Relative systoles of relative-essential 2–complexes

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We prove a systolic inequality for a ϕ–relative systole of a ϕ–essential 2–complex X, where $\phi\colon \pi_1(X) \to G$ is a homomorphism to a finitely presented group G. Thus, we show that universally for any ϕ–essential Riemannian 2–complex X, and any G, the following inequality is satisfied: $\mathrm{sys}(X,\phi)^2 \le 8\mathrm{Area}(X)$. Combining our results with a method of L Guth, we obtain new quantitative results for certain 3–manifolds: in particular for the Poincaré homology sphere Σ, we have $\mathrm{sys}(\Sigma)^3 \le 24\mathrm{Vol}(\Sigma)$.

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1 Relative systoles

We prove a systolic inequality for a ϕ–relative systole of a ϕ–essential 2–complex X, where $\phi\colon \pi_1(X) \to G$ is a homomorphism to a finitely presented group G. Thus, we show that universally for any ϕ–essential Riemannian 2–complex X, and any G, we have $\mathrm{sys}(X,\phi)^2 \le 8\,\mathrm{Area}(X)$. Combining our results with a method of L Guth, we obtain new quantitative results for certain 3–manifolds: in particular for the Poincaré homology sphere Σ, we have $\mathrm{sys}(\Sigma)^3 \le 24\,\mathrm{Vol}(\Sigma)$. To state the results more precisely, we need the following definition.

Let X be a finite connected 2–complex. Let $\phi\colon \pi_1(X) \to G$ be a group homomorphism. Recall that ϕ induces a classifying map (defined up to homotopy) $X \to K(G,1)$.

Definition 1.1 The complex X is called *ϕ–essential* if the classifying map $X \to K(G,1)$ cannot be homotoped into the 1–skeleton of $K(G,1)$.

Definition 1.2 Given a piecewise smooth Riemannian metric on X, the ϕ–relative systole of X, denoted $\mathrm{sys}(X,\phi)$, is the least length of a loop of X whose free homotopy class is mapped by ϕ to a nontrivial class.

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When ϕ is the identity homomorphism of the fundamental group, the relative systole is simply called the systole, and denoted $\operatorname{sys}(X)$.

Definition 1.3 The ϕ–systolic area $\sigma_\phi(X)$ of X is defined as

$$\sigma_\phi(X) = \frac{\operatorname{Area}(X)}{\operatorname{sys}(X,\phi)^2}.$$

Furthermore, we set

$$\sigma_*(G) = \inf_{X,\phi} \sigma_\phi(X),$$

where the infimum is over all ϕ–essential piecewise Riemannian finite connected 2–complexes X, and homomorphisms ϕ with values in G.

In the present text, we prove a systolic inequality for the ϕ–relative systole of a ϕ–essential 2–complex X. More precisely, in the spirit of Guth's text [18], we prove a stronger, *local* version of such an inequality, for almost extremal complexes with minimal first Betti number. Namely, if X has a minimal first Betti number among all ϕ–essential piecewise Riemannian 2–complexes satisfying $\sigma_\phi(X) \le \sigma_*(G) + \varepsilon$ for an $\varepsilon > 0$, then the area of a suitable disk of X is comparable to the area of a Euclidean disk of the same radius, in the sense of the following result.

Theorem 1.4 *Let $\varepsilon > 0$. Suppose X has a minimal first Betti number among all ϕ–essential piecewise Riemannian 2–complexes satisfying $\sigma_\phi(X) \le \sigma_*(G) + \varepsilon$. Then each ball centered at a point x on a ϕ–systolic loop in X satisfies the area lower bound*

$$\operatorname{Area} B(x,r) \ge \frac{(r - \varepsilon^{1/3})^2}{2 + \varepsilon^{1/3}}$$

whenever r satisfies $\varepsilon^{1/3} \le r \le \frac{1}{2}\operatorname{sys}(X,\phi)$.

A more detailed statement appears in Proposition 8.2. The theorem immediately implies the following systolic inequality.

Corollary 1.5 *Every finitely presented group G satisfies*

$$\sigma_*(G) \ge \frac{1}{8},$$

so that every piecewise Riemannian ϕ–essential 2–complex X satisfies the inequality

$$\operatorname{sys}(X,\phi)^2 \le 8 \operatorname{Area}(X).$$

In the case of the absolute systole, we prove a similar lower bound with a Euclidean exponent for the area of a suitable disk, when the radius is smaller than half the systole, without the assumption of near-minimality. Namely, we will prove the following theorem.

Theorem 1.6 *Every piecewise Riemannian essential* 2*–complex* X *admits a point* $x \in X$ *such that the area of the* r*–ball centered at* x *is at least* r^2*, that is,*

$$\text{Area}(B(x,r)) \geq r^2, \tag{1-1}$$

for all $r \leq \frac{1}{2}\,\text{sys}(X)$*.*

We conjecture a bound analogous to (1-1) for the area of a suitable disk of a ϕ–essential 2–complex X, with the ϕ–relative systole replacing the systole; cf the GG–property below. The application we have in mind is in the case when $\phi\colon \pi_1(X) \to \mathbb{Z}_p$ is a homomorphism from the fundamental group of X to a finite cyclic group. Note that the conjecture is true in the case when ϕ is a homomorphism to $\mathbb{Z}_2$, by Guth's result [18].

Definition 1.7 (GG–property[1]) Let $C > 0$. Let X be a finite connected 2–complex, and $\phi\colon \pi_1(X) \to G$, a group homomorphism. We say that X has the GG_C–property for ϕ if every piecewise smooth Riemannian metric on X admits a point $x \in X$ such that the r–ball of X centered at x satisfies the bound

$$\text{Area}\, B(x,r) \geq C r^2, \tag{1-2}$$

for every $r \leq \frac{1}{2}\,\text{sys}(X,\phi)$.

Note that if the 2–complex X is ε–almost minimal, ie, satisfies the bound $\sigma_\phi(X) \leq G_*(G) + \varepsilon$, and has least first Betti number among all such complexes, then it satisfies (1-2) for some $C > 0$ and for $r \geq \varepsilon^{1/3}$ by Theorem 1.4.

Modulo such a conjectured bound, we prove a systolic inequality for closed 3–manifolds with finite fundamental group.

