Algebraic & Geometric Topology 11 (2011) 747–838 **747**

Iterated bar complexes of E–infinity algebras and homology theories

BENOIT FRESSE

We proved in a previous article that the bar complex of an E_∞–algebra inherits a natural E_∞–algebra structure. As a consequence, a well-defined iterated bar construction $\mathtt{B}^n(A)$ can be associated to any algebra over an E_∞–operad. In the case of a commutative algebra A, our iterated bar construction reduces to the standard iterated bar complex of A.

The first purpose of this paper is to give a direct effective definition of the iterated bar complexes of E_∞–algebras. We use this effective definition to prove that the n–fold bar construction admits an extension to categories of algebras over E_n–operads.

Then we prove that the n–fold bar complex determines the homology theory associated to the category of algebras over an E_n–operad. In the case $n = \infty$, we obtain an isomorphism between the homology of an infinite bar construction and the usual Γ–homology with trivial coefficients.

57T30; 55P48, 18G55, 55P35

Introduction

The standard reduced bar complex $\mathtt{B}(A)$ is basically defined as a functor from the category of associative differential graded algebras (associative dg-algebras for short) to the category of differential graded modules (dg-modules for short). In the case of a commutative dg-algebra, the bar complex $\mathtt{B}(A)$ inherits a natural multiplicative structure and still forms a commutative dg-algebra. This observation implies that an iterated bar complex $\mathtt{B}^n(A)$ is naturally associated to any commutative dg-algebra A, for every $n \in \mathbb{N}$. In this paper, we use techniques of modules over operads to study extensions of iterated bar complexes to algebras over E_n–operads. Our main result asserts that the n–fold bar complex $\mathtt{B}^n(A)$ determines the homology theory associated to an E_n–operad.

For the purpose of this work, we take the category of dg-modules as a base category and we assume that all operads belong to this category. An E_n–operad refers to a dg-operad

Published: 12 March 2011 DOI: 10.2140/agt.2011.11.747

equivalent to the chain operad of little n–cubes. Many models of E_n–operads come in nested sequences

$$(*) \qquad \mathsf{E}_1 \subset \cdots \subset \mathsf{E}_n \subset \cdots \subset \operatorname{colim}_n \mathsf{E}_n = \mathsf{E}$$

such that $\mathsf{E} = \operatorname{colim}_n \mathsf{E}_n$ is an E_∞–operad, an operad equivalent in the homotopy category of dg-operads to the operad of commutative algebras C. Recall that an E_1–operad is equivalent to the operad of associative algebras A and forms, in another usual terminology, an A_∞–operad. The structure of an algebra over an A_∞–operad includes a product $\mu\colon A \otimes A \to A$ and a full set of homotopies that make this product associative. The structure of an algebra over an E_∞–operad includes a product $\mu\colon A \otimes A \to A$ and a full set of homotopies that make this product associative and commutative. The intermediate structure of an algebra over an E_n–operad includes a product $\mu\colon A \otimes A \to A$ which is fully homotopy associative, but homotopy commutative up to some degree only.

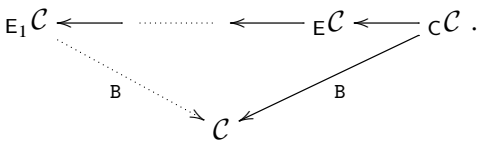
Throughout the paper, we use the letter $\mathcal{C}$ to denote the base category of dg-modules and the notation ${}_{\mathsf{P}}\mathcal{C}$, where P is any operad, refers to the category of P–algebras in dg-modules. The category of commutative dg-algebras, identified with the category of algebras over the commutative operad C, is denoted by ${}_{\mathsf{C}}\mathcal{C}$.

Recall that an operad morphism $\phi\colon \mathsf{P} \to \mathsf{Q}$ gives rise to a restriction functor $\phi^*\colon {}_{\mathsf{Q}}\mathcal{C} \to {}_{\mathsf{P}}\mathcal{C}$ since the restriction of a Q–action through ϕ provides any Q–algebra with a natural P–action. The category of P–algebras is also equipped with an obvious forgetful functor towards the base category $\mathcal{C}$. For a nested sequence of E_n–operads, we have a chain of restriction functors

$$ {}_{\mathsf{E}_1}\mathcal{C} \longleftarrow \cdots\cdots \longleftarrow {}_{\mathsf{E}_n}\mathcal{C} \longleftarrow \cdots\cdots \longleftarrow {}_{\mathsf{E}}\mathcal{C} \longleftarrow {}_{\mathsf{C}}\mathcal{C} $$

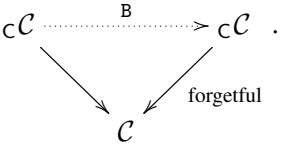
starting from the category of commutative algebras.

The bar construction is basically defined as a functor from the category of associative dg-algebras to the category of dg-modules. This usual bar construction has an extension to any category of algebras over an A_∞–operad, equivalently to any category of algebras over an E_1–operad. For a nested sequence of E_n–operads, the restriction of the usual bar construction to commutative dg-algebras and the extended bar complex $\mathsf{B}\colon {}_{\mathsf{E}_1}\mathcal{C} \to \mathcal{C}$ form a commutative diagram

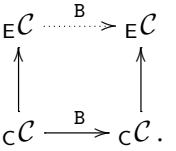


Recall that the bar construction of a commutative dg-algebra inherits a commutative algebra structure. Thus, the restriction of the bar complex functor to commutative

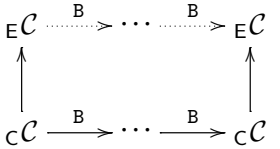
dg-algebras admits a factorization



Suppose that the E_∞–operad E is cofibrant as a dg-operad. In [24] we prove that the bar construction of E–algebras admits a factorization through the category of E–algebras so that the functor $\mathtt{B}\colon {}_{\mathsf{C}}\mathcal{C} \to {}_{\mathsf{C}}\mathcal{C}$ admits an extension



As a byproduct, we can form a composite functor



to extend the iterated bar construction of commutative dg-algebras to the category of E–algebras. The definition of the iterated bar construction $\mathtt{B}^n\colon A \mapsto \mathtt{B}^n(A)$ can be generalized to any E_∞–operad E (not necessarily cofibrant as a dg-operad), because we can pick a cofibrant replacement $\epsilon\colon \mathsf{Q} \xrightarrow{\sim} \mathsf{E}$ and use the associated restriction functor $\epsilon^*\colon {}_{\mathsf{E}}\mathcal{C} \to {}_{\mathsf{Q}}\mathcal{C}$ to map E–algebras into the category of algebras over Q.

The cochain complex $C^*(X)$ of a topological space (or simplicial set) X forms an instance of an algebra over an E_∞–operad. By theorems of [24], the iterated bar complex $\mathtt{B}^n(C^*(X))$ is equivalent to $C^*(\Omega^n X)$, the cochain algebra of the n–fold loop space of X, provided that the space X satisfies mild finiteness and completeness assumptions. This result gives the original motivation for the extension of iterated bar complexes to categories of algebras over E_∞–operads.

The topological interpretation of iterated bar complexes (the actual objective of [24]) uses the full multiplicative structure of the bar complex, but the definition of the iterated bar construction as a composite functor $\mathtt{B}^n\colon {}_{\mathsf{E}}\mathcal{C} \to {}_{\mathsf{E}}\mathcal{C}$ involves much more information than necessary for the determination of the differential of $\mathtt{B}^n(A)$. Moreover, the structure of a minimal cofibrant E_∞–operad models all secondary operations which occur on the homology of E_∞–algebras (see Chataur and Livernet [14]). For this

reason, the determination of the iterated bar complex by naive iterations of the bar construction carries deep difficulties.

The first purpose of this paper is to give a direct construction of the iterated bar complex, within the framework of [24], but so that we avoid the iteration process of the definition and the use of cofibrant replacements of E_∞–operads. Roughly, we show that the definition of the iterated bar complex $\mathtt{B}^n(A)$ can be reduced to a construction of linear homological algebra in the context of operads.

Let R be any operad. In [23], we show that a functor $\mathtt{S}_{\mathsf{R}}(M,-)\colon {}_{\mathsf{R}}\mathcal{C}\to\mathcal{C}$ is naturally associated to any right R–module M and all functors on R–algebras which are defined by composites of colimits and tensor products have this form.

Let $\mathsf{R}=\mathsf{E}$ or $\mathsf{R}=\mathsf{C}$. In [24], we check that the bar construction is an instance of a functor of this form $\mathtt{B}(-)=\mathtt{S}_{\mathsf{R}}(B_{\mathsf{R}},-)$ for a certain right R–module B_{R}. In fact, we prove the existence of multiplicative structures on the bar construction at the module level. An assertion of [23] implies that the iterated bar construction $\mathtt{B}^n(A)$, defined by a composite of functors associated to right modules over operads, forms itself a functor determined by a right R–module. This observation gives the starting point of the construction of the present article.

Throughout the paper, we use the notation B^n_{R} for the right R–module which represents the iterated bar complex $\mathtt{B}^n\colon {}_{\mathsf{R}}\mathcal{C}\to\mathcal{C}$. We study the structure of B^n_{R}. We check that B^n_{R} is defined by a pair $B^n_{\mathsf{R}}=(T^n\circ\mathsf{R},\partial_{\mathsf{R}})$, where (see Section 2):

- the composite $T^n\circ\mathsf{R}$ represents a free right R–module, whose associated functor $\mathtt{S}_{\mathsf{R}}(T^n\circ\mathsf{R})\colon {}_{\mathsf{R}}\mathcal{C}\to\mathcal{C}$ is the iterated tensor coalgebra underlying the iterated bar construction;
- the term ∂_{R} refers to a twisting homomorphism of right R–modules $\partial_{\mathsf{R}}\colon T^n\circ\mathsf{R}\to T^n\circ\mathsf{R}$; the differential of the iterated bar construction $\mathtt{B}^n(A)$ is defined by the addition of the twisting homomorphism

$$\underbrace{\mathtt{S}_{\mathsf{R}}(T^n\circ\mathsf{R},A)}_{=\mathtt{B}^n(A)}\xrightarrow{\mathtt{S}_{\mathsf{R}}(\partial_{\mathsf{R}},A)}\underbrace{\mathtt{S}_{\mathsf{R}}(T^n\circ\mathsf{R},A)}_{=\mathtt{B}^n(A)}$$

  induced by $\partial_{\mathsf{R}}\colon T^n\circ\mathsf{R}\to T^n\circ\mathsf{R}$ to the natural differential of the iterated tensor coalgebra.

For the commutative operad $\mathsf{R}=\mathsf{C}$, the definition of the twisting homomorphism ∂_{C} arises from the standard definition of the iterated bar construction of commutative

algebras. For an E_∞–operad $\mathsf{R} = \mathsf{E}$, the twisting homomorphism ∂_{E} solves a lifting problem

$$(**) \qquad \begin{array}{ccc} T^n \circ \mathsf{E} & \overset{\partial_{\mathsf{E}}}{\dashrightarrow} & T^n \circ \mathsf{E} \\ \downarrow & & \downarrow \\ T^n \circ \mathsf{C} & \xrightarrow[\partial_{\mathsf{C}}]{} & T^n \circ \mathsf{C}\,. \end{array}$$

We will observe that any quasifree module $B^n_{\mathsf{E}} = (T^n \circ \mathsf{E}, \partial_{\mathsf{E}})$ associated to a twisting homomorphism ∂_{E} satisfying $(**)$ determines a functor $\mathtt{B}^n(-) = \mathtt{S}_{\mathsf{E}}(B^n_{\mathsf{E}}, -)\colon {}_{\mathsf{E}}\mathcal{C} \to \mathcal{C}$ which extends of the n–fold bar complex of commutative algebras $\mathtt{B}^n(-)\colon {}_{\mathsf{C}}\mathcal{C} \to \mathcal{C}$. We prove that all functors of this form $\mathtt{B}^n(-) = \mathtt{S}_{\mathsf{E}}(B^n_{\mathsf{E}}, -)$ are weakly-equivalent (Theorem 1.22). The desired direct definition of the functor $\mathtt{B}^n\colon {}_{\mathsf{E}}\mathcal{C} \to \mathcal{C}$ is deduced from this homotopy uniqueness result and from the observation that a solution of the lifting problem $(**)$ can be obtained by direct effective arguments (Sections 2.4–2.5 and Theorem 2.9). To provide this effective solution, we use an easy generalization of classical techniques of linear homological algebra in the context of right modules over operads.

In this construction we lose multiplicative structures, but we can apply the argument of [24, Theorem 2.A] to redefine an E_∞–multiplicative structure on $\mathtt{B}^n(A)$, in full or in part, at any stage of iteration of the bar construction. The uniqueness argument of [24] ensures that we retrieve the good multiplicative structure anyway.

The first main objective of this paper (achieved by Theorem 5.5) is to prove that the n–fold bar complex $\mathtt{B}^n\colon {}_{\mathsf{E}}\mathcal{C} \to \mathcal{C}$ extends to the category of E_n–algebras (but not further)

$$\begin{array}{ccccccc} {}_{\mathsf{E}_1}\mathcal{C} & \longleftarrow & \cdots & \longleftarrow & {}_{\mathsf{E}_n}\mathcal{C} & \longleftarrow & \cdots \longleftarrow {}_{\mathsf{E}}\mathcal{C} \\ & & & & & & \downarrow \mathtt{B}^n \\ & & & & & \overset{\mathtt{B}^n}{\searrow} & \mathcal{C} \end{array}$$

for certain E_∞–operads E equipped with a filtration of the form $(*)$. Then our main result (given by Theorem 8.22) asserts that the n–fold desuspension of the iterated bar complex $\Sigma^{-n}\,\mathtt{B}^n(A)$ determines the homology of E_n–algebras, the homology theory $\mathtt{H}_*^{\mathsf{E}_n}(A)$ defined abstractly as the homology of a derived indecomposable functor $L\,\mathtt{Indec}\colon \mathtt{Ho}({}_{\mathsf{E}_n}\mathcal{C}) \to \mathtt{Ho}(\mathcal{C})$. Since the publication of the present preprint, a topological version of this result has been obtained by Basterra and Mandell [5], independently from our work, by relying, at some point, on the relationship between topological little n–cubes operads and iterated loop spaces (see Boardman and Vogt [11] and May [39]). In the dg-context, we follow another, more direct, approach.

First, to define the iterated bar complex of E_n–algebras, we observe (Theorem 5.4) that the twisting homomorphism $\partial_{\mathsf{E}}\colon T^n \circ \mathsf{E} \to T^n \circ \mathsf{E}$ admits a restriction

$$\begin{array}{ccc} T^n \circ \mathsf{E}_n & \overset{\partial_{\mathsf{E}_n}}{\dashrightarrow} & T^n \circ \mathsf{E}_n \\ \downarrow & & \downarrow \\ T^n \circ \mathsf{C} & \xrightarrow[\partial_{\mathsf{C}}]{} & T^n \circ \mathsf{C} \end{array}$$

for a good choice of ∂_{E}. Hence, we can form a twisted right E_n–module $B^n_{\mathsf{E}_n} = (T^n \circ \mathsf{E}_n, \partial_{\mathsf{E}_n})$ and the associated functor $\mathtt{S}_{\mathsf{E}_n}(B^n_{\mathsf{E}_n})$ gives the desired extension of the iterated bar complex. Then we prove that $B^n_{\mathsf{E}_n} = (T^n \circ \mathsf{E}_n, \partial_{\mathsf{E}_n})$ forms a cofibrant resolution of a unit object I in right E_n–modules (Theorem 8.21) and we use this observation to conclude that the functor $\mathtt{S}_{\mathsf{E}_n}(B^n_{\mathsf{E}_n})$ determines the E_n–homology $\mathtt{H}_*^{\mathsf{E}_n}(-)$ as announced (Theorem 8.22).

The iterated bar complexes are connected by suspension morphisms

$$\sigma\colon \Sigma^{1-n}\,\mathtt{B}^{n-1}(A) \to \Sigma^{-n}\,\mathtt{B}^n(A)$$

and we can perform a colimit to associate an infinite bar complex

$$\Sigma^{-\infty}\,\mathtt{B}^{\infty}(A) = \operatorname{colim}_n \Sigma^{-n}\,\mathtt{B}^n(A)$$

to any E–algebra A. The relationship $\mathtt{H}_*^{\mathsf{E}_n}(A) = \mathtt{H}_*(\Sigma^{-n}\,\mathtt{B}^n(A))$ also holds in the case $n = \infty$ (Theorem 9.5).

The Γ–homology of Robinson [43] and the E_∞ André–Quillen homology of [35] are other definitions of the homology theory associated to an E_∞–operad. Our result implies that the Γ–homology with trivial coefficient agrees with the homology of the infinite bar complex $\Sigma^{-\infty}\,\mathtt{B}^{\infty}(A)$, for any E_∞ algebra A (provided that A is cofibrant as a dg-module), and similarly as regards the dg-version of the E_∞ André–Quillen homology of Mandell [35]. This relationship between Γ–homology and iterated bar complexes does not seem to occur in the literature even in the case of commutative algebras, for which we can apply the classical definition of iterated bar constructions.

The identity $\mathtt{H}_*^{\mathsf{E}_n}(A) = \mathtt{H}_*(\Sigma^{-n}\,\mathtt{B}^n(A))$ enables us to deduce the E_n–homology of usual commutative algebras (polynomial algebras, exterior algebras, divided power algebras, ...) from results of Cartan et al [13]. In the case $n = \infty$, this approach could be used to give explicit representatives of Γ–homology classes and to improve on results of Richter [42].

In general, we have a natural spectral sequence

$$\mathtt{E}^1 = \Sigma^{-n}\,\mathtt{B}^n(\mathtt{H}_*(A)) \Rightarrow \mathtt{H}_*(\Sigma^{-n}\,\mathtt{B}^n(A)) = \mathtt{H}_*^{\mathsf{E}_n}(A),$$

whose E^1–term reduces to the usual n–fold bar construction of a commutative algebra for any $n > 1$ (the homology of an E_n–algebra forms a commutative algebra for $n > 1$, an associative algebra for $n = 1$). In the case of the cochain algebra $A = C^*(X)$ of a space X, we conjecture that this spectral sequence agrees, from E^2–stage, with a spectral sequence of Ahearn and Kuhn [1] which is defined with Goodwillie's calculus of functors.

On one hand, one might gain quantitative information on the cohomology of iterated loop spaces from the study of such spectral sequences, arising from filtrations of iterated bar complexes. On the other hand, the connection between the homology of iterated bar complexes and the homology of E_n–algebras could be used to gain qualitative information of a new nature on the cohomology of iterated loop spaces – notably, our result implies that certain groups of homotopy automorphisms of E_n–operads act on this cohomology and we conceive, from insights of Kontsevich [31], that this gives an action of higher versions of the Grothendieck–Teichmüller group on the cohomology of iterated loop spaces. Besides, for a sphere $X = S^{n-m}$, we have a chain complex, defined purely algebraically (in terms of characteristic structures of E_n–operads), computing the cohomology $\mathrm{H}_*^{\mathsf{E}_n}(C^*(S^{n-m}))$ (see our paper [16]). Thus spheres are first examples of spaces for which our approach seems appropriate and for which we plan to study the applications of our results.

Since the publication of this preprint, an interpretation of the homology of iterated bar complexes in terms of a homology of functors has been discovered (see our paper [18] and Livernet and Richter [32]). This new connection gives an additional approach for the study of the homology of E_n–algebras by means of iterated bar complexes. Notably, we explain in the follow-up [18] (see also [22]), that the twisting homomorphism ∂_{E}, obtained by our lifting construction, is formally equivalent to a twisting cochain on a poset formed from Batanin's category of pruned trees (see [7; 6]) with values in the operad E.

Throughout the article, we study the application of our constructions to the Barratt–Eccles operad, a nice combinatorial E_∞–operad E equipped with a filtration of the form $(*)$. In fact, we will observe that this E_∞–operad has all extra structures needed for the constructions of the article. Besides, with Berger we proved in [9] that the Barratt–Eccles acts on cochain complexes of spaces. Thus the Barratt–Eccles operad is also well suited for the topological applications of our results.

Plan

This paper is in part a sequel of [24], but the reader is essentially supposed to have some familiarity with the theory of modules over operads, which gives the background of our

constructions, and with the Koszul duality of operads (over a ring), which is used in homology calculations. The overall setting is also reviewed in the preliminary part of the paper, "Background".

The next part, "The construction of iterated bar complexes", is devoted to the definition of the iterated bar complex of algebras over E_∞–operads and to the extension of the construction to algebras over E_n–operads. This part includes sections on iterated bar modules (Section 1), iterated bar modules as quasifree modules (Section 2), operads shaped on complete graph posets (Section 3), the structure of iterated bar modules (Section 4) and the restriction of iterated bar modules to E_n–operads (Section 5).

