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Short geodesics in hyperbolic 3–manifolds

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For each $g \geq 2$, we prove existence of a computable constant $\epsilon(g) > 0$ such that if S is a strongly irreducible Heegaard surface of genus g in a complete hyperbolic 3–manifold M and γ is a simple geodesic of length less than $\epsilon(g)$ in M, then γ is isotopic into S.

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1 Introduction

Let M be a hyperbolic 3–manifold and let S be a surface embedded in M. A simple closed curve γ in M is said to be *unknotted* with respect to S if γ can be isotoped into S. A finite collection $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ of simple closed curves is *unlinked* with respect to S if there is a collection of disjoint embedded surfaces $S_1, \dots, S_n$ which are isotopic to S and with $\gamma_i \subset S_i$ for all i. One can ask if short geodesics are unknotted or unlinked with respect to fibers, Heegaard surfaces or leaves of a foliation.

It follows from work of Otal [5] that short geodesics in a hyperbolic mapping torus are unlinked with respect to the fiber, where "short" depends only on the genus of the fiber.

Theorem 1 (Otal) *For every g there is a constant $\epsilon > 0$ such that the following holds: If M is a closed 3–dimensional hyperbolic mapping torus with genus g fiber and Γ is the collection of simple closed geodesics in M which are shorter than ϵ, then Γ is unlinked with respect to S.*

In an unpublished paper [10], Souto proved that short geodesics in hyperbolic compression bodies are unlinked with respect to the boundary of the compression body.

Theorem 2 (Souto [10]) *If $\bar{N}$ is a compression body then there is a constant $\epsilon > 0$ which depends only on $\chi(\bar{N})$ such that for every complete hyperbolic metric on the interior N of $\bar{N}$ we have: every finite collection of simple geodesics which are shorter than ϵ is unlinked with respect to $\partial\bar{N}$.*

In the same paper Souto sketched a proof that short geodesics in hyperbolic 3–manifolds are unlinked with respect to a strongly irreducible Heegaard surface:

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Theorem 3 (Souto [10]) *For every g there is a constant $\epsilon > 0$ such that the following holds: if M is a closed hyperbolic 3–manifold, S is a strongly irreducible Heegaard surface of genus g in M, and Γ is the collection of simple closed geodesics in M which are shorter than ϵ, then Γ is unlinked with respect to S.*

The constants from Theorem 1 and Theorem 2 are known to be computable. However Souto only sketches a proof of Theorem 3 and does not produce an explicit constant. In this paper, we develop a new approach to Theorem 3. Our proof is more topological than geometric and is more elementary than Souto's proof. Moreover, we prove existence of a *computable* constant ϵ (depending only on the genus g) such that primitive geodesics of length less than ϵ are unknotted with respect to a strongly irreducible Heegaard surface of genus g:

Theorem 4 *For each $g \geq 2$ there exists a computable constant $\epsilon := \epsilon(g) > 0$ such that if S is a strongly irreducible Heegaard surface of genus g in a complete hyperbolic 3–manifold M and γ is a simple geodesic of length less than ϵ in M, then γ is isotopic into S.*

See the remark after the proof of Lemma 2 for a description of the constant $\epsilon(g)$.

The proof of Theorem 4 uses three main tools: bounded area sweepouts provided by work of Pitts and Rubinstein [6], an argument using the Rubinstein–Scharlemann graphic similar to an argument of Johnson [2] used to prove that spines of strongly irreducible Heegaard splittings are locally unknotted, and a lemma of Schultens from [9]. In Section 2, we use bounded area sweepouts and the Rubinstein–Scharlemann graphic to prove existence of an embedded annulus connecting a Heegaard surface to the boundary of a Margulis tube around a short geodesic. First, using a bounded area sweepout and the fact that Margulis tubes around very short geodesics are very fat, we show that the intersection of one of the sweepout surfaces with the Margulis tube around a short geodesic contains a simple loop which is homotopic to a power of the short geodesic. This is the content of Lemma 2 and is the only step which uses a geometric argument. Next, we use the Rubinstein–Scharlemann graphic and Lemma 2 to show that there exists an embedded annulus connecting a Heegaard surface to the boundary of a Margulis tube around the short geodesic. This is the content of Lemma 1. In Section 3, a thin position argument of Schultens is used to show that the existence of the annulus provided by Lemma 1 implies that the short geodesic is isotopic into a Heegaard surface.

