Phi'}) = -18\alpha + 17$. Now the number of singular fiber germs of $f_{\Phi'}$ is

$$34\alpha - 41 - 2 \cdot (2 - 2 \cdot 4) = 34\alpha - 29,$$

hence the number of singular fiber germs of type I is $34\alpha - 30$. By the global signature formula we have

$$-18\alpha + 17 = (34\alpha - 30) \cdot (-9/17) + \sigma_{\mathcal{R}^4}(\mathcal{F}_{2,2}).$$

Finally the signature of $\bar{S}_\Phi$ is -1. This completes the proof.   □

Example 4.3.7 Let

$$\Phi(z, a_0, a_1, b_0, b_1) = (a_0 b_1 - a_1 b_0)^3 + z^3 \varphi^0(a_0, a_1, b_0, b_1).$$

We write $Y = \Delta \times \mathbb{P}_1 \times \mathbb{P}_1$ and let $S_\Phi \subset Y$ be the zero locus of Φ. Here $\varphi^0 \in V_{3,3}$ is a generic $(3,3)$ homogeneous polynomial.

Let $\Gamma = \{z = 0\} \subset Y$ and we denote by D the fiber at 0 of the first projection $S_\Phi \to \Delta$. The fiber D is the diagonal locus in $\Gamma \cong \mathbb{P}_1 \times \mathbb{P}_1$. Let $\tau_1 \colon Y_1 \to Y$ be the blow up along D. Let $E_0 \subset Y_1$ be the proper transform of Γ, E_1 the exceptional set of τ_1, and $\widetilde{S}_\Phi \subset Y_1$ the proper transform of S_Φ. Note that E_1 is isomorphic to the Hirzebruch surface $\mathbb{F}_2$ of degree 2.

Then we see that $\widetilde{S}_\Phi$ is nonsingular and $\widetilde{S}_\Phi \cap E_0 = \varnothing$. We write $f_\Phi \colon \widetilde{S}_\Phi \to \Delta$ the natural projection. Then we have a $\mathcal{R}^4$–degeneration $\mathcal{F}'_R := (\widetilde{S}_\Phi, f_\Phi, \Delta, 0)$. We see that $f_\Phi^{-1}(0) = E_1 \cap S_1$ is a smooth curve of genus 4. This curve is nonhyperelliptic, but not a curve of rank 4. This can be seen as follows. First we can contract E_0. Let $\bar{\tau} \colon Y_1 \to \bar{Y}$ be the contraction of E_0. The projection $Y_1 \to \Delta$ induces the projection $\widetilde{f} \colon \bar{Y} \to \Delta$, whose central fiber $\widetilde{f}^{-1}(0) = \bar{\tau}(E_1)$ is $\mathbb{F}_2$ and the other fibers are isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$. Thus, we can think $f_\Phi^{-1}(0)$ is contained in $\mathbb{F}_2$. Contracting the negative section of $\mathbb{F}_2$, we get a quadric Q_3 of rank 3 in $\mathbb{P}_3$. If we map $f_\Phi^{-1}(0)$ into $\mathbb{P}_3$ by this contraction, then $f_\Phi^{-1}(0)$ can be realized as a $(2,3)$–complete intersection: the intersection of Q_3 and some cubic surface. Thus $f_\Phi^{-1}(0)$ is nonhyperelliptic but not of rank 4. Topologically $f_\Phi \colon \widetilde{S}_\Phi \to \Delta$ is a trivial Σ_4–bundle.

Proposition 4.3.8 *Let* $\mathcal{F}'_R = (\widetilde{S}_\Phi, f, \Delta, 0)$ *be the fiber germ as above. Then we have* $\sigma_{\mathcal{R}^4}(\mathcal{F}'_R) = \phi_{\mathcal{R}^4}(x_{\mathcal{F}'_R}) = 4/17$.

Proof The proof proceeds as the same before. Let α, φ, Φ', $S_{\Phi'}$, and $\widetilde{S}_{\Phi'}$ be the same as in the proof of Proposition 4.3.4. We use the same notation Y and Y_1 for $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$ and the blow up along D, respectively.