In "Iterated bar complexes and homology theories", we prove that the n–fold bar complex determines the homology of algebras over an E_n–operad. This part includes sections on iterated bar complexes of usual commutative algebras (Section 6), the homology of algebras over operads and operadic $\mathtt{Tor}$–functors (Section 7), iterated bar complexes and homology of algebras over E_n–operads (Section 7) and infinite bar complexes (Section 9).

In the concluding part, "Applications to the cohomology of iterated loop spaces", we explain the applications of our results to iterated loop space cohomology.

We refer to the introduction of each part for a more detailed outline.

The appendix "Iterated bar modules and level tree posets" of a previous version of this work has been deleted and will give the matter of an independent publication [22], whose purpose is to explain the relationship between Batanin's categories of pruned trees (see [7; 6]) and iterated bar complexes and to give an interpretation of our constructions in this formalism. The references to the former appendix are now directed to this addendum [22].

Main theorems

Theorem 1.22: Definition of the iterated bar construction of algebras over an E_∞–operad E by a certain cofibrant replacement in the category of right E–modules.

Theorem 2.9: Definition of the iterated bar module as a lifting of quasifree modules.

Theorems 5.4–5.5: Extension of the n–fold bar construction to algebras over an E_n–operad.

Theorems 8.21–8.22: The n–fold bar complex determines the homology theory associated to E_n–operads $\mathrm{H}_*^{\mathsf{E}_n}(-)$.

Theorem 9.5: The infinite bar complex determines the homology theory associated to E_∞–operads, equivalently the Γ–homology with trivial coefficients.

In the Afterword: For the cochain algebra of a space X, the homology $\mathrm{H}_*^{\mathsf{E}_n}(C^*(X))$ determines, under mild finiteness and completeness assumptions, the cohomology of the n–fold loop space $\Omega^n X$.

Acknowledgements This research was supported in part by ANR grant JCJC06-0042 OBTH. I am grateful to the editorial board and the AGT team for the efficient processing of this article.

Background

The structure of a module over an operad is used to model functors on algebras over operads. The purpose of this part is to review this theory.

The point of view adopted in the present paper is borrowed from the book [23] to which we refer for a comprehensive account of the background of our constructions. The résumé of this part would be sufficient to make the conceptual setting of the paper accessible to readers who are already familiar with usual definitions of the theory of operads.

In [23; 24], we only use the standard definition of an operad, according to which the elements of an operad model operations with r inputs indexed by integers $i \in \{1, \ldots, r\}$, for any $r \in \mathbb{N}$. But there is another usual definition of the structure of an operad in which the inputs of operations are indexed by any finite set $\underline{e} = \{e_1, \ldots, e_r\}$. The indexing by finite sets is more natural for certain constructions of the article. Therefore, we revisit a few definitions of [23; 24] in this formalism.

But, first of all, we recall the categorical settings of [24] that we keep for this article.

0.1 Categorical settings A commutative ground ring $\Bbbk$ is fixed once and for all. In applications, we take $\Bbbk = \mathbb{Q}$, the field of rationals, or $\Bbbk = \mathbb{F}_p$, a finite primary field, or $\Bbbk = \mathbb{Z}$, the ring of integers, but no assumption on the ground ring is really required (apart from commutativity).

We take the category of differential graded modules over $\Bbbk$ as a base symmetric monoidal category and we use the notation $\mathcal{C}$ to refer to this category. For us, a differential graded module (a dg-module for short) consists of a $\mathbb{Z}$–graded module C equipped with a differential $\delta\colon C \to C$ that lowers degrees by 1.

The letter $\mathcal{E}$ refers to a symmetric monoidal category over $\mathcal{C}$, whose structure includes a unit object $\mathbf{1} \in \mathcal{E}$, an external tensor product $\otimes\colon \mathcal{C} \times \mathcal{E} \to \mathcal{E}$, and an internal tensor product $\otimes\colon \mathcal{E} \times \mathcal{E} \to \mathcal{E}$, which satisfy the usual relations of symmetric monoidal categories. In this paper, we take either $\mathcal{E} = \mathcal{C}$, the category of dg-modules itself, or

$\mathcal{E} = \mathcal{M}$, the category of Σ_*–objects in $\mathcal{C}$, or $\mathcal{E} = \mathcal{M}_{\mathsf{R}}$, the category of right modules over an operad R. The definition of the second-mentioned categories is recalled next.

Other examples include the categories of dg-modules over a graded $\Bbbk$–algebra R (see Section 9.6).

In the paper, we use an external hom-functor $\mathrm{Hom}_{\mathcal{E}}(-,-)\colon \mathcal{E}^{op} \times \mathcal{E} \to \mathcal{C}$ characterized by the adjunction relation $\mathrm{Mor}_{\mathcal{E}}(C \otimes E, F) = \mathrm{Mor}_{\mathcal{C}}(C, \mathrm{Hom}_{\mathcal{E}}(E, F))$, where $C \in \mathcal{C}$, $E, F \in \mathcal{E}$. The elements of the dg-module $\mathrm{Hom}_{\mathcal{E}}(E, F)$ are called homomorphisms to be distinguished from the actual morphisms of $\mathcal{E}$. In the case $\mathcal{E} = \mathcal{C}$, the dg-module $\mathrm{Hom}_{\mathcal{E}}(E, F)$ is spanned in degree d by the morphisms of $\Bbbk$–modules $f\colon E \to F$ which raise degree by d. This explicit definition has a straightforward generalization for the other categories $\mathcal{E} = \mathcal{M}, \mathcal{M}_{\mathsf{R}}$ whose definition is recalled next.

0.2 Functors on finite sets and symmetric modules The category of Σ_*–modules $\mathcal{M}$ consists of collections $M = \{M(r)\}_{r \in \mathbb{N}}$ whose term $M(r)$ is an object of the base category $\mathcal{C}$ (for us, the category of dg-modules) equipped with an action of the symmetric group on r letters Σ_r.

In certain applications, we use that the category of Σ_*–modules $\mathcal{M}$ is equivalent to the category of functors $F\colon \mathit{Bij} \to \mathcal{C}$, where Bij refers to the category formed by finite sets as objects and bijective maps as morphisms (this equivalence is borrowed from Getzler and Jones [25] and Ginzburg and Kapranov [27]; see also the surveys of our paper [20, Section 1.1.8] and Markl, Shnider and Stasheff [38, Section 1.7]).

In one direction, for a functor $F\colon \mathit{Bij} \to \mathcal{C}$, the dg-module $F(\{1, \ldots, r\})$ associated to the set $\underline{\mathrm{e}} = \{1, \ldots, r\}$ inherits an action of the symmetric group Σ_r since a permutation $w \in \Sigma_r$ is equivalent to a bijection $w\colon \{1, \ldots, r\} \to \{1, \ldots, r\}$. Hence, the collection $F(r) = F(\{1, \ldots, r\})$, $r \in \mathbb{N}$, forms a Σ_*–module naturally associated to $F\colon \mathit{Bij} \to \mathcal{C}$.

In the other direction, for a given Σ_*–module M, we set

$$M(\underline{\mathrm{e}}) = \mathit{Bij}(\{1, \ldots, r\}, \underline{\mathrm{e}}) \otimes_{\Sigma_r} M(r),$$

for any set with r elements $\underline{\mathrm{e}} = \{e_1, \ldots, e_r\}$. The tensor product $S \otimes C$ of a dg-module C with a finite set S is defined as the sum of copies of C indexed by the elements of S. The quotient over Σ_r makes the natural Σ_r–action on $M(r)$ agree with the action of permutations by right translations on $\mathit{Bij}(\{1, \ldots, r\}, \underline{\mathrm{e}})$. The map $M\colon \underline{\mathrm{e}} \mapsto M(\underline{\mathrm{e}})$ defines a functor naturally associated to M.

Intuitively, an element $x \in M(r)$, where $r \in \mathbb{N}$, can be viewed as an operation with r inputs indexed by $(1, \ldots, r)$; an element $x \in M(\underline{\mathrm{e}})$, where $\underline{\mathrm{e}}$ is a finite set $\underline{\mathrm{e}} = \{e_1, \ldots, e_r\}$, represents an operation $x = x(e_1, \ldots, e_r)$ whose inputs are indexed

by the elements of $\underline{e}$. In the definition of the functor associated to a Σ_*–module M, we use a bijection $u \in Bij(\{1,\dots,r\},\underline{e})$ to reindex the inputs of an operation $x \in M(r)$ by elements of $\underline{e}$.

Now, the subtlety is that we may use morphisms $f\colon M(r) \to N(r)$, like chain-homotopies, which do not preserve symmetric group actions. In this context, we have to assume that the finite set $\underline{e}$ comes equipped with a bijection $u\colon \{1,\dots,r\} \to \underline{e}$ in order to define a morphism $f\colon M(\underline{e}) \to N(\underline{e})$ associated to f. In applications, the bijection $u\colon \{1,\dots,r\} \to \underline{e}$ is often determined by an ordering $\underline{e} = \{e_1 < \cdots < e_r\}$ that $\underline{e}$ inherits from a larger set $\underline{e} \subset \underline{f} = \{f_1 < \cdots < f_s\}$.

In many usual situations, we specify the bijection $u\colon \{1,\dots,r\} \to \underline{e}$ by the sequence of values $(u(1),\dots,u(r))$.

0.3 Indexing by finite sets and the tensor product The definition of the tensor product of Σ_*–modules is recalled in [23, Sections 2.1.5–2.1.7]. The expansion given in this reference has a nice reformulation in terms of finite set indexings: the functor equivalent to the tensor product $M \otimes N$ of Σ_*–modules $M, N \in \mathcal{M}$ is defined on any finite set $\underline{e}$ by the direct sum

$$(M \otimes N)(\underline{e}) = \bigoplus_{\underline{u} \amalg \underline{v} = \underline{e}} M(\underline{u}) \otimes N(\underline{v}),$$

where the pair $(\underline{u},\underline{v})$ runs over partitions of $\underline{e}$. In this light, the elements of $(M \otimes N)(\underline{e})$ can be viewed as tensors $x(u_1,\dots,u_r) \otimes y(v_1,\dots,v_s)$ of elements $x \in M$ and $y \in N$ together with a sharing $\underline{e} = \{u_1,\dots,u_r\} \amalg \{v_1,\dots,v_s\}$ of the indices of $\underline{e}$. Throughout this paper, we adopt this representation of elements in tensor products, but we usually drop indices to simplify notation: $x \otimes y = x(u_1,\dots,u_r) \otimes y(v_1,\dots,v_s)$.

In [23], we represent the elements of $M \otimes N$ by point-tensors $w \cdot x \otimes y \in (M \otimes N)(r+s)$, where $w \in \Sigma_{r+s}$, $x \otimes y \in M(r) \otimes N(s)$ (see more specifically Section 0.5 and Section 2.1.9 of [23]). In the formalism of this paragraph, the point-tensor $w \cdot x \otimes y$ is equivalent to the tensor

$$x(u_1,\dots,u_r) \otimes y(v_1,\dots,v_s) \in (M \otimes N)(\{1,\dots,r+s\})$$

such that $(u_1,\dots,u_r) = (w(1),\dots,w(r))$ and $(v_1,\dots,v_s) = (w(r+1),\dots,w(r+s))$.

Note that each summand of a partition $\underline{e} = \{u_1,\dots,u_r\} \amalg \{v_1,\dots,v_s\}$ inherits a canonical ordering if $\underline{e}$ forms itself an ordered set.

The tensor product of Σ_*–modules is equipped with a symmetry isomorphism

$$\tau\colon M \otimes N \to N \otimes M,$$

which can be defined componentwise by the symmetry isomorphism $\tau\colon M(\underline{u})\otimes N(\underline{v}) \to N(\underline{v})\otimes M(\underline{u})$ inherited from the category of dg-modules. The tensor product of Σ_*–modules is also obviously associative and has the Σ_*–module such that

$$\mathbf{1}(r) = \begin{cases} \Bbbk & \text{if } r = 0, \\ 0 & \text{otherwise,} \end{cases}$$

as a unit object. Besides, we have an exterior tensor product $\otimes\colon \mathcal{C}\times\mathcal{M}\to\mathcal{M}$ such that $(C\otimes M)(r) = C\otimes M(r)$ for any $C\in\mathcal{C}$, $M\in\mathcal{M}$. Thus we finally obtain that the category of Σ_*–modules forms a symmetric monoidal category over the base category of dg-modules.

0.4 The composition structure of symmetric modules and operads Let $\mathcal{E}$ be any symmetric monoidal category over $\mathcal{C}$. Each Σ_*–module $M\in\mathcal{M}$ gives rise to a functor $\mathtt{S}(M)\colon \mathcal{E}\to\mathcal{E}$ which maps an object $E\in\mathcal{E}$ to the module of symmetric tensors with coefficients in M:

$$\mathtt{S}(M,E) = \bigoplus_{r=0}^{\infty} (M(r)\otimes E^{\otimes r})_{\Sigma_r}.$$

In this construction, the coinvariants $(-)_{\Sigma_r}$ identify the natural action of permutations on tensors with the natural Σ_r–action on $M(r)$. The map $\mathtt{S}\colon M\mapsto \mathtt{S}(M)$ defines a functor from the category of Σ_*–objects $\mathcal{M}$ to the category of functors $F\colon \mathcal{E}\to\mathcal{E}$.

The tensor product of the category of Σ_*–modules $\otimes\colon \mathcal{M}\times\mathcal{M}\to\mathcal{M}$ reflects the pointwise tensor product of functors $\mathtt{S}(M\otimes N,E) = \mathtt{S}(M,E)\otimes\mathtt{S}(N,E)$, and similarly as regards the tensor product over dg-modules.

The category of Σ_*–modules is additionally equipped with a composition product $\circ\colon \mathcal{M}\times\mathcal{M}\to\mathcal{M}$, characterized by the relation $\mathtt{S}(M\circ N) = \mathtt{S}(M)\circ\mathtt{S}(N)$, for $M,N\in\mathcal{M}$, and we have a unit object

$$\mathsf{I}(r) = \begin{cases} \Bbbk & \text{if } r = 1, \\ 0 & \text{otherwise,} \end{cases}$$

such that $\mathtt{S}(\mathsf{I}) = \mathrm{Id}$. The composition product can be defined by the formula $M\circ N = \bigoplus_{r=0}^{\infty}(M(r)\otimes N^{\otimes r})_{\Sigma_r}$, where we form the tensor power of N within the category of Σ_*–modules and we use the tensor product of Σ_*–modules over dg-modules to form the tensor product with $M(r)$. Equivalently, we have a formula $M\circ N = \mathtt{S}(M,N)$, where we apply the definition of the functor $\mathtt{S}(M)\colon \mathcal{E}\to\mathcal{E}$ to the category Σ_*–modules $\mathcal{E} = \mathcal{M}$.

The structure of an operad P is defined by a composition product $\mu\colon \mathsf{P}\circ\mathsf{P}\to\mathsf{P}$ together with a unit morphism $\eta\colon \mathsf{I}\to\mathsf{P}$ that satisfy the usual relations of monoid objects.

Modules over operads are defined naturally by using the composition structure of Σ_*–modules: a left module over an operad P consists of a Σ_*–module N equipped with a left P–action determined by a morphism $\lambda\colon \mathsf{P}\circ N \to N$; a right module over an operad R is defined symmetrically as a Σ_*–module M equipped with a right R–action determined by a morphism $\rho\colon M\circ\mathsf{R}\to M$; a bimodule over operads (P,R) is a Σ_*–module N equipped with both a right R–action $\rho\colon N\circ\mathsf{R}\to N$ and a left P–action $\lambda\colon \mathsf{P}\circ N\to N$ that commute to each other.

We refer to [23] for a comprehensive study of modules over operads and further bibliographical references on recollections of this paragraph.

0.5 Pointwise composition products In the original definition of the structure of an operad [39], the composition product $\mu\colon \mathsf{P}\circ\mathsf{P}\to\mathsf{P}$ is determined by a collection of morphisms

$$\mu\colon \mathsf{P}(r)\otimes\mathsf{P}(n_1)\otimes\cdots\otimes\mathsf{P}(n_r)\to\mathsf{P}(n_1+\cdots+n_r),$$

where $r,n_1,\ldots,n_r\in\mathbb{N}$. This definition is also used in the paper. The equivalence between the latter definition and the definition of Section 0.4 comes from an explicit expansion of the composition product of Σ_*–modules (see for instance [23, Section 2.2], brief recollections are also given in Section 0.8).

The image of $p\in\mathsf{P}(r)$ and $q_1\in\mathsf{P}(n_1),\ldots,q_r\in\mathsf{P}(n_r)$ under the termwise composition product is usually denoted by $p(q_1,\ldots,q_r)$. In the paper, we also use a generalization of the definition of the composite $p(q_1,\ldots,q_r)$ for elements $q_1\in\mathsf{P}(\underline{e}_1),\ldots,q_r\in\mathsf{P}(\underline{e}_r)$, where $(\underline{e}_1,\ldots,\underline{e}_r)$ are any finite sets. In this situation, the composite $p(q_1,\ldots,q_r)$ returns an element of $\mathsf{P}(\underline{e}_1\amalg\cdots\amalg\underline{e}_r)$.

Similar definitions apply to modules over operads.

0.6 Categories of modules and operads Throughout the paper, we use the notation $\mathcal{M}_{\mathsf{R}}$ for the category of right R–modules, the notation ${}_{\mathsf{P}}\mathcal{M}$ for the category of left P–modules, and the notation ${}_{\mathsf{P}}\mathcal{M}_{\mathsf{R}}$ for the category of (P,R)–bimodules.

The usual definitions of linear algebra (free objects, extension and restriction functors) have a natural extension in the context of modules over operads (see relevant sections of [23]). In particular, a relative composition product $M\circ_{\mathsf{R}}N$ is associated to any pair (M,N) such that M is equipped with a right R–action and N is equipped with a left R–action. The object $M\circ_{\mathsf{R}}N$ is defined by the reflexive coequalizer of the morphisms $d_0,d_1: M\circ\mathsf{R}\circ N\rightrightarrows M\circ N$ induced by the right R–action on M and the left R–action on N (see for instance [20, Section 5.1.5, Section 9.2.4] or [23, Section 2.1.7]; we also refer to the bibliography of [20] for further references on this definition).

The extension and restriction functors associated to an operad morphism $\phi\colon \mathsf{P} \to \mathsf{Q}$, respectively $\psi\colon \mathsf{R} \to \mathsf{S}$, are denoted by

$$\phi_!\colon {}_{\mathsf{P}}\mathcal{M} \rightleftarrows {}_{\mathsf{Q}}\mathcal{M} :\phi^*, \quad \text{respectively} \quad \psi_!\colon \mathcal{M}_{\mathsf{R}} \rightleftarrows \mathcal{M}_{\mathsf{S}} :\psi^*.$$

In the context of bimodules, we have extensions and restrictions of structure on the left and on the right. These extension and restriction functors are also denoted by

$$\phi_!\colon {}_{\mathsf{P}}\mathcal{M}_{\mathsf{R}} \rightleftarrows {}_{\mathsf{Q}}\mathcal{M}_{\mathsf{R}} :\phi^*, \quad \text{respectively} \quad \psi_!\colon {}_{\mathsf{P}}\mathcal{M}_{\mathsf{R}} \rightleftarrows {}_{\mathsf{P}}\mathcal{M}_{\mathsf{S}} :\psi^*.$$

The extension functors are given by relative composition products of the form

$$\phi_! N = \mathsf{Q} \circ_{\mathsf{P}} N, \quad \text{respectively} \quad \psi_! M = M \circ_{\mathsf{R}} \mathsf{S}.$$

In formulas, we usually omit specifying structure restrictions and we use the expression of the relative composite to denote structure extensions rather than the functor notation. Nevertheless we do use the functor notation $(\phi_!, \phi^*)$ to refer to the extension and restriction of structure as functors between modules categories.

For a Σ_*–module K, the composite $\mathsf{P} \circ K$ inherits a left P–action and represents the free object generated by K in the category of left P–modules. The symmetrical composite $K \circ \mathsf{R}$ represents the free object generated by K in the category of right R–modules.

Recall that the composition product is not symmetric and does not preserve colimits on the right. For that reason, categories of left modules differ in nature from categories of right modules over operads.

0.7 Algebras over operads By definition, an algebra over an operad P consists of an object $A \in \mathcal{E}$ together with an evaluation product $\lambda\colon \mathbb{S}(\mathsf{P}, A) \to A$ which satisfies natural associativity and unit relations with respect to the composition product and the unit of P. The evaluation product $\lambda\colon \mathbb{S}(\mathsf{P}, A) \to A$ is equivalent to a collection of morphisms $\lambda\colon \mathsf{P}(r) \otimes A^{\otimes r} \to A$.

In the context of dg-modules, the evaluation morphism associates an actual operation $p\colon A^{\otimes r} \to A$ to every element of the operad $p \in \mathsf{P}(r)$, and we use the notation $p(a_1, \ldots, a_r)$ to refer to the image of a tensor $a_1 \otimes \cdots \otimes a_r \in A^{\otimes r}$ under this operation $p\colon A^{\otimes r} \to A$.