Definitions Let M be a closed connected orientable 3–manifold. Let S be a closed connected orientable surface embedded in M which bounds handlebodies H_1 and H_2 on either side. We call (S, H_1, H_2) a *Heegaard splitting* of M. A Heegaard splitting

is *weakly reducible* if there are properly embedded essential disks in H_1, H_2 whose boundaries are disjoint. A Heegaard splitting is *strongly irreducible* if it is not weakly reducible.

2 Finding an annulus

Let S be a strongly irreducible Heegaard surface of genus $g \geq 2$ in a complete hyperbolic 3–manifold M. The goal of this section is to prove the following Lemma.

Lemma 1 *There exists a computable constant $\epsilon := \epsilon(g)$ such that if γ is a simple closed geodesic of length less than ϵ in M, then there is a regular neighborhood N of γ and an embedded annulus in $M \setminus N$ with boundary $\alpha \cup \alpha'$, where α is a simple essential nonmeridional loop in the boundary of N, and α' is contained in a surface isotopic to S.*

Sweepouts and Pitts–Rubinstein A sweepout of a 3–manifold M with a Heegaard surface S is a smooth degree one map $f\colon S \times [0,1] \to M$ such that $S_t := f(S \times \{t\})$ is a surface isotopic to S for each $t \in (0,1)$ and $f(S \times \{0\})$, $f(S \times \{1\})$ are spines of the handlebodies bounded by $S_{1/2}$. By work of Pitts and Rubinstein [6], there is a constant $A(g)$ such that if M has a genus g strongly irreducible Heegaard surface S then there exists a sweepout $f\colon S \times [0,1] \to M$ of M such that $\text{area}(S \times \{t\}) \leq A(g)$ for each $t \in [0,1]$. We can use $A(g) = 4\pi(g-1) + \delta$ for any $\delta > 0$.

We will use bounded area sweepouts to prove the following Lemma.

Lemma 2 *There exists a computable constant $\epsilon := \epsilon(g) > 0$ such that the following holds: if γ is a simple closed geodesic of length less than ϵ in M and T is a Margulis tube about γ, then S may be isotoped in M so that $S \cap T$ contains a simple loop which is essential in T.*

Proof Let $f\colon S \times [0,1] \to M$ be a sweepout of M with $\text{area}(S \times \{t\}) \leq A := A(g)$ for each $t \in [0,1]$. Let $\epsilon = \epsilon(g) > 0$ be so small that the Margulis tube about any geodesic in a complete hyperbolic 3–manifold of length at most epsilon has totally geodesic meridian disks with area at least $3A$. Note that the constant ϵ is computable. Let γ be a geodesic of length at most ϵ.

The set $M \setminus S_{1/2}$ is a union of two handlebodies. Let $H_{1/2}$ and $W_{1/2}$ be the closures of these handlebodies. For $t \in (0,1)$, let H_t be the closure of the component of $M \setminus S_t$ which changes into $H_{1/2}$ as S_t isotopes to $S_{1/2}$ and let W_t be the closure of the other component of $M \setminus S_t$. For $t \in (0,1)$ near 0, one of the handlebodies, say H_t, is a small neighborhood of a spine. Since f has degree one, the handlebody W_t is a small neighborhood of a spine for $t \in (0,1)$ near 1.

If $S_t \cap T$ contains a loop which is essential in T for some $t \in [0,1]$, then we are done proving Lemma 2. Assume that $S_t \cap T$ does not contain a loop which is essential in T for any $t \in [0,1]$.

