Tunnel complexes of 3–manifolds

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For each closed 3–manifold M and natural number t, we define a simplicial complex $\mathcal{T}_t(M)$, the *t–tunnel complex*, whose vertices are knots of tunnel number at most t. These complexes have a strong relation to disk complexes of handlebodies. We show that the complex $\mathcal{T}_t(M)$ is connected for M the 3–sphere or a lens space. Using this complex, we define an invariant, the *t–tunnel complexity*, for tunnel number t knots. These invariants are shown to have a strong relation to toroidal bridge numbers and the hyperbolic structures.

57M25; 57M15, 57M27

Introduction

In the study of low-dimensional topology, it is often constructed a simplicial complex whose vertices are the set of topological objects and where every collection of distinct vertices spans a simplex if they share a certain topological property. Such a property is basically explained by a "mutually disjoint realization" or by a "topological move".

Primary examples of the first type are the *curve complexes* of surfaces as in Hempel [11] and the *disk complexes* of handlebodies as in McCullough [21]. Note that a path in the 1–skeleton of such complexes is also understood as a sequence elementary moves.

The second type are essentially defined to be 1–dimensional simplicial complexes and then sometimes extended to higher dimension requiring to be *flag complexes*, that is, any collection of $k+1$ vertices such that any two of them are adjacent are required to span a k–simplex. The *Gordian complex* for knots (see Hirasawa and Uchida [12]) and the *IH–complex* for spatial trivalent graphs (see Ishii and Kishimoto [14]) are examples of such complexes.

These complexes are used to give global viewpoints for the sets of topological objects. When the complex is connected, we may define a distance of two topological objects using the simplicial distance of the corresponding two vertices. The distance gives topological and geometric information of the objects.

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In this paper, based on the philosophy of the first type, we construct simplicial complexes for knots with bounded tunnel number. These turn out to have a deep relation to the complex of nonseparating t–tuple of disks in a genus $t+1$ handlebody.

A *θ–curve of order n*, or simply a *θ_n–curve* is the graph on two vertices and n edges $\alpha_1, \alpha_2, \dots, \alpha_n$ joining them. Let M be a closed orientable 3–manifold. The image $\widetilde{\theta}_n$ of a θ_n–curve under an embedding $\theta_n \to M$ is called a *spatial θ_n–curve*. Then each simple loop $\widetilde{\alpha}_i \cup \widetilde{\alpha}_j$, $1 \leqslant i < j \leqslant n$, forms a knot in M which is called a *constituent knot* of $\widetilde{\theta}_n$. The definition of a constituent knot of a general spatial graph will be reviewed in Section 1.

Given a knot K in a closed orientable 3–manifold M, an *unknotting tunnel system* for K is a set of mutually disjoint simple arcs in M with their endpoints in K such that the complement of a regular neighborhood of the union of K and the arcs is a handlebody. The *tunnel number* of K is the minimum number of arcs over all unknotting tunnel systems for K.

For each closed orientable 3–manifold M and a natural number t, we define a simplicial complex $\mathcal{T}_t(M)$ whose vertices are knots in M of tunnel number at most t, and where distinct vertices $K_0, K_1, \dots, K_k$ span a k–simplex of $\mathcal{T}_t(M)$ if there exists a spatial θ_{t+2}–curve $\widetilde{\theta}_{t+2}$ such that the complement of the interior of a regular neighborhood of $\widetilde{\theta}_{t+2}$ is a handlebody and that for each $0 \leqslant l \leqslant k$, the vertex K_l is a constituent knot of $\widetilde{\theta}_{t+2}$. We call this complex the *t–tunnel complex* and denote it by $\mathcal{T}_t(M)$. Our first main theorem is the following:

Theorem 0.1 (1) *If M is the 3–sphere or a lens space, then the tunnel complex $\mathcal{T}_t(M)$ is connected for any natural number t.*

(2) *Let M be a closed orientable 3–manifold admitting finitely many genus g Heegaard splittings. Let n be the numbers of genus g Heegaard splittings of M up to ambient isotopy. Then the number of connected components of $\mathcal{T}_{g-1}(M)$ is at most $2n$.*

Recently, Cho and McCullough [4; 5; 3; 6] provided a very strong method for the study of unknotting tunnels of tunnel number one knots in S^3. They considered a simplicial complex $\mathcal{D}(H_2)/\mathcal{G}_2$, where H_2 is an unknotted genus two handlebody embedded in S^3, $\mathcal{D}(H_2)$ is the complex of nonseparating disks in H_2, and $\mathcal{G}_2$ is the genus two Goeritz group, ie the group of isotopy classes of orientation-preserving automorphisms of S^3 preserving H_2. In Section 3.1, we will observe that there exists a simplicial surjection from $\mathcal{D}(H_2)/\mathcal{G}_2$ to $\mathcal{T}_1(S^3)$.

Moreover, we will generalize this observation defining the complex $\mathcal{D}_t(H_{t+1})$ of *nonseparating t–tuples* of disks in a genus $t+1$ handlebody H_{t+1}. This complex is defined to be a simplicial complex whose vertices are the *nonseparating t–tuple* (see Section 3.2) of vertices of $\mathcal{D}(H_{t+1})$ such that a collection of $k+1$ vertices spans a k–simplex if and only if all vertices in the $k+1$ t–tuples are realized at the same time as pairwise disjoint (but some pairs possibly be parallel) disks in H_{t+1}.

Theorem 0.2 *For any t, the simplicial complex $\mathcal{D}_t(H_{t+1})$ is connected.*

Then we show that these complexes are connected and there exists a surjection from the 1–skeleton of $\mathcal{D}_t(H_{t+1})$ to that of $T_t(M)$, where $t \geqslant 1$ and M is the 3–sphere or a lens space.

A knot in a closed orientable 3–manifold is called a *$(g,1)$–knot* if there exists a genus g Heegaard splitting of the 3–manifold such that the Heegaard surface splits the knot into two trivial arcs with respect to the splitting. Note that every knot in a closed orientable 3–manifold is a $(g,1)$–knot for a certain g greater than or equal to the Heegaard genus of the 3–manifold.

Let M be a closed orientable 3–manifold. We define the *t–distance* $d_{\mathcal{T}_t(M)}$ of two knots K_1 and K_2 of tunnel number at most t to be the simplicial distance of the corresponding two vertices in the 1–skeleton of $\mathcal{T}_t(M)$. If the two vertices K_1 and K_2 belong to different components, then we set $d_{\mathcal{T}_t(M)}(K_1, K_2) = \infty$. Let the unknot U be a vertex of $\mathcal{T}_t(M)$. Then the *t–tunnel complexity*, $\mathrm{tc}_t(K)$, of a knot $K \subset M$ of tunnel number at most t is then defined to be the distance from the unknot to K.

Recall that a *$(1,t)$–decomposition* of a knot K in a closed orientable 3–manifold is a decomposition of K by a genus one Heegaard splitting of the 3–manifold such that the Heegaard surface splits the knot in t trivial arcs with respect to the splitting.

Proposition 0.3 (1) *Let M be the 3–sphere or a lens space. Let $K \subset M$ be a tunnel number one knot. Let C be the core of a solid torus appearing in a genus one Heegaard splitting of M. Then $d_{\mathcal{T}_1(M)}(C, K) \leqslant 1$ if and only if K is a $(1,1)$–knot. In the case of $M \cong S^3$, we have, in particular, $\mathrm{tc}_1(K) \leqslant 1$ if and only if K is a $(1,1)$–knot.*

(2) *Let M and C be as in* (1)*. Then $d_{\mathcal{T}_t(M)}(C, K) \leqslant 1$ if K admits a $(1,t)$–decomposition.*

(3) *Let $K \subset M$ be a tunnel number t nontrivial knot in a closed 3–manifold M. Then $\mathrm{tc}_{t+1}(K) = 1$.*

It follows from the above proposition that the tunnel complexity of all *Morimoto–Sakuma–Yokota knots* is two (see Section 4).

We can say more about the tunnel complexity in the case of $M \cong S^3$. In Section 4 we prove that the diameter of the tunnel complex $\mathcal{T}_1(S^3)$ is infinite, as is for $\mathcal{D}(H_2)/\mathcal{G}_2$:

Theorem 0.4 *For every integer n, there exists a tunnel number one knot $K \subset S^3$ such that $\mathrm{tc}_1(K) > n$, which implies that the diameter of the tunnel complex $\mathcal{T}_1(S^3)$ is infinite.*

We see that all nonhyperbolic tunnel number one knots lie in the neighborhood of the unknot while the tunnel complex $\mathcal{T}_1(S^3)$ turns out to be unbounded:

Theorem 0.5 *Let K be a tunnel number one knot in S^3. If $\mathrm{tc}_1(K) \geqslant 2$, then K is hyperbolic.*

By definition, both $\mathcal{D}(H_2)$ and $\mathcal{D}(H_2)/\mathcal{G}_2$ are *flag complexes*, that is, any cycles of length three in $\mathcal{D}(H_2)$ or $\mathcal{D}(H_2)/\mathcal{G}_2$ spans a 2–simplex. On the contrary, we have the following in Section 5:

Theorem 0.6 *The simplicial complex $\mathcal{T}_1(S^3)$ is not a flag complex. Moreover, there exist infinitely many distinct cycles of length three in $\mathcal{T}_1(S^3)$ that do not span 2–simplices.*

Notation Let X be a subset of a given topological space or a manifold Y. Throughout the paper, we will denote the interior of X by $\mathrm{Int}\, X$, the closure of X by $\overline{X}$ and the number of components of X by $\#X$. We will use $N(X;Y)$ to denote a regular neighborhood of X in Y. If the ambient space Y is clear from the context, we simply denote it by $N(X)$. By a 3–*manifold*, we always mean a *connected, compact* and *orientable* one without boundary, unless otherwise mentioned. By a *graph*, we mean a finite, connected multigraph possibly with loops. Let X be a simplicial complex. We denote by $X^{(k)}$ the k–skeleton of X. By a *knot*, we mean a simple closed curve embedded in a closed 3–manifold. We say that two knots are the *same* if they are ambient isotopic, and *distinct* otherwise.

1 Preliminaries

Let M be a closed 3–manifold and G be a graph. Throughout the paper, we denote by $\widetilde{G}$ the image of an embedding $f\colon G \to M$, and we call it a *spatial graph*. Let v (e, respectively) be a vertex (an edge, respectively) of G then we denote its image under the embedding f by $\widetilde{v}$ ($\widetilde{e}$, respectively). A cycle of $\widetilde{G}$ is called a *constituent knot* of $\widetilde{G}$ when we regard it as a knot.

1.1 Definition of tunnel complexes

To begin with, we recall the following classical theorem by Kinoshita:

Theorem 1.1 [18] *Let $n \geqslant 3$ be an integer. For any collection $\{K_{i,j}\}_{1 \leqslant i < j \leqslant n}$ of knots in S^3, there exists a spatial θ_n–curve $\widetilde{\theta}_n$ whose constituent knot $\widetilde{\alpha}_i \cup \widetilde{\alpha}_j$ is ambient isotopic to $K_{i,j}$ for each pair (i,j).*

This theorem implies that the embeddings of θ_n are quite varied. Focusing on unknotting tunnels of knots, we may consider the following question analogous to Theorem 1.1.