First we have $\chi(\mathcal{O}_{S_{\Phi'}}) = 4\alpha - 3$ and $\omega_{S_{\Phi'}}{}^2 = 14\alpha - 24$. We need to compute $\chi(\mathcal{O}_{\widetilde{S}_{\Phi'}})$ and $\omega_{\widetilde{S}_{\Phi'}}{}^2$. By $K_{Y_1} \sim \tau_1^* K_Y + E_1$ (linear equivalence) and $S_1 \sim \tau_1^* S - 3E_1$, we have $K_{Y_1} + S_1 \sim \tau_1^*(K_Y + S) - 2E_1$. By restricting to S_1, we have $\tau_1^* \omega_S = \omega_{S_1} + 2E_1|_{S_1}$. We get

$$\omega_S{}^2 = (\tau_1^* \omega_S)^2 = (\omega_{S_1} + 2E_1|_{S_1})^2$$
$$= \omega_{S_1}{}^2 + 4\omega_{S_1} \cdot E_1|_{S_1} + 4(E_1|_{S_1})^2 = \omega_{S_1}{}^2 + 24,$$

since $\omega_{S_1} \cdot E_1|_{S_1} = 2g(E_1 \cap S_1) - 2 - (E_1|_{S_1})^2 = 6 - (E_1|_{S_1})^2$ by the adjunction formula. Therefore, $\omega_{\widetilde{S}_{\Phi'}}{}^2 = 14\alpha - 48$. We next compute

$$\chi(\mathcal{O}_{S_1}) = \chi(\mathcal{O}_{Y_1}) - \chi(\mathcal{O}_{Y_1}(-S_1))$$
$$= \chi(\mathcal{O}_{Y_1}) - \chi(\mathcal{O}_{Y_1}(-S_1 - E_1)) - \chi(\mathcal{O}_{E_1}(-S_1|_{E_1}))$$
$$\vdots$$
$$= \chi(\mathcal{O}_{Y_1}) - \chi(\mathcal{O}_{Y_1}(-S_1 - 3E_1)) - \sum_{i=0}^{2} \chi(\mathcal{O}_{E_1}(-S_1|_{E_1} - iE_1|_{E_1})).$$

We have $\chi(\mathcal{O}_{Y_1}) - \chi(\mathcal{O}_{Y_1}(-S_1 - 3E_1)) = \chi(\mathcal{O}_{Y_1}) - \chi(\mathcal{O}_{Y_1}(-\tau_1^* S)) = \chi(\mathcal{O}_Y) - \chi(\mathcal{O}_Y(-S)) = \chi(\mathcal{O}_S)$. To compute the remaining term, we use

$$\chi(\mathcal{O}_{E_1}(-S_1|_{E_1} - iE_1|_{E_1})) = \chi(\mathcal{O}_{E_1}(-iE_1|_{E_1})) - \chi(\mathcal{O}_{E_1 \cap S_1}(-iE_1|_{E_1 \cap S_1})).$$

Note that the divisor $(E_0 + E_1)|_{E_1}$ is trivial on E_1, since $E_0 + E_1$ is a fiber of $Y_1 \to \Delta$; $C_\infty = E_0 \cap E_1$ is the negative section of $E_1 \cong \mathbb{F}_2$, so $C_\infty{}^2 = -2$; $E_0 \cap E_1 \cap S_1 = \varnothing$. From these we have $\mathcal{O}_{E_1}(-E_1|_{E_1}) \cong \mathcal{O}_{E_1}(C_\infty)$ and $\mathcal{O}_{E_1 \cap S_1}(-E_1) \cong \mathcal{O}_{E_1 \cap S_1}$. Using the Riemann–Roch formula and $\chi(\mathcal{O}_{E_1}) = 1$, we get $\chi(\mathcal{O}_{E_1}(-S_1|_{E_1})) = 4$; $\chi(\mathcal{O}_{E_1}(-S_1|_{E_1} - E_1|_{E_1})) = 3$; $\chi(\mathcal{O}_{E_1}(-S_1|_{E_1} - 2E_1|_{E_1})) = 0$.

In summary, we have $\chi(\mathcal{O}_{S_1}) = \chi(\mathcal{O}_S) - 7$. Therefore, $\chi(\mathcal{O}_{\widetilde{S}_{\Phi'}}) = 4\alpha - 10$.