Theorem 1.8 *Let* $p \geq 2$ *be a prime. Assume that every* ϕ*–essential* 2*–complex has the* GG_C*–property* (1-2) *for each homomorphism* ϕ *into* $\mathbb{Z}_p$ *and for some universal constant* $C > 0$*. Then every orientable closed Riemannian* 3*–manifold* M *with finite fundamental group of order divisible by* p *satisfies the bound*

$$\text{sys}(M)^3 \leq 24 C^{-1}\ \text{Vol}(M).$$

More precisely, there is a point $x \in M$ *such that the volume of every* r*–ball centered at* x *is at least* $(C/3)r^3$*, for all* $r \leq \frac{1}{2}\,\text{sys}(M)$*.*

[1] GG–property stands for the property analyzed by M Gromov and L Guth.

A slightly weaker bound can be obtained modulo a weaker GG–property, where the point x is allowed to depend on the radius r.

Since the GG–property is available for $p = 2$ and $C = 1$ by Guth's article [18], we obtain the following corollary.

Corollary 1.9 *Every closed Riemannian* 3*–manifold* M *with fundamental group of even order satisfies*

$$\mathrm{sys}(M)^3 \leq 24\,\mathrm{Vol}(M). \tag{1-3}$$

For example, the Poincaré homology 3–sphere satisfies the systolic inequality (1-3).

In the next section, we present related developments in systolic geometry and compare some of our arguments in the proof of Theorem 1.8 to Guth's in [18]; cf Remark 2.1. Additional recent developments in systolic geometry include Ambrosio and Katz [1], Babenko and Balacheff [3], Balacheff [4], Bangert et al [5], Belolipetsky and Thomson [6], Berger [7], Brunnbauer [9; 9; 10], Dranishnikov, Katz and Rudyak [12], Dranishnikov and Rudyak [13], El Mir [14], El Mir and Lafontaine [15], Guth [18], Katz and Katz [22; 21], Katz [24], Katz and Rudyak [25], Katz, Schaps and Vishne [27], Katz and Shnider [28], Nabutovsky and Rotman [30], Parlier [31], Rotman [33], Rudyak and Sabourau [34], and Sabourau [35; 36].

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2 Recent progress on Gromov's inequality

M Gromov's upper bound for the 1–systole of an essential manifold M [16] is a central result of systolic geometry. Gromov's proof exploits the Kuratowski imbedding of M in the Banach space L^∞ of bounded functions on M. A complete analytic proof of Gromov's inequality [16], but still using the Kuratowski imbedding in L^∞, was recently developed by L Ambrosio and the second author [1]. See also Ambrosio and Wenger [2].

S Wenger [40] gave a complete analytic proof of an isoperimetric inequality between the volume of a manifold M, and its filling volume, a result of considerable independent interest. On the other hand, his result does not directly improve or simplify the proof of Gromov's main filling inequality for the filling radius. Note that both the filling inequality and the isoperimetric inequality are proved simultaneously by Gromov, so that proving the isoperimetric inequality by an independent technique does not directly simplify the proof of either the filling radius inequality, or the systolic inequality.

L Guth [17] gave a new proof of Gromov's systolic inequality in a strengthened *local* form. Namely, he proved Gromov's conjecture that every essential manifold with unit systole contains a ball of unit radius with volume uniformly bounded away from zero.

Most recently, Guth [18] reproved a significant case of Gromov's systolic inequality [16] for essential manifolds, without using Gromov's filling invariants.

Actually, in the case of surfaces, Gromov himself had proved better estimates, without using filling invariants, by sharpening a technique independently due to Y Burago and V Zalgaller [11, page 43] and J Hebda [20]. Here the essential idea is the following.

Let $\gamma(s)$ be a minimizing noncontractible closed geodesic of length L in a surface S, where the arclength parameter s varies through the interval $[-L/2, L/2]$. We consider metric balls (metric disks) $B(p,r) \subset S$ of radius $r < L/2$ centered at $p = \gamma(0)$. The two points $\gamma(r)$ and $\gamma(-r)$ lie on the boundary sphere (boundary curve) $\partial B(p,r)$ of the disk. If the points lie in a common connected component of the boundary (which is necessarily the case if S is a surface and $L = \mathrm{sys}(S)$, but may fail if S is a more general 2–complex), then the boundary curve has length at least $2r$. Applying the coarea formula

$$\text{Area}\, B(p,r) = \int_0^r \text{length}\, \partial B(p,\rho)\, d\rho, \tag{2-1}$$

we obtain a lower bound for the area which is quadratic in r.

Guth's idea is essentially a higher-dimensional analogue of Hebda's, where the minimizing geodesic is replaced by a minimizing hypersurface. Some of Guth's ideas go back to the even earlier texts by Schoen and Yau [37; 38].

The case handled in [18] is that of n–dimensional manifolds of maximal $\mathbb{Z}_2$–cuplength, namely n. Thus, Guth's theorem covers both tori and real projective spaces, directly generalizing the systolic inequalities of Loewner and Pu; see Pu [32] and Katz [23] for details.

Remark 2.1 To compare Guth's argument in his text [18] and our proof of Theorem 1.8, we observe that the topological ingredient of Guth's technique exploits the multiplicative structure of the cohomology ring $H^*(\mathbb{Z}_2;\mathbb{Z}_2) = H^*(\mathbb{RP}^\infty;\mathbb{Z}_2)$. This ring is generated by the 1–dimensional class. Thus, every n–dimensional cohomology class decomposes into the cup product of 1–dimensional classes. This feature enables a proof by induction on n.

Meanwhile, for p odd, the cohomology ring $H^*(\mathbb{Z}_p;\mathbb{Z}_p)$ is not generated by the 1–dimensional class; see Proposition 9.1 for a description of its structure. Actually, the square of the 1–dimensional class is zero, which seems to yield no useful geometric information.

Another crucial topological tool used in the proof of [18] is Poincaré duality which can be applied to the manifolds representing the homology classes in $H_*(\mathbb{Z}_2;\mathbb{Z}_2)$. For p odd, the homology classes of $H_{2k}(\mathbb{Z}_p;\mathbb{Z}_p)$ cannot be represented by manifolds. One could use D Sullivan's notion of $\mathbb{Z}_p$–manifolds (cf [39; 29]) to represent these homology class, but they do not satisfy Poincaré duality.

Finally, we mention that, when working with cycles representing homology classes with torsion coefficients in $\mathbb{Z}_p$, we exploit a notion of volume which ignores the multiplicities in $\mathbb{Z}_p$; cf Definition 10.3. This is a crucial feature in our proof. Note that minimal cycles with torsion coefficients were studied by B White [41].