Question Let $n \geqslant 3$ be an integer and M be a closed 3–manifold. For any collection $\{K_{i,j}\}_{1 \leqslant i < j \leqslant n}$ of knots of tunnel number at most $n-2$ in M, does there exist a spatial θ_n–curve $\widetilde{\theta}_n$ in M such that

(1) $M \setminus \operatorname{Int} N(\widetilde{\theta}_n)$ is a handlebody; and

(2) the constituent knot $\widetilde{\alpha}_i \cup \widetilde{\alpha}_j$ is ambient isotopic to the knot $K_{i,j}$ for each pair (i,j)?

Note that the first condition of the above question is crucial at least in the case of $M \cong S^3$ because Theorem 1.1 gives a positive answer without the condition. The above question asks if we have a result analogous to Theorem 1.1 after restricting the knots to have tunnel number at most $n-2$ and the embedding of θ_n to have a handlebody complement. In the case of θ_3–curve embedded in the 3–sphere, Kinoshita's theorem implies that for any correction, say $\{J, K, L\}$, of knots in the 3–sphere, one can find a spatial θ_3–curve whose constituent knots are precisely the specified knots $\{J, K, L\}$. The above question asks whether this can be achieved for any collection $\{J, K, L\}$ of knots of tunnel number at most one, while simultaneously requiring the complement to be a handlebody. In Section 4, we give a negative answer to this question. In fact, we prove that even when $J = K = L$ is a trefoil, such a embedding of the θ_3–curve does not exist. (See the example after Proposition 3.3.) The aim of the paper is to understand how far the reality is from the positive answer to the above question.

Definition 1.2 Let M be a closed 3–manifold and t be a natural number. The *t–tunnel complex $\mathcal{T}_t(M)$ of M* is a simplicial complex such that

(1) the set of vertices of $\mathcal{T}_t(M)$ consists of knots of tunnel number at most t in M; and

(2) a collection $\{K_0, K_1, \ldots, K_k\}$ of distinct vertices spans a k–simplex of $\mathcal{T}_t(M)$ if there exists a spatial θ_{t+2}–curve $\widetilde{\theta}_{t+2}$ such that

 (a) $M \setminus \operatorname{Int} N(\widetilde{\theta}_{t+2})$ is a handlebody; and

 (b) each K_l $(0 \leqslant l \leqslant k)$ is a constituent knot of $\widetilde{\theta}_{t+2}$.

The following is straightforward from the definition.

Lemma 1.3 *For a closed 3–manifold M, the following holds:*

(1) *The dimension of $\mathcal{T}_t(M)$ is at most $(t+2)(t+1)/2-1$.*

(2) *If M is of Heegaard genus g $(\geqslant 3)$, we have*

$$\mathcal{T}_1(M) = \mathcal{T}_2(M) = \cdots = \mathcal{T}_{g-2}(M) = \varnothing.$$

(3) *There exists a sequence of simplicial embeddings*

$$\mathcal{T}_1(M) \hookrightarrow \mathcal{T}_2(M) \hookrightarrow \cdots \hookrightarrow \mathcal{T}_n(M) \hookrightarrow \cdots.$$

In particular, the dimension of $\mathcal{T}_t(M)$ is less than or equal to that of $\mathcal{T}_{t+1}(M)$ for any natural number t.

2 Connectivity of tunnel complexes

For a spatial graph $\widetilde{G}$ in a closed 3–manifold, We denote by $g(\widetilde{G})$ the genus of the handlebody $N(\widetilde{G})$.

Let $n \geqslant 2$ be a natural number. An n–bouquet B_n is a topological space obtained by gluing together a collection of n circles along a single point, which is called the *vertex* of B_n.

Lemma 2.1 *Let $\widetilde{G}$ be a spatial trivalent graph with $g(\widetilde{G}) = g$ embedded in a closed 3–manifold M. Let K_1 and K_2 be constituent knots of $\widetilde{G}$. Then there exists a spatial bouquet $\widetilde{B}_g$ such that*

(1) *$N(\widetilde{G})$ is ambient isotopic to $N(\widetilde{B}_g)$ in M; and*

(2) *both K_1 and K_2 are constituent knots of $\widetilde{B}_g$.*

Proof Let $\widetilde{e}$ be an edge of $K_1 \setminus K_2$. Let $\widetilde{v}$ be an endpoint of $\widetilde{e}$. Let C be a simple loop embedded in a solid torus $N(K_1)$ such that

(1) $C \cap \widetilde{G} = \widetilde{v}$; and

(2) C is ambient isotopic to K_1 in $N(K_1)$.

Set $\widetilde{G}' := (\widetilde{G} \setminus \operatorname{Int} \widetilde{e}) \cup C$. Then $N(\widetilde{G})$ is ambient isotopic to $N(\widetilde{G}')$ in M and both K_1 and K_2 are constituent knots of $\widetilde{G}'$.

Let K_2 consists of the edges $\widetilde{e}_1, \widetilde{e}_2, \ldots, \widetilde{e}_n$. Let $\widetilde{B}_g$ be a spatial bouquet obtained by edge contractions along the edges $\widetilde{e}_2, \widetilde{e}_3, \ldots, \widetilde{e}_n$ and some other edges except $\widetilde{e}_1$. Then $\widetilde{B}_g$ satisfies the required conditions. $\square$

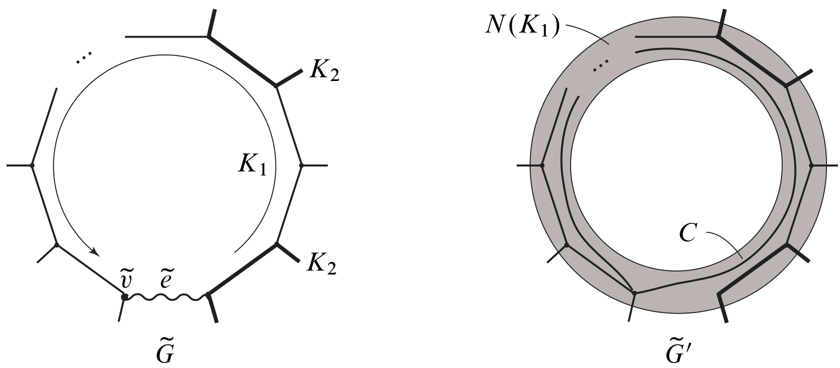

Figure 1

Lemma 2.2 *Let $\widetilde{G}$ be a spatial trivalent graph with $g(\widetilde{G}) = g$ in a closed 3–manifold M such that $M \setminus \operatorname{Int} N(\widetilde{G})$ be a handlebody. Let K be a constituent knot of $\widetilde{G}$. Then K is of tunnel number at most $g-1$.*

Proof By Lemma 2.1, there exists a spatial bouquet $\widetilde{B}_g$ such that

(1) $M \setminus \operatorname{Int} N(\widetilde{B}_g)$ is a handlebody; and

(2) K is a constituent knot of $\widetilde{B}_g$.

Let $\tilde{v}$ be the unique vertex of $\widetilde{B}_g$. Then $\overline{\widetilde{B}_g \setminus (N(\tilde{v}) \cup K)}$ consists of $g-1$ arcs which form an unknotting tunnel system of K. □

Lemma 2.3 *Let $\widetilde{G}$ be a spatial trivalent graph with $g(\widetilde{G}) = g$ embedded in a closed 3–manifold M such that $M \setminus \operatorname{Int} N(\widetilde{G})$ is a handlebody. Let K_1 and K_2 be constituent knots of $\widetilde{G}$. Then K_1 and K_2 are adjacent in $\mathcal{T}_{g-1}(M)$.*

Proof By Lemma 2.1, there exists a spatial bouquet $\widetilde{B}_g$ such that

(1) $M \setminus \operatorname{Int} N(\widetilde{B}_g)$ is a handlebody; and

(2) K_1 and K_2 are constituent knots of $\widetilde{B}_g$.

Let $\tilde{v}$ be the unique vertex of $\widetilde{B}_g$. Let $\widetilde{G}$ be the spatial graph obtained by slightly modifying $\widetilde{B}_g$ in $N(\tilde{v})$ as shown in Figure 2. Let $\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_{g-2}$ be the path in $\widetilde{G}$ shown in the figure. By contracting the edges $\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_{g-2}$, we get a spatial θ_{g+1}–curve $\tilde{\theta}_{g+1}$ such that

(1) $M \setminus \operatorname{Int} N(\tilde{\theta}_{g+1})$ is a handlebody; and

(2) K_1 and K_2 are constituent knots of $\tilde{\theta}_{g+1}$.

Since both K_1 and K_2 are vertices of the tunnel complex $\mathcal{T}_{g-1}(M)$ by Lemma 2.2, this means that K_1 and K_2 are adjacent in $\mathcal{T}_{g-1}(M)$. □

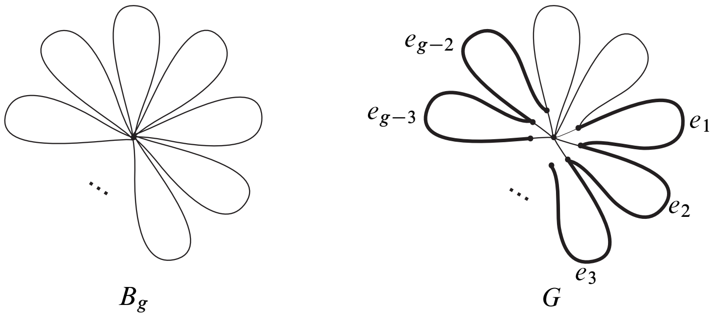

Figure 2: From B_g to G

The above proof of Lemma 2.3 gives rise to the following proposition.

Proposition 2.4 *Let K_0, $K_1, \ldots, K_{g-1}$ be the set of constituent knots of a spatial bouquet $\widetilde{B}_g$ with $M \setminus \mathrm{Int}\, N(\widetilde{B}_g)$ a handlebody. Then there exists a spatial θ_{g+1}–curve $\widetilde{\theta}_{g+1}$ such that*

(1) *$M \setminus \mathrm{Int}\, N(\widetilde{\theta}_{g+1})$ is a handlebody; and*

(2) *all K_0, $K_1, \ldots, K_{g-1}$ are constituent knots of $\widetilde{\theta}_{g+1}$.*

Remark Let M be a closed 3–manifold and t be a natural number. Let $\mathcal{B}_t(M)$ be a simplicial complex defined by:

(1) the set of vertices of $\mathcal{B}_t(M)$ consists of the set of knots in M of tunnel number at most t; and

(2) a collection of distinct vertices $K_0, K_1, \ldots, K_k$ spans a k–simplex of $\mathcal{B}_t(M)$ if there exists a spatial bouquet $\widetilde{B}_{t+1}$ such that
   (a) $M \setminus \mathrm{Int}\, N(B_{t+1})$ is a handlebody; and
   (b) each K_l $(0 \leqslant l \leqslant k)$ is a constituent knot of $\widetilde{B}_{t+1}$.

Then Proposition 2.4 implies that there exists a canonical simplicial injection from $\mathcal{B}_t(M)$ to the t–tunnel complex $\mathcal{T}_t(M)$.