Note that the definition of a P–algebra makes sense in any symmetric monoidal category $\mathcal{E}$ over $\mathcal{C}$, and not only in the category of dg-modules itself. We adopt the notation ${}_{\mathsf{P}}\mathcal{E}$ to refer to the category of P–algebras in $\mathcal{E}$.

In the next paragraphs, we explain that this definition of the category of P–algebras in a symmetric monoidal category over $\mathcal{C}$ applies, besides the category of dg-modules

itself, to the category of Σ_*–objects and to categories of right modules over an operad. In this paper, we only use these particular symmetric monoidal categories over $\mathcal{C}$, but we need the general notion to give a conceptual framework for our constructions.

Recall that an operad morphism $\phi\colon \mathsf{P} \to \mathsf{Q}$ determines a pair of adjoint extension and restriction functors $\phi_!\colon {}_{\mathsf{P}}\mathcal{E} \rightleftarrows {}_{\mathsf{Q}}\mathcal{E} :\phi^*$, defined like the extension and restriction functors of modules over operads (see [23, Section 3.3]).

0.8 Tensor products and composition structures We recall in Section 0.3 that the category of Σ_*–modules comes equipped with a tensor product and forms a symmetric monoidal category over the base category of dg-modules. As a consequence, we can associate a category of algebras in Σ_*–modules to any operad P. In fact, since we have an identity $\mathtt{S}(M,N) = M \circ N$ between the functor $\mathtt{S}(M)\colon \mathcal{E} \to \mathcal{E}$ associated to M and the composition with M in Σ_*–modules, we have a formal equivalence between the structure of a P–algebra in Σ_*–modules, determined by a morphism $\lambda\colon \mathtt{S}(\mathsf{P}, N) \to N$, and the structure of a left P–module, which is itself determined by a morphism $\lambda\colon \mathsf{P} \circ N \to N$.

The tensor product of Σ_*–modules satisfies a distribution relation $(M \otimes N) \circ P \simeq (M \circ P) \otimes (N \circ P)$ with respect to the composition product. From this observation, we deduce that a tensor product of right R–modules inherits a natural right R–action (see [23, Section 6]) and we obtain that the category of right modules $\mathcal{M}_{\mathsf{R}}$ forms a symmetric monoidal category over $\mathcal{C}$, like the category of Σ_*–modules $\mathcal{M}$. As a consequence, we can also apply the ideas of Section 0.7 to the category right R–modules $\mathcal{E} = \mathcal{M}_{\mathsf{R}}$ and we can associate a category of algebras in right modules R to any operad P. Actually, the composite $M \circ N$ inherits a natural right R–action when N is a right R–modules and the identity $\mathtt{S}(M,N) = M \circ N$ holds in the category of right R–modules, where we apply the definition of the functor $\mathtt{S}(M)\colon \mathcal{E} \to \mathcal{E}$ to the category of right R–modules $\mathcal{E} = \mathcal{M}_{\mathsf{R}}$. From these observations, we readily deduce that the category of P–algebras in right R–modules is formally equivalent to the category of (P, R)–bimodules.

The associativity of the composition product of Σ_*–modules is equivalent to the distribution relation $\mathtt{S}(M,N) \circ P \simeq \mathtt{S}(M, N \circ P)$ on the module of symmetric tensors $\mathtt{S}(M,N)$.

The definition of a P–algebra in terms of evaluation morphisms $\lambda\colon \mathsf{P}(r) \otimes A^{\otimes r} \to A$ and pointwise operations applies in the context of Σ_*–modules and right-modules over operads too. In the case of a Σ_*–module N, the evaluation morphisms $\lambda\colon \mathsf{P}(r) \otimes N^{\otimes r} \to N$ are formed by using the tensor product of Σ_*–modules, whose definition is recalled in Section 0.3. Thus, the operation $p\colon N^{\otimes r} \to N$ associated to an element $p \in \mathsf{P}(r)$

consists of a collection of dg-module homomorphisms

$$p\colon N(\underline{e}_1)\otimes\cdots\otimes N(\underline{e}_r)\to N(\underline{e}_1\amalg\cdots\amalg\underline{e}_r)$$

indexed by partitions $\underline{e}=\underline{e}_1\amalg\cdots\amalg\underline{e}_r$. We also use the notation $p(a_1,\ldots,a_r)$ to represent the evaluation of operations $p\in\mathsf{P}(r)$ on point-tensors in Σ_*–modules.

In the case of a right R–module N, the operation $p\colon N^{\otimes r}\to N$ associated to an element $p\in\mathsf{P}(r)$ is simply assumed to preserve right R–actions.

The equivalence between P–algebras in Σ_*–modules, respectively in right R–modules, and left P–modules, respectively of a (P,R)–bimodules, is pointed out in [23, Sections 3.2.9–3.2.10, Section 9]. Depending on the context, we use either the idea of algebras in symmetric monoidal categories or the language of modules of operads, because each point of view has its own interests.

An operad P forms obviously an algebra over itself in the category of right modules over itself. In this case, the evaluation operation $p\colon \mathsf{P}(\underline{e}_1)\otimes\cdots\otimes\mathsf{P}(\underline{e}_r)\to\mathsf{P}(\underline{e}_1\amalg\cdots\amalg\underline{e}_r)$ is nothing but the composition product of Section 0.5.

0.9 Free algebras over operads The object $\mathsf{S}(\mathsf{P},E)$ associated to any $E\in\mathcal{E}$ inherits a natural P–algebra structure and represents the free object generated by E in the category of P–algebras ${}_{\mathsf{P}}\mathcal{E}$. In the paper, we use the notation $\mathsf{P}(E)=\mathsf{S}(\mathsf{P},E)$ to refer to the object $\mathsf{S}(\mathsf{P},E)$ equipped with this free P–algebra structure.

In the case $\mathcal{E}=\mathcal{M}$, we have an identity between the free P–algebra and the free left P–module generated by a Σ_*–module M, and similarly in the context of right modules over an operad. In the paper, we use both representations $\mathsf{P}(M)=\mathsf{P}\circ M$ for these free objects since each representation has its own interests.

The associativity of composition products is equivalent to a distribution relation $\mathsf{P}(M)\circ N=\mathsf{P}(M\circ N)$.

0.10 Modules over operads and functors Recall that a Σ_*–module M determines a functor $\mathsf{S}(M)\colon\mathcal{E}\to\mathcal{E}$. For a right R–module M, we have a functor $\mathsf{S}_{\mathsf{R}}(M)\colon{}_{\mathsf{R}}\mathcal{E}\to\mathcal{E}$ from the category of R–algebras ${}_{\mathsf{R}}\mathcal{E}$ to the underlying category $\mathcal{E}$. The object $\mathsf{S}_{\mathsf{R}}(M,A)\in\mathcal{E}$ associated to an R–algebra $A\in{}_{\mathsf{R}}\mathcal{E}$ is defined by the reflexive coequalizers of morphisms $d_0,d_1\colon\mathsf{S}(M\circ\mathsf{R},A)\rightrightarrows\mathsf{S}(M,A)$ induced by the right R–action on M and the left R–action on A (see [23, Section 5.1]).

For a left P–module $N\in{}_{\mathsf{P}}\mathcal{M}$, the objects $\mathsf{S}(N,E)$ inherit a natural left P–action so that the map $\mathsf{S}(N)\colon E\mapsto\mathsf{S}(N,E)$ defines a functor $\mathsf{S}(N)\colon\mathcal{E}\to{}_{\mathsf{P}}\mathcal{E}$ from the underlying category $\mathcal{E}$ to the category of P–algebras ${}_{\mathsf{P}}\mathcal{E}$ (see [23, Section 3.2]). For a

(P,R)–bimodule $N \in {}_{\mathsf{P}}\mathcal{M}_{\mathsf{R}}$, we have a functor $\mathbb{S}_{\mathsf{R}}(N)\colon {}_{\mathsf{R}}\mathcal{E} \to {}_{\mathsf{P}}\mathcal{E}$, from the category of R–algebras ${}_{\mathsf{R}}\mathcal{E}$ to the category of P–algebras ${}_{\mathsf{P}}\mathcal{E}$ (see [23, Section 9.2]).

The relative composition product $M \circ_{\mathsf{R}} N$ reflects the composition of functors associated to modules over operads: for a (P,R)–bimodule M and an (R,Q)–bimodule N, we have a natural isomorphism

$$\mathbb{S}_{\mathsf{Q}}(M \circ_{\mathsf{R}} N) \simeq \mathbb{S}_{\mathsf{R}}(M) \circ \mathbb{S}_{\mathsf{Q}}(N),$$

and similarly if we assume that M or N has a right (respectively, left) R–module structure only (see for instance [23, Section 9.2]). The relative composition product $M \circ_{\mathsf{R}} N$ can also be identified with the object $\mathbb{S}_{\mathsf{R}}(M, N)$ associated to N by the functor $\mathbb{S}_{\mathsf{R}}(M)\colon \mathcal{E} \to \mathcal{E}$ determined by M on the category of right Q–modules $\mathcal{E} = \mathcal{M}_{\mathsf{Q}}$.

As an illustration, recall that an operad morphism $\phi\colon \mathsf{P} \to \mathsf{Q}$ defines adjoint extension and restriction functors between categories of algebras over operads: $\phi_!\colon {}_{\mathsf{P}}\mathcal{E} \rightleftarrows {}_{\mathsf{Q}}\mathcal{E} :\phi^*$. By [23, Theorem 7.2.2], we have an identity $\phi_! A = \mathbb{S}_{\mathsf{P}}(\phi^* \mathsf{Q}, A)$, for any P–algebra $A \in {}_{\mathsf{P}}\mathcal{E}$, where we use a restriction of structure on the right to identify the operad Q with a (Q,P)–bimodule. Symmetrically, we have an identity $\phi^* B = \mathbb{S}_{\mathsf{Q}}(\phi^* \mathsf{Q}, B)$, for any Q–algebra $B \in {}_{\mathsf{Q}}\mathcal{E}$, where we use a restriction of structure on the left to identify the operad Q with a (P,Q)–bimodule.

In [23, Section 7.2], we also prove that extensions and restrictions of modules over operads correspond, at the functor level, to composites with extension and restriction functors on categories of algebras over operads. To be explicit, we have a functor identity $\mathbb{S}_{\mathsf{R}}(N, -) = \mathbb{S}_{\mathsf{R}}(N, \psi_! -)$, for every right S–module N, and a functor identity $\mathbb{S}_{\mathsf{S}}(M \circ_{\mathsf{R}} \mathsf{S}, -) = \mathbb{S}_{\mathsf{R}}(M, \psi^* -)$ for every right R–module M where we consider the extension and restriction functors associated to an operad morphism $\psi\colon \mathsf{R} \to \mathsf{S}$. Similar commutation formulas hold when we perform extensions and restrictions of modules on the left.

0.11 The characterization of functors associated to modules over operads The composition of functors and the extension and restriction operations are not the only categorical operations on functors which can be represented at the module level. In [23, Sections 5–7], we prove that the functor $\mathbb{S}_{\mathsf{R}}\colon M \mapsto \mathbb{S}_{\mathsf{R}}(M)$ commutes with colimits and tensor products of right R–modules, like the functor $\mathbb{S}\colon M \mapsto \mathbb{S}(M)$ on Σ_*–modules. Similar statements occur when we consider functors associated to left and bimodules over operads.

To retrieve the right module underlying a functor $F = \mathbb{S}_{\mathsf{R}}(M)$, we use the simple idea that the application of F to the operad R, viewed as an algebra over itself in the category of right modules over itself, defines a right R–module $F(\mathsf{R}) = \mathbb{S}_{\mathsf{R}}(M, \mathsf{R})$

which is naturally isomorphic to M. In terms of relative composition product, this identity reads $M \circ_{\mathsf{R}} \mathsf{R} = M$.

In what follows, we often switch from modules to functors when a construction becomes easier in the functor setting. In any case, we only deal with functors formed by tensor products, colimits, and operations which have a representative at the module level so that all our functors are properly modeled by modules over operads.

In positive characteristic, one might consider divided power algebras, which do not have the form of a functor $\mathtt{S}(M, E)$ associated to a Σ_*–module M. In characteristic 2, we also have the exterior algebra functor $\Lambda(E) = \bigoplus_{r=0}^{\infty} E^{\otimes r}/(x^2 \equiv 0)$ which occurs in the standard definition of the Chevalley–Eilenberg homology and is not a functor of the form $\mathtt{S}(M, E)$ too. In what follows, we do not use such constructions. In particular, when we deal with the Chevalley–Eilenberg complex, we tacitly consider the most usual definition involving a symmetric algebra $\mathtt{S}(\Sigma G) = \bigoplus_{r=0}^{\infty}((\Sigma G)^{\otimes r})_{\Sigma_r}$ on a suspension of the Lie algebra G, and not an exterior algebra.

In fact, we only apply the Chevalley–Eilenberg complex to Lie algebras belonging to the category of connected Σ_*–modules (the definition of a connected Σ_*–module is recalled next, in Section 0.14) and we deduce from observations of [20, Section 1.2] that symmetric algebras $\mathtt{S}(E)$ like all functors of the form $\mathtt{S}(M, E)$ behave well in homology when the variable E ranges over connected Σ_*–modules (see again Section 0.14) – even when the ground ring is not a field of characteristic zero. Therefore the generalized symmetric algebra functors associated to Σ_*–modules $\mathtt{S}(M)$ and the functors associated to modules over operads $\mathtt{S}_{\mathsf{R}}(M)$ are sufficient for our purpose.

0.12 Model categories We use the theory of model categories to give an abstract characterization of iterated bar constructions.

Recall that the category of dg-modules $\mathcal{C}$ has a standard model structure in which a morphism is a weak-equivalence if it induces an isomorphism in homology, a fibration if it is degreewise surjective, a cofibration if it has the right lifting property with respect to acyclic fibrations (see for instance [30, Section 2.3]). The category of Σ_*–modules $\mathcal{M}$ inherits a model structure so that a morphism $f\colon M \to N$ forms a weak-equivalence, respectively a fibration, if each of its components $f\colon M(n) \to N(n)$ defines a weak-equivalence, respectively a fibration, in the category of dg-modules. The category of right modules over an operad $\mathcal{M}_{\mathsf{R}}$ has a similarly defined model structure (see [23, Section 14]). In all cases, the cofibrations are characterized by the right lifting property with respect to acyclic fibrations, where an acyclic fibration refers to a morphism which is both a fibration and a weak-equivalence. By convention, we say that a morphism of Σ_*–modules (respectively, of right R–modules) $f\colon M \to N$ forms a $\mathcal{C}$–cofibration

if its components $f\colon M(n) \to N(n)$ are cofibrations of dg-modules. We also say that a morphism of right R–modules $f\colon M \to N$ forms a Σ_*–cofibration when it defines a cofibration in the category of Σ_*–modules. Any cofibration of Σ_*–modules forms a $\mathcal{C}$–cofibration, but the converse implication does not hold. According to [23, Proposition 14.1.1], a cofibration of right R–modules is a $\mathcal{C}$–cofibration (respectively, a Σ_*–cofibration) when the operad R forms itself a $\mathcal{C}$–cofibrant (respectively, Σ_*–cofibrant) object in some natural sense (see Section 0.13).

The model categories $\mathcal{E} = \mathcal{C}, \mathcal{M}, \mathcal{M}_{\mathsf{R}}$ are all cofibrantly generated and monoidal in the sense of [23, Section 11.3.3] (the unit object is cofibrant and the tensor product satisfies the pushout-product axiom). These assertions imply that the category of P–algebras in $\mathcal{C}$, the category of P–algebras in $\mathcal{M}$ (equivalently, the category of left P–modules), and the category of P–algebras in $\mathcal{M}_{\mathsf{R}}$ (equivalently, the category of (P, R)–bimodules), inherit a natural semimodel structure when the operad P is Σ_*–cofibrant (see [23, Section 12]). In all cases $\mathcal{E} = \mathcal{C}, \mathcal{M}, \mathcal{M}_{\mathsf{R}}$, the forgetful functor $U\colon {}_{\mathsf{P}}\mathcal{E} \to \mathcal{E}$ preserves cofibrations with a cofibrant domain, but a morphism of P–algebras which forms a cofibration in the underlying category $\mathcal{E}$ does not form a cofibration in the category of P–algebras in general. By convention, we say that a morphism of P–algebras in $\mathcal{E}$ forms an $\mathcal{E}$–cofibration if it defines a cofibration in the underlying category $\mathcal{E}$, a P–algebra $A \in {}_{\mathsf{P}}\mathcal{E}$ is $\mathcal{E}$–cofibrant if the unit morphism $\eta\colon \mathsf{P}(0) \to A$ forms an $\mathcal{E}$–cofibration.

In the next sections, we only deal with operads such that $\mathsf{P}(0) = 0$. In this context, a P–algebra A is $\mathcal{E}$–cofibrant if and only if A defines a cofibrant object in the underlying category $\mathcal{E}$. In main theorems, we prefer to use this full phrase. The model categories of P–algebras are used extensively in [24], but in this paper we mostly use model structures on ground categories.

0.13 Model structures and operads The category of operads $\mathcal{O}$ comes also equipped with a semimodel structure so that a morphism $f\colon \mathsf{P} \to \mathsf{Q}$ forms a weak-equivalence, respectively a fibration, if it defines a weak-equivalence, respectively a fibration, in the category of Σ_*–modules (see [23, Section 13] and further bibliographical references therein). Again, the cofibrations of the category of operads are characterized by the right lifting property with respect to acyclic fibration. The forgetful functor $U\colon \mathcal{O} \to \mathcal{M}$ preserves cofibrations with a cofibrant domain, but a morphism of operads which forms a cofibration in the category of Σ_*–modules does not form a cofibration in the category of operads in general.

By convention, a morphism of operads which defines a cofibration in the category of Σ_*–modules is called a Σ_*–cofibration, and an operad P is Σ_*–cofibrant if the unit morphism $\eta\colon \mathsf{I} \to \mathsf{P}$ is a Σ_*–cofibration.

Similarly, we say that a morphism of operads $\phi\colon \mathsf{P} \to \mathsf{Q}$ is a $\mathcal{C}$–cofibration if its components $f\colon \mathsf{P}(n) \to \mathsf{Q}(n)$ define cofibrations in the underlying category of dg-modules, we say that an operad P is $\mathcal{C}$–cofibrant if the unit morphism $\eta\colon \mathsf{I} \to \mathsf{P}$ is a $\mathcal{C}$–cofibration. In what follows, we tacitely assume that any operad P is at least $\mathcal{C}$–cofibrant whenever we deal with model structures. This assumption is necessary in most homotopical constructions.

In [24], we need the notion of a cofibrant operad to define the multiplicative structure of the bar construction. For this reason, we still have to deal with cofibrant operads in Section 1, where we study the iterated bar complex deduced from the construction of [24]. Otherwise, we only deal with the model structure of Σ_*–modules.

0.14 The Künneth isomorphism The composite of Σ_*–modules has an expansion of the form

$$M \circ N = \mathtt{S}(M, N) = \bigoplus_{r=0}^{\infty} (M(r) \otimes N^{\otimes r})_{\Sigma_r}$$

and we have a natural Künneth morphism

$$\mathtt{H}_*(M) \circ \mathtt{H}_*(N) \to \mathtt{H}_*(M \circ N),$$

for all Σ_*–modules $M, N \in \mathcal{M}$. In the paper, we also use Künneth morphisms $\mathsf{P}(\mathtt{H}_*(N)) \to \mathtt{H}_*(\mathsf{P}(N))$ associated to composites of the form $\mathsf{P}(N) = \mathsf{P} \circ N$, where P is a graded operad (equipped with a trivial differential).

In [20, Sections 1.3.7–1.3.9], we observe that the composite $M \circ N$ has a reduced expansion in which no coinvariant occurs when $N(0) = 0$ (we say that the Σ_*–module N is connected). As a byproduct, if M, N are connected $\mathcal{C}$–cofibrant and the homology modules $\mathtt{H}_*(M), \mathtt{H}_*(N)$ consist of projective $\mathbb{k}$–modules, then the Künneth morphism $\mathtt{H}_*(M) \circ \mathtt{H}_*(N) \to \mathtt{H}_*(M \circ N)$ is an isomorphism.

In what follows, we apply most constructions to connected Σ_*–modules for which such good properties hold.

The construction of iterated bar complexes

In this part, we explain the construction of iterated bar complexes

$$\mathtt{B}^n\colon {}_{\mathsf{R}}\mathcal{E} \to \mathcal{E}$$

as functors $\mathtt{B}^n(-) = \mathtt{S}_{\mathsf{R}}(B^n_{\mathsf{R}}, -)$ associated to modules B^n_{R} over an operad R.

First, in Section 1, we review the definition of the bar complex of A_∞–algebras, of E_∞–algebras, and we prove that the construction of [24], in the context of algebras over an

E_∞–operad E, gives an iterated bar complex of the required form $\mathtt{B}^n(-) = \mathtt{S}_{\mathsf{E}}(B^n_{\mathsf{E}}, -)$. In Section 2, we give a simple construction of the iterated bar module B^n_{E} as a lifting of quasifree modules over operads.