3 Area of balls in 2–complexes

It was proved in [16] and [26] that a finite 2–complex admits a systolic inequality if and only if its fundamental group is nonfree, or equivalently, if it is ϕ–essential for $\phi = \mathrm{Id}$.

In [26], we used an argument by contradiction, relying on an invariant called *tree energy*, to prove a bound for the systolic ratio of a 2–complex. We present an alternative short proof which yields a stronger result and simplifies the original argument.

Theorem 3.1 *Let X be a piecewise Riemannian finite essential 2–complex. There exists $x \in X$ such that the area of every r–ball centered at x is at least r^2 for every $r \leq \frac{1}{2}\,\mathrm{sys}(X)$.*

As mentioned in the introduction, we conjecture that this result still holds for ϕ–essential complexes and with the ϕ–relative systole in place of sys.

Proof We can write the Grushko decomposition of the fundamental group of X as

$$\pi_1(X) = G_1 * \cdots * G_r * F,$$

where F is free, while each group G_i is nontrivial, nonisomorphic to $\mathbb{Z}$, and not decomposable as a nontrivial free product.

Consider the equivalence class $[G_1]$ of G_1 under external conjugation in $\pi_1(X)$. Let γ be a loop of least length representing a nontrivial class $[\gamma]$ in $[G_1]$. Fix $x \in \gamma$ and a copy of $G_1 \subset \pi_1(X, x)$ containing the homotopy class of γ. Let $\overline{X}$ be the cover of X with fundamental group G_1.

Lemma 3.2 *We have* $\operatorname{sys}(\bar{X}) = \operatorname{length}(\gamma)$.

Proof The loop γ lifts to $\bar{X}$ by construction of the subgroup G_1. Thus, $\operatorname{sys}(\bar{X}) \leq \operatorname{length}(\gamma)$. Now, the cover $\bar{X}$ does not contain noncontractible loops δ shorter than γ, because such loops would project to X so that the nontrivial class $[\delta]$ maps into $[G_1]$, contradicting our choice of γ. □

Continuing with the proof of the theorem, let $\bar{x} \in \bar{X}$ be a lift of x. Consider the level curves of the distance function from $\bar{x}$. Note that such curves are necessarily connected, for otherwise one could split off a free-product-factor $\mathbb{Z}$ in $\pi_1(\bar{X}) = G_1$ (cf [26, Proposition 7.5]) contradicting our choice of G_1. In particular, the points $\gamma(r)$ and $\gamma(-r)$ can be joined by a path contained in the curve at level r. Applying the coarea formula (2-1), we obtain a lower bound $\operatorname{Area} B(\bar{x}, r) \geq r^2$ for the area of an r–ball $B(\bar{x}, r) \subset \bar{X}$, for all $r \leq \frac{1}{2}\operatorname{length}(\gamma) = \frac{1}{2}\operatorname{sys}(\bar{X})$.

If, in addition, we have $r \leq \frac{1}{2}\operatorname{sys}(X)$ (which apriori might be smaller than $\frac{1}{2}\operatorname{sys}(\bar{X})$), then the ball projects injectively to X, proving that

$$\operatorname{Area}(B(x, r) \subset X) \geq r^2$$

for all $r \leq \frac{1}{2}\operatorname{sys}(X)$. □

4 Outline of argument for relative systole

Let X be a piecewise Riemannian connected 2–complex, and assume X is ϕ–essential for a group homomorphism $\phi\colon \pi_1(X) \to G$. We would like to prove an area lower bound for X, in terms of the ϕ–relative systole as in Theorem 3.1. Let $x \in X$. Denote by $B = B(x, r)$ and $S = S(x, r)$ the open ball and the sphere (level curve) of radius r centered at x with $r < \frac{1}{2}\operatorname{sys}(X, \phi)$. Consider the interval $I = [0, L/2]$, where $L = \operatorname{length}(S)$.

Definition 4.1 We consider the complement $X \setminus B$, and attach to it a buffer cylinder along each connected component S_i of S. Here a buffer cylinder with base S_i is the quotient

$$S_i \times I/\sim$$

where the relation $\sim$ collapses each subset $S_i \times \{0\}$ to a point x_i. We thus obtain the space

$$(S_i \times I/\sim) \cup_f (X \setminus B),$$

where the attaching map f identifies $S_i \times \{L/2\}$ with $S_i \subset X \setminus B$. To ensure the connectedness of the resulting space, we attach a cone CA over the set of points $A = \{x_i\}$. We set the length of the edges of the cone CA equal to $\operatorname{sys}(X, \phi)$. We will denote by

$$Y = Y(x, r) \tag{4-1}$$

the resulting 2–complex. The natural metrics on $X \setminus B$ and on the buffer cylinders induce a metric on Y.

In Section 5, we show that Y is ψ–essential for some homomorphism $\psi\colon \pi_1(Y) \to G$ derived from ϕ. The purpose of the buffer cylinder is to ensure that the relative systole of Y is at least as large as the relative systole of X. Note that the area of the buffer cylinder is $L^2/2$.

We normalize X to unit relative systole and take a point x on a relative systolic loop of X. Suppose X has a minimal first Betti number among the complexes essential in $K(G, 1)$ with almost minimal systolic area (up to epsilon). We sketch below the proof of the local relative systolic inequality satisfied by X.

If for every r, the space $Y = Y(x, r)$ has a greater area than X, then

$$\text{Area}\, B(r) \leq \tfrac{1}{2}(\text{length}\, S(r))^2$$

for every $r < \frac{1}{2} \operatorname{sys}(X, \phi)$. Using the coarea inequality, this leads to the differential inequality $y(r) \leq \frac{1}{2} y'(r)^2$. Integrating this relation shows that the area of $B(r)$ is at least $r^2/2$, and the conclusion follows.

If for some r, the space Y has a smaller area than X, we argue by contradiction. We show that a ϕ–relative systolic loop of X (passing through x) meets at least two connected components of the level curve $S(r)$. These two connected components project to two endpoints of the cone CA connected by an arc of $Y \setminus CA$. Under this condition, we can remove an edge e from CA so that the space $Y' = Y \setminus e$ has a smaller first Betti number than X. Here Y' is still essential in $K(G, 1)$, and its relative systolic area is better than the relative systolic area of X, contradicting the definition of X.