Let $\widetilde{G}$ be a trivalent spatial graph in a compact 3–manifold (possibly with nonempty boundary). An *IH–move* for $\widetilde{G}$ is a local change of $\widetilde{G}$ as illustrated in Figure 3. Note that the inverse of an IH–move is also an IH–move. A *trivalent spine* of a handlebody H is a spatial trivalent graph $\widetilde{G}$ embedded in H such that a regular neighborhood of $\widetilde{G}$ is a deformation retract of H.

Lemma 2.5 [20; 13] *Given two trivalent spines of a handlebody, there exists a finite sequence of IH–moves and ambient isotopies transforming one into the other.*

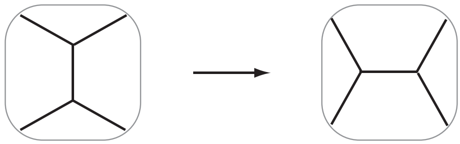

Figure 3: IH–move

Lemma 2.6 *Let $\widetilde{G}_1$ and $\widetilde{G}_2$ be spatial trivalent graphs embedded in a closed 3–manifold M such that*

(1) *$M \setminus \operatorname{Int} N(\widetilde{G}_i)$ $(i = 1, 2)$ is a genus g handlebody; and*

(2) *$\widetilde{G}_2$ is obtained from $\widetilde{G}_1$ by an IH–move.*

Let K_i $(i = 1, 2)$ be a constituent knot of $\widetilde{G}_i$. Then K_1 and K_2 belong to the same component of $\mathcal{T}_{g-1}(M)$.

Proof Let an IH–move for $\widetilde{G}_1$ is performed along an edge $\widetilde{e}$, and let $\widetilde{G}_2$ be the resulting trivalent spatial graph. Note that $N(\widetilde{G}_1)$ is ambient isotopic to $N(\widetilde{G}_2)$ in M. To prove the lemma, it suffices to show that there exists a constituent knot of $\widetilde{G}_1$ which is also a constituent knot of $\widetilde{G}_2$.

Let $\widetilde{v}_1$ and $\widetilde{v}_2$ be the endpoints of $\widetilde{e}$ and let $\widetilde{e}_1$, $\widetilde{e}_2$, $\widetilde{e}_3$ and $\widetilde{e}_4$ be the edges shown in Figure 4.

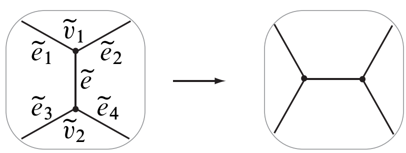

Figure 4

Case 1 $\widetilde{e}$ is a cut edge of $\widetilde{G}_1$.

In this case, if there exists a cycle $\widetilde{C}$ in $\widetilde{G}_1$ which contains both $\widetilde{e}_1$ and $\widetilde{e}_2$ and does not contain neither $\widetilde{e}_3$ nor $\widetilde{e}_4$. Then $\widetilde{C}$ is also a constituent knot of $\widetilde{G}_2$. Suppose that there exists no such a cycle $\widetilde{C}$. Since $\widetilde{G}_1$ is trivalent, we may easily check that there must be a cycle $\widetilde{C}'$ which contains none of the edges $\widetilde{e}_1$, $\widetilde{e}_2$, $\widetilde{e}_3$ and $\widetilde{e}_4$, by an argument of Euler Characteristics of $\widetilde{G}_1$. Then it is clear that $\widetilde{C}'$ is also a constituent knot of $\widetilde{G}_2$.

Case 2 $\widetilde{e}$ is not a cut edge of $\widetilde{G}_1$.

In this case, there exists a path $\widetilde{P}$ in $\widetilde{G} \setminus \operatorname{Int} \widetilde{e}$ connecting $\widetilde{v}_1$ and $\widetilde{v}_2$. Set $\widetilde{C} := \widetilde{P} \cup \widetilde{e}$. Then $\widetilde{C}$ is also a constituent knot of $\widetilde{G}_2$. □

Lemma 2.7 *Let $\widetilde{G}_i$ $(i = 1, 2)$ be a spatial trivalent graph with $g(\widetilde{G}_i) = g$ embedded in a closed 3–manifold M such that*

1. *$M \setminus \operatorname{Int} N(\widetilde{G}_i)$ $(i = 1, 2)$ is a handlebody; and*
2. *$N(\widetilde{G}_1)$ is ambient isotopic to $N(\widetilde{G}_2)$.*

Let K_i $(i = 1, 2)$ be a constituent knot of $\widetilde{G}_i$. Then K_1 and K_2 belong to the same component of $\mathcal{T}_{g-1}(M)$.

Proof By Lemma 2.5, $\widetilde{G}_1$ is transformed into $\widetilde{G}_2$ by a finite sequence of IH–moves and ambient isotopies. Let

$$\widetilde{G}_1 = \widetilde{T}_1 \xrightarrow{\text{IH–move}} \widetilde{T}_2 \xrightarrow{\text{IH–move}} \cdots \xrightarrow{\text{IH–move}} \widetilde{T}_n = \widetilde{G}_2$$

be such a sequence.

By Lemma 2.2, all constituent knots of $\widetilde{T}_i$ for $i = 1, 2, \dots, n$ are knots of tunnel number at most t. By Lemma 2.3, all constituent knots of $\widetilde{T}_i$ are adjacent in $\mathcal{T}_{g-1}(M)$ for each $1 \leqslant i \leqslant n$. Moreover, by Lemma 2.6, all constituent knots of $\widetilde{T}_i$ and $\widetilde{T}_{i+1}$ $(i = 1, 2, \dots, n-1)$ belong to the same component of $\mathcal{T}_{g-1}(M)$. This implies that K_1 and K_2 belong to the same component of $\mathcal{T}_{g-1}(M)$. □

Lemma 2.8 (Remark in Section 2 of [10]) *The two handlebodies of a genus g Heegaard splitting of a closed orientable 3–manifold can be interchanged by an ambient isotopy after stabilizing at most g times.*

The following classical results play an important role in the proof of Theorem 0.1.

Theorem 2.9 (1) (Waldhausen [26]) *Every Heegaard splitting of S^3 is standard, that is, every Heegaard splitting of S^3 is obtained by stabilizing the unique splitting of genus zero.*

(2) (Bonahon and Otal [2]) *Every Heegaard splitting of a lens space is standard, that is, every Heegaard splitting of a lens space is obtained by stabilizing the unique Heegaard splitting of genus one.*

Proof of Theorem 0.1 (1) Assume that M is the 3–sphere or a lens space and let t be a natural number. Let K_1 and K_2 be arbitrary knots of tunnel number at most t. Let $\{\tau_1, \tau_2, \dots, \tau_t\}$ ($\{\mu_1, \mu_2, \dots, \mu_t\}$, respectively) be an unknotting tunnel system of K_1 (K_2, respectively). Then both the union $\widetilde{G}_1 := K_1 \cup (\bigcup_{i=1}^{t} \tau_i)$ and $\widetilde{G}_2 := K_2 \cup (\bigcup_{i=1}^{t} \mu_i)$ are spatial trivalent graphs with handlebody complements.

By Theorem 2.9, $\widetilde{G}_1$ and $\widetilde{G}_2$ are trivalent spine of one of the two genus $t+1$ handlebodies appearing in the standard genus $t+1$ Heegaard splitting $M = V \cup W$. Since the Heegaard splitting $M = V \cup W$ is obtained by stabilizing the standard genus one Heegaard splitting of M, there exists an ambient isotopy interchanging V and W due to Lemma 2.8. Therefore we may regard $\widetilde{G}_1$ and $\widetilde{G}_2$ as trivalent spines of the handlebody V. This implies that $\widetilde{G}_1$ and $\widetilde{G}_2$ satisfy the assumption of Lemma 2.7, therefore K_1 and K_2 belong to the same component of $\mathcal{T}_t(M)$. This means that $\mathcal{T}_t(M)$ is connected, which concludes the proof of (1).

(2) Let $V_1 \cup W_1$, $V_2 \cup W_2$ and $V_n \cup W_n$ be the set of all genus g Heegaard splittings of M. It is clear that each knot of tunnel number at most $g-1$ belong to one of the constituent knots of a trivalent spine of some V_k or W_k $(1 \leqslant k \leqslant n)$. By Lemma 2.7, any constituent knots of any trivalent spines of V_k or W_k belong to the same component of $\mathcal{T}_{g-1}(M)$. This implies (2). □

Corollary 2.10 *Let M be a closed 3–manifold of Heegaard genus at least two. Let g be a natural number greater than two. If all of genus g Heegaard splitting of M is stabilized, then the tunnel complex $\mathcal{T}_{g-1}(M)$ is connected.*

Proof Let K be an arbitrary knot of tunnel number at most $g-1$. Let $\tau_1, \tau_2, \ldots, \tau_{g-1}$ be an unknotting tunnel system of K. Set $\widetilde{G} := K \cup (\bigcup_{i=1}^{g-1} \tau_i)$. Then the union $N(\widetilde{G}) \cup (M \setminus \operatorname{Int} N(\widetilde{G}))$ is a stabilized Heegaard splitting by the assumption. It follows that there exists a trivalent spine of $N(\widetilde{G})$ of which the unknot is a constituent knot. By Lemma 2.7, K and the unknot belong to the same component of $\mathcal{T}_{g-1}(M)$. This implies that $\mathcal{T}_{g-1}(M)$ is connected. □

3 Tunnel complexes $\mathcal{T}_t(M)$ and disk complexes

3.1 The tunnel complex $\mathcal{T}_1(S^3)$ and the Cho–McCullough complex

We briefly review the notion and a theory of Cho and McCullough. See [4; 5; 3; 6] for the details.

Let M be a compact, irreducible 3–manifold. The disk complex $\mathcal{K}(M)$ is the simplicial complex such that

(1) the set of vertices of $\mathcal{K}(M)$ consists of the ambient isotopy classes of essential properly embedded disks in M; and

(2) a collection of $k+1$ vertices spans a k–simplex if and only if they admit a set of pairwise-disjoint representatives.

Denote by $\mathcal{D}(M)$ the subcomplex of $\mathcal{K}(M)$ spanned by the nonseparating disks.

Theorem 3.1 [21] *The following holds:*

(1) *If ∂M is compressible, then $\mathcal{K}(M)$ is contractible.*

(2) *If M has a nonseparating compressing disk, then $\mathcal{D}(M)$ is contractible.*

Let H_2 be an unknotted genus two handlebody embedded in S^3. Let $\mathcal{G}_2$ be the Goeritz group, ie the group of isotopy classes of orientation-preserving automorphisms of S^3 preserving H_2. Then the group $\mathcal{G}_2$ acts on $\mathcal{D}(H_2)$ and the quotient $\mathcal{D}(H_2)/\mathcal{G}_2$ naturally inherits the structure of simplicial complex.