Then we aim at proving that the n–fold bar module B^n_{E} is an extension of a module $B^n_{\mathsf{E}_n}$ defined over a filtration layer $\mathsf{E}_n \subset \mathsf{E}$, for certain E_∞–operads E equipped with a filtration

$$(*) \qquad \mathsf{E}_1 \subset \cdots \subset \mathsf{E}_n \subset \cdots \subset \operatorname{colim}_n \mathsf{E}_n = \mathsf{E}$$

such that E_n is an E_n–operad. For this purpose, we use that usual E_∞–operads are equipped with a particular cell structure, reviewed in Section 3, which refines their filtration by E_n–operads. We observe that the iterated bar modules B^n_{C} of the commutative operad C are equipped with a cell structure of the same shape (Section 4) and so are the iterated bar modules B^n_{E} associated to any E_∞–operad E (Section 5). We draw our conclusion from the latter assertion.

The module $B^n_{\mathsf{E}_n}$ determines an extension of the iterated bar complex $\mathtt{B}^n\colon {}_{\mathsf{E}}\mathcal{E} \to \mathcal{E}$ to the category of E_n–algebras ${}_{\mathsf{E}_n}\mathcal{E} \supset {}_{\mathsf{E}}\mathcal{E}$.

Conventions From now on, we tacitely assume that any operad P satisfies $\mathsf{P}(0) = 0$.

This assumption $\mathsf{P}(0) = 0$ (we also say that the operad P is nonunitary) amounts to considering algebras over operads without 0–ary operation. This setting simplifies the definition of iterated bar complexes and is required in constructions and arguments of Section 8.

Unital algebras are more usually considered in the literature, but for the standard categories of algebras (associative algebras, commutative algebras, ...) we have an equivalence of categories between algebras without unit and algebras with unit and augmentation. In one direction, to any algebra A, we can associate the algebra $A_+ = \Bbbk 1 \oplus A$, where a unit is formally added. This algebra inherits an augmentation $\epsilon\colon A_+ \to \Bbbk$ defined by the projection onto the summand $\Bbbk 1$. In the other direction, we form the augmentation ideal $\overline{A} = \ker(A \to \Bbbk)$ to associate an algebra without unit $\overline{A}$ to any augmented algebra A. It is easy to check that these constructions are inverse to each other.

Similar observations hold for coalgebras and Hopf algebras, but in the context of Hopf algebras, the distribution relation between products and coproducts becomes

$$\begin{aligned}\Delta(a\cdot b) = \sum_{(a),(b)} a_{(1)}\cdot b_{(1)} \otimes a_{(2)}\cdot b_{(2)} + \sum_{(a)}(a_{(1)}\cdot b \otimes a_{(2)} + a_{(1)} \otimes a_{(2)}\cdot b) \\ + \sum_{(b)}(a\cdot b_{(1)} \otimes b_{(2)} + b_{(1)} \otimes a\cdot b_{(2)}) + (a\otimes b + b\otimes a),\end{aligned}$$

where we use the notation $\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$ and $\Delta(b) = \sum_{(b)} b_{(1)} \otimes b_{(2)}$ for the expansion of the coproduct of a and b (see [33, Section 2.1.4.1]). The standard distribution relation of Hopf algebras can be retrieved from this formula by adding terms $x \otimes 1 + 1 \otimes x$ to each application of the diagonal $\Delta(x)$. Formally, this does not change our constructions.

Throughout this article, we deal with constructions of the literature which apply to augmented algebras. Thus we just consider the equivalent construction on the augmentation ideal of the algebra. Actually certain constructions studied in the paper, like the bar complex, are naturally defined on the augmentation ideal, not on the algebra itself, and this observation gives the main motivation for the point of view which we adopt throughout the article.

1 Iterated bar modules

In this section, we study the structure of the iterated bar complex $\mathtt{B}^n(A)$ as it arises from the construction of [24] for algebras over an E_∞–operad: we check that the iterated bar construction $\mathtt{B}^n\colon {}_{\mathsf{R}}\mathcal{E} \to \mathcal{E}$, where R is either the commutative operad $\mathsf{R} = \mathsf{C}$ or an E_∞–operad $\mathsf{R} = \mathsf{E}$, is an instance of a functor associated to a right R–module, for which we adopt the notation B^n_{R}; we prove that the iterated bar module B^n_{E} associated to an E_∞–operad E defines a cofibrant replacement, in the category of right E–modules, of the iterated bar module B^n_{C} associated to the commutative operad C; we use this result to obtain a simple characterization of the iterated bar complex $\mathtt{B}^n\colon {}_{\mathsf{E}}\mathcal{E} \to \mathcal{E}$ for algebras over an E_∞–operad E.

First of all, we review the definition of an A_∞–operad, of an E_∞–operad, and we recall the definition of the bar complex $\mathtt{B}\colon {}_{\mathsf{R}}\mathcal{E} \to \mathcal{E}$ for algebras over such operads.

1.1 The associative and commutative operads Throughout the paper, we use the notation A for the operad of associative algebras without unit and the notation C for the operad of associative and commutative algebras without unit (for short, commutative algebras).

The associative operad A is defined by

$$\mathsf{A}(r) = \begin{cases} 0 & \text{if } r = 0, \\ \Bbbk[\Sigma_r] & \text{otherwise,} \end{cases}$$

where $\Bbbk[\Sigma_r]$ represents the free $\Bbbk$–module spanned by Σ_r. The element of A represented by the identity permutation $\mathrm{id} \in \Sigma_2$ is also denoted by $\mu \in \mathsf{A}(2)$. The operation $\mu\colon A^{\otimes 2} \to A$ determined by $\mu \in \mathsf{A}(2)$ represents the product of associative algebras.

The commutative operad C is defined by

$$\mathsf{C}(r) = \begin{cases} 0 & \text{if } r = 0, \\ \Bbbk & \text{otherwise,} \end{cases}$$

where the one dimensional $\Bbbk$–module $\Bbbk$ is equipped with a trivial Σ_r–action. The generating element $\mu \in \mathsf{C}(2)$ determines an operation $\mu\colon A^{\otimes 2} \to A$ which represents the product of commutative algebras. For a nonempty finite set $\underline{\mathrm{e}} = \{e_1, \ldots, e_r\}$, we also have $\mathsf{C}(\underline{\mathrm{e}}) = \Bbbk$. The expression of the commutative monomial $e_1 \cdots e_r$ can conveniently be used to denote the generator of $\mathsf{C}(\underline{\mathrm{e}})$ when necessary.

The augmentations $\alpha\colon \Bbbk[\Sigma_r] \to \Bbbk$, $r \in \mathbb{N}$, define an operad morphism $\alpha\colon \mathsf{A} \to \mathsf{C}$ such that the restriction functor $\alpha^*\colon {}_{\mathsf{C}}\mathcal{E} \to {}_{\mathsf{A}}\mathcal{E}$ represents the usual embedding from the category of commutative algebras into the category of associative algebras.

Recall that an A_∞–operad (respectively, of an E_∞–operad) refers to an operad equivalent to the associative operad A (respectively, to the commutative operad C) in the homotopy category of operads. In the next paragraphs, we give a more precise definition of these structures in a form suitable for our needs.

1.2 On A_∞–operads For us, an A_∞–operad consists of an operad K augmented over the associative operad A so that the augmentation $\epsilon\colon \mathsf{K} \to \mathsf{A}$ defines an acyclic fibration in the category of operads. The restriction functor $\epsilon^*\colon {}_{\mathsf{A}}\mathcal{E} \to {}_{\mathsf{K}}\mathcal{E}$, naturally associated to the augmentation of an A_∞–operad, defines an embedding of categories from the category of associative algebras to the category of K–algebras.

In our work on the bar complex, we only use a particular A_∞–operad (apart from the associative operad), namely the chain operad of Stasheff's associahedra (Stasheff's operad for short) and the letter K will only refer to this A_∞–operad. The Stasheff operad is quasifree and can be defined by a pair $\mathsf{K} = (\mathsf{F}(M), \partial)$, where $\mathsf{F}(M)$ is a free operad and $\partial\colon \mathsf{F}(M) \to \mathsf{F}(M)$ is an operad derivation that determines the differential of K. The generating Σ_*–module M of Stasheff's operad K is defined by

$$M(r) = \begin{cases} 0 & \text{if } r = 0, 1, \\ \Sigma_r \otimes \Bbbk\, \mu_r & \text{otherwise,} \end{cases}$$

where μ_r is homogeneous of degree $r - 2$. The derivation $\partial\colon \mathsf{F}(M) \to \mathsf{F}(M)$ is determined on generating operations by the formula

$$\partial(\mu_r) = \sum_{s+t-1=r} \left(\sum_{i=1}^{s} \pm \mu_s \circ_i \mu_t \right),$$

for some sign $\pm$ (see [37]). This operad is cofibrant as an operad.

The augmentation $\epsilon\colon \mathsf{K} \to \mathsf{A}$ cancels the generating operations $\mu_r \in \mathsf{K}(r)$ such that $r > 2$ and maps $\mu_2 \in \mathsf{K}(2)$ to the generating operation of the associative operad $\mu \in \mathsf{A}(2)$, the operation which represents the product $\mu\colon A^{\otimes 2} \to A$ in the category of associative algebras.

The structure of an algebra over Stasheff's operad K is equivalent to a collection of homomorphisms $\mu_r\colon A^{\otimes r} \to A$, $r \in \mathbb{N}$, which give the action of the generating operations $\mu_r \in \mathsf{K}(r)$ on A, so that we have the differential relation

$$\delta(\mu_r)(a_1, \ldots, a_r) = \sum_{s+t-1=r} \left(\sum_{i=1}^{s} \pm \mu_s(a_1, \ldots, \mu_t(a_i, \ldots, a_{i+t-1}), \ldots, a_r) \right),$$

where $\delta(\mu_r)$ refers to the differential of homomorphisms. In this way, we retrieve the usual definition of the structure of an A_∞–algebra.

An associative algebra is equivalent to a K–algebra such that $\mu_r(a_1, \ldots, a_r) = 0$, for $r > 2$.

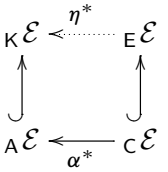
1.3 On E_∞–operads An E_∞–operad is an operad E augmented over the commutative operad C so that the augmentation $\epsilon\colon \mathsf{E} \to \mathsf{C}$ defines an acyclic fibration in the category of operads. An E_∞–operad is usually assumed to be Σ_*–cofibrant. This requirement ensures that the category of E–algebras is equipped with a semimodel structure. Moreover, different Σ_*–cofibrant E_∞–algebras have equivalent homotopy categories of algebras.

The restriction functor $\epsilon^*\colon {}_{\mathsf{C}}\mathcal{E} \to {}_{\mathsf{E}}\mathcal{E}$, associated to the augmentation of an E_∞–operad, defines an embedding of categories from the category of commutative algebras to the category of E–algebras (as in the context of A_∞–operads). For any E_∞–operad E, we can pick a lifting in the diagram

$$\begin{array}{ccc} \mathsf{K} & \overset{\eta}{\dashrightarrow} & \mathsf{E} \\ \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\ \mathsf{A} & \xrightarrow[\alpha]{} & \mathsf{C} \end{array}$$

to make E an object of the comma category $\mathcal{O} \setminus \mathsf{K}$ of operads under K. The commutativity of the diagram implies that the restriction functor $\eta^*\colon {}_{\mathsf{E}}\mathcal{E} \to {}_{\mathsf{K}}\mathcal{E}$ associated to the

morphism $\eta\colon \mathsf{K} \to \mathsf{E}$ fits in a commutative diagram of functors



and hence extends the usual embedding from the category of commutative algebras to the category of associative algebras.

1.4 The example of the Barratt–Eccles operad The Barratt–Eccles operad is a classical instance of an E_∞–operad introduced by M Barratt and P Eccles in the simplicial setting [4]. Throughout the paper, we use this nice combinatorial operad to illustrate our constructions. For our purpose, we consider a dg-version of the Barratt–Eccles operad which is defined by the normalized chain complexes

$$\mathsf{E}(r) = N_*(E\Sigma_r),$$

where $E\Sigma_r$ denotes the acyclic homogeneous bar construction of the symmetric group Σ_r. By convention, we remove the 0–component $N_*(E\Sigma_0)$ to assume $\mathsf{E}(0) = 0$ and to have a nonunitary analogue of this operad. In [9], we observe that the Barratt–Eccles acts on the cochain complex of spaces. Therefore this instance of E_∞–operad is also well suited when we tackle topological applications of iterated bar constructions. In Section 3, we recall that the Barratt–Eccles is also equipped with a filtration of the form $(*)$ which refines into a particularly nice cell structure. Therefore the Barratt–Eccles is also particularly well suited for the connection, studied in the present article, between iterated bar complexes and the homology of E_n–algebras.

The dg-module $\mathsf{E}(r)$ is spanned in degree d by the d–simplices of permutations

$$(w_0, \ldots, w_d) \in \Sigma_r \times \cdots \times \Sigma_r$$

divided out by the submodule spanned by degenerate simplices

$$s_j(w_0, \ldots, w_{d-1}) = (w_0, \ldots, w_j, w_j, \ldots, w_{d-1}), \quad j = 0, \ldots, d-1.$$

The differential of a simplex $(w_0, \ldots, w_d) \in \mathsf{E}(r)$ is defined by the alternate sum

$$\delta(w_0, \ldots, w_d) = \sum_{i=0}^{d} \pm(w_0, \ldots, \widehat{w_i}, \ldots, w_d).$$

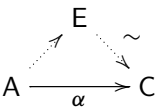
The symmetric group Σ_r operates diagonally on $\mathsf{E}(r)$ and the composition products of E are yielded by an explicit substitution process on permutations. The reader is

referred to [9] for a detailed definition and a comprehensive study of the Barratt–Eccles operad in the dg-setting.

If we extend the definition of the dg-modules $\mathsf{E}(r)$ to finite sets $\underline{\mathsf{e}} = \{e_1, \ldots, e_r\}$ as explained in Section 0.2, then we obtain a dg-module $\mathsf{E}(\underline{\mathsf{e}})$ spanned by nondegenerate simplices $(w_0, \ldots, w_d)$ such that w_i is a bijection $w_i\colon \{1, \ldots, r\} \xrightarrow{\sim} \{e_1, \ldots, e_r\}$. In applications, we use that such a bijection $w\colon \{1, \ldots, r\} \to \{e_1, \ldots, e_r\}$ is equivalent to an ordering of $\underline{\mathsf{e}}$. At some places, we specify such a bijection w by the associated sequence of values $(w(1), \ldots, w(r))$.

The operad equivalence $\epsilon\colon \mathsf{E} \xrightarrow{\sim} \mathsf{C}$ giving an E_∞–operad structure to E is defined by the usual augmentations $\epsilon\colon N_*(E\Sigma_r) \xrightarrow{\sim} \Bbbk$. In applications, we use the standard section of this augmentation $\iota\colon \mathsf{C}(r) \to \mathsf{E}(r)$, which identifies $\mathsf{C}(r) = \Bbbk$ with the component of $\mathsf{E}(r)$ spanned by the identity permutation id in degree 0, and the usual contracting chain homotopy $\nu\colon \mathsf{E}(r) \to \mathsf{E}(r)$ defined by $\nu(w_0, \ldots, w_d) = (w_0, \ldots, w_d, \mathrm{id})$ for any simplex of permutations $(w_0, \ldots, w_d) \in \mathsf{E}(r)$. Naturally, these definitions have a natural generalization giving a section $\iota\colon \mathsf{C}(\underline{\mathsf{e}}) \to \mathsf{E}(\underline{\mathsf{e}})$ and a contracting chain homotopy $\nu\colon \mathsf{E}(\underline{\mathsf{e}}) \to \mathsf{E}(\underline{\mathsf{e}})$ for any finite set $\underline{\mathsf{e}}$ as long as we have a fixed ordering $\underline{\mathsf{e}} = \{e_1 < \cdots < e_r\}$, equivalent to a bijection $\sigma\colon \{1, \ldots, r\} \xrightarrow{\sim} \underline{\mathsf{e}}$, associated to $\underline{\mathsf{e}}$. In this context, the section $\iota\colon \mathsf{C}(\underline{\mathsf{e}}) \to \mathsf{E}(\underline{\mathsf{e}})$ is the map sending a commutative word $e_1 \cdots e_r \in \mathsf{C}(\underline{\mathsf{e}})$ to the bijection $\iota(e_1 \cdots e_r) = \sigma$, defined by the given ordering of $\underline{\mathsf{e}}$, and the chain-homotopy $\nu\colon \mathsf{E}(\underline{\mathsf{e}}) \to \mathsf{E}(\underline{\mathsf{e}})$ is simply given by the formula $\nu(w_0, \ldots, w_d) = (w_0, \ldots, w_d, \sigma)$ for any simplex $(w_0, \ldots, w_d) \in \mathsf{E}(\underline{\mathsf{e}})$.

The associative operad A can be identified with the degree 0 part of E and forms a suboperad of the Barratt–Eccles operad. The restriction of the augmentation $\epsilon\colon \mathsf{E} \xrightarrow{\sim} \mathsf{C}$ to $\mathsf{A} \subset \mathsf{E}$ agrees with the morphism $\alpha\colon \mathsf{A} \to \mathsf{C}$ defined in Section 1.1. Thus we have a factorization



and the lifting construction of Section 1.3 is not necessary for the Barratt–Eccles operad.

1.5 The bar complex of algebras over Stasheff's operad The bar complex is defined naturally for K–algebras.

Let A be a K–algebra in $\mathcal{E}$, where $\mathcal{E} = \mathcal{C}$ the category of dg-modules, or $\mathcal{E} = \mathcal{M}$ the category of Σ_*–objects, or $\mathcal{E} = \mathcal{M}_{\mathsf{R}}$ the category of right modules over an operad R. The (reduced and normalized) bar complex of A is defined by a pair

$\mathtt{B}(A) = (\mathtt{T}^c(\Sigma A), \partial)$ formed by the (nonaugmented) tensor coalgebra

$$\mathtt{T}^c(\Sigma A) = \bigoplus_{d=1}^{\infty} (\Sigma A)^{\otimes d},$$

where ΣA is the suspension of A in $\mathcal{E}$, together with a homomorphism of degree -1

$$\partial \in \mathrm{Hom}_{\mathcal{E}}(\mathtt{T}^c(\Sigma A), \mathtt{T}^c(\Sigma A)),$$

called abusively the bar differential, defined pointwise by the formula

$$\partial(a_1 \otimes \cdots \otimes a_d) = \sum_{r=2}^{d} \Big(\sum_{i=1}^{d-r+1} \pm a_1 \otimes \cdots \otimes \mu_r(a_i, \ldots, a_{i+r-1}) \otimes \cdots \otimes a_d \Big).$$

The differential of the bar complex $\mathtt{B}(A)$ is the sum $\delta + \partial$ of the natural differential of the tensor coalgebra $\delta\colon \mathtt{T}^c(\Sigma A) \to \mathtt{T}^c(\Sigma A)$, which is induced by the internal differential of A, together with the bar differential $\partial\colon \mathtt{T}^c(\Sigma A) \to \mathtt{T}^c(\Sigma A)$, which determined by the K–action on A. Note that the identity $(\delta + \partial)^2 = 0$ holds in $\mathtt{B}(A)$ for every algebra over Stasheff's operad. The term "differential" is abusive for $\partial\colon \mathtt{T}^c(\Sigma A) \to \mathtt{T}^c(\Sigma A)$, because the relation $(\delta + \partial)^2 = 0$ does not hold for the isolated term ∂. Usually, we call twisting homomorphisms the homomorphisms of dg-modules $\partial\colon C \to C$, like the bar differentials, whose addition to the internal differential of C satisfies the equation of differentials $(\delta + \partial)^2 = 0$. More recollections on this notion are given next (in Section 2.1) when we tackle the definition of an iterated bar complex by the direct construction of a twisting homomorphism.

The definition of generalized point-tensors in Section 0.3 allows us to give a sense to the formula of the bar differential in the context of Σ_*–objects $\mathcal{E} = \mathcal{M}$ and right modules over an operad $\mathcal{E} = \mathcal{M}_{\mathsf{R}}$, and not only in the context of dg-modules $\mathcal{E} = \mathcal{C}$.

1.6 Coalgebra structures The tensor coalgebra $\mathtt{T}^c(\Sigma A)$ is equipped with a diagonal $\Delta\colon \mathtt{T}^c(\Sigma A) \to \mathtt{T}^c(\Sigma A) \otimes \mathtt{T}^c(\Sigma A)$ defined by the deconcatenation of tensors

$$\Delta(a_1 \otimes \cdots \otimes a_d) = \sum_{e=1}^{d} (a_1 \otimes \cdots \otimes a_e) \otimes (a_{e+1} \otimes \cdots \otimes a_d).$$

This diagonal makes $\mathtt{T}^c(\Sigma A)$ a coassociative coalgebra. Recall that we tacitely consider nonaugmented coalgebras only. Therefore, in our definition of the bar complex, we take a nonaugmented version of the tensor coalgebra $\mathtt{T}^c(\Sigma A)$, where the component of tensors of order 0 is removed.