5 First Betti number and essentialness of Y

Let G be a fixed finitely presented group. We are mostly interested in the case of a finite group $G = \mathbb{Z}_p$. Unless specified otherwise, all group homomorphisms have values in G, and all complexes are assumed to be finite. Consider a homomorphism $\phi\colon \pi_1(X) \to G$ from the fundamental group of a piecewise Riemannian finite connected 2–complex X to G.

Definition 5.1 A loop γ in X is said to be ϕ–contractible if the image of the homotopy class of γ by ϕ is trivial, and ϕ–noncontractible otherwise. Thus, the ϕ–systole of X, denoted by $\mathrm{sys}(X,\phi)$, is defined as the least length of a ϕ–noncontractible loop in X. Similarly, the ϕ–systole based at a point x of X, denoted by $\mathrm{sys}(X,\phi,x)$, is defined as the least length of a ϕ–noncontractible loop based at x.

The following elementary result will be used repeatedly in the sequel.

Lemma 5.2 *If $r < \frac{1}{2}\,\mathrm{sys}(X,\phi,x)$, then the π_1–homomorphism i_* induced by the inclusion $B(x,r) \subset X$ is trivial when composed with ϕ, that is $\phi \circ i_* = 0$. More specifically, every loop in $B(x,r)$ is homotopic to a composition of loops based at x of length at most $2r+\varepsilon$, for every $\varepsilon > 0$.*

Without loss of generality, we may assume that the piecewise Riemannian metric on X is piecewise flat. Let $x_0 \in X$. The piecewise flat 2–complex X can be embedded into some $\mathbb{R}^N$ as a semialgebraic set and the distance function f from x_0 is a continuous semialgebraic function on X (cf [8]). Thus, (X,B) is a CW–pair when B is a ball centered at x_0 (see also [26, Corollary 6.8]). Furthermore, for almost every r, there exists a $\eta > 0$ such that the set

$$\{x \in X \mid r-\eta < f(x) < r+\eta\}$$

is homeomorphic to $S(x_0,r)\times(r-\eta,r+\eta)$ where $S(x_0,r)$ is the r–sphere centered at x_0 and the t–level curve of f corresponds to $S(x_0,r)\times\{t\}$; see [8, § 9.3] and [26] for a precise description of level curves on X. In such case, we say that r is a *regular value* of f.

Consider the connected 2–complex $Y = Y(x_0,r)$ introduced in Definition 4.1, with $r < \frac{1}{2}\,\mathrm{sys}(X,\phi)$ and r regular. Since r is a regular value, there exists $r_- \in (0,r)$ such that $B \setminus B(x_0,r_-)$ is homeomorphic to the product

$$S\times[r_-,r) = \coprod_i S_i \times [r_-,r).$$

Consider the map

$$\pi\colon X \to Y \tag{5-1}$$

which leaves $X\setminus B$ fixed, takes $B(x_0,r_-)$ to the vertex of the cone CA, and sends $B\setminus B(x_0,r_-)$ to the union of the buffer cylinders and CA. This map induces an epimorphism between the first homology groups. In particular,

$$b_1(Y) \le b_1(X). \tag{5-2}$$

Lemma 5.3 *If $r < \frac{1}{2}\,\mathrm{sys}(X,\phi)$, then Y is ψ–essential for some homomorphism $\psi\colon \pi_1(Y) \to G$ such that*

$$\psi \circ \pi_* = \phi \tag{5-3}$$

where π_* is the π_1–homomorphism induced by $\pi\colon X \to Y$.

Proof Consider the CW–pair (X, B) where $B = B(x_0, r)$. By Lemma 5.2, the restriction of the classifying map $\varphi\colon X \to K(G,1)$ induced by ϕ to B is homotopic to a constant map. Thus, the classifying map φ extends to $X \cup CB$ and splits into

$$X \hookrightarrow X \cup CB \to K(G,1),$$

where CB is a cone over $B \subset X$ and the first map is the inclusion map. Since $X \cup CB$ is homotopy equivalent to the quotient X/B (see Hatcher [19, Example 0.13]), we obtain the following decomposition of φ up to homotopy:

$$X \xrightarrow{\pi} Y \to X/B \to K(G,1). \tag{5-4}$$

Hence, $\psi \circ \pi_* = \phi$ for the π_1–homomorphism $\psi\colon \pi_1(Y) \to G$ induced by the map $Y \to K(G,1)$. If the map $Y \to K(G,1)$ can be homotoped into the 1–skeleton of $K(G,1)$, the same is true for

$$X \to Y \to K(G,1)$$

and so for the homotopy equivalent map φ, which contradicts the ϕ–essentialness of X. $\square$

6 Exploiting a "fat" ball

We normalize the ϕ–relative systole of X to one, ie $\mathrm{sys}(X,\phi) = 1$. Choose a fixed $\delta \in (0, \frac{1}{2})$ (close to 0) and a real parameter $\lambda > \frac{1}{2}$ (close to $\frac{1}{2}$).

Proposition 6.1 *Suppose there exist a point $x_0 \in X$ and a value $r_0 \in (\delta, \frac{1}{2})$ regular for f such that*

$$\mathrm{Area}\, B > \lambda(\mathrm{length}\, S)^2 \tag{6-1}$$

where $B = B(x_0, r_0)$ and $S = S(x_0, r_0)$. Then there exists a piecewise flat metric on $Y = Y(x_0, r_0)$ such that the systolic areas (see Definition 1.3) satisfy

$$\sigma_\psi(Y) \le \sigma_\phi(X).$$

Proof Consider the metric on Y described in Definition 4.1. Strictly speaking, the metric on Y is not piecewise flat since the connected components of S are collapsed to points, but it can be approximated by piecewise flat metrics.

Due to the presence of the buffer cylinders, every loop of Y of length less than $\mathrm{sys}(X,\phi)$ can be deformed into a loop of $X \setminus B$ without increasing its length. Thus, by (5-3), one obtains

$$\mathrm{sys}(Y,\psi) \geq \mathrm{sys}(X,\phi) = 1.$$

Furthermore, we have

$$\mathrm{Area}\, Y \leq \mathrm{Area}\, X - \mathrm{Area}\, B + \tfrac{1}{2}(\mathrm{length}\, S)^2.$$

Combined with the inequality (6-1), this leads to

$$\sigma_\psi(Y) < \sigma_\phi(X) - (\lambda - \tfrac{1}{2})(\mathrm{length}\, S)^2. \tag{6-2}$$

Hence, $\sigma_\psi(Y) \leq \sigma_\phi(X)$, since $\lambda > \frac{1}{2}$. □

7 An integration by separation of variables

Let X be a piecewise Riemannian finite connected 2–complex. Let $\phi\colon \pi_1(X) \to G$ be a nontrivial homomorphism to a group G. We normalize the metric to unit relative systole: $\mathrm{sys}(X,\phi) = 1$. The following area lower bound appeared in [34, Lemma 7.3].

