A note on cabling and L–space surgeries

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We prove that the (p,q)–cable of a knot $K \subset S^3$ admits a positive L–space surgery if and only if K admits a positive L–space surgery and $q/p \geq 2g(K)-1$, where $g(K)$ is the Seifert genus of K. The "if" direction is due to Hedden [1].

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1 Introduction

In [7], Ozsváth and Szabó introduced a powerful tool for studying closed 3–manifolds, Heegaard Floer homology, and later equipped this invariant with a filtration [6] (independently developed by Rasmussen in [10]) that defined an invariant for a knot in the 3–manifold. The relationship between the knot invariant and the Heegaard Floer homology of the 3–manifold obtained by Dehn surgery on that knot has been well studied (see Ozsváth and Szabó [8; 9; 3]), and can also be considered from the perspective of bordered Heegaard Floer homology as by Lipshitz, Ozsváth and Thurston [2].

In this note, we restrict our consideration to the simplest "hat" version of the theory, assuming that the reader is familiar with the finitely generated abelian groups

$$\widehat{HF}(Y) \quad \text{and} \quad \widehat{HFK}(Y,K)$$

associated with a 3–manifold Y and a null-homologous knot $K \subset Y$ (see [6]). We will work over the coefficient field $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ throughout, and we will write simply $\widehat{HFK}(K)$ when it is clear that the ambient 3–manifold is S^3. For the present purposes, we do not need to concern ourselves with the gradings on these groups. We focus our attention on a class of 3–manifolds with particularly simple Heegaard Floer homology. For a rational homology sphere Y, Proposition 5.1 of [7] tells us that

$$\operatorname{rk} \widehat{HF}(Y) \geq |H_1(Y,\mathbb{Z})|.$$

An *L–space* is a rational homology sphere Y for which the above bound is sharp. The name comes from the fact that lens spaces are L–spaces, which can be seen by examining the Heegaard Floer complex associated to a standard genus one Heegaard decomposition of a lens space.

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We call a knot $K \subset S^3$ an *L–space knot* if there exists $n \in \mathbb{Z}$, $n > 0$, such that n surgery on K yields an L–space. We will denote the resulting 3–manifold by $S^3_n(K)$. Torus knots are a convenient source of L–space knots, since $pq \pm 1$ surgery on the (p,q)–torus knot yields a lens space. It was proved in [8, Theorem 1.2] that if a knot K is an L–space knot, then the knot Floer complex associated to K has a particularly simple form that can be deduced from the Alexander polynomial of K, $\Delta_K(t)$. Thus, knowing that a knot K admits a lens space (or L–space) surgery yields a remarkable amount of information about the Heegaard Floer invariants associated to both the knot K, and manifolds obtained by Dehn surgery on K. In particular, [8, Theorem 1.2] combined with [3, Theorem 1.1] allows one to compute the Heegaard Floer invariants of any Dehn surgery on an L–space knot K from the Alexander polynomial of K.

Recall that the (p,q)–cable of a knot K, denoted $K_{p,q}$, is the satellite knot with pattern the (p,q)–torus knot. More precisely, we can construct $K_{p,q}$ by equipping the boundary of a tubular neighborhood of K with the (p,q)–torus knot, where the knot traverses the longitudinal direction p times and the meridional direction q times. We will assume throughout that $p > 1$. (This assumption does not cause any loss of generality, since $K_{-p,-q} = -K_{p,q}$, where $-K_{p,q}$ denotes $K_{p,q}$ with the opposite orientation, and since $K_{1,q} = K$.)

It is natural to ask how satellite operations affect various properties of a knot. We will focus on the operation of cabling. In [4], Ozsváth and Szabó define an integer-valued concordance invariant $\tau(K)$. Hedden [1] and Van Cott [11] have studied the behavior of τ under cabling, giving bounds and, in special cases, formulas for $\tau(K_{p,q})$. These results will play a key role later in this note. In a forthcoming paper, we will use bordered Heegaard Floer homology to completely describe the behavior of τ under cabling, in terms of the cabling parameters, $\tau(K)$, and a second knot Floer concordance invariant, $\varepsilon(K)$.