Theorem 3.2 [4] *The simplicial complex $\mathcal{D}(H_2)/\mathcal{G}_2$ is contractible.*

Recall that an essential properly embedded disk D in H_2 is said to be *primitive* if there exists a properly embedded disk D' in the complementary handlebody $S^3 \setminus \operatorname{Int} H_2$ such that the circles ∂D and $\partial D'$ in ∂H_2 intersect transversely in a single point. Denote by π_0, μ_0 and θ_0 the unique $\mathcal{G}_2$–orbit of the primitive disks, the pairs of primitive disks, and the triples of primitive disks, respectively. Note that the set of tunnels of the tunnel number one or zero knots embedded in S^3 bijectively correspond to the set of vertices except μ_0 and θ_0 of the quotient $\mathcal{D}(H_2)/\mathcal{G}_2$. In [4], Cho and McCullough called the "upper" and "lower" tunnels of 2–bridge knots *simple* tunnels. Denote by Σ the set of all 2–simplices spanned by π_0, μ_0 and a simple tunnel (these 2–simplices are called *half simplices* in [4]). Denote also by σ the set of all 1–simplices spanned by μ_0 and simple tunnels.

Consider the set

$$\mathcal{S} = \{\theta_0, \mu_0, \langle \pi_0, \mu_0 \rangle, \langle \mu_0, \theta_0 \rangle, \langle \theta_0, \pi_0 \rangle, \langle \pi_0, \mu_0, \theta_0 \rangle\} \cup \sigma \cup \Sigma$$

of (open) simplices in $\mathcal{D}(H_2)/\mathcal{G}_2$. Set $\mathcal{X} := (\mathcal{D}(H_2)/\mathcal{G}_2) \setminus \mathcal{S}$. Observe that $\mathcal{D}(H_2)/\mathcal{G}_2$ collapses onto $\mathcal{X}$, hence $\mathcal{X}$ is connected. Observe also that the set of vertices of $\mathcal{X}$ exactly corresponds to the set of tunnels of tunnel number one or zero knots embedded in S^3. Let $p_1 \colon \mathcal{D}(H_2) \to \mathcal{X}$ be the simplicial map defined by the composition of the quotient by $\mathcal{G}_2$ and the above collapsing. Let $p_2 \colon \mathcal{X} \to \mathcal{T}_1(S^3)$ be the simplicial map which sends each tunnel τ to its associated knot. In this way, we get a sequence of simplicial surjections

$$\mathcal{D}(H_2) \xrightarrow{p_1} \mathcal{X} \xrightarrow{p_2} \mathcal{T}_1(S^3).$$

Observe that the map p_2 is complicated, since knots can have more than one unknotting tunnels and there will be identifications made under this map that reflects the nature of tunnel number one knots with multiple unknotting tunnels. Note that many torus knots

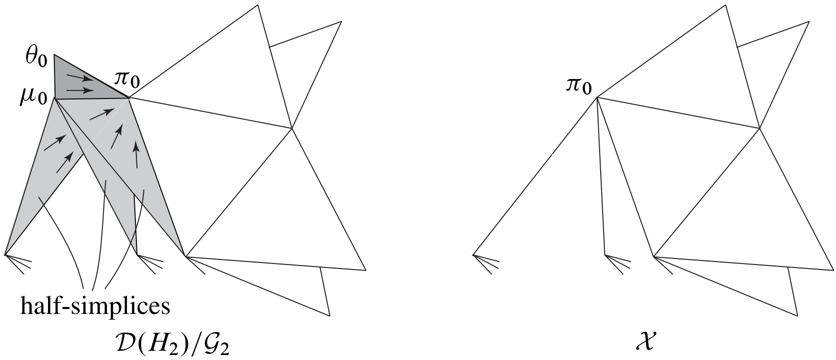

Figure 5: $\mathcal{D}(H_2)$ and $\mathcal{X}$

have both non–$(1,1)$– and $(1,1)$–tunnels (see [3]). Note also that Goda and Hayashi gave an example of non–torus knot having both non–$(1,1)$– and $(1,1)$–tunnels (see Figure 18 in [7]).

Let M be a lens space. Let $M = V \cup W$ be a genus two Heegaard splitting of M. Since there exists an ambient isotopy in M interchanging V and W, we may see in the similar argument as above that there exists a simplicial surjection from the disk complex $\mathcal{D}(H_2)$.

Summing up the above argument, we have the following:

Proposition 3.3 *Let M be the 3–sphere or a lens space. Let $M = V \cup W$ be the unique genus two Heegaard splitting of M. Then there exists a simplicial surjection*

$$\mathcal{D}(H_2) \to \mathcal{T}_1(M).$$

In particular, if M is the 3–sphere, there exists a factorization

$$\mathcal{D}(H_2) \to \mathcal{X} \to \mathcal{T}_1(S^3)$$

of simplicial surjections.

Since $\mathcal{D}(H_2)$ is connected, Theorem 0.1 (1) in the case of $t = 1$ follows from the above proposition.

Example There exists no edge in $\mathcal{T}_1(S^3)$ connecting the (right or left-handed) trefoil knot with the figure-eight knot. In fact, if they are adjacent in $\mathcal{T}_1(S^3)$, there would be an edge in $\mathcal{X}$ connecting one of the two unknotting tunnels of the figure-eight knot (see Kobayashi [19]) to the unique tunnel of the trefoil (see Boileau, Rost and Zieschang [1] and Moriah [22]). However, the argument in [4] shows that no such an edge exists.

Recall Question. We see that even when $J = K = L$ is a trefoil, such a embedding of the θ_3–curve does not exist. Assume for contradiction that such an embedding of the θ_3–curve exists. Then there exist a collection of mutually disjoint, mutually nonparallel, essential nonseparating disks D_0, D_1, D_2 in $H_2 \subset S^3$. Note that one of D_0, D_1, D_2 is primitive. Since the trefoil has a unique unknotting tunnel, the three disks is contained in the same $\mathcal{G}_2$–orbit. Then there must be a vertex corresponding to the $\mathcal{G}_2$–orbit of the pair $\{D_0, D_1\}$. However, the argument in [4] shows that no such a vertex exists.

3.2 Complex of nonseparating t–tuples of disks in a genus $(t+1)$ handlebody

In this subsection, we generalize the observation in the above subsection.

Let g be a natural number. Let $t \leqslant g$ be a natural number and let H_g be a genus g handlebody. A t–tuple $\{[D_1], [D_2], \dots, [D_t]\}$ of pairwise distinct vertices of the disk complex $\mathcal{D}(H_g)$ is said to be *nonseparating* if the exterior $H_g \setminus \mathrm{Int}(\bigcup_{i=1}^t N(D_i))$ is a solid torus.

The complex $\mathcal{D}_t(H_{t+1})$ of nonseparating t–tuples of disks is defined to be a simplicial complex whose vertices are the nonseparating t–tuples of vertices of $\mathcal{D}(H_{t+1})$ such that a collection of $k+1$ vertices spans a k–simplex if and only if all vertices in all of the $(k+1)$ t–tuples are realized at the same time as pairwise disjoint (but some pairs possibly be parallel) disks in H_{t+1}. Note that $\mathcal{D}(H_2) = \mathcal{D}_1(H_2)$.

Lemma 3.4 *Let $\{v_1, v_2, \dots, v_t\}$ and $\{w_1, w_2, \dots, w_t\}$ be vertices of $\mathcal{D}_t(H_{t+1})$. Assume that v_1 and w_1 are adjacent in $\mathcal{D}(H_{t+1})$ and that $\{v_1, w_1\}$ is a nonseparating pair of H_{t+1}. Then $\{v_1, v_2, \dots, v_t\}$ and $\{w_1, w_2, \dots, w_t\}$ belong to the same component of $\mathcal{D}_t(H_{t+1})$.*

Proof The proof proceeds by induction on t. There is nothing to prove the case of $t = 1$ since $\mathcal{D}(H_2) = \mathcal{D}_1(H_2)$ is connected.

Assume that $t > 1$. Let $H_t^{v_1}$ be the handlebody obtained by cutting H_{t+1} along the disk v_1. Since $\{v_1, v_2, \dots, v_t\}$ is a nonseparating t–tuple, $\{v_2, v_3, \dots, v_t\}$ becomes a nonseparating $(t-1)$–tuple of distinct vertices of $\mathcal{D}(H_t^{v_1})$. In fact, if v_i and v_j $i \neq j$ are parallel in $H_t^{v_1}$, $\{v_1, v_i, v_j\}$ would be a separating triple of H_{t+1} and this is a contradiction. Also, in $H_t^{v_1}$, w_1 becomes a vertex of $\mathcal{D}(H_t^{v_1})$. Then there exists a vertices $v'_3, v'_4, \dots, v'_t$ of $\mathcal{D}(H_t^{v_1})$ such that $\{w_1, v'_3, v'_4, \dots, v'_t\}$ is a $(t-1)$–tuple of distinct vertices of $\mathcal{D}(H_t^{v_1})$. By the induction assumption, there exists a sequence of nonseparating $(t-1)$–tuples $(\{x_2^i, x_3^i, \dots, x_t^i\})_{i=1}^{n_1}$ in $\mathcal{D}_{t-1}(H_t^{v_1})$ such that

(1) $\{v_2, v_3, \dots, v_t\} = \{x^1_2, x^1_3, \dots, x^1_t\}$;

(2) $\{w_1, v'_3, \dots, v'_t\} = \{x^{n_1}_2, x^{n_1}_3, \dots, x^{n_1}_t\}$; and

(3) $\{x^i_2, x^i_3, \dots, x^i_t\}$ and $\{x^{i+1}_2, x^{i+1}_3, \dots, x^{i+1}_t\}$ are adjacent in $\mathcal{D}_{t-1}(H^{v_1}_t)$ for $1 \leqslant i \leqslant n_1$.

This implies that $\{v_1, v_2, \dots, v_t\}$ and $\{v_1, w_1, v'_3, \dots, v'_t\}$ belong to the same component of $\mathcal{D}_t(H_{t+1})$.

Let $H^{w_1}_t$ be the handlebody obtained by cutting H_{t+1} along the disk w_1. Then by the same reason as above, $\{v_1, v'_3, \dots, v'_t\}$ and $\{w_2, w_3, \dots, w_t\}$ become nonseparating $(t-1)$–tuples of distinct vertices of $\mathcal{D}_{t-1}(H^{w_1}_t)$. Applying again the induction assumption for these two $(t-1)$–tuple as above, we get a path of nonseparating $(t-1)$–tuples in $\mathcal{D}_{t-1}(H^{w_1}_t)$ which connects $\{v_1, v'_3, \dots, v'_t\}$ and $\{w_2, w_3, \dots, w_t\}$. This implies that $\{v_1, w_1, v'_3, \dots, v'_t\}$ and $\{w_1, w_2, \dots, w_t\}$ belong to the same component of $\mathcal{D}_t(H_{t+1})$. This completes the proof. □

Lemma 3.5 *Let $g > 2$. Let v_1 and w_1 be vertices of $\mathcal{D}(H_g)$. Then there exists a path $(p_i)^n_{i=1}$ from v_1 to w_1 in $\mathcal{D}(H_g)$ such that $p_1 = v_1$, $p_n = w_1$ and each pair $\{p_i, p_{i+1}\}$ is nonseparating.*

Proof Since the complex $\mathcal{D}(H_g)$ is connected, there exists a path $(p'_i)^m_{i=1}$ from v_1 to w_1 in $\mathcal{D}(H_g)$ such that $p'_1 = v_1$, $p'_m = w_1$ and each pair $\{p'_i, p'_{i+1}\}$ is nonseparating. Assume that the pair $\{p'_i, p'_{i+1}\}$ is separating for H_g. Since each of p'_i and p'_{i+1} is nonseparating, there exists a vertex q_i of $\mathcal{D}(H_g)$ such that both pairs $\{p'_i, q_i\}$ and $\{p'_{i+1}, q_i\}$ are nonseparating. Hence, substituting an edge (p'_i, p'_{i+1}) in the sequence by the two edges $(p'_i, q_i) \cup (p'_{i+1}, q_i)$ for all of such pairs $\{p'_i, p'_{i+1}\}$, we get a required path. □

Proof of Theorem 0.2 There is nothing to prove the case of $t = 1$ since $\mathcal{D}(H_2) = \mathcal{D}_1(H_2)$ is already shown to be connected.