Lemma 7.1 *Let $x \in X$, $\lambda > 0$ and $\delta \in (0, \frac{1}{2})$. If*

$$\mathrm{Area}\, B(x,r) \leq \lambda(\mathrm{length}\, S(x,r))^2 \tag{7-1}$$

for almost every $r \in (\delta, \frac{1}{2})$, then

$$\mathrm{Area}\, B(x,r) \geq \frac{1}{4\lambda}(r-\delta)^2$$

for every $r \in (\delta, \frac{1}{2})$.

In particular,
$$\mathrm{Area}(X) \geq \frac{1}{16\lambda}\,\mathrm{sys}(X,\phi)^2.$$

Proof By the coarea formula, we have

$$a(r) := \mathrm{Area}\, B(x,r) = \int_0^r \ell(s)\, ds$$

where $\ell(s) = \text{length}\, S(x,s)$. Since the function $\ell(r)$ is piecewise continuous, the function $a(r)$ is continuously differentiable for all but finitely many r in $(0, \frac{1}{2})$ and $a'(r) = \ell(r)$ for all but finitely many r in $(0, \frac{1}{2})$. By hypothesis, we have

$$a(r) \leq \lambda a'(r)^2$$

for all but finitely many r in $(\delta, \frac{1}{2})$. That is,

$$(\sqrt{a(r)})' = \frac{a'(r)}{2\sqrt{a(r)}} \geq \frac{1}{2\sqrt{\lambda}}.$$

We now integrate this differential inequality from δ to r, to obtain

$$\sqrt{a(r)} \geq \frac{1}{2\sqrt{\lambda}}(r-\delta).$$

Hence, for every $r \in (\delta, \frac{1}{2})$, we obtain

$$a(r) \geq \frac{1}{4\lambda}(r-\delta)^2,$$

completing the proof. □

8 Proof of relative systolic inequality

We prove that if X is a ϕ–essential piecewise Riemannian 2–complex which is almost minimal (up to ε), and has least first Betti number among such complexes, then X possesses an r–ball of large area for each $r < \frac{1}{2}\,\text{sys}(X,\phi)$. We have not been able to find such a ball for an arbitrary ϕ–essential complex (without the assumption of almost minimality), but at any rate the area lower bound for almost minimal complexes suffices to prove the ϕ–systolic inequality for all ϕ–essential complexes, as shown below.

Remark 8.1 We do not assume at this point that $\sigma_*(G)$ is nonzero; cf Definition 1.3. In fact, the proof of $\sigma_*(G) > 0$ does not seem to be any easier than the explicit bound of Corollary 1.5.

Theorem 1.4 and Corollary 1.5 are consequences of the following result.

Proposition 8.2 *Let $\varepsilon > 0$. Suppose X has a minimal first Betti number among all ϕ–essential piecewise Riemannian 2–complexes satisfying*

$$\sigma_\phi(X) \leq \sigma_*(G) + \varepsilon. \tag{8-1}$$

Then each ball centered at a point x on a ϕ–systolic loop in X satisfies the area lower bound

$$\operatorname{Area} B(x,r) \geq \frac{(r-\delta)^2}{2+\varepsilon/\delta^2}$$

for every $r \in (\delta, \frac{1}{2}\operatorname{sys}(X,\phi))$, where $\delta \in (0, \frac{1}{2}\operatorname{sys}(X,\phi))$. In particular, we obtain the bound

$$\sigma_*(G) \geq \frac{1}{8}.$$

Proof We will use the notation and results of the previous sections. Choose $\lambda > 0$ such that

$$\varepsilon < 4(\lambda - \tfrac{1}{2})\delta^2. \tag{8-2}$$

That is,

$$\lambda > \frac{1}{2} + \frac{\varepsilon}{4\delta^2} \quad \Big(\text{close to } \frac{1}{2} + \frac{\varepsilon}{4\delta^2}\Big).$$

We normalize the metric on X so that its ϕ–systole is equal to one. Choose a point $x_0 \in X$ on a ϕ–systolic loop γ of X.

If the balls centered at x_0 are too "thin", ie, the inequality (7-1) is satisfied for x_0 and almost every $r \in (\delta, \frac{1}{2})$, then the result follows from Lemma 7.1.

We can therefore assume that there exists a "fat" ball centered at x_0, ie, the hypothesis of Proposition 6.1 holds for x_0 and some regular f–value $r_0 \in (\delta, \frac{1}{2})$, where f is the distance function from x_0. (Indeed, almost every r is regular for f.) Arguing by contradiction, we show that the assumption on the minimality of the first Betti number rules out this case.

We would like to construct a ψ–essential piecewise flat 2–complex Y' with $b_1(Y') < b_1(X)$ such that $\sigma_\psi(Y') \leq \sigma_\phi(X)$ and therefore

$$\sigma_\psi(Y') \leq \sigma_*(G) + \varepsilon \tag{8-3}$$

for some homomorphism $\psi\colon \pi_1(Y') \to G$.

By Lemma 5.3 and Proposition 6.1, the space $Y = Y(x_0, r_0)$, endowed with the piecewise Riemannian metric of Proposition 6.1, satisfies

$$\sigma_*(G) \leq \sigma_\psi(Y) \leq \sigma_\phi(X).$$

Combined with the inequalities (6-2) in the proof of Proposition 6.1 and (8-1), this yields

$$\left(\lambda - \frac{1}{2}\right)(\operatorname{length} S)^2 < \varepsilon.$$

From $\varepsilon < 4(\lambda - \frac{1}{2})\delta^2$ and $\delta \leq r_0$, we deduce that

$$\operatorname{length} S < 2r_0.$$

Now, by Lemma 5.2, the ϕ–systolic loop $\gamma \subset X$ does not entirely lie in B. Therefore, there exists an arc α_0 of γ passing through x_0 and lying in B with endpoints in S. We have

$$\operatorname{length}(\alpha_0) \geq 2r_0.$$

If the endpoints of α_0 lie in the same connected component of S, then we can join them by an arc $\alpha_1 \subset S$ of length less than $2r_0$. By Lemma 5.2, the loop $\alpha_0 \cup \alpha_1$, lying in B, is ϕ–contractible. Therefore, the loop $\alpha_1 \cup (\gamma \setminus \alpha_0)$, which is shorter than γ, is ϕ–noncontractible. Hence a contradiction.