Now $\chi(\widetilde{S}_{\Phi'}) = 34\alpha - 72$, and $\mathrm{Sign}(\widetilde{S}_{\Phi'}) = -18\alpha + 32$. The number of topologically singular fibers is

$$34\alpha - 72 - 2 \cdot (2 - 2 \cdot 4) = 34\alpha - 60.$$

By the global signature formula we have

$$-18\alpha + 32 = (34\alpha - 60) \cdot (-9/17) + \sigma_{\mathcal{R}^4}(\mathcal{R}_3).$$

Finally, the signature of $\widetilde{S}_\Phi$ is zero. This completes the proof. $\qquad\square$

Example 4.3.9 In this last example we do not use the global signature formula (4.1.7) directly. Let $q_1 = x_0{}^2 + x_1{}^2 + x_3{}^2$ and $q_2 = x_1{}^2 + x_2{}^2 - x_3{}^2$, and $h(x_0, x_1, x_2, x_3)$ a generic cubic polynomial. Let

$$S_h = \{(x, z) \in \mathbb{P}_3 \times \Delta : h(x) = (q_1 + zq_2)(x) = 0\},$$

and let $f\colon S_h \to \Delta$ be the second projection. Note that $q_1 + zq_2$ defines a smooth quadric except for $z = 0$ and $f^{-1}(0)$ is contained in the singular quadric $\{q_1 = 0\}$. Thus $\mathcal{F}_R := (S_h, f, \Delta, 0)$ is a $\mathcal{R}^4$–degeneration. Topologically $f\colon S_h \to \Delta$ is a trivial Σ_4–bundle.

Proposition 4.3.10 $\sigma_{\mathcal{R}^4}(\mathcal{F}_R) = \phi_{\mathcal{R}^4}(x_{\mathcal{F}_R}) = 2/17$.

Proof We start from describing the associated principal $\mathcal{G}$–bundle $P(\xi_{\mathcal{F}_R})$.

Let ω_z be the basis of $\Omega^1(f^{-1}(z)) \subset \mathbb{P}_3$ corresponding to the homogeneous coordinates of $\mathbb{P}_3$ (see the paragraph before Theorem 4.2.1). Any frame of $\Omega^1(f^{-1}(z))$ (modulo $\mathbb{C}^*$) is written as the form $A \cdot \omega_z$, $A \in \mathrm{PGL}(4)$. Then

$$P(\xi_{\mathcal{F}_R}) \cong \left\{ (z, A) \in (\Delta \setminus \{0\}) \times \mathrm{PGL}(4); (A^{-1})^t B(z) A^{-1} = H \right\},$$

where
$$B(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+z & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & 1-z \end{pmatrix}.$$

Let $\kappa\colon \Delta \setminus \{0\} \to \Delta \setminus \{0\}$ be the map defined by $w \mapsto z^2$ and consider the pullback $\kappa^* P(\xi_{\mathcal{F}_R})$. Then the principal $\mathcal{G}$–bundle $\kappa^* P(\xi_{\mathcal{F}_R}) \to \Delta \setminus \{0\}$ has a section given by $w \mapsto (w, A(w))$ where

$$A(w) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \sqrt{-1}r_1(w) & 0 & 0 \\ 0 & 0 & w & \sqrt{-1}r_2(w) \\ 0 & 0 & -w & \sqrt{-1}r_2(w) \\ 1 & -\sqrt{-1}r_1(w) & 0 & 0 \end{pmatrix}.$$

Here, $r_1(w) = \sqrt{1 + w^2}$ and $r_2(w) = \sqrt{1 - w^2}$. Thus, as a candidate for $g_{\kappa^* \mathcal{F}_R}$ (see the proof of Proposition 4.1.8), we can take the map given by $w \mapsto [e, A(w) \cdot f(a_0 b_0, a_0 b_1, a_1 b_0, a_1 b_1)]$. Since the diagram

$$\begin{array}{ccc} & & \pi_1(U_{\mathcal{G}}^X) \\ & {\scriptstyle g_{\kappa^* \mathcal{F}_R}} \nearrow & \uparrow {\scriptstyle g_{\mathcal{F}_R}} \\ \pi_1(\Delta \setminus \{0\}) & \xrightarrow{\times 2} & \pi_1(\Delta \setminus \{0\}) \end{array}$$

is commutative up to conjugacy, $x_{\mathcal{F}_R}{}^2$ is conjugate to $x_{\kappa^* \mathcal{F}_R} = g_{\kappa^* \mathcal{F}_R}(\partial \Delta)$.