Claim 1 *For each $t \in [0,1]$ either the interior of $W_t \cap T$ or the interior of $H_t \cap T$ contains a loop which is essential in T.*

Proof We are assuming that $S_t \cap T$ does not contain a loop which is essential in T. If $S_t \cap T$ is empty then we are done with Claim 1, so we will assume that $S_t \cap T$ is nonempty. Thus we have that $S_t \cap T$ is nonempty and that any loop in $S_t \cap T$ is trivial in T. Let $\widetilde{T}$ be a lift of T to the universal cover $\mathbb{H}^3$ of M. Since any loop in $S_t \cap T$ is trivial in T, there is a lift $\widetilde{S}$ of $S_t \cap T$ contained in $\widetilde{T}$ which is homeomorphic to $S_t \cap T$. Let $\widetilde{S}_0$ be a connected component of $\widetilde{S}$. Since $\widetilde{T}$ is a ball, $\widetilde{S}_0$ must separate $\widetilde{T}$. We claim that $\widetilde{T} \setminus \widetilde{S}_0$ contains a component which has compact closure which does not separate the ends of $\widetilde{T}$.

Let D be a totally geodesic meridian disk in $\widetilde{T}$ such that D is orthogonal to $\partial\widetilde{T}$. Consider the projection $p\colon \widetilde{T} \to D$ of $\widetilde{T}$ to D along lines equidistant from the geodesic core of $\widetilde{T}$. The area of S_t is at least the area of $p(\widetilde{S})$. Since the area of S_t is less than the area of D, the interior of D contains a point x which is not in $p(\widetilde{S})$. The preimage $p^{-1}(x)$ of x is disjoint from $\widetilde{S}_0$ and therefore contained in one component of $\widetilde{T} \setminus \widetilde{S}_0$. Thus $\widetilde{S}_0$ does not separate the ends of $\widetilde{T}$ Also, $\widetilde{S}_0$ is compact, hence contained in some compact subset K of $\widetilde{T}$. The complement $\widetilde{T} \setminus K$ of K in $\widetilde{T}$ is contained in the component of $\widetilde{T} \setminus \widetilde{S}_0$ which contains $p^{-1}(x)$. Thus the other component of $\widetilde{T} \setminus \widetilde{S}_0$ is contained in the compact set K and therefore has compact closure. We have shown that every component of $\widetilde{S}$ splits off a connected piece of $\widetilde{T}$ which does not separate the ends of $\widetilde{T}$ and which has compact closure.

Let D_1 and D_2 be distinct meridian disks in $\widetilde{T}$ which project to the same disk in T. There is one component of $\widetilde{T} \setminus (D_1 \cup D_2)$ whose closure is compact. Let $\mathcal{F}$ be the closure of this component. Since S_t is compact, there are finitely many lifts $\widetilde{S}_1, \dots, \widetilde{S}_k$ of $S_t \cap T$ which intersect the compact set $\mathcal{F}$. For each $i = 1, \dots, k$, $\widetilde{S}_i$ splits a piece from $\widetilde{T}$ which has compact closure so each component of $\widetilde{S}_1 \cup \cdots \cup \widetilde{S}_k$ splits off a piece of $\widetilde{T}$ which does not separate the ends of $\widetilde{T}$ and which has compact closure. Thus the set $\widetilde{T} \setminus (\widetilde{S}_1 \cup \cdots \cup \widetilde{S}_k)$ contains a connected component which intersects both D_1 and D_2. Therefore we can find an arc in $\widetilde{T} \setminus (\widetilde{S}_1 \cup \cdots \cup \widetilde{S}_k)$ with endpoints in D_1 and D_2. We have shown that the complete preimage of $S_t \cap T$ does not separate the two ends of $\widetilde{T}$ and thus some component $\widetilde{C}$ of the preimage of $T \setminus S_t$ in $\widetilde{T}$ is noncompact. This noncompact component $\widetilde{C} \subset \widetilde{T}$ projects to a set C in $T \setminus S_t$. Since $\widetilde{C}$ is noncompact, the set C has nontrivial image in $\pi_1(T)$ under the map induced by

inclusion. Thus there is a loop contained in C (which is contained in either the interior of $W_t \cap T$ or the interior of $H_t \cap T$) which is essential in T. This completes the proof of Claim 1. □

The following Claim will complete the proof of Lemma 2.