Let $g(K)$ denote the Seifert genus of K. In Theorem 1.10 of [1], Hedden proves that if K is an L–space knot and $q/p \geq 2g(K) - 1$, then $K_{p,q}$ is an L–space knot. The goal of this note is to prove the converse:

Theorem *The (p,q)–cable of a knot $K \subset S^3$ is an L–space knot if and only if K is an L–space knot and $q/p \geq 2g(K) - 1$.*

It was already known that if $K_{p,q}$ is an L–space knot, then $q > 0$ and $\tau(K) = g(K)$ [11, Corollary 6]. We prove our theorem by methods similar to those used in [1, Theorem 1.10]. An interesting question to consider is whether there are other satellite constructions that also yield L–space knots.

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2 Proof of Theorem

An L–space Y can be thought of as rational homology sphere with the "smallest" possible Heegaard Floer invariants, ie $\mathrm{rk}\,\widehat{HF}(Y) = |H_1(Y,\mathbb{Z})|$. In a similar spirit, an L–space knot K can be thought of as a knot with the "smallest" possible knot Floer invariants. For example, since

$$\Delta_K(t) = \sum_{m,s} (-1)^m \,\mathrm{rk}\, \widehat{HFK}_m(K,s)\, t^s,$$

so we see immediately that the total rank of $\widehat{HFK}(K)$ is bounded below by the sum of the absolute value of the coefficients of the Alexander polynomial of K, $\Delta_K(t)$. A necessary, but not sufficient, condition for a knot K to be an L–space knot is for this bound to be sharp; see [8, Theorem 1.2] for the complete statement. The spirit of our proof is that when either K is not an L–space knot, or $q/p < 2g(K)-1$, the knot Floer invariants of $K_{p,q}$ are not "small" enough for $K_{p,q}$ to be an L–space knot. We will determine this by looking at the rank of $\widehat{HF}(S^3_{pq}(K_{p,q}))$.

Recall that $\tau(K)$ is the integer-valued concordance invariant defined by Ozsváth and Szabó in [4]. Let $\mathcal{P}$ denote the set of all knots K for which $g(K) = \tau(K)$. We begin by assembling the following collection of facts.

(1) If K is an L–space knot, then $K \in \mathcal{P}$. This follows from [8, Theorem 1.2] combined with the fact that knot Floer homology detects genus [5, Theorem 1.2]

(2) Let

$$s_K = \sum_{s\in\mathbb{Z}} (\mathrm{rk}\, H_*(\hat{A}_s(K)) - 1),$$

where $\hat{A}_s(K)$ is the subquotient complex of $CFK^\infty(K)$ defined in [9, Section 4.3]. We may think of $CFK^\infty(K)$ as generated over $\mathbb{F}[U,U^{-1}]$ by $\widehat{CFK}(K)$, in which case $\mathrm{rk}\,\hat{A}_s(K) = \mathrm{rk}\,\widehat{CFK}(K)$ for all s. Recall that $\mathrm{rk}\,\widehat{CFK}(K)$ is always odd, since the graded Euler characteristic of $\widehat{CFK}(K)$ is the Alexander polynomial of K. Therefore, $\mathrm{rk}\,H_*(\hat{A}_s(K))$ is odd, hence greater than or equal to 1, and so s_K is always nonnegative. Let