But now $A(w) \cdot f(a_0 b_0, a_0 b_1, a_1 b_0, a_1 b_1)$ is equal to

$$f\left(\frac{a_0 b_0 + a_1 b_1}{\sqrt{2}}, \frac{a_0 b_0 - a_1 b_1}{\sqrt{-1}r_1(w)}, \frac{a_0 b_1 - a_1 b_0}{w}, \frac{a_0 b_1 + a_1 b_0}{\sqrt{-1}r_2(w)} \right).$$

Modulo $\mathbb{C}^*$ this can be written as

$$(a_0 b_1 - a_1 b_0)^3 + w^3 (\varphi^0 + w\varphi^1 + \text{higher term with respect to } w)$$

where φ^i is some $(3,3)$ homogeneous polynomial. This shows that $x_{\kappa^* \mathcal{F}_R}$ is homotopic to $x_{\mathcal{F}'_R}$ in Proposition 4.3.8. Since $f\colon S_h \to \Delta$ is topologically trivial,

$$4/17 = \phi_{\mathcal{R}^4}(x_{\mathcal{F}'_R}) = \phi_{\mathcal{R}^4}(x_{\kappa^* \mathcal{F}_R}) = \phi_{\mathcal{R}^4}(x_{\mathcal{F}_R}{}^2) = 2\phi_{\mathcal{R}^4}(x_{\mathcal{F}_R}).$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Compare the above proof with [3, Example 7.5], where the same fiber germ is considered.

4.4 Fibrations of nontrigonal curves of genus 5

Let C be a nonhyperelliptic Riemann surface of genus 5. By the Enriques–Petri theorem [10, page 535], the canonical image of C is a $(2, 2, 2)$ complete intersection if and only if C is nontrigonal, ie, there is no holomorphic map $C \to \mathbb{P}_1$ of degree 3. Let $\mathcal{NT}^5 \subset \mathbb{M}_5$ be the set of nonhyperelliptic and nontrigonal Riemann surface of genus 5. $\mathcal{NT}^5$ is Zariski open in $\mathbb{M}_5$.

We denote by $[\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3 : \alpha_4]$ the homogeneous coordinates of $\mathbb{P}_4$ and let S_5 be the space of 5×5 symmetric matrices. The Veronese map $v_2\colon \mathbb{P}_4 \to \mathbb{P}(S_5)$ is given by

$$v_2([\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3 : \alpha_4]) := [(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)^t \cdot (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)].$$

This map is equivariant with respect to the action of $\mathcal{G} = \mathrm{PGL}(5)$, where the action of $\mathcal{G}$ on $\mathbb{P}_4$ is induced by

$$A \cdot (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) A^t$$

for $A \in \mathrm{GL}_5(\mathbb{C})$ and $\alpha_i \in \mathbb{C}$, and the action of $\mathcal{G}$ on $\mathbb{P}(S_5)$ is induced by $A \cdot B = ABA^t$ for $A \in \mathrm{GL}_5(\mathbb{C})$ and $B \in S_5$. Set

$$X := v_2(\mathbb{P}_4).$$

The action of $\mathcal{G}$ on $G_{11}(\mathbb{P}(S_5))$ induced from the $\mathcal{G}$–action on $\mathbb{P}(S_5)$ preserves D_X and $U^X = G_{11}(\mathbb{P}(S_5)) \setminus D_X$, and the projection $p_X\colon \mathcal{C}^X \to U^X$ is $\mathcal{G}$–equivariant. Note that for $W \in U^X$, the fiber $p_X^{-1}(W)$ is isomorphic to a smooth complete intersection in $\mathbb{P}_4$ of type $(2, 2, 2)$. Thus $\xi^X = (\mathcal{C}^X, p_X, U^X)$ is a $\mathcal{NT}^5$–family.