Claim 2 *The Heegaard surface S_β contains a simple loop in $S_\beta \cap T$ which is essential in T for some $\beta \in (0,1)$.*

Proof Since H_t is a small neighborhood of a spine for $t \in (0,1)$ near 0, the interior of $W_t \cap T$ contains a loop which is essential in T for $t \in (0,1)$ near 0. Fix $\delta > 0$ so that the interior of $W_\delta \cap T$ contains a loop which is essential in T. If $H_\delta \cap T$ contains a loop which is essential in T, then let $\beta = \delta$. If $H_\delta \cap T$ does not contain a loop which is essential in T, then let $\sigma = \inf\{t \in (\delta, 1) | H_t \cap T$ contains a loop which is essential in $T\}$. Note that σ exists since W_t bounds a very small neighborhood of a graph for t near 1.

If the interior of $H_\sigma \cap T$ contains a loop which is essential in T, then $H_t \cap T$ contains a loop which is essential in T for t near σ, contradicting the definition of σ. Thus the interior of $H_\sigma \cap T$ does not contain a loop which is essential in T and therefore Claim 1 implies that the interior of $W_\sigma \cap T$ contains a loop which is essential in T. So $W_t \cap T$ contains a loop which is essential in T for t near σ. This implies that for some $\beta > \sigma$ near σ, both $H_\beta \cap T$ and $W_\beta \cap T$ contain loops $l_W \subset W$ and $l_H \subset H$ which are essential in T. For some natural numbers m, n we have $(l_W)^m$ is homotopic in T to $(l_H)^n$. Thus there is an immersed annulus A in T with boundary components equal to $(l_W)^m$ and $(l_H)^n$. If $A \cap S_\beta$ does not contain any loops which are essential in A, then there is an arc in A with endpoints in l_W and l_H which is disjoint from S_β. This contradicts the fact that l_W and l_H are in different components of $T \setminus S_\beta$. Thus some loop α in $A \cap S_\beta$ is essential in A. Since the boundary components of A are essential in T, the loop α must be essential in T. Thus S_β contains a loop which is essential in T, so S_β contains a simple loop which is essential in T. This completes the proof of Claim 2, and hence of Lemma 2. □

Remark The constant $\epsilon(g)$ in Lemma 2 is the constant we will use in Theorem 4. From the proof of Lemma 2, $\epsilon(g)$ should be so small that a meridian disk in a Margulis tube around a closed geodesic of length less than $\epsilon(g)$ has area greater than $A(g) = 4\pi(g-1)$. The area of a totally geodesic disk of radius r in hyperbolic space is $2\pi(\cosh(r) - 1)$. Thus we want the radius of the Margulis tube to be greater than $\operatorname{arccosh}(2g-1)$. Meyerhoff [4] proved that the radius r of a Margulis tube around a closed geodesic of length l less than 0.107 satisfies

$$\sinh^2(r) = 1/2(\sqrt{1-2k}/k - 1) \quad \text{where } k = \cosh(\sqrt{4\pi l/\sqrt{3}}) - 1.$$

Thus choosing $\epsilon(g)$ so that

$$\operatorname{arcsinh}(\sqrt{1/2(\sqrt{1-2k}/k-1)}) > \operatorname{arccosh}(2g-1)$$

where $k = \cosh(\sqrt{4\pi\epsilon(g)/\sqrt{3}}) - 1$ will suffice.

By Lemma 2, we may isotope S so that $S \cap T$ contains a simple loop ω which is essential in T. We will show that we may isotope S so that $S \cap \partial T$ contains a simple loop which is essential in T by using an argument of Johnson used in [2] to prove that spines of strongly irreducible Heegaard splittings are locally unknotted. The idea is to use the Rubinstein–Scharlemann graphic to show that some sweepout surface S' intersects ∂T in loops which are inessential in S'. Since S and S' cobound a thickened copy of S, there is an embedded annulus with one boundary component equal to ω and the other boundary component disjoint from T. The intersection of this annulus with ∂T must contain a simple loop which is essential in the annulus, providing us with an embedded annulus to isotope ω into ∂T.