This shows that the ϕ–systolic loop γ of X meets two connected components of S.

Since a ϕ–systolic loop is length-minimizing, the loop γ intersects S exactly twice. Therefore, the complementary arc $\alpha = \gamma \setminus \alpha_0$, joining two connected components of S, lies in $X \setminus B$. The two endpoints of α are connected by a length-minimizing arc of $Y \setminus (X \setminus \bar{B})$ passing exactly through two edges of the cone CA.

Let Y' be the 2–complex obtained by removing the interior of one of these two edges from Y. The complex $Y' = Y \setminus e$ is clearly connected and the space Y, obtained by gluing back the edge e to Y, is homotopy equivalent to $Y' \vee S^1$. That is,

$$Y \simeq Y' \vee S^1. \tag{8-4}$$

Thus, Y' is ψ–essential if we still denote by ψ the restriction of the homomorphism $\psi\colon \pi_1(Y) \to G$ to $\pi_1(Y')$. Furthermore, we clearly have

$$\sigma_\psi(Y') = \sigma_\psi(Y) \leq \sigma_\phi(X).$$

Combined with (5-2), the homotopy equivalence (8-4) also implies

$$b_1(Y') < b_1(Y) \leq b_1(X).$$

Hence the result. □

Remark 8.3 We could use round metrics (of constant positive Gaussian curvature) on the "buffer cylinders" of the space Y in the proof of Proposition 6.1. This would allow us to choose λ close to $1/(2\pi)$ and to derive the lower bound of $\pi/8$ for $\sigma_\phi(X)$ in Corollary 1.5. We chose to use flat metrics for the sake of simplicity.

9 Cohomology of Lens spaces

Let p be a prime number. The group $G = \mathbb{Z}_p$ acts freely on the contractible sphere $S^{2\infty+1}$ yielding a model for the classifying space

$$K = K(\mathbb{Z}_p, 1) = S^{2\infty+1}/\mathbb{Z}_p.$$

The following facts are well-known; see Hatcher [19].

Proposition 9.1 *The cohomology ring $H^*(\mathbb{Z}_p;\mathbb{Z}_p)$ for p an odd prime is the algebra $\mathbb{Z}_p(\alpha)[\beta]$ which is exterior on one generator α of degree 1, and polynomial with one generator β of degree 2. Thus,*

- *α is a generator of $H^1(\mathbb{Z}_p;\mathbb{Z}_p) \simeq \mathbb{Z}_p$, satisfying $\alpha^2 = 0$;*
- *β is a generator of $H^2(\mathbb{Z}_p;\mathbb{Z}_p) \simeq \mathbb{Z}_p$.*

Here the 2–dimensional class is the image under the Bockstein homomorphism of the 1–dimensional class. The cohomology of the cyclic group is generated by these two classes. The cohomology is periodic with period 2 by Tate's theorem. Every even-dimensional class is proportional to β^n. Every odd-dimensional class is proportional to $\alpha \cup \beta^n$.

Furthermore, the reduced integral homology is $\mathbb{Z}_p$ in odd dimensions and vanishes in even dimensions. The integral cohomology is $\mathbb{Z}_p$ in even positive dimensions, generated by a lift of the class β above to $H^2(\mathbb{Z}_p;\mathbb{Z})$.

Proposition 9.2 *Let M be a closed 3–manifold M with $\pi_1(M) = \mathbb{Z}_p$. Then its classifying map $\varphi\colon M \to K$ induces an isomorphism*

$$\varphi_i\colon H_i(M;\mathbb{Z}_p) \simeq H_i(K;\mathbb{Z}_p)$$

for $i = 1, 2, 3$.

Proof Since M is covered by the sphere, for $i = 2$ the isomorphism is a special case of Whitehead's theorem. Now consider the exact sequence (of Hopf type)

$$\pi_3(M) \xrightarrow{\times p} H_3(M;\mathbb{Z}) \to H_3(\mathbb{Z}_p;\mathbb{Z}) \to 0$$

since $\pi_2(M) = 0$. Since the homomorphism $H_3(M;\mathbb{Z}) \to H_3(\mathbb{Z}_p;\mathbb{Z})$ is onto, the result follows by reduction modulo p. □

10 Volume of a ball

Our Theorem 1.8 is a consequence of the following result.

Theorem 10.1 *Assume the* GG_C*–property* (1-2) *is satisfied for some universal constant* $C > 0$ *and every homomorphism* ϕ *into a finite group* G*. Then every closed Riemannian* 3*–manifold* M *with fundamental group* G *contains a metric ball* $B(R)$ *of radius* R *satisfying*

$$\mathrm{Vol}\, B(R) \geq \frac{C}{3} R^3, \tag{10-1}$$

for every $R \leq \frac{1}{2}\,\mathrm{sys}(M)$.

We will first prove Theorem 10.1 for a closed 3–manifold M of fundamental group $\mathbb{Z}_p$, with p prime. We assume that p is odd (the case $p = 2$ was treated by L Guth). In particular, M is orientable. Let D be a 2–cycle representing a nonzero class $[D]$ in

$$H_2(M;\mathbb{Z}_p) \simeq H_1(M;\mathbb{Z}_p) \simeq \mathbb{Z}_p.$$

Denote by D_0 the finite 2–complex of M given by the support of D. Without loss of generality, we can assume that D_0 is connected. The restriction of the classifying map $\varphi\colon M \to K$ to D_0 induces a homomorphism $\phi\colon \pi_1(D_0) \to \mathbb{Z}_p$.

Lemma 10.2 *The cycle* D *induces a trivial relative class in the homology of every metric* R*–ball* B *in* M *relative to its boundary, with* $R < \frac{1}{2}\,\mathrm{sys}(M)$*. That is,*

$$[D \cap B] = 0 \in H_2(B, \partial B; \mathbb{Z}_p).$$

Proof Suppose the contrary. By the Lefschetz–Poincaré duality theorem, the relative 2–cycle $D \cap B$ in B has a nonzero intersection with an (absolute) 1–cycle c of B. Thus, the intersection between the 2–cycle D and the 1–cycle c is nontrivial in M. Now, by Lemma 5.2, the 1–cycle c is homotopically trivial in M. Hence a contradiction. □

We will exploit the following notion of volume for cycles with torsion coefficients.