$$t_K^{a/b} = 2\,\max(0, (2g(K)-1)b - a),$$

for a pair of relatively prime integers a and b, $b > 0$. Notice that

$$t_K^{a/b} = 0 \quad \text{if and only if} \quad a/b \geq 2g(K)-1.$$

For $K \in \mathcal{P}$ and a, b as above,

$$\operatorname{rk} \widehat{HF}(S^3_{a/b}(K)) = a + bs_K + t_K^{a/b}.$$

This is a special case of Proposition 9.5 of [3]. In particular, the term $\nu(K)$ appearing in Proposition 9.5 is bounded below by $\tau(K)$ [4, Proposition 3.1] and above by $g(K)$ [6, Theorem 5.1], so $K \in \mathcal{P}$ implies $\nu(K) = g(K)$. We notice that

$$K \text{ admits a positive } L\text{–space surgery if and only if } s_K = 0.$$

Indeed, if $s_K = 0$, then p surgery on K yields an L–space, for any integer $p \geq 2g(K) - 1$. Conversely, if K is an L–space knot, then there exists some integer $p > 0$ such that p surgery on K is an L–space, in which case s_K, which is always nonnegative, must be 0.

(3) Recall our convention that p, q are relatively prime integers, with $p > 1$. If $K_{p,q} \in \mathcal{P}$, then $K \in \mathcal{P}$, and if $K \in \mathcal{P}$, then $\tau(K_{p,q}) = p\tau(K) + \frac{1}{2}(p-1)(q-1)$. These facts are Corollaries 4 and 3, respectively, in [11]. Therefore, if $K_{p,q} \in \mathcal{P}$, we have

$$\begin{aligned}(2g(K)-1)p - q &= (2\tau(K)-1)p - q \\ &= 2\bigl(p\tau(K) + (p-1)(q-1)/2\bigr) - 1 - pq \\ &= 2\tau(K_{p,q}) - 1 - pq \\ &= 2g(K_{p,q}) - 1 - pq,\end{aligned}$$

or equivalently,

$$\text{if } K_{p,q} \in \mathcal{P}, \text{ then } t_K^{q/p} = t_{K_{p,q}}^{pq}.$$

(4) It is well-known that pq surgery on $K_{p,q}$ is the manifold $L(p,q) \# S^3_{q/p}(K)$ (see [1, Proof of Theorem 1.10] for a nice proof of this fact). We also have from [7, Proposition 6.1] that

$$\operatorname{rk} \widehat{HF}(Y_1 \# Y_2) = \operatorname{rk} \widehat{HF}(Y_1) \cdot \operatorname{rk} \widehat{HF}(Y_2).$$

Then

$$\begin{aligned}\operatorname{rk} \widehat{HF}(S^3_{pq}(K_{p,q})) &= \operatorname{rk} \widehat{HF}(L(p,q)) \cdot \operatorname{rk} \widehat{HF}(S^3_{q/p}(K)) \\ &= p \cdot \operatorname{rk} \widehat{HF}(S^3_{q/p}(K)).\end{aligned}$$

With these facts in place, we are ready to prove the theorem. Assume $K_{p,q}$ is an L–space knot. Then by (1) and (3), $K_{p,q} \in \mathcal{P}$ and $t_{K_{p,q}}^{pq} = t_K^{q/p}$, and by (2),

$$\operatorname{rk} \widehat{HF}(S^3_{pq}(K_{p,q})) = pq + s_{K_{p,q}} + t_{K_{p,q}}^{pq} \quad \text{and} \quad \operatorname{rk} \widehat{HF}(S^3_{q/p}(K)) = q + ps_K + t_K^{q/p}.$$

Then by (4), $\operatorname{rk}\widehat{HF}(S^3_{pq}(K_{p,q})) = p \cdot \operatorname{rk}\widehat{HF}(S^3_{q/p}(K))$, and $s_{K_{p,q}} = 0$, since $K_{p,q}$ is an L–space knot. So we find that

$$p^2 s_K + (p-1)t_K^{q/p} = 0.$$

Therefore, since $p > 1$, we have that s_K and $t_K^{q/p}$ must both be zero, or equivalently, K is an L–space knot and $q/p \geq 2g(K)-1$. This completes the proof of the theorem.

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