The twisting homomorphism of the bar complex $\partial\colon \mathtt{T}^c(\Sigma A) \to \mathtt{T}^c(\Sigma A)$ satisfies the coderivation relation $\Delta\partial = (\partial \otimes \mathrm{id} + \mathrm{id} \otimes \partial)\Delta$ with respect to the diagonal of the

tensor coalgebra. From this observation, we conclude that the bar complex $\mathtt{B}(A) = (\mathtt{T}^c(\Sigma A), \partial)$ forms a coalgebra in the category of dg-modules $\mathcal{E} = \mathcal{C}$ (respectively, in the category of Σ_*–modules $\mathcal{E} = \mathcal{M}$, in the category of right modules over an operad $\mathcal{E} = \mathcal{M}_{\mathsf{R}}$) in which $\mathtt{B}(A)$ is defined.

This structure is used in Section 8 to simplify calculations of differentials in iterated bar complexes.

1.7 The suspension morphism The bar differential vanishes on the summand

$$\Sigma A \subset \bigoplus_{d=1}^{\infty} (\Sigma A)^{\otimes d} = \mathtt{T}^c(\Sigma A).$$

Thus, the canonical embedding $\Sigma A \hookrightarrow \mathtt{T}^c(\Sigma A)$ defines a natural morphism $\sigma_A\colon \Sigma A \to \mathtt{B}(A)$. This morphism is called the suspension morphism. For short, we can set $\sigma = \sigma_A$.

1.8 The bar complex and restriction of algebra structures The bar complex defines a functor $\mathtt{B}\colon {}_{\mathsf{K}}\mathcal{E} \to \mathcal{E}$ from the category of algebras over Stasheff's operad to the underlying category.

Let R be an operad together with a fixed morphism $\eta\colon \mathsf{K} \to \mathsf{R}$, so that R forms an object of the category of operads under K. The composite of the bar construction $\mathtt{B}\colon {}_{\mathsf{K}}\mathcal{E} \to \mathcal{E}$ with the restriction functor $\eta^*\colon {}_{\mathsf{R}}\mathcal{E} \to {}_{\mathsf{K}}\mathcal{E}$ defines a bar construction on the category of R–algebras. To simplify, we omit marking restriction functors in the notation of the bar complex $\mathtt{B}(A)$ associated to an R–algebra A.

The bar complex of an R–algebra is given by the same construction as in the context of a K–algebra. In the definition of the bar differential, we simply take the image of the generating operations of Stasheff's operad $\mu_r \in \mathsf{K}(r)$ under the morphism $\eta\colon \mathsf{K} \to \mathsf{R}$ to determine the action of μ_r on A.

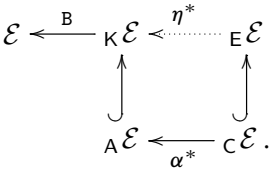
In the case of the associative operad A, the bar differential reduces to terms

$$\partial(a_1 \otimes \cdots \otimes a_d) = \sum_{i=1}^{d-1} \pm a_1 \otimes \cdots \otimes \mu_2(a_i, a_{i+1}) \otimes \cdots \otimes a_d.$$

Thus the restriction of the functor $\mathtt{B}\colon {}_{\mathsf{K}}\mathcal{E} \to \mathcal{E}$ to the category of associative algebras ${}_{\mathsf{A}}\mathcal{E}$ gives the usual bar complex of associative algebras.

In the case of an E_∞–operad E, the bar construction gives a functor $\mathtt{B}\colon {}_{\mathsf{E}}\mathcal{E} \to \mathcal{E}$ which extends the usual bar construction of commutative algebras since restriction functors

assemble to give a commutative diagram

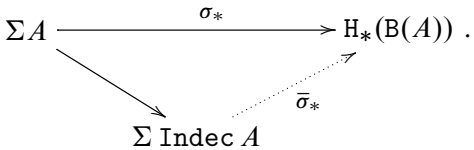


1.9 The suspension morphism and indecomposables The indecomposable quotient of an associative (respectively, commutative) algebra A is defined by the cokernel

$$A/A^2 = \operatorname{coker}(\mu\colon A \otimes A \to A),$$

where μ refers to the product of A. In what follows, we also use the notation $\mathtt{Indec}\, A = A/A^2$ for this quotient. The functor $\mathtt{Indec}\colon {}_{\mathsf{R}}\mathcal{E} \to \mathcal{E}$, for $\mathsf{R} = \mathsf{A}, \mathsf{C}$, is left adjoint to the obvious functor $\mathtt{Ab}\colon \mathcal{E} \to {}_{\mathsf{R}}\mathcal{E}$ which identifies any object $E \in \mathcal{E}$ with an R–algebra in $\mathcal{E}$ equipped with a trivial algebra structure.

Suppose that A has a trivial internal differential. Then the suspension morphism $\sigma\colon \Sigma A \to \mathtt{B}(A)$ maps $A^2 = \operatorname{im}(\mu\colon A \otimes A \to A)$ to boundaries of $\mathtt{B}(A)$. Hence, we obtain that the morphism $\sigma_*\colon A \to \mathtt{H}_*(\mathtt{B}(A))$ induced by the suspension in homology admits a factorization



1.10 The bar module In the context of right modules over an operad R, the bar complex defines a functor $\mathtt{B}\colon {}_{\mathsf{K}}\mathcal{M}_{\mathsf{R}} \to \mathcal{M}_{\mathsf{R}}$ from the category of K–algebras in right R–modules to the category of right R–modules.

Recall that an operad R forms an algebra over itself in the category of right modules over itself. By restriction of structure on the left, an operad under K forms an algebra over K in the category of right modules over itself. We adopt the notation $B_{\mathsf{R}} = \mathtt{B}(\mathsf{R})$ for the bar complex of this K–algebra. The object B_{R} is the bar module associated to R.

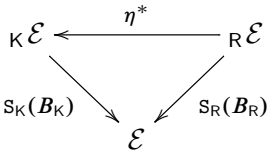
Note that a morphism $\psi\colon \mathsf{R} \to \mathsf{S}$ in the category of operads under K defines a morphism of K–algebras in right R–modules and hence induce a natural morphism $\psi_\sharp\colon B_{\mathsf{R}} \to B_{\mathsf{S}}$ in the category of right R–modules. This morphism has an adjoint $\psi_\flat\colon B_{\mathsf{R}} \circ_{\mathsf{R}} \mathsf{S} \to B_{\mathsf{S}}$.

The next assertions are proved in [24]:

1.11 Proposition (See [24, Section 1.4].) *Let* R *be an operad under* K. *We have a natural isomorphism* $\mathtt{B}(A) = \mathtt{S}_{\mathsf{R}}(B_{\mathsf{R}}, A)$, *for all* $A \in {}_{\mathsf{R}}\mathcal{E}$.

The bar module B_{R} *satisfies the relation* $B_{\mathsf{R}} \simeq B_{\mathsf{K}} \circ_{\mathsf{K}} \mathsf{R}$. *More generally, the natural morphism* $\psi_\flat\colon B_{\mathsf{R}} \circ_{\mathsf{R}} \mathsf{S} \to B_{\mathsf{S}}$ *associated to any morphism* $\psi\colon \mathsf{R} \to \mathsf{S}$ *in the category of operads under* K *forms an isomorphism.* □

Recall that the extension of right modules over operads represents the composition with restriction functors at the functor level. The relation $B_{\mathsf{R}} \simeq B_{\mathsf{K}} \circ_{\mathsf{K}} \mathsf{R}$ implies that the diagram of functors



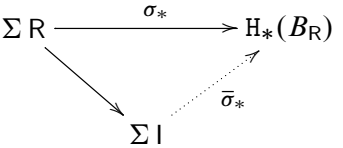
commutes up to a natural isomorphism. Hence the isomorphism $B_{\mathsf{R}} \simeq B_{\mathsf{K}} \circ_{\mathsf{K}} \mathsf{R}$ reflects the definition of the bar construction on the category of R–algebras by the restriction of a functor $\mathtt{B}\colon {}_{\mathsf{K}}\mathcal{E} \to \mathcal{E}$ on the category of K–algebras.

1.12 The representation of suspension morphisms The suspension morphism $\sigma_A\colon \Sigma A \to \mathtt{B}(A)$ can naturally be identified with a morphism of functors

$$\mathtt{S}_{\mathsf{R}}(\sigma_{\mathsf{R}})\colon \underbrace{\mathtt{S}_{\mathsf{R}}(\Sigma\,\mathsf{R})}_{=\,\Sigma\,\mathrm{Id}} \to \mathtt{S}_{\mathsf{R}}(B_{\mathsf{R}})$$

associated to a morphism of right R–modules $\sigma = \sigma_{\mathsf{R}}\colon \Sigma\,\mathsf{R} \to B_{\mathsf{R}}$, which is nothing but the suspension morphism of the operad R viewed as a K–algebra in the category of right modules over itself.

The observation of Section 1.9 implies that this suspension morphism $\sigma\colon \Sigma\,\mathsf{R} \to B_{\mathsf{R}}$ factors through $\mathtt{Indec}\,\mathsf{R}$ in the case $\mathsf{R} = \mathsf{A}, \mathsf{C}$. The operads $\mathsf{R} = \mathsf{A}, \mathsf{C}$ are equipped with an augmentation $\epsilon\colon \mathsf{R} \to \mathsf{I}$ which provides the composition unit I with the structure of a right R–module. In both cases $\mathsf{R} = \mathsf{A}, \mathsf{C}$, we have an obvious isomorphism $\mathtt{Indec}\,\mathsf{R} \simeq \mathsf{I}$ in the category of right R–modules. Consequently, the observation of Section 1.9 implies the existence of a factorization



in the category of right R–modules, for $\mathsf{R} = \mathsf{A}, \mathsf{C}$.

1.13 The bar complex for algebras over the commutative operad Recall that the bar complex of a commutative algebra is equipped with a commutative algebra structure (see [13, Exposé 4] or [34, Section X.12]). The product $\mu\colon \mathtt{B}(A)\otimes \mathtt{B}(A)\to \mathtt{B}(A)$, called the shuffle product, is defined on components of the tensor coalgebra by sums of tensor permutations

$$(\Sigma A)^{\otimes d}\otimes(\Sigma A)^{\otimes e}\xrightarrow{\sum_w w_*}(\Sigma A)^{\otimes d+e}$$

such that w ranges over the set of (d,e)–shuffles in Σ_{d+e} (see [34, Section X.12] for details). The shuffle product is naturally associative and commutative, but the derivation relation $\partial\mu(\alpha,\beta)=\mu(\partial\alpha,\beta)+\pm\mu(\alpha,\partial\beta)$ with respect to the bar differential $\partial\colon \mathtt{T}^c(\Sigma A)\to \mathtt{T}^c(\Sigma A)$ holds for commutative algebras only. The definition of the shuffle product is clearly functorial. Hence the restriction of the bar construction to the category of commutative algebras defines a functor $\mathtt{B}\colon {}_{\mathtt{C}}\mathcal{E}\to {}_{\mathtt{C}}\mathcal{E}$.

The definition of the shuffle product makes sense not only in the context of dg-modules $\mathcal{E}=\mathcal{C}$, but also in the context of Σ_*–modules $\mathcal{E}=\mathcal{M}$, and in the context of right modules over an operad $\mathcal{E}=\mathcal{M}_{\mathtt{R}}$. In particular, since we consider the commutative operad $\mathtt{C}$ as a commutative algebra in the category of right modules over itself, we obtain that the bar module of the commutative operad $B_{\mathtt{C}}$ forms a commutative algebra in right $\mathtt{C}$–modules, equivalently a bimodule over the commutative operad.

In [24, Section 2.1], we observe that the identity $\mathtt{B}(A)=\mathtt{S}_{\mathtt{C}}(B_{\mathtt{C}},A)$ holds in the category of functors $F\colon {}_{\mathtt{C}}\mathcal{E}\to {}_{\mathtt{C}}\mathcal{E}$.

1.14 The Hopf algebra structure In Section 1.13, we recall that the shuffle product preserves the differential of the bar complex. One proves further (see for instance [41, Section 0.2, Section 1.4]) that the shuffle product $\mu\colon \mathtt{T}^c(\Sigma E)\otimes \mathtt{T}^c(\Sigma E)\to \mathtt{T}^c(\Sigma E)$ commutes with the diagonal of the tensor coalgebra $\mathtt{T}^c(\Sigma E)$, for every object $E\in\mathcal{E}$ in a symmetric monoidal category $\mathcal{E}$. (In the nonunital setting, we simply have to use the expression given in the introduction of this part for this commutation relation.) As a byproduct, the bar complex $\mathtt{B}(A)=(\mathtt{T}^c(\Sigma A),\partial)$ inherits a natural commutative Hopf algebra structure, for every commutative algebra A. Again, this assertion holds in the context of dg-modules $\mathcal{E}=\mathcal{C}$, in the context of Σ_*–modules $\mathcal{E}=\mathcal{M}$, and in the context of right modules over an operad $\mathcal{E}=\mathcal{M}_{\mathtt{R}}$.

The Hopf algebra structure is used in Section 6 to determine the homology of $\mathtt{B}(A)$ for generalizations of usual commutative algebras.

The next statement is proved in [24, Section 2] in order to extend the multiplicative structure of the bar construction to the category of algebras over any E_∞–operad:

1.15 Theorem (See [24, Theorem 2.A].) *Let* E *be an* E_∞*–operad. Suppose* E *is cofibrant as an operad. The bar module* B_{E} *can be equipped with the structure of an* E *algebra in right* E*–modules so that the natural isomorphism of right* C*–modules* $\epsilon_\flat\colon B_{\mathsf{E}} \circ_{\mathsf{E}} \mathsf{C} \xrightarrow{\simeq} B_{\mathsf{C}}$ *defines an isomorphism of* E*–algebras in right* C*–modules, where we use a restriction of structure on the left to make* B_{C} *an* E*–algebra in right* C*–modules.* □

This Theorem implies that the functor $\mathtt{B}(A) = \mathtt{S}_{\mathsf{E}}(B_{\mathsf{E}}, A)$ lands in the category of E–algebras. The relation $B_{\mathsf{E}} \circ_{\mathsf{E}} \mathsf{C} \simeq B_{\mathsf{C}}$ in the category ${}_{\mathsf{E}}\mathcal{M}_{\mathsf{C}}$ implies that the diagram of functors

$$\begin{array}{ccc} {}_{\mathsf{E}}\mathcal{E} & \xrightarrow{\mathtt{B}(-) = \mathtt{S}_{\mathsf{E}}(B_{\mathsf{E}})} & {}_{\mathsf{E}}\mathcal{E} \\ \uparrow & & \uparrow \\ {}_{\mathsf{C}}\mathcal{E} & \xrightarrow[\mathtt{B}(-) = \mathtt{S}_{\mathsf{C}}(B_{\mathsf{C}})]{} & {}_{\mathsf{C}}\mathcal{E} \end{array}$$

commutes up to a natural isomorphism in the category of E–algebras, because extensions on the right at the module level correspond to restrictions on the source at the functor level, restrictions on the left at the module level correspond to restrictions on the target at the functor level.

Thus Theorem 1.15 implies that the bar construction of commutative algebras extends to a functor from E–algebras to E–algebras. As a byproduct, we have a well defined iterated bar complex $\mathtt{B}^n\colon {}_{\mathsf{R}}\mathcal{E} \to {}_{\mathsf{R}}\mathcal{E}$ defined by the n–fold composite of the bar construction $\mathtt{B}\colon {}_{\mathsf{R}}\mathcal{E} \to {}_{\mathsf{R}}\mathcal{E}$, for $\mathsf{R} = \mathsf{C}$ and $\mathsf{R} = \mathsf{E}$, so that the diagram

$$\begin{array}{ccc} {}_{\mathsf{E}}\mathcal{E} & \xrightarrow{\mathtt{B}^n} & {}_{\mathsf{E}}\mathcal{E} \\ \uparrow & & \uparrow \\ {}_{\mathsf{C}}\mathcal{E} & \xrightarrow[\mathtt{B}^n]{} & {}_{\mathsf{C}}\mathcal{E} \end{array}$$

commutes. We use the theory of modules over operads to determine the structure of this iterated bar construction. Observe first:

1.16 Proposition *Let* $\mathsf{R} = \mathsf{C}$ *or* $\mathsf{R} = \mathsf{E}$.

The n*–fold bar complex* $\mathtt{B}^n\colon {}_{\mathsf{R}}\mathcal{E} \to {}_{\mathsf{R}}\mathcal{E}$*, defined by the* n*–fold composite*

$$ {}_{\mathsf{R}}\mathcal{E} \xrightarrow{\mathtt{B}} {}_{\mathsf{R}}\mathcal{E} \xrightarrow{\mathtt{B}} \cdots \xrightarrow{\mathtt{B}} {}_{\mathsf{R}}\mathcal{E}, $$

is isomorphic to the functor $\mathtt{S}(B_{\mathsf{R}}^n)\colon {}_{\mathsf{R}}\mathcal{E} \to {}_{\mathsf{R}}\mathcal{E}$ *associated to the composite module*

$$ B_{\mathsf{R}}^n = B_{\mathsf{R}} \circ_{\mathsf{R}} \cdots \circ_{\mathsf{R}} B_{\mathsf{R}}. $$

Proof This is an immediate consequence of [23, Proposition 9.2.5] (see recollections in Section 0.10). □

We have further:

1.17 Proposition *The iterated bar module B^n_{R} is cofibrant as a right R–module, for $\mathsf{R} = \mathsf{C}$ and $\mathsf{R} = \mathsf{E}$. Moreover we have the relation $B^n_{\mathsf{E}} \circ_{\mathsf{E}} \mathsf{C} \simeq B^n_{\mathsf{C}}$.*

Proof The bar module B_{R} is cofibrant as a right R–module by [24, Proposition 1.4.6]. The functor $\mathbb{S}_{\mathsf{R}}(M)\colon {}_{\mathsf{R}}\mathcal{E} \to \mathcal{E}$ associated to a cofibrant right R–module M maps the R–algebras which are cofibrant in $\mathcal{E}$ to cofibrant objects of $\mathcal{E}$ by [23, Lemma 15.1.1]. In the case $M = B_{\mathsf{R}}$ and $\mathcal{E} = \mathcal{M}_{\mathsf{R}}$, we obtain that the relative composition product $B_{\mathsf{R}} \circ_{\mathsf{R}} N$ is cofibrant as a right R–module if N is so (recall that $\mathbb{S}_{\mathsf{R}}(M, N) = M \circ_{\mathsf{R}} N$ for $\mathcal{E} = \mathcal{M}_{\mathsf{R}}$). By induction, we conclude that B^n_{R} forms a cofibrant right R–module as asserted in the Proposition.

For a commutative algebra in right C–modules N, we have relations

$$B_{\mathsf{E}} \circ_{\mathsf{E}} (\epsilon^* N) \simeq (B_{\mathsf{E}} \circ_{\mathsf{E}} \mathsf{C}) \circ_{\mathsf{C}} N \simeq \epsilon^*(B_{\mathsf{C}} \circ_{\mathsf{C}} N)$$

(we prefer to mark the restriction of structure $\epsilon^*\colon {}_{\mathsf{E}}\mathcal{M}_{\mathsf{C}} \to {}_{\mathsf{E}}\mathcal{M}_{\mathsf{C}}$ in this formula). The relation $B^n_{\mathsf{E}} \circ_{\mathsf{E}} \mathsf{C} \simeq B^n_{\mathsf{C}}$ follows by induction. □

1.18 Iterated suspension morphisms The definition of the suspension morphism $\sigma_A\colon \Sigma A \to \mathtt{B}(A)$ can be applied to iterated bar complexes. In this way, we obtain a natural transformation

$$\sigma_A\colon \Sigma\, \mathtt{B}^{n-1}(A) \to \mathtt{B}^n(A),$$

for every $n \geq 1$, and for any R–algebra A, where $\mathsf{R} = \mathsf{C}, \mathsf{E}$. (By convention, in the case $n = 1$, we set $\mathtt{B}^0 = \mathrm{Id}$.)

The application of this construction to the operad itself $\mathsf{R} = \mathsf{C}, \mathsf{E}$ gives a morphism of right R–modules

$$\sigma_{\mathsf{R}}\colon \Sigma B^{n-1}_{\mathsf{R}} \to B^n_{\mathsf{R}}$$

such that $\mathbb{S}_{\mathsf{R}}(\sigma_{\mathsf{R}}, A) = \sigma_A$, for every $A \in {}_{\mathsf{R}}\mathcal{E}$. The diagram

$$\begin{array}{ccc} \Sigma B^{n-1}_{\mathsf{E}} & \xrightarrow{\sigma_{\mathsf{E}}} & B^n_{\mathsf{E}} \\ \downarrow & & \downarrow \\ \Sigma B^{n-1}_{\mathsf{C}} & \xrightarrow[\sigma_{\mathsf{C}}]{} & B^n_{\mathsf{C}} \end{array}$$

commutes by functoriality of the suspension. We have equivalently $\sigma_{\mathsf{E}} \circ_{\mathsf{E}} \mathsf{C} = \sigma_{\mathsf{C}}$.