Definition 10.3 Let D be a k–cycle with coefficients in $\mathbb{Z}_p$ in a Riemannian manifold M. We have

$$D = \sum_i n_i \sigma_i \tag{10-2}$$

where each σ_i is a k–simplex, and each $n_i \in \mathbb{Z}_p^*$ is assumed nonzero. We define the notion of k–area Area for cycles as in (10-2) by setting

$$\mathrm{Area}(D) = \sum_i |\sigma_i|, \tag{10-3}$$

where $|\sigma_i|$ is the k–area induced by the Riemannian metric of M.

Remark 10.4 The nonzero coefficients n_i in (10-2) are ignored in defining this notion of volume.

Proof of Theorem 10.1 We continue the proof of Theorem 10.1 when the fundamental group of M is isomorphic to $\mathbb{Z}_p$, with p an odd prime. We will use the notation introduced earlier. Suppose now that D is a piecewise smooth 2–cycle area minimizing in its homology class $[D] \neq 0 \in H_2(M;\mathbb{Z}_p)$ up to an arbitrarily small error term $\varepsilon > 0$, for the notion of volume (area) as defined in (10-3).

Recall that $\phi\colon \pi_1(D_0) \to \mathbb{Z}_p$ is the homomorphism induced by the restriction of the classifying map $\varphi\colon K \to M$ to the support D_0 of D. By Proposition 9.2, the 2–complex D_0 is ϕ–essential. Thus, by hypothesis of Theorem 10.1, we can choose a point $x \in D_0$ satisfying the GG_C–property (1-2), ie, the area of R–balls in D_0 centered at x grows at least as CR^2 for $R < \frac{1}{2}\operatorname{sys}(D_0,\phi)$. Therefore, the intersection of D_0 with the R–balls of M centered at x satisfies

$$\mathrm{Area}(D_0 \cap B(x,R)) \geq CR^2 \tag{10-4}$$

for every $R < \frac{1}{2}\operatorname{sys}(D_0,\phi)$. The idea of the proof is to control the area of distance spheres (level surfaces of the distance function) in M, in terms of the areas of the distance disks in D_0.

Let $B = B(x,R)$ be the metric R–ball in M centered at x with $R < \frac{1}{2}\operatorname{sys}(M)$. We subdivide and slightly perturb D first, to make sure that $D \cap \overline{B}$ is a subchain of D. Write

$$D = D_- + D_+,$$

where D_- is a relative 2–cycle of $\overline{B}$, and D_+ is a relative 2–cycle of $M \setminus B$. By Lemma 10.2, D_- is homologous to a 2–chain $\mathcal{C}$ contained in the distance sphere $\partial B = S(x,R)$ with

$$\partial\mathcal{C} = \partial D_- = -\partial D_+.$$

We subdivide and perturb $\mathcal{C}$ in $S(x,R)$ so that the interiors of its 2–simplices either agree or have an empty intersection. Here the simplices of the 2–chain $\mathcal{C}$ may have nontrivial multiplicities. Such multiplicities necessarily affect the volume of a chain if one works with integer coefficients. However, these multiplicities are ignored for the notion of 2–volume (10-3). This special feature allows us to derive the following: the 2–volume (10-3) of the chain $\mathcal{C}$ is a lower bound for the usual area of the distance sphere $S(x,R)$.

Note that the homology class $[\mathcal{C} + D_+] = [D] \in H_2(M;\mathbb{Z}_p)$ stays the same. We chose D to be area minimizing up to ε in its homology class in M for the notion of volume (10-3). Hence we have the following bound:

(10-5) $$\operatorname{Area}(S(x,R)) \geq \operatorname{Area}(\mathcal{C}) \geq \operatorname{Area}(D_-) - \varepsilon \geq \operatorname{Area}(D_0 \cap B) - \varepsilon.$$

Now, clearly $\operatorname{sys}(M) \leq \operatorname{sys}(D_0,\phi)$. Combining the estimates (10-4) and (10-5), we obtain

(10-6) $$\operatorname{Area}(S(x,R)) \geq CR^2 - \varepsilon$$

for every $R < \frac{1}{2}\operatorname{sys}(M)$. Integrating the estimate (10-6) with respect to R and letting ε go to zero, we obtain a lower bound of $\frac{C}{3}R^3$ for the 3–volume of some R–ball in the closed manifold M, proving Theorem 10.1 for closed 3–manifolds with fundamental group $\mathbb{Z}_p$.

Suppose now that M is a closed 3–manifold with finite (nontrivial) fundamental group. Choose a prime p dividing the order $|\pi_1(M)|$ and consider a cover N of M with fundamental group cyclic of order p. This cover satisfies $\operatorname{sys}(N) \geq \operatorname{sys}(M)$, and we apply the previous argument to N.

Note that the reduction to a cover could not have been done in the context of M Gromov's formulation of the inequality in terms of the global volume of the manifold. Meanwhile, in our formulation using a metric ball, following L Guth, we can project injectively the ball of sufficient volume, from the cover to the original manifold. Namely, the proof above exhibits a point $x \in N$ such that the volume of the R–ball $B(x,R)$ centered at x is at least $(C/3)R^3$ for every $R < \frac{1}{2}\operatorname{sys}(M)$. Since R is less than half the systole of M, the ball $B(x,R)$ of N projects injectively to an R–ball in M of the required volume, completing the proof of Theorem 10.1. □

References

[1] **L Ambrosio**, **M G Katz**, *Flat currents modulo p in metric spaces and filling radius inequalities*, to appear in Comm. Math. Helv. `arXiv:1004.1374`

[2] **L Ambrosio**, **S Wenger**, *Rectifiability of flat chains in Banach spaces with coefficients in* $\mathbb{Z}_p$, to appear in Math. Z. `arXiv:1004.1374`

[3] **I Babenko**, **F Balacheff**, *Distribution of the systolic volume of homology classes* `arXiv:1009.2835`

[4] **F Balacheff**, *A local optimal diastolic inequality on the two-sphere*, J. Topol. Anal. 2 (2010) 109–121 MR2646992