Let $\{v_1, v_2, \dots, v_t\}$ and $\{w_1, w_2, \dots, w_t\}$ be two vertices of $\mathcal{D}_t(H_{t+1})$. By Lemma 3.5, there exists a path $(p_i)^n_{i=1}$ from v_1 to w_1 in $\mathcal{D}_t(H_{t+1})$ such that $p_1 = v_1$, $p_n = w_1$ and each pair $\{p_i, p_{i+1}\}$ is nonseparating. For each p_i $(2 \leqslant i \leqslant n-1)$, there exists a nonseparating t–tuple $\{p_i, v^i_2, v^i_3, \dots, v^i_t\}$ of vertices of $\mathcal{D}_t(H_{t+1})$. Set $v_i = v^{(1)}_i$ and $w_i = v^{(n)}_i$ for $2 \leqslant i \leqslant t$. By Lemma 3.4, there exists a path from $\{p_i, v^i_2, v^i_3, \dots, v^i_t\}$ to $\{p_{i+1}, v^{i+1}_2, v^{i+1}_3, \dots, v^{i+1}_t\}$ in $\mathcal{D}_t(H_{t+1})$ for $1 \leqslant i \leqslant n-1$. This completes the proof. □

Let M be the 3–sphere or a lens space. Let H_{t+1} be an embedded genus $t+1$ handlebody in M such that $M \setminus \operatorname{Int} N(H_{t+1})$ is a handlebody. Define a vertex map $\phi_t\colon \mathcal{D}_t^{(0)}(H_{t+1}) \to \mathcal{T}_t^{(0)}(S^3)$ by sending each vertex $\{[D_1],[D_2],\ldots,[D_t]\}$ to the equivalent class of the core knot K of the solid torus $H_{t+1} \setminus \operatorname{Int}(\bigcup_{i=1}^{t} N(D_i))$.

Let $\boldsymbol{v} := \{v_1, v_2, \ldots, v_t\}$ and $\boldsymbol{w} := \{w_1, w_2, \ldots, w_t\}$ be adjacent vertices of $\mathcal{D}_t(H_{t+1})$. Then the union $\boldsymbol{v} \cup \boldsymbol{w}$ can be realized as a pairwise disjoint (but some pairs possibly be parallel) disks in H_{t+1}. Let $\{z_1, z_2, \ldots, z_s\}$ be the subset of $\boldsymbol{v} \cup \boldsymbol{w}$ obtained by removing one of the parallel disks for each parallel pairs. Adding some more disks $z_{s+1}, z_{s+2}, \ldots, z_{3t}$, we get a set $\{z_1, z_2, \ldots, z_{3t}\}$ of pairwise disjoint, pairwise nonparallel essential disks which gives a pants decomposition of H_{t+1}. Let $\widetilde{G} \subset H_{t+1}$ be a spatial trivalent spine corresponds to this pants decomposition. Set $K_1 := \phi_t(\{v_1, v_2, \ldots, v_t\})$ and $K_2 := \phi_t(\{w_1, w_2, \ldots, w_t\})$. Then both K_1 and K_2 are constituent knots of $\widetilde{G}$.

Corollary 3.6 *The vertex map ϕ_t can be extended to a simplicial map*

$$\phi_t\colon \mathcal{D}_t^{(1)}(H_{t+1}) \to \mathcal{T}_t^{(1)}(M)$$

and it is a surjection.

Proof By Lemma 2.1 and Lemma 2.3, $\phi_t\colon \mathcal{D}_t^{(1)}(H_{t+1}) \to \mathcal{T}_t^{(1)}(S^3)$ a simplicial map. By Lemma 2.2 it is a surjection for the vertices. Let K_1 and K_2 be adjacent in $\mathcal{T}_t^{(1)}(M)$. Let then there exist an spatial θ_{t+2}–curve $\widetilde{\theta}_{t+2}$ such that $M \setminus \operatorname{Int} N(\widetilde{\theta}_{t+2})$ is a handlebody and both K_1 and K_2 are constituent knots of $\widetilde{\theta}_{t+2}$. Then slightly modifying $\widetilde{\theta}_{t+2}$ as shown in Figure 6, we may get a trivalent spine $\widetilde{G}$ of H_{t+1} containing K_1 and K_2 as constituent knots. Taking t edges from $\{\alpha_1, \alpha_2, \ldots, \alpha_{t+2}\}$, we obtain vertices

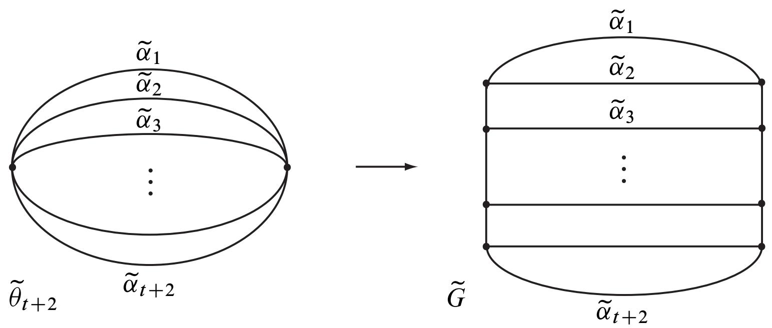

Figure 6

$\{v_1, v_2, \ldots, v_t\}$ and $\{w_1, w_2, \ldots, w_t\}$ of $\mathcal{D}_t(H_{t+1})$ where $\phi_t(\{v_1, v_2, \ldots, v_t\}) = K_1$ and $\phi_t(\{w_1, w_2, \ldots, w_t\}) = K_2$. Now, the edge $(\{v_1, v_2, \ldots, v_t\}, \{w_1, w_2, \ldots, w_t\})$ is mapped by ϕ_t to the edge (K_1, K_2) of $\mathcal{T}_t(M)$. This completes the proof. □

Observe that the above corollary gives an alternative proof (from a relative viewpoint) of Theorem 0.1.

4 Tunnel complexities

4.1 Definition of the tunnel complexities

Let M be a closed 3–manifold. Let K be a knot of tunnel number $t \in \mathbb{N}$ embedded in M. Define a distance $d_{\mathcal{T}_t(M)}$, *t–distance*, of two knots K_1 and K_2 of tunnel number at most t belonging to the same component of $\mathcal{T}_t(M)$ to be the simplicial distance of the two vertices K_1 and K_2 in the 1–skeleton $\mathcal{T}_t^{(1)}(M)$ of $\mathcal{T}_t(M)$. More precisely, $\mathcal{T}_t^{(1)}(M)$ becomes a geodesic space by letting each edge have length 1, and $d_{\mathcal{T}_t(M)}$ is defined by taking the minimal of the length of geodesics between given two vertices K_1 and K_2. We define $d_{\mathcal{T}_t(M)}(K_1, K_2) = \infty$ if K_1 and K_2 belong to different component of $\mathcal{T}_t(M)$. Note that Theorem 0.1 implies that $d_{\mathcal{T}_t(M)}(K_1, K_2) < \infty$ for any two vertices K_1 and K_2 in $\mathcal{T}_t(M)$ if M is the 3–sphere or a lens space.

Definition 4.1 The *t–tunnel complexity* $\mathrm{tc}_t(K)$ of K is the simplicial distance between K and the unknot in the 1–skeleton $\mathcal{T}_t^{(1)}(S^3)$ of $\mathcal{T}_t(S^3)$, that is, the minimal number of edges contained in paths in $\mathcal{T}_t^{(1)}(S^3)$ connecting K and the unknot.

Proposition 4.2 *Let M be a closed 3–manifold. Let K_1 and K_2 be adjacent vertices of $\mathcal{T}_t(M)$. Then there exists a spatial θ_{t+2}–curve $\widetilde{\theta}_{t+2}$ in M such that $\widetilde{\alpha}_1 \cup \widetilde{\alpha}_2$ is ambient isotopic to K_1 and $\widetilde{\alpha}_2 \cup \widetilde{\alpha}_3$ is ambient isotopic to K_2.*

Proof This follows from the argument in Lemma 2.1 and Lemma 2.3. □

Remark Let M be a closed 3–manifold. By Proposition 4.2, a path

$$(K_1, K_2, \ldots, K_n) \quad (n \geqslant 2)$$

in $\mathcal{T}_t^{(1)}(M)$ can be recognized as the following operations:

Step 0 Set $i = 1$.

Step 1 Take an unknotting tunnel system $\{\tau_1^i, \tau_2^i, \ldots, \tau_t^i\}$ for the knot K_i. (Here, $\partial\tau_1^i$ separates K_i into two arcs β_i and γ_i.)

Step 2 Set $K_{i+1} := \beta_i \cup \tau_1^i$.

Step 3 Back to Step 1 unless $i = n-1$, adding 1 to i.

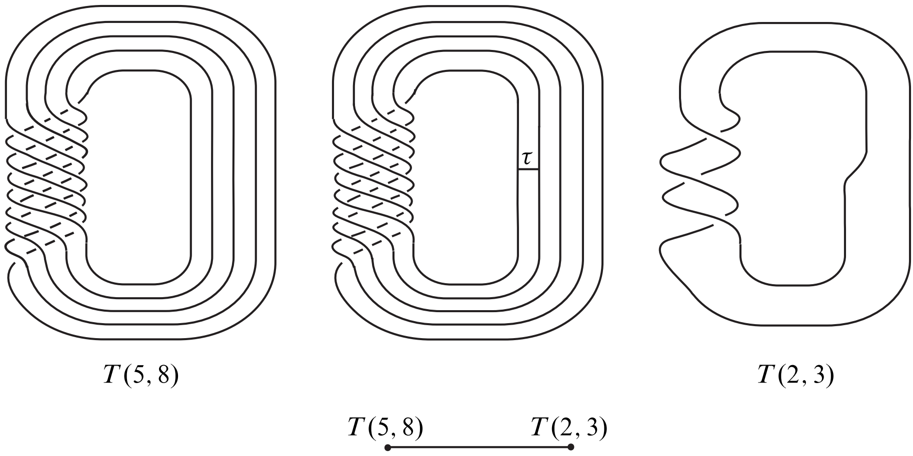

Figure 7: A path in $\mathcal{T}_1(S^3)$

Proof of Proposition 0.3 (1) Since the union of a tunnel number one knot K and its $(1,1)$–tunnel is a spatial θ_3–curve which guarantees the existence of an edge in $\mathcal{T}_1(M)$ whose endpoints are the core of a genus one handlebody of a Heegaard splitting of M and K, we easily see the "if" part. We will prove the "only if" part.