1.19 The iterated bar module associated to noncofibrant E_∞–operads We can extend the definition of the iterated bar module B^n_{E} to any E_∞–operad, not necessarily cofibrant as an operad. Indeed, an E_∞–operad E has a cofibrant replacement $\phi\colon \mathsf{Q} \xrightarrow{\sim} \mathsf{E}$ which forms a cofibrant E_∞–operad and hence has an associated iterated bar module B^n_{Q}. Define the iterated bar module of E by the extension of structure $B^n_{\mathsf{E}} = B^n_{\mathsf{Q}} \circ_{\mathsf{Q}} \mathsf{E}$. By transitivity of relative composition products, we still have $B^n_{\mathsf{E}} \circ_{\mathsf{E}} \mathsf{C} \simeq B^n_{\mathsf{C}}$. Moreover, the object B^n_{E} is cofibrant in the category of right E–modules since the functor of extension of structure $- \circ_{\mathsf{Q}} \mathsf{E}$ is the left adjoint of a Quillen equivalence (see [23, Theorem 16.B]).

The functor $\mathbb{S}_{\mathsf{E}}(B^n_{\mathsf{E}}) = \mathbb{S}_{\mathsf{E}}(B^n_{\mathsf{Q}} \circ_{\mathsf{Q}} \mathsf{E})$ is identified with the composite

$$ {}_{\mathsf{E}}\mathcal{E} \xrightarrow{\phi^*} {}_{\mathsf{Q}}\mathcal{E} \xrightarrow{\mathrm{B}^n} {}_{\mathsf{Q}}\mathcal{E}, $$

where $\phi^*\colon {}_{\mathsf{E}}\mathcal{E} \to {}_{\mathsf{Q}}\mathcal{E}$ is the restriction functor associated to $\phi\colon \mathsf{Q} \xrightarrow{\sim} \mathsf{E}$.

We deduce a simple homotopical characterization of the iterated bar module B^n_{E} from Proposition 1.17. Recall that:

1.20 Fact (See [23, Theorem 16.B].) *Let E be any E_∞–operad, together with an augmentation $\epsilon\colon \mathsf{E} \xrightarrow{\sim} \mathsf{C}$. The extension and restriction functors*

$$ \epsilon_!\colon \mathcal{M}_{\mathsf{E}} \rightleftarrows \mathcal{M}_{\mathsf{C}} :\epsilon^* $$

define Quillen adjoint equivalences of model categories.

Consequently:

1.21 Proposition *Let M^n_{E} be any cofibrant right E–module equipped with a weak-equivalence*

$$ f_\flat\colon M^n_{\mathsf{E}} \circ_{\mathsf{E}} \mathsf{C} \xrightarrow{\sim} B^n_{\mathsf{C}} $$

in the category of right C–modules. The morphism of right E–modules

$$ f_\sharp\colon M^n_{\mathsf{E}} \to B^n_{\mathsf{C}} $$

adjoint to $f_\flat$ defines a weak-equivalence in the category of right E–modules. □

According to the assertion of Proposition 1.17, this Proposition applies to the iterated bar module B^n_{E}. As a byproduct, every module M^n_{E} which satisfies the requirement of Proposition 1.21 is connected to the iterated bar module B^n_{E} by weak-equivalences

$$ M^n_{\mathsf{E}} \xrightarrow{\sim} B^n_{\mathsf{C}} \xleftarrow{\sim} B^n_{\mathsf{E}} $$

in the category of right E–modules. Thus the homotopy type of the iterated bar module B^n_{E} is fully characterized by the result of Propositions 1.17 and 1.21.

In this chain of weak-equivalences $M^n_{\mathsf{E}} \xrightarrow{\sim} B^n_{\mathsf{C}} \xleftarrow{\sim} B^n_{\mathsf{E}}$ the right E–module M^n_{E} is cofibrant by assumption, and the iterated bar module B^n_{E} is cofibrant as well by Proposition 1.17, but the right C–module B^n_{C} is not cofibrant as a right E–module. Nevertheless, as usual in a model category, we can replace our chain of weak-equivalences by a chain of weak-equivalences

$$M^n_{\mathsf{E}} \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xrightarrow{\sim} B^n_{\mathsf{E}}$$

so that all intermediate objects are cofibrant right E–modules.

By [23, Theorem 15.1.A], the existence of such weak-equivalences at the module level implies:

1.22 Theorem *Suppose we have a cofibrant right E–module M^n_{E}, where E is any E_∞–operad, together with a weak-equivalence*

$$f_\flat\colon M^n_{\mathsf{E}} \circ_{\mathsf{E}} \mathsf{C} \xrightarrow{\sim} B^n_{\mathsf{C}}$$

in the category of right C–modules.

The functor $\mathtt{S}_{\mathsf{E}}(M^n_{\mathsf{E}})\colon {}_{\mathsf{E}}\mathcal{E} \to \mathcal{E}$ determined by M^n_{E} is connected to the iterated bar complex $\mathtt{B}^n\colon {}_{\mathsf{E}}\mathcal{E} \to \mathcal{E}$ by a chain of natural morphisms

$$\mathtt{S}_{\mathsf{E}}(M^n_{\mathsf{E}}, A) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xrightarrow{\sim} \mathtt{S}_{\mathsf{E}}(B^n_{\mathsf{E}}, A) = \mathtt{B}^n(A),$$

which are weak-equivalences as long as the E–algebra A defines a cofibrant object in the underlying category $\mathcal{E}$.

Proof By [23, Theorem 15.1.A], a weak-equivalence $f\colon M \xrightarrow{\sim} N$, where M, N are cofibrant right E–modules, induces a weak-equivalence at the functor level

$$\mathtt{S}_{\mathsf{E}}(f, A)\colon \mathtt{S}_{\mathsf{E}}(M, A) \xrightarrow{\sim} \mathtt{S}_{\mathsf{E}}(N, A),$$

for all $\mathcal{E}$–cofibrant E–algebras A. Hence, in our context, we have a chain of weak-equivalences

$$\mathtt{S}_{\mathsf{E}}(M^n_{\mathsf{E}}, A) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xrightarrow{\sim} \mathtt{S}_{\mathsf{E}}(B^n_{\mathsf{E}}, A) = \mathtt{B}^n(A)$$

between $\mathtt{S}_{\mathsf{E}}(M^n_{\mathsf{E}}, A)$ and the iterated bar complex $\mathtt{B}^n(A) = \mathtt{S}_{\mathsf{E}}(B^n_{\mathsf{E}}, A)$. □

According to this Theorem, the definition of a proper iterated bar complex $\mathtt{B}^n\colon {}_{\mathsf{E}}\mathcal{E} \to \mathcal{E}$, where E is any E_∞–operad, reduces to the construction of a cofibrant right E–module M^n_{E} together with a weak-equivalence $M^n_{\mathsf{E}} \circ_{\mathsf{E}} \mathsf{C} \xrightarrow{\sim} B^n_{\mathsf{C}}$. In the next section, we give an effective construction of such a cofibrant right E–module M^n_{E} starting from the iterated bar module over the commutative operad B^n_{C}.

2 Iterated bar modules as quasifree modules

Recall that the iterated bar module B^n_{C} is given by the iterated bar complex of the commutative operad, viewed as a commutative algebra in the category of right modules over itself. The bar construction $\mathtt{B}(A)$ is defined by a twisted complex $\mathtt{B}(A) = (\mathtt{T}^c(\Sigma A), \partial)$ and so does any of its composite. From this statement, we deduce that the n–fold bar module B^n_{C} is identified with a twisted right C–module of the form $B^n_{\mathsf{C}} = ((\mathtt{T}^c\,\Sigma)^n(\mathsf{C}), \partial_\gamma)$, where we take the n–fold composite of the functor $\mathtt{T}^c(\Sigma -)$ underlying the bar complex $\mathtt{B}(-)$.

We prove in this section that the right C–module $(\mathtt{T}^c\,\Sigma)^n(\mathsf{C})$ is isomorphic to a composite $(\mathtt{T}^c\,\Sigma)^n(\mathsf{C}) \simeq T^n \circ \mathsf{C}$, for some free Σ_*–module T^n. We use this structure result to lift the twisting homomorphism $\partial_\gamma\colon T^n \circ \mathsf{C} \to T^n \circ \mathsf{C}$ to the right E–module $T^n \circ \mathsf{E}$, for any E_∞–operad E. We obtain from this construction a cofibrant right E–module $B^n_{\mathsf{E}} = (T^n \circ \mathsf{E}, \partial_\epsilon)$ such that $B^n_{\mathsf{E}} \circ_{\mathsf{E}} \mathsf{C} = B^n_{\mathsf{C}}$. Hence, the lifting construction produces a right E–module which satisfies the requirements of Theorem 1.22 and, as such, determines a good iterated bar complex on the category of E–algebras.

We can make our construction effective if we assume that the E_∞–operad E is equipped with an effective contracting chain homotopy $\nu\colon \mathsf{E} \to \mathsf{E}$ such that $\delta(\nu) = \mathrm{id} - \iota\epsilon$, for some section $\iota\colon \mathsf{C} \to \mathsf{E}$ of the augmentation morphism $\epsilon\colon \mathsf{E} \to \mathsf{C}$. We need this effective construction in Section 8 in order to obtain a homotopy interpretation of iterated bar complexes.

The composite $K \circ \mathsf{R}$ represents a free object in the category of right R–modules. The twisted objects $M = (K \circ \mathsf{R}, \partial)$ associated to a free right R–module $K \circ \mathsf{R}$ are called quasifree. To begin this section, we review the definition and usual properties of these quasifree modules.

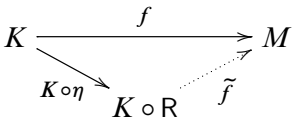
2.1 Twisting cochains in the category of right R–modules In certain constructions, the natural differential of a dg-module E is twisted by a homomorphism $\partial\colon E \to E$ of degree -1, called a twisting homomorphism, to produce a new dg-module $M = (E, \partial)$, which has the same underlying graded object as E, but whose differential is given by the sum $\delta + \partial\colon E \to E$. The bar complex $\mathtt{B}(A) = (\mathtt{T}^c(\Sigma A), \partial)$ gives an application of this construction.

To ensure that the homomorphism $\delta + \partial\colon E \to E$ satisfies the equation of a differential $(\delta + \partial)^2 = 0$, we simply have to require that a twisting homomorphism $\partial\colon E \to E$ satisfies the equation $\delta(\partial) + \partial^2 = 0$ in $\mathrm{Hom}_{\mathcal{E}}(E, E)$.

The construction of twisted objects makes sense in the category of Σ_*–modules and in the category of right modules over an operad, for a twisting homomorphism ∂

in the hom-object of the concerned category $\mathcal{E} = \mathcal{M}, \mathcal{M}_{\mathsf{R}}$. The bar module $B_{\mathsf{R}} = (\mathrm{T}^c(\Sigma\,\mathsf{R}), \partial)$ is an instance of a twisted object in the category of right R–modules.

2.2 Free modules over operads and free symmetric modules Let K be any Σ_*–module. The composite Σ_*–module $K \circ \mathsf{R}$ inherits a right R–action, defined by the morphism $K \circ \mu\colon K \circ \mathsf{R} \circ \mathsf{R} \to K \circ \mathsf{R}$ induced by the operad composition product $\mu\colon \mathsf{R} \to \mathsf{R}$, and forms naturally a right R–module. This object $K \circ \mathsf{R}$ is identified with a free right R–module associated to K, in the sense that any morphism of Σ_*–modules $f\colon K \to M$, where $M \in \mathcal{M}_{\mathsf{R}}$, has a unique extension



such that $\widetilde{f}\colon K \circ \mathsf{R} \to M$ is a morphism of right R–modules. Equivalently, the map $K \mapsto K \circ \mathsf{R}$ defines a left adjoint of the functor $U\colon \mathcal{M}_{\mathsf{R}} \to \mathcal{M}$.

Intuitively, the object $K \circ \mathsf{R}$ is spanned by formal composites $\xi(p_1, \ldots, p_r)$ of a generating element $\xi \in K(r)$ with operations $p_1, \ldots, p_r \in \mathsf{R}$. The extension to the free right R–module $K \circ \mathsf{R}$ of a morphism of Σ_*–modules $f\colon K \to M$ is determined by the formula $\widetilde{f}(\xi(p_1, \ldots, p_r)) = f(\xi)(p_1, \ldots, p_r)$.

Let $\mathcal{C}^{\mathbb{N}}$ denote the category of collections $G = \{G(r)\}_{r \in \mathbb{N}}$, where $G(r) \in \mathcal{C}$. The forgetful functor $U\colon \mathcal{M} \to \mathcal{C}^{\mathbb{N}}$ has a left adjoint which maps a collection G to an associated free Σ_*–module, denoted by $\Sigma_* \otimes G$. This Σ_*–module is represented by the external tensor products

$$(\Sigma_* \otimes G)(r) = \Sigma_r \otimes G(r).$$

The adjunction unit $\eta\colon G \to \Sigma_* \otimes G$ identifies $G(r)$ with the summand $\mathrm{id} \otimes G(r)$ of $\Sigma_* \otimes G$. The functor on finite sets equivalent to $\Sigma_* \otimes G$ satisfies

$$(\Sigma_* \otimes G)(\underline{\mathsf{e}}) = Bij(\{1, \ldots, r\}, \underline{\mathsf{e}}) \otimes G(r),$$

for every set $\underline{\mathsf{e}}$ such that $\underline{\mathsf{e}} = \{e_1, \ldots, e_r\}$.

In the case of a free Σ_*–module $K = \Sigma_* \otimes G$, the composite $K \circ \mathsf{R}$ has an expansion of the form

$$(\Sigma_* \otimes G) \circ \mathsf{R} = \mathsf{S}(\Sigma_* \otimes G, \mathsf{R}) = \bigoplus_{r=0}^{\infty} G(r) \otimes \mathsf{R}^{\otimes r},$$

where no coinvariant occurs.

2.3 Quasifree modules By convention, a quasifree right R–module M refers to a twisted object $M = (K \circ \mathsf{R}, \partial)$ formed from a free right R–module $K \circ \mathsf{R}$. In [23, Section 14.2], we prove that a quasifree R–module $M = (K \circ \mathsf{R}, \partial)$ is cofibrant if we have $K = \Sigma_* \otimes G$ for a collection of free graded $\Bbbk$–modules $G = \{G(r)\}_{r \in \mathbb{N}}$ equipped with a good filtration (see [23] for details). The filtration condition is automatically satisfied if we assume that each $G(r)$ is nonnegatively graded (in this case we can apply the arguments of [23] to the degreewise filtration).

The goal of this section is to prove that the iterated bar module B^n_{R}, is defined by a cofibrant quasifree module of this form $B^n_{\mathsf{R}} = (T^n \circ \mathsf{R}, \partial)$, for some free Σ_*–module T^n. For this purpose, we use that the twisting homomorphism $\partial\colon K \circ \mathsf{R} \to K \circ \mathsf{R}$ is determined by a homomorphism $\alpha\colon G \to K \circ \mathsf{R}$ in the category of collections $\mathcal{E} = \mathcal{C}^{\mathbb{N}}$ whenever $K = \Sigma_* \otimes G$. Indeed, the adjunction relations $\mathcal{C}^{\mathbb{N}} \rightleftarrows \mathcal{M} \rightleftarrows \mathcal{M}_{\mathsf{R}}$ yield isomorphisms of dg-modules

$$\mathrm{Hom}_{\mathcal{C}^{\mathbb{N}}}(G, K \circ \mathsf{R}) \simeq \mathrm{Hom}_{\mathcal{M}}(K, K \circ \mathsf{R}) \simeq \mathrm{Hom}_{\mathcal{M}_{\mathsf{R}}}(K \circ \mathsf{R}, K \circ \mathsf{R}),$$

for $K = \Sigma_* \otimes G$.

Let $\partial_\alpha\colon K \circ \mathsf{R} \to K \circ \mathsf{R}$ denote the homomorphism of right R–modules equivalent to $\alpha\colon G \to K \circ \mathsf{R}$. The composite $\partial_\alpha \partial_\beta\colon K \circ \mathsf{R} \to K \circ \mathsf{R}$ is necessarily associated to a homomorphism of collections for which we adopt the notation $\alpha \star \beta\colon G \to K \circ \mathsf{R}$. The equation of twisting homomorphisms $\delta(\partial_\alpha) + \partial_\alpha^2 = 0$ is equivalent to the equation $\delta(\alpha) + \alpha \star \alpha = 0$ in $\mathrm{Hom}_{\mathcal{C}^{\mathbb{N}}}(G, K \circ \mathsf{R})$.

The homomorphism $\alpha \star \beta\colon G \to K \circ \mathsf{R}$ can be identified with the composite

$$G \xrightarrow{\beta} K \circ \mathsf{R} \xrightarrow{\partial_\alpha} K \circ \mathsf{R}.$$

Intuitively, the twisting homomorphism ∂_α associated to a homomorphism $\alpha\colon G \to K \circ \mathsf{R}$ is determined by the relation $\partial_\alpha(\xi(p_1, \ldots, p_r)) = \alpha(\xi)(p_1, \ldots, p_r)$ for any formal composite $\xi(p_1, \ldots, p_r) \in K \circ \mathsf{R}$, where $\xi \in G(r)$ and $p_1, \ldots, p_r \in \mathsf{R}$.

2.4 Lifting twisting homomorphisms of quasifree modules Suppose we have a quasifree module $N = (K \circ \mathsf{S}, \partial_\beta)$ such that $K = \Sigma_* \otimes G$, for a collection of free graded $\Bbbk$–modules $G = \{G(r)\}_{r \in \mathbb{N}}$. Suppose further that each $G(r)$ is nonnegatively graded.

Let $\psi\colon \mathsf{R} \to \mathsf{S}$ be an acyclic fibration of operads. Suppose we have a section $\iota\colon \mathsf{S}(\underline{\mathrm{e}}) \to \mathsf{R}(\underline{\mathrm{e}})$ and a contracting chain homotopy $\nu\colon \mathsf{R}(\underline{\mathrm{e}}) \to \mathsf{R}(\underline{\mathrm{e}})$ such that $\psi \cdot \iota = \mathrm{id}$, $\psi \cdot \nu = 0$ and $\delta(\nu) = \mathrm{id} - \iota \cdot \psi$, for every finite set $\underline{\mathrm{e}}$ equipped with an ordering $\underline{\mathrm{e}} = \{e_1 < \cdots < e_r\}$. Note that such maps ι and ν are ensured to exist when R and S are nonnegatively graded operads.

The section $\iota\colon \mathsf{S}(\underline{e}) \to \mathsf{R}(\underline{e})$ and the contracting chain homotopy $\nu\colon \mathsf{S}(\underline{e}) \to \mathsf{S}(\underline{e})$ have natural extensions $\widetilde{\iota}\colon \mathsf{S}^{\otimes r}(\underline{e}) \to \mathsf{R}^{\otimes r}(\underline{e})$ and $\widetilde{\nu}\colon \mathsf{R}^{\otimes r}(\underline{e}) \to \mathsf{R}^{\otimes r}(\underline{e})$ given by the tensor products

$$\widetilde{\iota} = \iota^{\otimes r} \quad \text{and} \quad \widetilde{\nu} = \sum_{i=1}^{r} (-1)^{i-1} (\iota\psi)^{\otimes i-1} \otimes \nu \otimes \mathrm{id}^{\otimes r-i+1}$$

on the summands of the tensor power

$$\mathsf{S}^{\otimes r}(\underline{e}) = \bigoplus_{\underline{e}_1 \amalg \cdots \amalg \underline{e}_r = \underline{e}} \mathsf{S}(\underline{e}_1) \otimes \cdots \otimes \mathsf{S}(\underline{e}_r).$$

In this definition, we use that each subset $\underline{e}_i \subset \underline{e}$ inherits a natural ordering from $\underline{e}$.

Since $K \circ M = \bigoplus_{r=0}^{\infty} G(r) \otimes M^{\otimes r}$ for every Σ_*–object M, the extension of ι and ν to tensor powers gives also rise to a section $\widetilde{\iota}\colon K \circ \mathsf{S}(\underline{e}) \to K \circ \mathsf{R}(\underline{e})$ and a contracting chain homotopy $\widetilde{\nu}\colon K \circ \mathsf{R}(\underline{e}) \to K \circ \mathsf{R}(\underline{e})$ such that $K \circ \psi \cdot \widetilde{\iota} = \mathrm{id}$, $K \circ \psi \cdot \widetilde{\nu} = 0$ and $\delta(\widetilde{\nu}) = \mathrm{id} - \widetilde{\iota} \cdot K \circ \psi$.