We will now define a new sweepout of M. Assume that we have isotoped S so that $S \cap T$ contains a simple loop ω which is essential in T. The Heegaard surface S splits M into two handlebodies H_1 and H_2. Let $f\colon M \to [-1,1]$ be a smooth function such that $f^{-1}(-1)$ is a spine of H_1, $f^{-1}(1)$ is a spine of H_2, $f^{-1}(t)$ is a surface isotopic to S for each $t \in (-1,1)$, and $f^{-1}(0) = S$ (ie, the map f provides a sweepout of M by disjoint Heegaard surfaces, one of which is S). For each $t \in (-1,1)$, let $S_t = f^{-1}(t)$. Note that the surfaces in this sweepout do not necessarily have area bounded in terms of g.

Lemma 3 *Let Y be a solid torus in M. One of the following holds:*

(1) *$S_t \cap \partial Y$ contains a loop which is essential in ∂Y and does not bound a meridian disk in Y for some $t \in (-1,1)$.*

(2) *For all $t \in (-1,1)$, if $S_t \cap \partial Y$ contains a loop which is essential in S_t, then $S_t \cap \partial Y$ contains a loop which bounds a properly embedded, essential disk in one of the handlebodies bounded by S_t.*

Proof Assume that $S_t \cap \partial Y$ does not contain a loop which is both essential in ∂Y and does not bound a meridian disk in Y for all $t \in (-1,1)$. Thus any loop in $S_t \cap \partial Y$ is either trivial in ∂Y or a meridian for Y. In particular, any loop in $S_t \cap \partial Y$ bounds a properly embedded disk in M. Scharlemann's No Nesting Lemma [8] implies that any loop which is essential in S_t and bounds a disk in M bounds a properly embedded, essential disk in one of the handlebodies bounded by S_t. Thus if $S_t \cap \partial Y$ contains a loop which is essential in S_t, then $S_t \cap \partial Y$ contains a loop which bounds a properly embedded, essential disk in one of the handlebodies bounded by S_t. □

The Rubinstein–Scharlemann graphic Let $g\colon T \to [0,1]$ be a smooth function such that $g^{-1}(0) = \gamma$ and $g^{-1}(1) = \partial T$ and $g^{-1}(t)$ is a surface isotopic to ∂T for each t in $(0,1)$. For each $t \in [-1,1]$ let $g_t = g|_{S_t \cap T}$. We say a function is *near-Morse* if there is a single degenerate critical point or there are two nondegenerate critical points at the same level. By work of Cerf [1], we can isotope f and g so that g_t is a Morse function for all but finitely many t and g_t is near-Morse for the remaining values of t. The Rubinstein–Scharlemann graphic, G, is the set of points $(t, s_t) \in [-1,1] \times [0,1]$ such that s_t is a critical value of the function g_t. Rubinstein and Scharlemann originally used the graphic to compare two sweepouts by Heegaard surfaces [7]. We are using a sweepout of the 3–manifold be Heegaard surfaces and a sweepout of a solid torus by tori. The properties of the graphic we need to use follow from the same arguments Rubinstein and Scharlemann use in their paper. Rubinstein and Scharlemann showed that this set of points is a graph with vertices of valence 2 and 4 in the interior of $[-1,1] \times [0,1]$ and valence 1 and 2 in the edges. A valence–2 vertex at (t, s_t) occurs when the map g_t has a degenerate critical point. A valence-4 vertex at (t, s_t) occurs when g_t has critical points at the same level. We will use the fact that if (t_1, s_1) and (t_2, s_2) are in the same component of $[-1,1] \times [0,1] \setminus G$, then the surface S_{t_1} is isotopic to S_{t_2} via an isotopy which takes the loops in $g_{t_1}^{-1}(s_1)$ to the loops in $g_{t_2}^{-1}(s_2)$. See Rubinstein and Scharlemann [7] or Johnson [2] for more on the Rubinstein–Scharlemann graphic.