Assume that $d_{\mathcal{T}_1(M)}(C,K)=1$. Then there exists a spatial θ_3–curve $\widetilde{\theta}_3 \subset M$ such that $\widetilde{\alpha}_1 \cup \widetilde{\alpha}_2 = C$ and $\widetilde{\alpha}_2 \cup \widetilde{\alpha}_3 = K$. Set $N_1 := N(C)$ and $N_2 := M \setminus \operatorname{Int} N_1$. Note that both N_1 and N_2 are solid tori, and $\alpha'_3 := \overline{\widetilde{\alpha}_3 \setminus N_1}$ is an arc properly embedded in the solid torus N_2. Since $W := N_2 \setminus \operatorname{Int} N(\widetilde{\alpha}'_3)$ is a handlebody, the cocore loop in ∂W of the arc α'_3 turns out to be primitive due to Theorem $1'$ in [9]. This implies that the arc α'_3 is unknotted in N_2 and hence the Heegaard splitting $N_1 \cup N_2$ gives a $(1,1)$–decomposition of K. This completes the proof of (1).

(2) Let K admits a $(1,t)$–decomposition. Using a $(1,t)$–decomposition $M = V \cup W$ of K, we can find an unknotting tunnel system $\{\tau_1, \tau_2, \dots, \tau_t\}$ of K as shown in Figure 8. Then the spatial trivalent graph $\widetilde{G} := K \cup (\bigcup_i \tau_i)$ contains a constituent knot

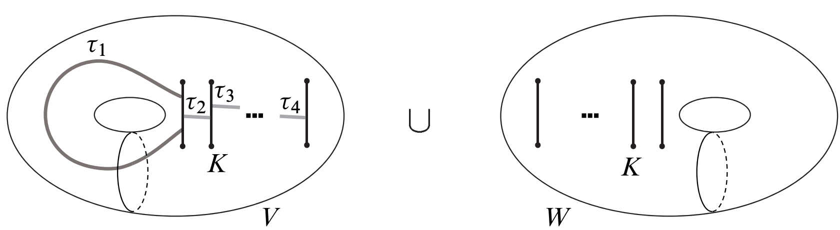

Figure 8: A unknotting tunnel system $\{\tau_1, \tau_2, \dots, \tau_t\}$ of K

consisting of τ_1 and a subarc of K that is ambient isotopic to the core C of a solid torus V_1 appearing in the shown Heegaard splitting. This implies that $\widetilde{G}$ contains C and K as constituent knots. Hence by Lemma 2.3, C and K are adjacent in $\mathcal{T}_t(M)$.

(3) Let $\{\tau_1, \tau_2, \dots, \tau_t\}$ be an unknotting tunnel system of K. Let τ_{t+1} be a tunnel parallel to τ_1. Then there exists an embedded square $I \times I$ such that

(1) $\{0\} \times I \subset \tau_1$;

(2) $\{1\} \times I \subset \tau_{t+1}$; and

(3) $I \times \{0, 1\} \subset K$.

Note that $\{\tau_1, \tau_2, \dots, \tau_t, \tau_{t+1}\}$ is also an unknotting tunnel system of K and the unknot ∂S is a constituent knot of the spatial trivalent graph $\widetilde{T}_1 := K_1 \cup (\bigcup_{i=1}^{t} \tau_i)$.

By Lemma 2.3, there exists a spatial θ_{t+1}–curve of which both the unknot and K are constituent. This concludes the proof of (3). $\square$

Consider *twisted torus knots* $K(p, q; r) \subset S^3$ shown in Figure 9. Morimoto, Sakuma

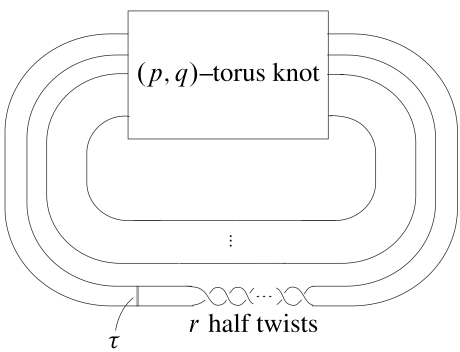

Figure 9: Twisted torus knots $K(p, q; r)$

and Yokota [24] showed that the knot $K(7, 17; 10m-4)$ ($m \in \mathbb{N}$) admits no $(1, 1)$–decompositions. We call these knots *Morimoto–Sakuma–Yokota knots*. Since every $K(p, q; r)$ admits an unknotting tunnel τ shown in Figure 9, $K(p, q; r)$ is connected with a torus knot in $\mathcal{T}_1(S^3)$. By this fact and Proposition 0.3, we immediately get the following corollary.

Corollary 4.3 *The tunnel complexity of every Morimoto–Sakuma–Yokota knot is two.*

Note that in [16], Johnson and Thompson show that for any natural number n there exists a tunnel number one knot in S^3 that does not admit a $(1, n)$–decomposition.

4.2 Unboundedness of tunnel complexity

Cho and McCullough defined an invariant of an unknotting tunnel τ of a tunnel number one knot in S^3, called the *depth*. Recall that each unknotting tunnel τ is a vertex of the simplicial complex $\mathcal{X}$. Then the *depth* $\mathrm{depth}(\tau)$ of τ is the simplicial distance of τ and π_0 in the 1–skeleton $\mathcal{X}^{(1)}$ of $\mathcal{X}$. Define $\mathrm{depth}(K)$ to be the minimum of the depths of all unknotting tunnels of K.

Proposition 4.4 *Let K be a tunnel number one knot. Then we have $\mathrm{tc}_1(K) \leqslant \mathrm{depth}(K)$. In particular, $\mathrm{tc}_1(K) = \mathrm{depth}(K) = 1$ for a $(1,1)$–knot K.*

Proof This follows from the existence of the simplicial map $p_2\colon \mathcal{X} \to \mathcal{T}_1(M)$ which we observed in Section 3.1. □

In [6], it is obtained as a corollary of Tunnel Leveling Addendum (see also [8]) the following lower bound of the bridge number.

Theorem 4.5 [6] *Let τ be an unknotting tunnel of a tunnel number one knot K in the 3–sphere. Then the bridge number of K is bounded below by*

$$\frac{(1+\sqrt{2})^{\mathrm{depth}(\tau)}}{\sqrt{2}} - \frac{(1-\sqrt{2})^{\mathrm{depth}(\tau)}}{\sqrt{2}}.$$

We can deduces the above theorem and Proposition 4.4 that the bridge number of a knot in S^3 is bounded below by

$$\frac{(1+\sqrt{2})^{\mathrm{tc}_1(K)}}{\sqrt{2}} - \frac{(1-\sqrt{2})^{\mathrm{tc}_1(K)}}{\sqrt{2}}.$$

Observe that this lower bound is weaker than that in Theorem 4.5 since there exists a torus knot having unknotting tunnels both with depth one and an arbitrary number, however, it is still deserved to be mentioned.

Let Σ be a closed orientable surface. The *curve complex*, denoted by $\mathcal{C}(\Sigma)$, is a simplicial complex whose vertices are ambient isotopy classes of essential simple closed curves and whose simplices are pairwise disjoint sets of simple closed curves. The *distance* between simple closed curves ℓ_1 and ℓ_2 in Σ is defined as the length of the shortest edge path in $\mathcal{C}(\Sigma)$ between the vertices representing them. Let $K \subset S^3$ be a tunnel number one knot and τ be its unknotting tunnel. Set $H := N(K \cup \tau)$. Then the *distance* of τ, denoted by $\mathrm{dist}(\tau)$, is defined to be the distance in the curve complex $\mathcal{C}(\partial H)$ from the co-core of τ to the boundaries of properly embedded essential disks in $S^3 \setminus \operatorname{Int} H$. Define $\operatorname{dist}(K)$ to be the minimum of the distances of all unknotting tunnels of K.

To show that the 1–tunnel complexity of S^3 is unbounded, we use the following very strong result given by Scharlemann and Tomova [25] and Johnson [15]. (Note that Scharlemann and Tomova [25] proved a much more general case and Johnson [15] deduced the following from it.)

Theorem 4.6 [25; 15] *Let τ be an unknotting tunnel of a tunnel number one knot $K \subset S^3$. If $\operatorname{dist}(\tau) > 5$, then τ is the unique unknotting tunnel of K.*

In [15] Johnson also showed the following:

Theorem 4.7 [15] *For every integer n, there exists an unknotting tunnel τ of a tunnel number one knot $K \subset S^3$ such that $\operatorname{dist}(\tau) > n$.*

Proposition 4.8 [6] *Let τ be an unknotting tunnel of a tunnel number one knot $K \subset S^3$. Then the following holds:*

(1) $\operatorname{dist}(\tau) = 1$ *if and only if* $\tau = \pi_0$.

(2) $\operatorname{dist}(\tau) = 2$ *if* $\operatorname{depth}(\tau) = 1$.

(3) $\operatorname{dist}(\tau) - 1 \leqslant \operatorname{depth}(\tau)$.

The following is immediately deduced from the above proposition:

Corollary 4.9 *Let $K \subset S^3$ be a tunnel number one knot.*

(1) $\operatorname{dist}(K) = 1$ *if and only if K is the unknot.*

(2) $\operatorname{dist}(K) = 2$ *if* $\operatorname{tc}_1(K) = 1$.

(3) $\operatorname{dist}(K) - 1 \leqslant \operatorname{depth}(K)$.

Proof of Theorem 0.4 We shall use the same notation as in Section 3. Let $\mathcal{Y}$ be the subcomplex of $\mathcal{X}$ spanned by the tunnels of distance more than 5. Theorem 4.6 implies that restriction $p_2|_{\mathcal{Y}}\colon \mathcal{Y} \to \mathcal{T}_1(S^3)$ is an embedding.

Sending each unknotting tunnel τ to $\max\{5, \operatorname{dist}(\tau)\}$ defines a simplicial map $s\colon \mathcal{X} \to [5, \infty)$, where $\mathbb{R}$ has the simplicial structure with vertices the integers. By Theorem 4.6, all of the nontrivial identifications made by the map $p_2\colon \mathcal{X} \to \mathcal{T}_1(S^3)$ occur in $s^{-1}(5)$. Consequently, s factors through $p_2\colon \mathcal{X} \to \mathcal{T}_1(S^3)$, giving a simplicial map $\overline{s}\colon \mathcal{T}_1(S^3) \to [5, \infty)$. By definition, $\overline{s}$ maps a tunnel number one knot to

$\min\{5, \mathrm{dist}(\tau) \mid \tau$ is an unknotting tunnel of $K\}$. Since s is a surjection by Theorem 4.7, $\overline{s}$ is also a surjection.