We claim that ξ^X and the $\mathcal{G}$–action on it satisfies the conditions in Proposition 4.1.3. The proof is similar to the case of $\mathcal{R}^4$, so we only describe the way of constructing $\mathcal{G}$–bundles and $\mathcal{G}$–equivariant maps.

Let $C \in \mathcal{NT}^5$. By Max Noether's theorem, the natural map $t_2 \colon \text{Sym}^2 \Omega^1(C) \to H^0(C; K_C^{\otimes 2})$ is surjective hence the kernel is 3–dimensional. By taking the dual, we get the codimension 3 subspace $\text{Ker}(t_2)^* \subset \text{Sym}^2 \Omega^1(C)^*$. If we take a basis $\omega = (\omega_0, \omega_1, \omega_2, \omega_3, \omega_4)$ of $\Omega^1(C)$, $\text{Sym}^2 \Omega^1(C)^*$ is identified with S_5 by assigning $B \in S_5$ with the quadratic function

$$\Omega^1(C) \to \mathbb{C}, \ x_0\omega_0 + \cdots + x_4\omega_4 \mapsto (x_0, \ldots, x_4) B (x_0, \ldots, x_4)^t,$$

and we have the plane $B(\omega) \subset \mathbb{P}(S_5)$ of codimension 3 corresponding to $\text{Ker}(t_2)^*$. Note that the image $v_2 \circ \iota_\omega(C)$, where $\iota_\omega \colon C \to \mathbb{P}_4$ is the canonical map given by $c \mapsto [\omega_0(c) : \cdots : \omega_4(c)]$, is the smooth intersection of X and $B(\omega)$.

Let $P(C)$ be the set of all frames of $\Omega^1(C)$ modulo $\mathbb{C}^*$. By assigning $\omega \in P(C)$ with $B(\omega)$, we have the map $E_C \colon P(C) \to U^X$. Moreover, given the left action of $\mathcal{G}$ on $P(C)$ by the same formula as the action of $\text{PO}_4^H(\mathbb{C})$ on $P(C)$ in Section 4.2, we see that E_C is $\mathcal{G}$–equivariant.

Let $\xi = (\mathcal{C}, p, B)$ be a $\mathcal{NT}^5$–family. Applying the above construction to all the fibers, we get a principal $\mathcal{G}$–bundle $P(\xi)$ and a $\mathcal{G}$–equivariant map $E_\xi \colon P(\xi) \to U^X$ what we want.

Theorem 4.4.1 Let $\mathcal{NT}^5$ be the set of nonhyperelliptic and nontrigonal Riemann surfaces of genus 5 and X, $\mathcal{G}$ as above. Then $\xi_\mathcal{G}^X = (\mathcal{C}_\mathcal{G}^X, p_u, U_\mathcal{G}^X)$ is a universal $\mathcal{NT}^5$–family. We denote by $\rho_u \colon \pi_1(U_\mathcal{G}^X) \to \Gamma_5$ the topological monodromy of $p_u \colon \mathcal{C}_\mathcal{G}^X \to U_\mathcal{G}^X$. Then there exists a unique $\mathbb{Q}$–valued 1–cochain $\phi_{\mathcal{NT}^5} \colon \pi_1(U_\mathcal{G}^X) \to \mathbb{Q}$ whose coboundary equals $\rho_u^* \tau_5$.

Proof The proof is the same as the proof of Theorem 4.2.1 except that (4.2.2) is replaced with the exact sequence

$$\pi_1(\mathcal{G}) \cong \mathbb{Z}/5\mathbb{Z} \to \pi_1(U^X) \to \pi_1(U_\mathcal{G}^X) \to *. \qquad \square$$

Corollary 4.4.2 Let $\mathcal{NT}^5$ be the set of nonhyperelliptic and nontrigonal Riemann surfaces of genus 5. Then the formula

$$\sigma_{\mathcal{NT}^5}(\mathcal{F}) = \phi_{\mathcal{NT}^5}(x_\mathcal{F}) + \text{Sign}(S)$$

for $\mathcal{F} = (S, \pi, \Delta, b) \in \mathcal{NT}_{\text{loc}}^5$ (see (4.1.9)) gives a local signature with respect to $\mathcal{NT}^5$.

Lemma 4.4.3 Let $X = v_2(\mathbb{P}_4)$ be as above. Then D_X is a hypersurface and $\deg D_X = 40$. For a lasso σ_X around D_X, we have $\phi_X(\sigma_X) = -1/2$.