Lemma 4 *Either $S_t \cap \partial T$ contains a loop which is essential in ∂T and does not bound a meridian disk in T for some $t \in (-1,1)$ or there is a $\sigma \in (-1,1)$ such that $S_\sigma \cap T$ does not contain an essential loop of S_σ.*

Proof Assume that $S_t \cap \partial T$ does not contain a loop which is an essential nonmeridional loop in ∂T for any $t \in (-1,1)$. Suppose, for contradiction, that $S_t \cap T$ contains an essential loop of S_t for each $t \in (-1,1)$. For each $t \in (-1,1)$, let C_t be a component of $S_t \cap T$ that contains a loop which is essential in S_t. First we will show that this implies that for each $t \in (-1,1)$ such that g_t is a Morse function, there exists a simple loop in $S_t \cap \partial T$ which bounds a properly embedded, essential disk in one of the handlebodies bounded by S_t. Suppose g_t is a Morse function and suppose that there is no simple loop in $S_t \cap \partial T$ which is essential in S_t. In particular, each loop in $C_t \cap \partial T$ bounds a disk in S_t. We may isotope S_t to eliminate any of these disks which are disjoint from the interior of T and we still have that C_t contains a loop which is essential in S_t. If $C_t \cap \partial T$ is still nonempty, then C_t must be a disk since we have eliminated any disks bounded by loops in $C_t \cap \partial T$ which are disjoint from the interior of T. This contradicts our assumption that C_t contains a loop which is essential in S_t. If $C_t \cap \partial T$ is empty after eliminating disks outside T, then we have that S_t is isotopic into T, giving us a contradiction. Thus, for each $t \in (-1,1)$, if g_t is a Morse

function then there exists a simple loop in $S_t \cap \partial T$ which is essential in S_t. By Lemma 3 this implies that for each $t \in (-1,1)$ such that g_t is a Morse function, there exists a simple loop in $S_t \cap \partial T$ which bounds a properly embedded essential disk in one of the handlebodies bounded by S_t.

For each component A of $([-1,1]\times[0,1]) \setminus G$, choose a point (t_A, s_A) in the interior of A. If one of the loops in $g_{t_A}^{-1}(s_A)$ bounds a properly embedded essential disk in the handlebody $f^{-1}[-1,t_A]$, then label the component A with a 1. If one of the loops in $g_{t_A}^{-1}(s_A)$ bounds a properly embedded essential disk in the handlebody $f^{-1}[t_A,1]$, then label the component A with a 2. If some component A of $([-1,1)\times[0,1]) \setminus G$ has more than one label, then some loop in $g_{t_A}^{-1}(s_A)$ bounds disks in both $f^{-1}[-1,t_A]$ and $f^{-1}[t_A,1]$ implying that the Heegaard splitting is weakly reducible.

We have shown that for each $t \in (-1,1)$ such that g_t is a Morse function, there exists a simple loop in $S_t \cap \partial T = g_t^{-1}(1)$ which bounds a disk in one of the handlebodies bounded by S_t. In other words, any component of $([-1,1]\times[0,1]) \setminus G$ which meets $[-1,1]\times\{1\}$ is labeled.

For t near -1, the vertical line $\{t\}\times[0,1]$ must intersect a component labeled 1, because S_t is near the spine of $f^{-1}[-1,t]$. For t near 1, the vertical line $\{t\}\times[0,1]$ must intersect a component labeled 2, because S_t is near the spine of $f^{-1}[t,1]$. Thus there must be adjacent segments in $[-1,1]\times\{1\} \setminus G$ which have different labels. Since G cannot contain any vertical segments, this implies that some vertical line $\{t\}\times[0,1]$ intersects two labeled components with different labels. Then for some $s_1, s_2 \in [0,1]$, there is a loop in $g_t^{-1}(s_1)$ which bounds a disk in $f^{-1}[-1,t_A]$ and there is a loop in $g_t^{-1}(s_2)$ which bounds a disk in $f^{-1}[t_A,1]$. Since $g_t^{-1}(s_1)$ and $g_t^{-1}(s_2)$ are disjoint, the Heegaard splitting is weakly reducible, giving a contradiction.