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "s"] \arrow[dr, "p_2"'] & & {[5,\infty)} \\ & \mathcal{T}_1(S^3) \arrow[ur, "\overline{s}"'] & \end{tikzcd}$$

Let d be an arbitrary natural number greater than five. Then there exists a tunnel number one knot $K \subset S^3$ such that $\overline{s}(K) = d$. Since the map $\overline{s}$ is simplicial, we have

$$\mathrm{tc}_1(K) = \mathrm{dist}_{\mathcal{T}_1(S^3)}(U, K) \geqslant \overline{s}(K) - \overline{s}(U) = d - 5,$$

where U denotes the unknot. Since d can be arbitrary large, this completes the proof. □

4.3 Hyperbolicity

We denote by $T(p,q)$ the (p,q)–torus knot in S^3. Let T be a standard torus bounding a solid torus V in S^3 and let $T(p,q)$ lies in T. The unknotting tunnels of torus knots were classified by Boileau, Rost and Zieschang [1] and Moriah [22]. Recall that the *middle tunnel* of $T(p,q)$ is represented by an arc in the torus T that meets $T(p,q)$ only in its endpoints. The *lower* (*upper*, respectively) *tunnel* of $T(p,q)$ is represented by an arc α properly embedded in the solid torus V ($S^3 \setminus \mathrm{Int}\, V$, respectively), such that the circle which is the union of α with one of the two arcs of $T(p,q)$ with endpoints equal to the endpoints of α is a deformation retract of V ($S^3 \setminus \mathrm{Int}\, V$, respectively).

Recall that a nontrivial knot in S^3 is *hyperbolic* if it is neither a torus knot nor a satellite knot (see eg [17]).

Proof of Theorem 0.5 Given a nontrivial torus knot $T(p,q)$, since the union of $T(p,q)$ and its lower tunnel τ is a spatial θ–curve of which both the unknot and the torus knot itself are constituent knots, we have $\mathrm{tc}_1(T(p,q)) = 1$. On the other hand, Morimoto and Sakuma [23] gave the complete list of tunnel number one satellite knots, and they showed that all of them are $(1,1)$–knots. It follows from Proposition 0.3 that the 1–tunnel complexity of them are one. This completes the proof. □

5 2–simplices of the tunnel complex $\mathcal{T}_1(S^3)$

By Proposition 0.3, we get another example of knots which provide negative answer to the Question in Section 1 in the case of $M \cong S^3$ and $t = 1$. That is, there exists no

2–simplex whose vertices contain both the unknot and a Morimoto–Sakuma–Yokota knot. Moreover, we proved in Section 4 that the diameter of $\mathcal{T}_1(S^3)$ is infinite. This implies that the realization of three knots of tunnel number one as the constituent knots of the same spatial θ–curve is far from being success, and the shape of tunnel complexes is not too simple. How about if any two knots of the triple $\{K_0, K_1, K_2\}$ of tunnel number one knots in S^3 span 1–simplices. In this section, we focus on the 2–simplices of the tunnel complex $\mathcal{T}_1(S^3)$.

We first introduce some obvious sufficient conditions for a triple $\{K_0, K_1, K_2\}$ of tunnel number one knots in S^3 to span a 2–simplex in $\mathcal{T}_1(S^3)$.

Lemma 5.1 *Let $a_1, a_2, \dots, a_n$ be even integers. Then the triple $\{U, D(a_1, a_2, \dots, a_n), D(a_1 \pm 1, a_2, \dots, a_n)\}$ ($\{U, D(a_1, a_2, \dots, a_n), D(a_1, a_2, \dots, a_n \pm 1)\}$, respectively) of knots spans a 2–simplex in $\mathcal{T}_1(S^3)$, where U denotes the unknot and $D(\cdot, \cdot, \cdots, \cdot)$ denotes the Conway presentation of a two bridge knot.*

Proof This is due to the complete classification of unknotting tunnels for two bridge knots up to ambient isotopies and homeomorphisms in [19]. □

Lemma 5.2 *Given a twisted torus knot $K(p, q; r)$, where $(p, q) = 1$ and $r \in 2\mathbb{Z}_{\geqslant 0}$, let u and v be positive integers satisfying $pv - qu = 1$. Let $u < p$ and $v < q$. Then the triple $\{K(p, q; r), T(p, q), T(u, v)\}$ spans a 2–simplex in $\mathcal{T}_1(S^3)$.*

Proof This can be proved using the middle tunnels for given torus knots. □

Example The triple $\{T(2, 3), T(3, 5), T(5, 8)\}$ of torus knots spans a 2–simplex in $\mathcal{T}_1(S^3)$.

In fact, the spatial θ_3–curve $\widetilde{\theta}_3$ illustrated in Figure 10 shows that $\widetilde{\alpha}_1 \cup \widetilde{\alpha}_2 \approx T(5, 8)$, $\widetilde{\alpha}_2 \cup \widetilde{\alpha}_3 \approx T(3, 5)$ and $\widetilde{\alpha}_3 \cup \widetilde{\alpha}_1 \approx T(2, 3)$.

Recall that in [4], Cho and McCullough introduced a parametrization of all unknotting tunnels of all tunnel number one knots by sequences of rational numbers and ”binary” invariants and that they calculated these invariants for 2–bridge knots in [4] and torus knots in [3].

Proof of Theorem 0.6 For any $i \in \mathbb{N}$, set $T_i := T(3i + 4, 2i + 3)$. Let τ_i^m be the middle tunnel of T_i and τ_i^l (τ_i^u, respectively) be the lower (upper, respectively) tunnel of T_i. Note that some of τ_i^m, τ_i^l and τ_i^u coincide, see [1]. We give the Cho and McCullough's parameter of these unknotting tunnel using the argument in [3]. First, we

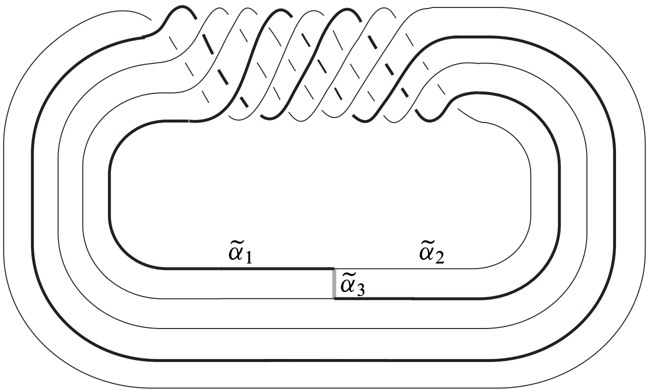

Figure 10: $T(5,8)$, $T(2,3)$ and $T(3,5)$ span a 2–simplex in $\mathcal{T}_1(S^3)$.

consider the middle slopes τ_i^m. Since the fraction $(3i+4)/(2i+3)$ can be represented using a continued fraction

$$\frac{3i+4}{2i+3} = [1, 2, i+1],$$

we get by an easy calculation that the parameter of the middle slope τ_i^m is

$$[1/3], \quad \{12k-7\}_{1\leqslant k\leqslant i+1}, \quad s_2 = 1, \quad s_3 = s_4 = \cdots = s_{i+1} = 0.$$

Next, we consider the lower tunnels. If $k = 3l-2$ $(l > 1)$, we have

$$(3l-2)(2i+3) = (2l-2)(3i+4) + (l+2i+2)$$

and $0 < l+2i+2 < 3i+4$. This implies that

$$\left\lceil \frac{(3l-2)(2i+3)}{3i+4} \right\rceil = 2l-1,$$

where $\lceil j \rceil := \min\{k \in \mathbb{Z} \mid j \leqslant k\}$ for a positive real number j. If $k = 3l-1$ $(l > 1)$, we have

$$(3l-1)(2i+3) = (2l-1)(3i+4) + (l+i+1)$$

and $0 < l+i+1 < 4+3i$. This implies that

$$\left\lceil \frac{(3l-1)(2i+3)}{3i+4} \right\rceil = 2l.$$

If $k = 3l$ $(l > 0)$, we have

$$3l(2i+3) = (2l-2)(3i+4) + l$$

and $0 < l < 4 + 3i$. This implies that

$$\left\lceil \frac{3l(2i+3)}{3i+4} \right\rceil = 2l + 1.$$

Hence the parameter of the lower slope τ_i^l is

$$[1/3], \quad \{p_k\}_{3 \leqslant k \leqslant 3i+3}, \quad s_2 = s_3 = \cdots = s_{3i+1} = 0,$$

where $p_{3l-2} = 4l - 3$ $(l > 1)$, $p_{3l-1} = 4l - 1$ $(l > 1)$ and $m_{3l} = 4l + 1$ $(l > 0)$.

Finally, we consider the upper tunnels. If $k = 2l - 1$ $(l > 1)$, we have

$$(2l-1)(3i+4) = (3l-2)(2i+3) + (-l+i+2)$$

and $0 < i - l + 2 < 2i + 3$. This implies that

$$\left\lceil \frac{(2l-1)(3i+4)}{2i+3} \right\rceil = 3l - 1.$$

If $k = 2l$ $(l > 0)$, we have

$$(2l)(3i+4) = (3l-1)(2i+3) + (-l+2i+3)$$

and $0 < -l + 2i + 3 < 3 + 2i$. This implies that

$$\left\lceil \frac{2l(3i+4)}{2i+3} \right\rceil = 3l.$$

Hence the parameter of the upper slope τ_i^u is

$$[1/3], \quad \{p'_k\}_{2 \leqslant k \leqslant 2i+2}, \quad s_2 = s_3 = \cdots = s_{2i+1} = 0,$$

where $p'_{2l-1} = 6l - 3$ $(l > 1)$ and $p'_{2l} = 6l - 1$ $(l > 0)$.

By the above argument, we obtain the following (see Figure 11):

(1) $\mathrm{depth}(\tau_i^\sigma) = 1$ for any $i \in \mathbb{N}$ and $\sigma \in \{l, u\}$.

(2) $\mathrm{depth}(\tau_i^m) = 2$ for any $i \in \mathbb{N}$.

(3) τ_i^m and τ_{i+1}^m are adjacent for $i \in \mathbb{N}$.

(4) $\tau_i^{\sigma_1}$ and $\tau_{i+1}^{\sigma_2}$ are not adjacent for any $i \in \mathbb{N}$ and $\sigma_1, \sigma_2 \in \{l, u\}$.