Proof This follows from Proposition 3.5.13. $\qquad \square$

Let $\iota\colon \Delta \to G_{11}(\mathbb{P}(S_5))$ be as in Proposition 2.3.8. Then we get a $\mathcal{NT}^5$–degeneration $\iota^*\mathcal{W} \to \Delta$, which we denote by $\mathcal{F}_I$ and call a singular fiber germ of type I.

Proposition 4.4.4 *Let $\mathcal{F}_I \in \mathcal{NT}^5_{\mathrm{loc}}$ be as above. Then*

$$\sigma_{\mathcal{NT}^5}(\mathcal{F}_I) = \phi_{\mathcal{NT}^5}(x_{\mathcal{F}_I}) = -1/2.$$

Proof The proof is similar to the proof of Proposition 4.3.2. □

This fiber germ is expected to play an important role when computing examples as $\mathcal{F}_I \in \mathcal{R}^4_{\mathrm{loc}}$ behaves like an "atomic" germ in Section 4.3, but at the present moment we don't have any example of element of $\mathcal{NT}^5$ other than $\mathcal{F}_I$ whose local signature has been computed.

Final remarks Although the construction of our local signature is purely topological, we have used some algebraic geometry to compute examples. It is an interesting problem to find and compute examples of fiber germs beyond the reach of algebraic geometry, or to give a formula for the Meyer functions ϕ_X as Meyer and Atiyah did. To do this we need a greater understanding of ρ_X or the topological monodromy ρ_u of a universal $\mathcal{A}$–family.

In the case of $\mathcal{A} = \mathcal{R}^4$ or $\mathcal{NT}^5$, $\mathcal{A}$ is Zariski open in the moduli space. Using this, we can prove that ρ_u is surjective. The proof is similar to the proof of [16, Proposition 6.3]. Here is an outline. Let $\mathcal{T}_g$ be the Teichmüller space of genus g, and let $\widetilde{\mathcal{A}} \subset \mathcal{T}_g$ be the inverse image of $\mathcal{A}$ by the quotient map $\mathcal{T}_g \to \mathbb{M}_g$. Then $\widetilde{\mathcal{A}}$ is Zariski open hence path connected, and is preserved by the action of Γ_g on $\mathcal{T}_g$. By a natural way we get an $\mathcal{A}$–family on the Borel construction $\widetilde{\mathcal{A}}_{\Gamma_g}$, which is easily seen to be universal. The homotopy exact sequence

$$\pi_1(\widetilde{\mathcal{A}}_{\Gamma_g}) \to \pi_1(B\Gamma_g) = \Gamma_g \to \pi_0(\widetilde{\mathcal{A}}) = *$$

of the $\widetilde{\mathcal{A}}$–bundle $\widetilde{\mathcal{A}}_{\Gamma_g} \to B\Gamma_g$ shows the desired surjectivity.

References

[1]   **T Ashikaga**, *Normal two-dimensional hypersurface triple points and the Horikawa type resolution*, Tohoku Math. J. (2) 44 (1992) 177–200   MR1161610

[2]   **T Ashikaga**, **K Konno**, *Global and local properties of pencils of algebraic curves*, from: "Algebraic geometry 2000, Azumino (Hotaka)", (S Usui, M Green, L Illusie, K Kato, E Looijenga, S Mukai, S Saito, editors), Adv. Stud. Pure Math. 36, Math. Soc. Japan, Tokyo (2002) 1–49   MR1971511

[3]   **T Ashikaga**, **K-I Yoshikawa**, *A divisor on the moduli space of curves associated to the signature of fibered surfaces (with an appendix by Kazuhiro Konno [2604074])*, from: "Singularities—Niigata–Toyama 2007", (J-P Brasselet, S Ishii, T Suwa, M Vaquie, editors), Adv. Stud. Pure Math. 56, Math. Soc. Japan, Tokyo (2009) 1–34   MR2604073

[4]   **M Atiyah**, *The logarithm of the Dedekind η–function*, Math. Ann. 278 (1987) 335–380   MR909232