We have shown that there is a $\sigma \in (-1,1)$ such that $S_\sigma \cap T$ does not contain an essential curve of S_σ. □

Proof of Lemma 1 We will show that we may isotope S so that $S \cap \partial T$ contains a loop which is essential and nonmeridional in ∂T. Let $f\colon M \to [-1,1]$ be a smooth function such that $f^{-1}(-1)$ is a spine of H_1, $f^{-1}(1)$ is a spine of H_2, $f^{-1}(t)$ is a surface isotopic to S for each $t \in (-1,1)$, and $f^{-1}(0) = S$. Suppose that $S_t \cap \partial T$ does not contain a loop which is essential and nonmeridional in ∂T for any $t \in (-1,1)$. We have isotoped $S = S_0$ so that $S \cap T$ contains a simple loop ω which is essential in T. We have also shown that there is a $\sigma \in (-1,1)$ such that $S_\sigma \cap T$ does not contain an essential loop of S_σ and therefore each loop in $S_\sigma \cap \partial T$ bounds a disk in S_σ. Let A be an annulus embedded in M with one boundary component equal to ω and the other boundary component l contained in S_σ. This annulus exists because S and S_σ

bound a surface times interval. Any loop in $S_\sigma \cap \partial T$ must bound a disk in S_σ since $S_\sigma \cap T$ does not contain a loop which is essential in S_σ. We may isotope A so that l is disjoint from $S_\sigma \cap \partial T$ since loops in $S_\sigma \cap \partial T$ bound disks in S_σ. The loop l is now either disjoint from T or contained in T since it is disjoint from ∂T. Since l is essential in the 3–manifold M, l must be essential in S_σ. Since $S_\sigma \cap T$ does not contain an essential loop of S_σ, we must have that l is contained in $M \setminus T$. We may isotope ω by a small isotopy so that it is contained in the interior of T. We now have an annulus A embedded in M such that $\partial A = \omega \cup l$ with $\omega \subset T$ and $l \subset (M \setminus T)$. Thus there must be a simple loop l' in $A \cap \partial T$ which is essential in A. The embedded annulus bounded by $l \cup l'$ can be used to isotope S_σ so that $S_\sigma \cap T$ contains a loop which is essential and nonmeridional in ∂T.

We have shown that we may isotope S so that $S \cap \partial T$ contains a loop which is essential and nonmeridional in ∂T. Now we can let N be a regular neighborhood of γ contained in T and disjoint from ∂T. Since $S \cap \partial T$ contains a loop which is essential and nonmeridional in ∂T, there is an embedded annulus in $M \setminus N$ with boundary $\alpha \cup \alpha'$, where α is a simple essential nonmeridional loop in the boundary of N, and α' is contained S. □

3 Finding an isotopy

Lemma 5 *Let γ be a simple loop in a 3–manifold M. Let $M = H_1 \cup_S H_2$ be a Heegaard splitting of M. Let α be a simple essential nonmeridional loop in the boundary of a regular neighborhood N of γ. If there is an embedded annulus A in M disjoint from the interior of N with boundary $\alpha \cup \alpha'$ where $\alpha' \subset S$, then γ is isotopic into S.*

The proof of Lemma 5 is a thin position argument used by Schultens in [9] to show that exceptional fibers in Seifert manifolds are isotopic into a Heegaard surface [9, Lemma 4.1]. See Johnson [3] for a use of this argument to classify genus-one Heegaard splittings of lens spaces. The only adjustment needed for our result is that the loop γ can be put into thin position while keeping the annulus A embedded.