For any $r \in \mathbb{N}$, we pick a lifting

$$\begin{array}{ccc} & & K \circ \mathsf{R}(r) \\ & \overset{\alpha_0}{\nearrow} & \big\downarrow {\scriptstyle K \circ \psi} \\ G(r) & \xrightarrow[\beta]{} & K \circ \mathsf{S}(r) \end{array}$$

by setting $\alpha_0 = \widetilde{\iota} \cdot \beta$, where we consider the section extension $\widetilde{\iota}\colon K \circ \mathsf{S}(\underline{r}) \to K \circ \mathsf{R}(\underline{r})$ defined with respect to the canonical ordering $\underline{r} = \{1 < \cdots < r\}$. We define a sequence of homomorphisms $\alpha_m\colon G(r) \to K \circ \mathsf{S}(r)$, $m \in \mathbb{N}$, by the inductive formula $\alpha_m = \sum_{p+q=m-1} \widetilde{\nu} \cdot (\alpha_p \star \alpha_q)$, where we consider the chain homotopy extension $\widetilde{\nu}\colon K \circ \mathsf{R}(\underline{r}) \to K \circ \mathsf{R}(\underline{r})$ defined with respect to the canonical ordering $\underline{r} = \{1 < \cdots < r\}$ too. Recall that $\alpha_p \star \alpha_q$ is given by the composite $\partial_{\alpha_p} \cdot \alpha_q\colon G \to K \circ \mathsf{R}$, where $\partial_{\alpha_p}\colon K \circ \mathsf{R} \to K \circ \mathsf{R}$ is the homomorphism of right R–modules extending $\alpha_p\colon G \to K \circ \mathsf{R}$.

We form the homomorphism $\alpha_* = \sum_{m=0}^{\infty} \alpha_m$. Note that α_m decreases the degree in K by m. Thus this infinite sum makes sense since K is supposed to vanish in degree $* < 0$.

This lifting process returns the following result:

2.5 Proposition *The homomorphism $\alpha_*\colon G \to K \circ \mathsf{R}$ determines a twisting homomorphism $\partial_\alpha\colon K \circ \mathsf{R} \to K \circ \mathsf{R}$ such that the diagram*

$$\begin{array}{ccc} K \circ \mathsf{R} & \overset{\partial_\alpha}{\dashrightarrow} & K \circ \mathsf{R} \\ {\scriptstyle K\circ\psi}\downarrow & & \downarrow{\scriptstyle K\circ\psi} \\ K \circ \mathsf{S} & \xrightarrow[\partial_\beta]{} & K \circ \mathsf{S} \end{array}$$

commutes. The quasifree module $M = (K \circ \mathsf{R}, \partial_\alpha)$ defined by this twisting homomorphism satisfies the extension relation $M \circ_{\mathsf{R}} \mathsf{S} \simeq N$ with respect to the given quasifree module $N = (K \circ \mathsf{S}, \partial_\beta)$.

Note that the composite $K \circ \psi$ defines a morphism of right R–modules

$$K \circ \psi\colon (K \circ \mathsf{R}, \partial_\alpha) \to (K \circ \mathsf{S}, \partial_\beta).$$

This assertion is an immediate consequence of the commutativity of the diagram. The isomorphism $M \circ_{\mathsf{R}} \mathsf{S} \simeq N$ corresponds to $K \circ \psi$ under the adjunction relation between extension and restriction functors.

Proof The equation of twisting homomorphisms $\delta(\alpha_*) + \alpha_* \star \alpha_* = 0$ follows from an immediate induction. The assumption $\psi\nu = 0$ implies $K \circ \psi \cdot \tilde{\nu} = 0$, from which we deduce $K \circ \psi \cdot \alpha_n = 0$ for $n > 0$. By definition, we also have $K \circ \psi \cdot \alpha_0 = \beta$. Hence we obtain $\psi_*\alpha_* = \beta$ and we deduce from this relation that the twisting homomorphisms equivalent to α and β satisfy the relation $K \circ \psi \cdot \partial_\alpha = \partial_\beta \cdot K \circ \psi$.

This commutation relation implies the identity $\partial_\alpha \circ_{\mathsf{R}} \mathsf{S} = \partial_\beta$ from which we deduce

$$(K \circ \mathsf{R}, \partial_\alpha) \circ_{\mathsf{R}} \mathsf{S} = ((K \circ \mathsf{R}) \circ_{\mathsf{R}} \mathsf{S}, \partial_\alpha \circ_{\mathsf{R}} \mathsf{S}) = (K \circ \mathsf{S}, \partial_\beta).$$

This assertion completes the proof of Proposition 2.5. □

2.6 The iterated bar complex of commutative algebras and the iterated bar module over the commutative operad The bar construction is a twisted dg-module by definition and, as a consequence, so does any of its composite. Thus, for a commutative algebra A, we have an identity

$$\mathtt{B}^n(A) = ((\mathtt{T}^c\,\Sigma)^n(A), \partial),$$

where the twisting homomorphism ∂ integrates all terms yielded by bar coderivations, occurring at each level of the composite $\mathtt{B}^n(A) = \mathtt{B} \circ \cdots \circ \mathtt{B}(A)$.

In the case of the commutative operad $A = \mathsf{C}$, we obtain that the iterated bar module B^n_{C} is identified with a twisted right C–module of the form $B^n_{\mathsf{C}} = ((\mathtt{T}^c\,\Sigma)^n(\mathsf{C}), \partial_\gamma)$. From this representation of B^n_{C}, we obtain:

2.7 Proposition *The iterated bar module B^n_{C} is identified with a quasifree right C–module of the form $B^n_{\mathsf{C}} = (T^n \circ \mathsf{C}, \partial_\gamma)$, where*

$$T^n = (\mathrm{T}^c\, \Sigma)^n(\mathsf{I})$$

is given by n iterations, within the category of Σ_*–modules, of the functor $\mathrm{T}^c(\Sigma -)$ applied to the composition unit I.

Proof Recall that we have a distribution relation

$$(M \otimes N) \circ P \simeq (M \circ P) \otimes (N \circ P)$$

between the tensor product and the composition product in the category of Σ_*–modules. For an iterated tensor coalgebra, we obtain an isomorphism of right C–modules

$$(\mathrm{T}^c\, \Sigma)^n(\mathsf{C}) \simeq (\mathrm{T}^c\, \Sigma)^n(\mathsf{I} \circ \mathsf{C}) \simeq (\mathrm{T}^c\, \Sigma)^n(\mathsf{I}) \circ \mathsf{C}\,.$$

The conclusion follows immediately. □

Note further:

2.8 Proposition *The Σ_*–module $T^n = (\mathrm{T}^c\, \Sigma)^n(\mathsf{I})$ is a free Σ_*–module*

$$T^n = \Sigma_* \otimes G^n$$

associated to a collection of nonnegatively graded $\Bbbk$–modules $G^n(r)$, $r \in \mathbb{N}$, defined inductively by

$$G^0(r) = \begin{cases} \Bbbk & \text{if } r = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$G^n(r) = \bigoplus_{d, r_1 + \cdots + r_d = r} \Sigma G^{n-1}(r_1) \otimes \cdots \otimes \Sigma G^{n-1}(r_d) \quad \text{for } n > 0.$$

The embedding $G^n(r) = \mathrm{id} \otimes G^n(r) \subset T^n(r)$ which yields the isomorphism $\Sigma_* \otimes G^n = T^n$ is defined in the proof of the Proposition.

Proof The identity $T^n = \Sigma_* \otimes G^n$ is obvious for $n = 0$.

For $n > 0$, we have a canonical morphism $\eta\colon G^n(r) \to T^n(r)$ formed inductively by tensor products

$$\begin{aligned} G^n(r) = &\bigoplus_{d, r_1+\cdots+r_d=r} \Sigma G^{n-1}(r_1) \otimes \cdots \otimes \Sigma G^{n-1}(r_d) \\ &\xrightarrow{\eta^{\otimes *}} \bigoplus_{d, r_1+\cdots+r_d=r} \Sigma T^{n-1}(r_1) \otimes \cdots \otimes \Sigma T^{n-1}(r_d) \\ &\simeq \bigoplus_{d, r_1+\cdots+r_d=r} \Sigma T^{n-1}(\underline{e}_1) \otimes \cdots \otimes \Sigma T^{n-1}(\underline{e}_d) \subset \mathrm{T}^c(\Sigma T^{n-1})(e), \end{aligned}$$

where we take the obvious ordered partition

$$\underline{e}_i = \{r_1 + \cdots + r_{i-1} + 1, \ldots, r_1 + \cdots + r_{i-1} + r_i\} \quad \text{for } i = 1, \ldots, d\,.$$

The identity $T^n = \Sigma_* \otimes G^n$ follows from an easy induction on n. □

From Proposition 2.7 and Proposition 2.8, we conclude:

2.9 Theorem *Let E be an E_∞–operad such that, for every finite ordered set $\underline{e} = \{e_1 < \cdots < e_r\}$, we have a section $\iota\colon \mathsf{C}(\underline{e}) \to \mathsf{E}(\underline{e})$ of the augmentation $\epsilon\colon \mathsf{E}(\underline{e}) \to \mathsf{C}(\underline{e})$ and a contracting chain homotopy $\nu\colon \mathsf{E}(\underline{e}) \to \mathsf{E}(\underline{e})$ satisfying $\epsilon \cdot \iota = \mathrm{id}$, $\epsilon \cdot \nu = 0$ and $\delta(\nu) = \mathrm{id} - \iota \cdot \epsilon$.*

Then the construction of Sections 2.4–2.5 can be applied to the iterated bar module $B^n_{\mathsf{C}} = (T^n \circ \mathsf{C}, \partial_\gamma)$ to produce a cofibrant quasifree module $B^n_{\mathsf{E}} = (T^n \circ \mathsf{E}, \partial_\epsilon)$ satisfying $B^n_{\mathsf{E}} \circ_{\mathsf{E}} \mathsf{C} \simeq B^n_{\mathsf{C}}$.

The Barratt–Eccles operad, whose definition is reviewed in Section 1.4, fulfils the requirement of this Theorem.

We have moreover:

2.10 Proposition *In the case of iterated bar modules, the twisting homomorphism $\partial_\epsilon\colon T^n \circ \mathsf{E} \to T^n \circ \mathsf{E}$ which arises from the construction of Sections 2.4–2.5 satisfies the coderivation relation $\Delta\partial_\epsilon = (\partial_\epsilon \otimes \mathrm{Id} + \mathrm{Id} \otimes \partial_\epsilon)\Delta$ so that $B^n_{\mathsf{E}} = (T^n \circ \mathsf{E}, \partial_\epsilon)$ forms a coalgebra.*

The diagonal $\Delta\colon T^n \circ \mathsf{E} \to T^n \circ \mathsf{E}$ comes from the deconcatenation coproduct of the first tensor coalgebra in the composite $T^n \circ \mathsf{E} = \mathrm{T}^c(\Sigma T^{n-1}) \circ \mathsf{E} = \mathrm{T}^c(\Sigma T^{n-1} \circ \mathsf{E})$.

Proof The twisting homomorphism $\partial_\gamma\colon T^n\circ\mathsf{C}\to T^n\circ\mathsf{C}$ associated to the commutative operad forms a coderivation with respect to the coproduct of $T^n\circ\mathsf{C}$ because the bar module B^n_{C} forms a coalgebra, like the bar complex of any commutative algebra.

The deconcatenation of tensors yield a diagonal

$$\Delta\colon G^n(t)\to\bigoplus_{r+s=t} G^n(r)\otimes G^n(s),$$

on the collection

$$G^n(r)=\bigoplus_{d,r_1+\cdots+r_d=r}\Sigma G^{n-1}(r_1)\otimes\cdots\otimes\Sigma G^{n-1}(r_d)$$

of Proposition 2.8. The deconcatenation coproduct of $T^n\circ\mathsf{R}=\mathrm{T}^c(\Sigma T^{n-1}\circ\mathsf{R})$, $\mathsf{R}=\mathsf{C},\mathsf{E}$, is clearly identified with the diagonal

$$G^n(t)\otimes\mathsf{R}^{\otimes t}\to\bigoplus_{r+s=t}(G^n(r)\otimes G^n(s))\otimes\mathsf{R}^{\otimes t}\simeq\bigoplus_{r+s=t}(G^n(r)\otimes\mathsf{R}^{\otimes r})\otimes(G^n(s)\otimes\mathsf{R}^{\otimes s})$$

induced by the deconcatenation coproduct of G^n.

One easily checks that the homomorphisms of Section 2.4

$$\widetilde{\iota}\colon G^n(r)\otimes\mathsf{S}^{\otimes r}\to G^n(r)\otimes\mathsf{R}^{\otimes r}\quad\text{and}\quad\widetilde{\nu}\colon G^n(r)\otimes\mathsf{R}^{\otimes r}\to G^n(r)\otimes\mathsf{R}^{\otimes r}$$

satisfy $\Delta\cdot\widetilde{\iota}=\widetilde{\iota}\otimes\widetilde{\iota}\cdot\Delta$ and $\Delta\cdot\widetilde{\nu}=(\widetilde{\nu}\otimes\mathrm{id}+\widetilde{\iota}\psi\otimes\widetilde{\nu})\cdot\Delta$. Then an easy induction shows each $\epsilon_*\colon G^n\to T^n\circ\mathsf{E}$ forms a coderivation. The conclusion follows immediately. □

The iterated bar modules B^n_{E} which arise from the construction of Section 1 are connected by suspension morphisms $\sigma_{\mathsf{E}}\colon\Sigma B^{n-1}_{\mathsf{E}}\to B^n_{\mathsf{E}}$. So do the iterated bar modules yielded by the construction of Theorem 2.9:

2.11 Proposition *The iterated bar modules of Theorem 2.9 are connected by suspension morphisms*

$$\sigma_{\mathsf{E}}\colon\Sigma B^{n-1}_{\mathsf{E}}\to B^n_{\mathsf{E}}$$

that fit in commutative diagrams

$$\begin{array}{ccc}\Sigma B^{n-1}_{\mathsf{E}} & \overset{\sigma_{\mathsf{E}}}{\dashrightarrow} & B^n_{\mathsf{E}}\\ \downarrow & & \downarrow\\ \Sigma B^{n-1}_{\mathsf{C}} & \xrightarrow[\sigma_{\mathsf{C}}]{} & B^n_{\mathsf{C}}\end{array}$$

for every $n>0$. We have moreover $\sigma_{\mathsf{E}}\circ_{\mathsf{E}}\mathsf{C}=\sigma_{\mathsf{C}}$.

This Proposition is an immediate consequence of the following Lemma:

2.12 Lemma *The canonical embedding*

$$\Sigma T^{n-1} \circ \mathsf{E} \hookrightarrow \mathrm{T}^c(\Sigma T^{n-1} \circ \mathsf{E}) \simeq T^n \circ \mathsf{E}$$

commutes with the twisting homomorphism of Theorem 2.9, for every $n > 0$.

Proof Observe that the embedding of the Lemma is realized at the level of the generating collection G^n of Proposition 2.8 and is clearly preserved by the homomorphisms $\widetilde{\tau}$ and $\widetilde{\nu}$ of Section 2.4. The Lemma follows by an easy induction from the definition of the twisting homomorphism ∂_ϵ. □

We have moreover:

2.13 Proposition *The suspension morphisms of Proposition 2.11*

$$\sigma_{\mathsf{E}}\colon \Sigma B^{n-1}_{\mathsf{E}} \to B^n_{\mathsf{E}}$$

are cofibrations in the category of right E–modules.

Proof Use the degreewise filtration of the Σ_*–module T^n to split the embedding of quasifree modules

$$\Sigma(T^{n-1} \circ \mathsf{E}, \partial_\epsilon) \hookrightarrow (T^n \circ \mathsf{E}, \partial_\epsilon)$$

into a sequence of generating cofibrations of right E–modules (use the overall ideas of [23, Section 11.2, Section 14.2]). □

In the next sections, we use the effective construction of Sections 2.4–2.5 to prove that, for some good E_∞–operads E equipped with a filtration of the form $(*)$, the twisting homomorphism $\partial_\epsilon\colon T^n \circ \mathsf{E} \to T^n \circ \mathsf{E}$ preserves the subobject $T^n \circ \mathsf{E}_n$ and restricts to a twisting homomorphism on this right E_n–module $T^n \circ \mathsf{E}_n$. For this aim, we use the existence of nice cell decompositions $\mathsf{E}(r) = \mathrm{colim}_{\kappa \in \mathcal{K}(r)} \mathsf{E}(\kappa)$ refining the filtration of E. Therefore we recall the overall definition of these cell structures before going further into the study of iterated bar complexes.

3 Interlude: Operads shaped on complete graph posets

For our analysis of iterated bar modules, we are going to use a particular cell structure of E_∞–operads, introduced in [8], and modeled by a certain operad in posets $\mathcal{K}$, the complete graph operad. The main purpose of this section is to revisit definitions of [8] in order to give an abstract formalization of the complete graph cell decompositions

of E_∞–operads and to extend the applications of these cell structures to iterated bar modules.

To begin with, we review the definition of the complete graph operad. For our needs, it is more convenient to define directly the functor on finite sets underlying the complete graph operad $\mathcal{K}$.

3.1 The complete graph posets Let $\underline{e}$ be any set with r elements $\{e_1, \ldots, e_r\}$. The complete graph poset $\mathcal{K}(\underline{e})$ consists of pairs $\kappa = (\mu, \sigma)$, such that μ is a collection of nonnegative integers $\mu_{ef} \in \mathbb{N}$ indexed by pairs $\{e, f\} \subset \underline{e}$ and σ is a bijection $\sigma\colon \{1, \ldots, r\} \to \underline{e}$, which amounts to the definition of an ordering on the set $\underline{e}$.

For a pair $\{e, f\} \subset \underline{e}$, we define the restriction $\sigma|_{ef}$ of an ordering σ as the ordering of $\{e, f\}$ defined by the occurrences of $\{e, f\}$ in the sequence $\sigma = (\sigma(1), \ldots, \sigma(r))$.

The elements of $\mathcal{K}(\underline{e})$ are represented by complete graphs on r–vertices, indexed by $\underline{e}$, whose edges are coherently oriented and equipped with a weight (see Figure 1). The

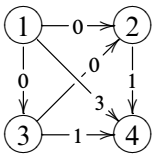


Figure 1: An element of the complete graph operad

weight of the edge $\{e, f\}$ is defined by the integer $\mu_{ef} \in \mathbb{N}$. The orientation of $\{e, f\}$ is defined by the ordering $\sigma|_{ef} \in \{(e, f), (f, e)\}$. The coherence of the orientations amounts to the requirement that the local orderings $\sigma|_{ef} \in \{(e, f), (f, e)\}$ assemble to a global ordering of the set $\{e_1, \ldots, e_r\}$.

For elements $(\mu, \sigma), (\nu, \tau) \in \mathcal{K}(r)$, we set $(\mu, \sigma) \leq (\nu, \tau)$ when we have

$$(\mu_{ef} < \nu_{ef}) \quad \text{or} \quad (\mu_{ef}, \sigma|_{ef}) = (\nu_{ef}, \tau|_{ef}),$$

for every pair $\{e, f\} \subset \underline{e}$. This relation clearly gives a poset structure on $\mathcal{K}(\underline{e})$.

The collection of posets $\mathcal{K}(\underline{e})$ defines a functor on the category of finite sets and bijections. The morphism $u_*\colon \mathcal{K}(\underline{e}) \to \mathcal{K}(\underline{f})$ induced by a bijection $u\colon \underline{e} \to \underline{f}$ is simply defined by reindexing the vertices of complete graphs.

In the case $\underline{e} = \{1, \ldots, r\}$, we simply replace the bijection $\sigma\colon \{1, \ldots, r\} \to \underline{e}$ by an equivalent permutation $\sigma \in \Sigma_r$ in the definition of $\mathcal{K}(r) = \mathcal{K}(\{1, \ldots, r\})$.

3.2 The complete graph operad The collection of posets

$$\mathcal{K}(r) = \mathcal{K}(\{1, \ldots, r\})$$

are equipped with an operad structure. The action of permutations $w_*\colon \mathcal{K}(r) \to \mathcal{K}(r)$ arises from the reindexing process of the previous paragraph.

The operadic composite $\kappa(\lambda_1, \ldots, \lambda_r) \in \mathcal{K}(\underline{e}_1 \amalg \cdots \amalg \underline{e}_r)$ is defined by the substitution of the vertices of $\kappa \in \mathcal{K}(r)$ by the complete graphs $\lambda_i \in \mathcal{K}(\underline{e}_i)$, $i = 1, \ldots, r$. Explicitly, the weight and the orientation of the edges of $\kappa(\lambda_1, \ldots, \lambda_r)$ are determined by the following rules: the edges of $\kappa(\lambda_1, \ldots, \lambda_r)$ between vertices $e, f \in \underline{e}_1 \amalg \cdots \amalg \underline{e}_r$ such that $e, f \in \underline{e}_i$, for some $i \in \{1, \ldots, r\}$, are copies of the edge $\{e, f\}$ of the graph λ_i; the edges of $\kappa(\lambda_1, \ldots, \lambda_r)$ between vertices $e, f \in \underline{e}_1 \amalg \cdots \amalg \underline{e}_r$ such that $e \in \underline{e}_i$ and $f \in \underline{e}_j$, for a pair $i \neq j$, are copies of the edge $\{i, j\}$ of the graph κ. An example is represented in Figure 2.