Then the existence of simplicial surjection $p_1 \circ p_2$ from $\mathcal{D}(H_2)/\mathcal{G}_2$ to $\mathcal{T}_1(M)$ implies that each T_i and the unknot U are adjacent in $\mathcal{T}_1(S^3)$ since τ_i^l and π_0 are adjacent in $\mathcal{D}(H_2)/\mathcal{G}_2$, and that each T_i and T_{i+1} are adjacent in $\mathcal{T}_1(S^3)$ since τ_i^m and τ_{i+1}^m are adjacent in $\mathcal{D}(H_2)/\mathcal{G}_2$. Therefore, there exist a cycle C_i of length three which contains the three vertices U, K_i and K_{i+1} for each $i \in \mathbb{N}$. Due to the classification

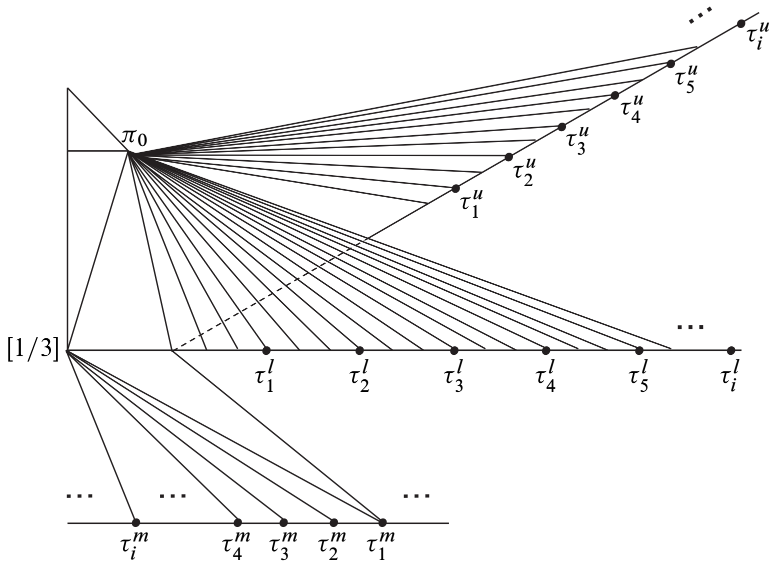

Figure 11: The tunnels of T_i $(i \in \mathbb{N})$ in $\mathcal{D}(H_2)/\mathcal{G}_2$

of torus knot, it is clear that these cycles are distinct. On the other hand, any triple $\{\pi_0, \tau_i^{\sigma_1}, \tau_{i+1}^{\sigma_2}\}$ $(\sigma_1, \sigma_2 \in \{m, l, u\})$ does not span a 2–simplex in $\mathcal{D}(H_2)/\mathcal{G}_2$ since there exists no cycle of length three in $\mathcal{D}(H_2)/\mathcal{G}_2$ that contains any of these triples. This implies that the cycle C_i does not span a 2–simplex in $\mathcal{T}_1(S^3)$ due to the fact that $p_2 \circ p_1$ is a simplicial surjection, which completes the proof. $\square$

6 Problems and perspectives

Recall that our first motivation was to consider a realization of tunnel number t knots as constituent knots of the same spatial θ_{t+1}–curve with complement a handlebody. In this viewpoint, the following is the most fundamental problem.

- Find the dimension of $\mathcal{T}_t(M)$ for $t > 1$.

By Theorem 0.1 (2), we see that if there exist a closed 3–manifold admitting finitely many Heegaard splittings of genus g then the number of component of $\mathcal{T}_{g-1}(M)$ is also finite. Hence the following question naturally arises:

- Does there exist a closed 3–manifold admitting infinitely many Heegaard splittings of genus g but having finitely many components of $\mathcal{T}_{g-1}(M)$?

On the t–distance of tunnel complexes, the following problems are fundamental.

- Is the diameter of any $\mathcal{T}_t(M)$ infinite?

In Proposition 4.4, we have that the 1–tunnel complexity of a tunnel number one knot K in S^3 is at most its depth.

- Find a tunnel number one knot $K \subset S^3$ with $\mathrm{tc}_1(K) < \mathrm{depth}(K)$.

In Theorem 0.5 we observed that knots in S^3 of 1–tunnel complexity at least two is hyperbolic.

- Can we have a similar result for t–tunnel complexity?
- Let $f(n)$ be the minimal hyperbolic volumes of knots in S^3 with 1–tunnel complexity n. Does $f(n)$ monotonically increase?

In [15], Johnson gave a lower bound for Seifert genus for tunnel number one knots with respect to the distance of unknotting tunnel.

- Can we estimate a Seifert genus of tunnel number one knots in S^3 using tunnel complexity?

In Section 5, we analyzed 2–simplices of $\mathcal{T}_1(S^3)$.

- Study interesting subspaces of $\mathcal{T}_1(S^3)$. In particular, determine the subcomplex of $\mathcal{T}_1(S^3)$ spanned by all torus knots or all hyperbolic knots, etc.
- What is $\pi_1(\mathcal{T}_1(S^3))$?

Since tunnel complexes are defined for closed 3–manifolds, the following is also very fundamental.

- Consider the relation between two tunnel complexes $\mathcal{T}_t(M_1)$ and $\mathcal{T}_t(M_2)$. Does there exist closed 3–manifolds M_1 and M_2 with $M_1 \not\cong M_2$ such that $\mathcal{T}_t(M_1) \cong \mathcal{T}_t(M_2)$ for any t?

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This paper is dedicated to Professor Ippei Ishii on the occasion of his 60–th birthday.

7 Appendix

In Section 2, we proved, in particular, that $\mathcal{T}_1(M)$ is connected, where M is S^3 or a lens space. In this case, we can prove the following stronger fact:

Theorem 7.1 *Let $\widetilde{G}_1$ and $\widetilde{G}_2$ be spatial θ_3–curves in a 3–manifold M having ambient isotopic regular neighborhoods. Then there exists a finite sequence of IH–moves and ambient isotopies transforming one into the other such that any spatial trivalent graphs appearing in the sequence is a spatial θ_3–curve.*

Remark This theorem implies not only that the tunnel complex $\mathcal{T}_1(M)$, where M is S^3 or a lens space, is connected but also the full-subcomplex of the IH–complex $\mathcal{C}_{\mathrm{IH}}$ (see [14]) spanned by spatial θ_3–curves is connected in each component of $\mathcal{C}_{\mathrm{IH}}$.

Proof By Lemma 2.5, the trivalent spatial graph $\widetilde{G}_1$ is transformed into $\widetilde{G}_2$ by a finite sequence of IH–moves as

$$\widetilde{G}_1 =: \widetilde{T}_1 \xrightarrow{\text{IH–move}} \widetilde{T}_2 \xrightarrow{\text{IH–move}} \widetilde{T}_3 \xrightarrow{\text{IH–move}} \cdots \xrightarrow{\text{IH–move}} \widetilde{T}_{n-1} \xrightarrow{\text{IH–move}} \widetilde{T}_n = \widetilde{G}_2.$$

Recall that there exist only two trivalent graphs of Betti number two, the θ_3–curve and the handcuff graph, shown in the left-hand side and the right-hand side of Figure 12, respectively.

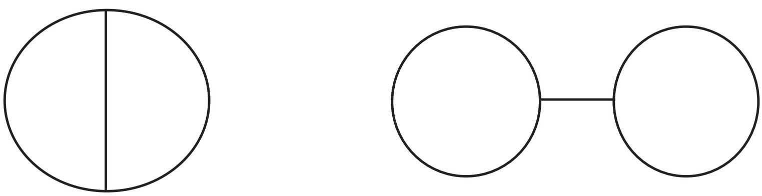

Figure 12: The θ_3–curve and the handcuff graph

Suppose that a spatial handcuff graph appears in the above sequence. Then there exists an integer $2 \leqslant i \leqslant n-1$ such that $\widetilde{T}_i$ is a spatial handcuff graph and $\widetilde{T}_j$ is the spatial θ_3–curve for all $1 \leqslant j < i$. An IH–move for a spatial trivalent graph is performed in a regular neighborhood of an edge of the graph with two distinct endpoints. Such an edge $\widetilde{e}$ for the spatial handcuff graph $\widetilde{T}_i$ is uniquely determined. Hence the graph $\widetilde{T}_{i+1}$ is a spatial θ_3–curve. Observe that the graph $\widetilde{T}_i$ might be modified by ambient isotopy before performing an IH–move. Indeed, we may possibly twist around the end point of the edge $\widetilde{e}$ as shown in the left hand side of Figure 13. We claim that the graph $\widetilde{T}_{i-1}$ can be related by $\widetilde{T}_{i+1}$ by a sequence of ambient isotopies and IH–moves which

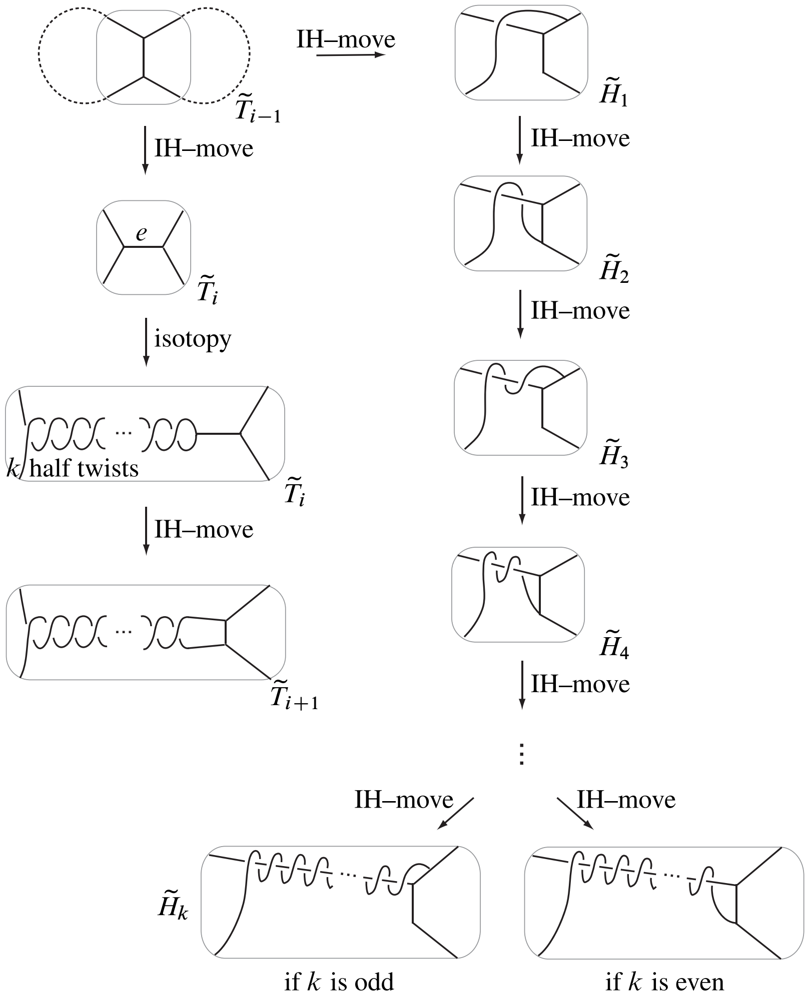

Figure 13: Bypass of sequences of IH–moves

produces only spatial θ_3–curves. Consider the sequence of spatial trivalent graphs $\widetilde{H}_1$, $\widetilde{H}_2, \ldots, \widetilde{H}_k$ shown in the right hand side of Figure 13. Note that all of these graphs are spatial θ_3–curves and they are obtained by a sequence of ambient isotopies and IH–moves from $\widetilde{T}_{i-1}$. Then it is clear from the figure that $\widetilde{T}_{i+1}$ is ambient isotopic to $\widetilde{H}_k$. Using the bypass

$$\widetilde{T}_{i-1} \xrightarrow{\text{IH–move}} \widetilde{H}_1 \xrightarrow{\text{IH–move}} \widetilde{H}_2 \xrightarrow{\text{IH–move}} \cdots \xrightarrow{\text{IH–move}} \widetilde{H}_k = \widetilde{T}_{i+1}$$

instead of

$$\widetilde{T}_{i-1} \xrightarrow{\text{IH–move}} \widetilde{T}_i \xrightarrow{\text{IH–move}} \widetilde{T}_{i+1},$$

we get another sequence of ambient isotopies and IH–moves such that all graphs appearing before $\widetilde{T}_{i+1}$ are spatial θ_3–curves.

Iterating this argument, we finally get a sequence of ambient isotopies and IH–moves from $\widetilde{G}_1$ to $\widetilde{G}_2$ such that all graphs appearing in the sequence are spatial θ_3–curve. This concludes the proof. $\square$

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