Proof Let $f\colon M \to [-1, 1]$ be a smooth function such that $f^{-1}(-1)$ is a spine of H_1, $f^{-1}(1)$ is a spine of H_2, and $f^{-1}(t)$ is a surface isotopic to S for each $t \in (-1, 1)$. Let $g\colon A \to M$ be an embedding of an annulus A into M with $\partial A = \alpha \cup \alpha'$ such that $g(\alpha)$ is an essential nonmeridional loop in the boundary of the regular neighborhood N of γ, $g(\alpha) \cap N = \varnothing$, and $g(\alpha') \subset S$. Extend g to an immersion (also called g) such that $g(\alpha) = \gamma^n$ and $g|_{A\setminus\alpha}$ is an embedding.

If $n = 1$, then we are done, so assume that $n \geq 2$. Let $h = f \circ g\colon A \to [-1, 1]$. We may assume that after a small isotopy $h|_{A\setminus\alpha}$ is a Morse function and that $h|_\alpha$ has no degenerate critical points. Since α' is mapped to the level surface S, we may also assume that the singular foliation $\mathcal{F}$ of A by level sets of h consists of parallel circles in a neighborhood of α'. Isotope γ and g so that γ is disjoint from the cores of the handlebodies in $M \setminus S$, while keeping $g|_{A\setminus\alpha}$ an embedding.

The singular foliation $\mathcal{F}$ contains an *essential saddle* if it contains a saddle singularity x such that the four arcs in the level set containing x emanating from x end on α. If x is an essential singularity, then x and the arcs emanating from x cut off three disks from A. If none of the disks cut off by an essential saddle x and the arcs emanating from x contain an essential saddle, then we call x an *outermost essential singularity*. If β is an outermost level arc in the foliation $\mathcal{F}$ of A, then it cuts off a disk D. Call D an *upper disk* if $h(x) > h(\beta)$ for all $x \in D \setminus \beta$ and call D a *lower disk* if $h(x) < h(\beta)$ for all $x \in D \setminus \beta$.

Suppose that a level set of $\mathcal{F}$ contains an outermost essential saddle which splits off an upper disk D_u between two lower disks D_1 and D_2 such that $g(D_1 \cap \alpha) \cap g(D_2 \cap \alpha) = \varnothing$. Note that since $\mathcal{F}$ consists of parallel circles in a neighborhood of α', we have that the disks D_u, D_1, and D_2 are disjoint from α'. We may isotope γ and the map g to push D_u below D_1 and D_2. We may similarly isotope γ and g to eliminate a lower disk D_l between two upper disks D_3 and D_4 such that $g(D_3 \cap \alpha) \cap g(D_4 \cap \alpha) = \varnothing$. Isotope γ and g until no such triples of upper and lower disks exist. We may choose these isotopies so that $g|_{A\setminus\alpha}$ is still an embedding since the upper and lower disks are disjoint from a neighborhood of α'. If $\gamma \subset S'$ for a level set S' of f, then we are done. Otherwise the map $f|_\gamma$ has at least two critical points so that $h|_\alpha$ has at least 4 critical points.

By Proposition 3.1 of [9], after an arbitrarily small isotopy of the map g near α, $\mathcal{F}$ contains an *outermost essential saddle*. An outermost essential saddle must cut off either an upper disk between two lower disks or a lower disk between two upper disks. Without loss of generality, assume it cuts off an upper disk D_u between two lower disks D_1 and D_2. We have isotoped γ and g so that we cannot have $g(D_1 \cap \alpha) \cap g(D_2 \cap \alpha) = \varnothing$. Thus we must have $g(D_1 \cap \alpha) = g(D_2 \cap \alpha)$. This implies that $g((D_1 \cap \alpha) \cup (D_u \cap \alpha)) = \gamma$, so that $g(D_1 \cup D_u)$ provides a disk to isotope γ to $g(\partial(D_1 \cup D_u) - \alpha)$ which is contained in a level surface of f. In other words, we may isotope γ into a Heegaard surface parallel to S. □

Proof of Theorem 4 This theorem now follows from Lemmas 1 and 5. □

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