[5] **V Bangert**, **M G Katz**, **S Shnider**, **S Weinberger**, *E_7, Wirtinger inequalities, Cayley 4–form, and homotopy*, Duke Math. J. 146 (2009) 35–70 MR2475399

[6] **M Belolipetsky**, **S Thomson**, *Systoles of hyperbolic manifolds* `arXiv:1008.2646`

[7] **M Berger**, *What is...a systole?*, Notices Amer. Math. Soc. 55 (2008) 374–376 MR2382824

[8] **J Bochnak**, **M Coste**, **M-F Roy**, *Real algebraic geometry*, Ergebnisse der Math. und ihrer Grenzgebiete (3) 36, Springer, Berlin (1998) MR1659509 Translated from the 1987 French original, Revised by the authors

[9] **M Brunnbauer**, *Filling inequalities do not depend on topology*, J. Reine Angew. Math. 624 (2008) 217–231 MR2456630

[10] **M Brunnbauer**, *On manifolds satisfying stable systolic inequalities*, Math. Ann. 342 (2008) 951–968 MR2443769

[11] **Y D Burago**, **V A Zalgaller**, *Geometric inequalities*, Grund. der Math. Wissenschaften 285, Springer, Berlin (1988) MR936419 Translated from the Russian by A B Sosinskiĭ, Springer Ser. in Soviet Math.

[12] **A N Dranishnikov**, **M G Katz**, **Y B Rudyak**, *Small values of the Lusternik–Schnirelmann category for manifolds*, Geom. Topol. 12 (2008) 1711–1727 MR2421138

[13] **A N Dranishnikov**, **Y B Rudyak**, *Stable systolic category of manifolds and the cup-length*, J. Fixed Point Theory Appl. 6 (2009) 165–177 MR2558490

[14] **C El Mir**, *The systolic constant of orientable Bieberbach 3–manifolds* `arXiv:0912.3894`

[15] **C El Mir**, **J Lafontaine**, *Sur la géométrie systolique des variétés de Bieberbach*, Geom. Dedicata 136 (2008) 95–110 MR2443345

[16] **M Gromov**, *Filling Riemannian manifolds*, J. Differential Geom. 18 (1983) 1–147 MR697984

[17] **L Guth**, *Volumes of balls in large Riemannian manifolds*, to appear in Ann. of Math. (2) `arXiv:math/0610212`

[18] **L Guth**, *Systolic inequalities and minimal hypersurfaces*, Geom. Funct. Anal. 19 (2010) 1688–1692 MR2594618

[19] **A Hatcher**, *Algebraic topology*, Cambridge Univ. Press (2002) MR1867354

[20] **J J Hebda**, *Some lower bounds for the area of surfaces*, Invent. Math. 65 (1981/82) 485–490 MR643566

[21] **K U Katz**, **M G Katz**, *Bi-Lipschitz approximation by finite-dimensional imbeddings*, to appear in Geometriae Dedicata `arXiv:0902.3126`

[22] **K U Katz**, **M G Katz**, *Hyperellipticity and Klein bottle companionship in systolic geometry* arXiv:0811.1717

[23] **M G Katz**, *Systolic geometry and topology*, Math. Surveys and Monogr. 137, Amer. Math. Soc. (2007) MR2292367 With an appendix by J P Solomon

[24] **M G Katz**, *Systolic inequalities and Massey products in simply-connected manifolds*, Israel J. Math. 164 (2008) 381–395 MR2391156

[25] **M G Katz**, **Y B Rudyak**, *Bounding volume by systoles of 3–manifolds*, J. Lond. Math. Soc. (2) 78 (2008) 407–417 MR2439632

[26] **M G Katz**, **Y B Rudyak**, **S Sabourau**, *Systoles of 2–complexes, Reeb graph, and Grushko decomposition*, Int. Math. Res. Not. (2006) Art. ID 54936 MR2250017

[27] **M G Katz**, **M Schaps**, **U Vishne**, *Hurwitz quaternion order and arithmetic Riemann surfaces*, to appear in Geom. Dedicata arXiv:math.RA/0701137

[28] **M G Katz**, **S Shnider**, *Cayley form, comass and triality isomorphisms*, Israel J. Math. 178 (2010) 187–208

[29] **J W Morgan**, **D P Sullivan**, *The transversality characteristic class and linking cycles in surgery theory*, Ann. of Math. (2) 99 (1974) 463–544 MR0350748

[30] **A Nabutovsky**, **R Rotman**, *The length of the second shortest geodesic*, Comment. Math. Helv. 84 (2009) 747–755 MR2534478

[31] **H Parlier**, *The homology systole of hyperbolic Riemann surfaces* arXiv:1010.0358

[32] **P M Pu**, *Some inequalities in certain nonorientable Riemannian manifolds*, Pacific J. Math. 2 (1952) 55–71 MR0048886

[33] **R Rotman**, *The length of a shortest geodesic loop at a point*, J. Differential Geom. 78 (2008) 497–519 MR2396252

[34] **Y B Rudyak**, **S Sabourau**, *Systolic invariants of groups and 2–complexes via Grushko decomposition*, Ann. Inst. Fourier (Grenoble) 58 (2008) 777–800 MR2427510

[35] **S Sabourau**, *Asymptotic bounds for separating systoles on surfaces*, Comment. Math. Helv. 83 (2008) 35–54 MR2365407

[36] **S Sabourau**, *Local extremality of the Calabi–Croke sphere for the length of the shortest closed geodesic*, J. London Math. Soc. 82 (2010) 549–562

[37] **R Schoen**, **S T Yau**, *Incompressible minimal surfaces, three-dimensional manifolds with nonnegative scalar curvature, and the positive mass conjecture in general relativity*, Proc. Nat. Acad. Sci. U.S.A. 75 (1978) 2567 MR496776

[38] **R Schoen**, **S T Yau**, *On the structure of manifolds with positive scalar curvature*, Manuscripta Math. 28 (1979) 159–183 MR535700

[39] **D P Sullivan**, *Triangulating and smoothing homotopy equivalences and homeomorphisms. Geometric Topology Seminar Notes*, from: “The Hauptvermutung book”, K–Monogr. Math. 1, Kluwer Acad. Publ., Dordrecht (1996) 69–103 MR1434103

[40] **S Wenger**, *A short proof of Gromov's filling inequality*, Proc. Amer. Math. Soc. 136 (2008) 2937–2941 MR2399061

[41] **B White**, *The deformation theorem for flat chains*, Acta Math. 183 (1999) 255–271 MR1738045

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