[5]   **W P Barth**, **K Hulek**, **C A M Peters**, **A Van de Ven**, *Compact complex surfaces*, second edition, Ergebnisse der Math. und ihrer Grenzgebiete. 3. Folge. 4, Springer, Berlin (2004)   MR2030225

[6]   **J A Carlson**, **D Toledo**, *Discriminant complements and kernels of monodromy representations*, Duke Math. J. 97 (1999) 621–648   MR1682991

[7]   **I Dolgachev**, **A Libgober**, *On the fundamental group of the complement to a discriminant variety*, from: "Algebraic geometry (Chicago, Ill., 1980)", (A Libgober, P Wagreich, editors), Lecture Notes in Math. 862, Springer, Berlin (1981) 1–25   MR644816

[8]   **H Endo**, *Meyer's signature cocycle and hyperelliptic fibrations*, Math. Ann. 316 (2000) 237–257   MR1741270

[9]   **I M Gel′fand**, **M M Kapranov**, **A V Zelevinsky**, *Discriminants, resultants, and multidimensional determinants*, Math.: Theory & Appl., Birkhäuser, Boston (1994)   MR1264417

[10]   **P Griffiths**, **J Harris**, *Principles of algebraic geometry*, Wiley Classics Library, Wiley, New York (1994)   MR1288523   Reprint of the 1978 original

[11]   **M Gromov**, *Volume and bounded cohomology*, Inst. Hautes Études Sci. Publ. Math. (1982) 5–99 (1983)   MR686042

[12]   **J Harer**, *The second homology group of the mapping class group of an orientable surface*, Invent. Math. 72 (1983) 221–239   MR700769

[13]   **E Horikawa**, *On deformations of quintic surfaces*, Proc. Japan Acad. 49 (1973) 377–379   MR0330173

[14]   **S Iida**, *Adiabatic limits of η–invariants and the Meyer functions*, Math. Ann. 346 (2010) 669–717   MR2578566

[15] **N Katz**, *Pinceaux de Lefschetz: théoréme d'existence*, from: "Groupes de monodromie en géométrie algébrique. II", Lecture Notes in Math. 340, Springer, Berlin (1973) 212–253 MR0354657 Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 II)

[16] **Y Kuno**, *The mapping class group and the Meyer function for plane curves*, Math. Ann. 342 (2008) 923–949 MR2443768

[17] **K Lamotke**, *The topology of complex projective varieties after S Lefschetz*, Topology 20 (1981) 15–51 MR592569

[18] **Y Matsumoto**, *On 4–manifolds fibered by tori*, Proc. Japan Acad. Ser. A Math. Sci. 58 (1982) 298–301 MR682687

[19] **Y Matsumoto**, *Lefschetz fibrations of genus two—a topological approach*, from: "Topology and Teichmüller spaces (Katinkulta, 1995)", (S Kojima, Y Matsumoto, K Saito, M Seppälä, editors), World Sci. Publ., River Edge, NJ (1996) 123–148 MR1659687

[20] **W Meyer**, *Die Signatur von Flächenbündeln*, Math. Ann. 201 (1973) 239–264 MR0331382

[21] **E Y Miller**, *The homology of the mapping class group*, J. Differential Geom. 24 (1986) 1–14 MR857372

[22] **T Morifuji**, *On Meyer's function of hyperelliptic mapping class groups*, J. Math. Soc. Japan 55 (2003) 117–129 MR1939188

[23] **S Morita**, *Characteristic classes of surface bundles*, Invent. Math. 90 (1987) 551–577 MR914849

[24] **D Mumford**, *Algebraic geometry. I. Complex projective varieties*, Grund. der Math. Wissenschaften 221, Springer, Berlin (1976) MR0453732

[25] **D Mumford**, *Towards an enumerative geometry of the moduli space of curves*, from: "Arithmetic and geometry, Vol. II", (M Artin, J Tate, editors), Progr. Math. 36, Birkhäuser, Boston (1983) 271–328 MR717614

[26] **I Shimada**, *Generalized Zariski–van Kampen theorem and its application to Grassmannian dual varieties*, Internat. J. Math. 21 (2010) 591–637 MR2650851

[27] **V G Turaev**, *First symplectic Chern class and Maslov indices*, J. Soviet Math. 37 (1987) 1115–1127 MR806563

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