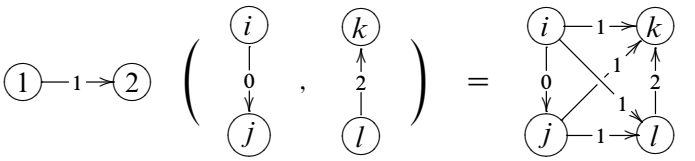


Figure 2: A composite in the complete graph operad

Since we only consider nonunitary operads in this article, we adopt the convention $\mathcal{K}(0) = \varnothing$, different from [8], for the complete graph operad.

The next observation is a simple consequence of the definition of the composition of complete graphs:

3.3 Observation *Define the restriction $\mu|_{\underline{e}_i}$ of an element $\mu \in \mathcal{K}(\underline{e}_1 \amalg \cdots \amalg \underline{e}_r)$ as the subgraph of μ generated by the vertices of $\underline{e}_i$.*

For a composite $\kappa(\lambda_1, \ldots, \lambda_r) \in \mathcal{K}(\underline{e}_1 \amalg \cdots \amalg \underline{e}_r)$, where $\kappa \in \mathcal{K}(r)$ and $\lambda_1 \in \mathcal{K}(\underline{e}_1), \ldots, \lambda_r \in \mathcal{K}(\underline{e}_r)$, we have $\kappa(\lambda_1, \ldots, \lambda_r) \leq \mu$ if and only if $\lambda_1 \leq \mu|_{\underline{e}_1}, \ldots, \lambda_r \leq \mu|_{\underline{e}_r}$ and $\kappa(\mu|_{\underline{e}_1}, \ldots, \mu|_{\underline{e}_r}) \leq \mu$.

3.4 Operads shaped on complete graph posets In [17, Section 4.1.2], we define a $\mathcal{K}$–operad as a collection of $\mathcal{K}(r)$–diagrams $\{\mathsf{P}(\kappa)\}_{\kappa \in \mathcal{K}(r)}$ together with Σ_r–actions

$$\mathsf{P}(\kappa) \xrightarrow{w_*} \mathsf{P}(w\kappa), \quad w \in \Sigma_*,$$

and composition products

$$\mathsf{P}(\kappa) \otimes (\mathsf{P}(\lambda_1) \otimes \cdots \otimes \mathsf{P}(\lambda_r)) \to \mathsf{P}(\kappa(\lambda_1, \ldots, \lambda_r))$$

which satisfy a natural extension of the standard axioms of operads. Naturally, the $\mathcal{K}(r)$–diagrams $\{\mathsf{P}(\kappa)\}_{\kappa\in\mathcal{K}(r)}$ are equivalent to $\mathcal{K}(\underline{\mathsf{e}})$–diagrams $\{\mathsf{P}(\kappa)\}_{\kappa\in\mathcal{K}(\underline{\mathsf{e}})}$, associated to all finite sets $\underline{\mathsf{e}}$, so that any bijection $u\colon \underline{\mathsf{e}}\to\underline{\mathsf{f}}$ defines a morphism of diagrams $u_*\colon \mathsf{P}(\kappa)\to\mathsf{P}(f_*\kappa)$.

We adopt the notation ${}_{\mathcal{K}}\mathcal{O}$ to refer to the category of $\mathcal{K}$–operads. Naturally, a morphism of $\mathcal{K}$–operads consists of a collection of $\mathcal{K}(r)$–diagram morphisms $\phi\colon \mathsf{P}(\kappa)\to\mathsf{Q}(\kappa)$ which commute with Σ_r–actions and composition structures.

The colimit of the underlying $\mathcal{K}(r)$–diagrams of a $\mathcal{K}$–operad

$$(\mathrm{colim}_{\mathcal{K}}\,\mathsf{P})(r)=\mathrm{colim}_{\kappa\in\mathcal{K}(r)}\,\mathsf{P}(\kappa)$$

inherits a natural operad structure. Hence we have a functor $\mathrm{colim}_{\mathcal{K}}\colon {}_{\mathcal{K}}\mathcal{O}\to\mathcal{O}$ from the category of $\mathcal{K}$–operads ${}_{\mathcal{K}}\mathcal{O}$ to the category of ordinary operads $\mathcal{O}$. This colimit functor is left adjoint to the obvious functor $cst\colon \mathcal{O}\to{}_{\mathcal{K}}\mathcal{O}$ which maps an ordinary operad P to the constant K–operad for which we have $\mathsf{P}(\kappa)=\mathsf{P}(r)$ for every $\kappa\in\mathcal{K}(r)$.

In [17, Section 4], we say that an operad $\mathsf{P}\in\mathcal{O}$ is equipped with a $\mathcal{K}$–structure when we have $\mathsf{P}(r)=\mathrm{colim}_{\kappa\in\mathcal{K}(r)}\,\mathsf{P}(\kappa)$ for a given $\mathcal{K}$–operad $\mathsf{P}(\kappa)$. In what follows, we assume that the components of this $\mathcal{K}$–operad $\mathsf{P}(\kappa)$ are given as dg-submodules $\mathsf{P}(\kappa)\subset\mathsf{P}(r)$, for all $\kappa\in\mathcal{K}(r)$ and all $r\in\mathbb{N}$. This convention $\mathsf{P}(\kappa)\subset\mathsf{P}(r)$ will simplify the presentation of the constructions of the present article.

In general, we use the same notation for the ordinary operad $\mathsf{P}\in\mathcal{O}$ and its underlying $\mathcal{K}$–operad $\mathsf{P}(\kappa)$ when we assume that the operad P is equipped with a $\mathcal{K}$–structure. In what follows, we simply use letters $r,s,t,\dots\in\mathbb{N}$ (or $\underline{\mathsf{e}},\underline{\mathsf{f}},\dots$) and $\kappa,\lambda,\dots\in\mathcal{K}(r)$ as dummy variables to mark the distinction between the components of these objects when necessary.

3.5 The example of the Barratt–Eccles operad The Barratt–Eccles operad is an instance of an operad equipped with a nice $\mathcal{K}$–structure. Recall that the dg-version of this operad consists of the normalized chain complexes $\mathsf{E}(r)=N_*(E\Sigma_r)$ so that the dg-module $\mathsf{E}(r)$ is spanned in degree d by the d–simplices of nondegenerate permutations $(w_0,\dots,w_d)\in\Sigma_r\times\cdots\times\Sigma_r$.

For $\kappa=(\mu,\sigma)\in\mathsf{K}(r)$, we form the module $\mathsf{E}(\kappa)\subset\mathsf{E}(r)$ spanned by the simplices of permutations $(w_0,\dots,w_d)$ such that, for every pair $\{i,j\}\subset\{1,\dots,r\}$, the sequence $(w_0|_{ij},\dots,w_d|_{ij})$ has less than μ_{ij} variations, or has exactly μ_{ij} variations and satisfies $w_d|_{ij}=\sigma|_{ij}$. This module is clearly preserved by the differential of the Barratt–Eccles operad and hence forms a dg-submodule of $\mathsf{E}(r)$. Moreover, we clearly have $\kappa\leq\lambda\Rightarrow\mathsf{E}(\kappa)\subset\mathsf{E}(\lambda)$ in $\mathsf{E}(r)$ and we can easily check that $\mathrm{colim}_{\kappa\in\mathcal{K}(r)}\,\mathsf{E}(\kappa)=\mathsf{E}(r)$, for every $r\in\mathbb{N}$.

The action of a permutation $w \in \Sigma_r$ on $\mathsf{E}(r)$ maps the subcomplex $\mathsf{E}(\kappa) \subset \mathsf{E}(r)$ into $\mathsf{E}(w\kappa) \subset \mathsf{E}(r)$ and the composition product of E restricts to morphisms

$$\mathsf{E}(\kappa) \otimes (\mathsf{E}(\lambda_1) \otimes \cdots \otimes \mathsf{E}(\lambda_r)) \xrightarrow{\mu} \mathsf{E}(\kappa(\lambda_1, \ldots, \lambda_r))$$

for every $\kappa \in \mathcal{K}(r)$, $\lambda_1, \ldots, \lambda_r \in \mathcal{K}$. Thus the collection of diagrams $\{\mathsf{E}(\kappa)\}_{\kappa \in \mathcal{K}(r)}$, inherits the structure of a $\mathcal{K}$–operad so that the inclusions $i\colon \mathsf{E}(\kappa) \hookrightarrow \mathsf{E}(r)$ yield an isomorphism of operads $i\colon \operatorname{colim}_{\kappa \in \mathcal{K}(r)} \mathsf{E}(\kappa) \xrightarrow{\simeq} \mathsf{E}(r)$.

Recall that we simply replace permutations $w_i \in \Sigma_r$ by bijections $w_i\colon \{1, \ldots, r\} \to \underline{\mathsf{e}}$ in the definition of the Barratt–Eccles operad to form the dg-module $\mathsf{E}(\underline{\mathsf{e}})$ associated to a finite set $\underline{\mathsf{e}}$. The extension of the definition to any finite indexing set works same for the dg-modules $\mathsf{E}(\kappa)$ when we assume $\kappa \in \mathcal{K}(\underline{\mathsf{e}})$.

3.6 Complete graph posets and cell decompositions of E_∞–operads The complete graph operad $\mathcal{K}$ has a nested sequence of suboperads

$$\mathcal{K}_1 \subset \cdots \subset \mathcal{K}_n \subset \cdots \subset \operatorname{colim}_n \mathcal{K}_n = \mathcal{K}$$

defined by bounding the weight of edges in complete graphs. Explicitly, the subposet $\mathcal{K}_n(\underline{\mathsf{e}}) \subset \mathcal{K}(\underline{\mathsf{e}})$ consists of complete graphs $\kappa = (\mu, \sigma) \in \mathcal{K}(\underline{\mathsf{e}})$ such that $\mu_{ef} < n$, for every pair $\{e, f\} \subset \underline{\mathsf{e}}$.

For any operad P equipped with a $\mathcal{K}$–structure, we have a sequence of operads

$$\mathsf{P}_1 \to \cdots \to \mathsf{P}_n \to \cdots \to \operatorname{colim}_n \mathsf{P}_n \xrightarrow{\simeq} \mathsf{P}$$

such that $\mathsf{P}_n(r) = \operatorname{colim}_{\kappa \in \mathcal{K}_n(r)} \mathsf{P}(\kappa)$.

The main Theorem of [8] implies that the sequence of dg-operads

$$\mathsf{P}_n(r) = \operatorname{colim}_{\kappa \in \mathcal{K}_n(r)} \mathsf{P}(\kappa)$$

is weakly-equivalent to the nested sequence of the chain operads of little n–cubes

$$C_*(\mathsf{D}_1) \to \cdots \to C_*(\mathsf{D}_n) \to \cdots \to \operatorname{colim}_n C_*(\mathsf{D}_n) = C_*(\mathsf{D}_\infty)$$

when we assume:

(K1) the collection $\mathsf{P}(\kappa)$ forms a cofibrant $\mathsf{K}(r)$–diagram in dg-modules (with respect to the standard model structure of diagrams in a cofibrantly generated model category), for every $r \in \mathbb{N}$;

(K2) we have a pointwise weak-equivalence of $\mathcal{K}$–operads $\epsilon\colon \mathsf{P}(\kappa) \xrightarrow{\sim} \mathsf{C}(\kappa)$, where $\mathsf{C}(\kappa)$ is the constant $\mathcal{K}$–operad defined by the commutative operad C.

Moreover, each morphism $\mathrm{colim}_{\kappa\in\mathcal{K}_{n-1}(r)}\mathsf{P}(\kappa)\to\mathrm{colim}_{\kappa\in\mathcal{K}_n(r)}\mathsf{P}(\kappa)$ is a cofibration of dg-modules whenever condition (K1) holds. In the remainder, we say that an operad P is a $\mathcal{K}$–cellular E_∞–operad when P is equipped with a $\mathcal{K}$–structure satisfying (K1)–(K2).

3.7 The example of the Barratt–Eccles operad (continued) The $\mathcal{K}$–structure $\mathsf{E}(\kappa)$ defined in Section 3.5 for the Barratt–Eccles operad satisfies the conditions (K1)–(K2). Hence, we have a nested sequence of operads $\mathsf{E}_n(r)=\mathrm{colim}_{\kappa\in\mathcal{K}_n(r)}\mathsf{E}(\kappa)$, formed from the Barratt–Eccles operad E, and weakly-equivalent to the nested sequence of the chain operads of little n–cubes (see [8]).

We refer to [17] for details about the cofibration condition (K1) in the dg-setting. We just recall the proof of the acyclicity condition (K2). We arrange the definition of the standard contracting chain homotopy $\nu\colon\mathsf{E}(\underline{e})\to\mathsf{E}(\underline{e})$ to prove that each $\mathsf{E}(\kappa)$ is contractible. Explicitly, for any simplex $(w_0,\ldots,w_d)\in\mathsf{E}(\kappa)$, where $\kappa=(\mu,\sigma)$, we set

$$\nu(w_0,\ldots,w_d)=(w_0,\ldots,w_d,\sigma).$$

Note that $(w_0,\ldots,w_d)\in\mathsf{E}(\mu,\sigma)\Rightarrow(w_0,\ldots,w_d,\sigma)\in\mathsf{E}(\mu,\sigma)$. Thus we have we well-defined homomorphism $\nu\colon\mathsf{E}(\mu,\sigma)\to\mathsf{E}(\mu,\sigma)$.

Recall that the standard section $\eta\colon\mathsf{C}(r)\to\mathsf{E}(r)$ of the augmentation $\epsilon\colon\mathsf{E}(r)\to\mathsf{C}(r)$ identifies $\mathsf{C}(r)$ with the summand of $\mathsf{E}(r)$ spanned by the identity permutation $\mathrm{id}\in\Sigma_r$. Since we have $(\mathrm{id})\in\mathsf{E}(\kappa)$ for every $\kappa\in\mathcal{K}(r)$, we immediately obtain a map $\iota\colon\mathsf{C}(\kappa)\to\mathsf{E}(\kappa)$ such that $\epsilon\iota=\mathrm{id}$. On the other hand, we easily check that the modified chain contraction ν satisfies $\delta(\nu)=\mathrm{id}-\iota\epsilon$ on each $\mathsf{E}(\kappa)$. Hence we conclude that the augmentation $\epsilon\colon\mathsf{E}\to\mathsf{C}$ gives rise to a pointwise equivalence of $\mathcal{K}$–operads $\epsilon\colon\mathsf{E}(\kappa)\xrightarrow{\sim}\mathsf{C}(\kappa)$.

3.8 Modules shaped on complete graph posets The definition of a $\mathcal{K}$–structure has an obvious generalization in the context of modules over an operad: a right $\mathcal{K}$–module over a $\mathcal{K}$–operad R consists of a collection of $\mathcal{K}(r)$–diagrams $\{M(\kappa)\}_{\kappa\in\mathcal{K}(r)}$ together with Σ_r–actions

$$M(\kappa)\xrightarrow{w_*}M(w\kappa),\quad w\in\Sigma_*,$$

and composition products

$$M(\kappa)\otimes(\mathsf{R}(\lambda_1)\otimes\cdots\otimes\mathsf{R}(\lambda_r))\xrightarrow{\mu}M(\kappa(\lambda_1,\ldots,\lambda_r))$$

which satisfy a natural extension of the standard axioms of modules over operads.

For a right $\mathcal{K}$–module M over a $\mathcal{K}$–operad R, the colimits $M(r)=\mathrm{colim}_{\kappa\in\mathcal{K}(r)}M(\kappa)$ naturally form a right module over the operad $\mathsf{R}(r)=\mathrm{colim}_{\kappa\in\mathcal{K}(r)}\mathsf{R}(\kappa)$. In the other direction, the constant diagrams $M(\kappa)=M(r)$, $\kappa\in\mathcal{K}(r)$, associated to a right module

over M forms a right $\mathcal{K}$–module over R and we have an adjunction between this constant functor and the colimit construction.

Say that a right R–module M is equipped with a $\mathcal{K}$–structure if we have a right $\mathcal{K}$–module $M(\kappa)$ over the right $\mathcal{K}$–operad R such that $\mathrm{colim}_{\kappa\in\mathcal{K}(r)} M(\kappa) \xrightarrow{\simeq} M(r)$ for the adjoint morphism of an embedding of the right $\mathcal{K}$–module $M(\kappa)$ into the constant object defined by M.

4 The structure of iterated bar modules

In the next section, we prove that the twisting homomorphism of the n–fold bar module $\partial_\epsilon\colon T^n\circ\mathsf{E}\to T^n\circ\mathsf{E}$ factors through $T^n\circ\mathsf{E}_n\subset T^n\circ\mathsf{E}$ when E is a $\mathcal{K}$–cell E_∞–operad, and we conclude from this verification that the n–fold bar module $B^n_{\mathsf{E}} = (T^n_{\mathsf{E}}, \partial_\epsilon)$ is defined over E_n.

To reach this objective, we study the iterated bar module of the commutative operad. First, we prove that B^n_{C} is equipped with a $\mathcal{K}$–structure over the constant $\mathcal{K}$–operad C. For this purpose, we essentially have to check the definition of the iterated bar complex $B^n_{\mathsf{C}} = \mathrm{B}^n(\mathsf{C})$.

4.1 The bar complex in symmetric modules Let M be any Σ_*–module. The definition of the tensor product of Σ_*–modules in Section 0.3 gives an expansion of the form

$$\mathrm{T}^c(\Sigma M)(\underline{\mathrm{e}}) = \bigoplus_{\substack{\underline{\mathrm{e}}_1\amalg\cdots\amalg\underline{\mathrm{e}}_r=\underline{\mathrm{e}}\\ r\geq 1}} \Sigma M(\underline{\mathrm{e}}_1)\otimes\cdots\otimes\Sigma M(\underline{\mathrm{e}}_r)$$

for the tensor coalgebra $\mathrm{T}^c(\Sigma M)$. The sum ranges over all integers $r\in\mathbb{N}^*$ and all partitions $\underline{\mathrm{e}}_1\amalg\cdots\amalg\underline{\mathrm{e}}_r=\underline{\mathrm{e}}$. If we assume that M is connected ($M(0)=0$), then the sum ranges over partitions $\underline{\mathrm{e}}_1\amalg\cdots\amalg\underline{\mathrm{e}}_r=\underline{\mathrm{e}}$ such that $\underline{\mathrm{e}}_i\neq\varnothing$, for $i=1,\ldots,r$.

Suppose A is a commutative algebra in the category of Σ_*–modules. The bar differential $\partial\colon \mathrm{T}^c(\Sigma A)\to\mathrm{T}^c(\Sigma A)$ and the shuffle product $\mu\colon \mathrm{T}^c(\Sigma A)\to\mathrm{T}^c(\Sigma A)$ have a transparent representation in terms of this expansion.

The product of A is defined by a collection of morphisms $\mu\colon A(\underline{\mathrm{e}})\otimes A(\underline{\mathrm{f}})\to A(\underline{\mathrm{e}}\amalg\underline{\mathrm{f}})$. The bar differential has a component

$$\partial\colon \Sigma A(\underline{\mathrm{e}}_1)\otimes\cdots\otimes\Sigma A(\underline{\mathrm{e}}_i)\otimes\Sigma A(\underline{\mathrm{e}}_{i+1})\otimes\cdots\otimes\Sigma A(\underline{\mathrm{e}}_r)\\ \to \Sigma A(\underline{\mathrm{e}}_1)\otimes\cdots\otimes\Sigma A(\underline{\mathrm{e}}_i\amalg\underline{\mathrm{e}}_{i+1})\otimes\cdots\otimes\Sigma A(\underline{\mathrm{e}}_r)$$

induced by the product $\mu\colon A(\underline{\mathrm{e}}_i)\otimes A(\underline{\mathrm{e}}_{i+1})\to A(\underline{\mathrm{e}}_i\amalg\underline{\mathrm{e}}_{i+1})$, together with the usual sign, for each merging $\underline{\mathrm{e}}_i,\underline{\mathrm{e}}_{i+1}\mapsto\underline{\mathrm{e}}_i\amalg\underline{\mathrm{e}}_{i+1}$.

The shuffle product $\mu\colon \mathrm{T}^c(\Sigma A)(\underline{\mathrm{e}}) \otimes \mathrm{T}^c(\Sigma A)(\underline{\mathrm{f}}) \to \mathrm{T}^c(\Sigma A)(\underline{\mathrm{e}} \amalg \underline{\mathrm{f}})$ has a component

$$\mu\colon (\Sigma A(\underline{\mathrm{e}}_1) \otimes \cdots \otimes \Sigma A(\underline{\mathrm{e}}_r)) \otimes (\Sigma A(\underline{\mathrm{f}}_1) \otimes \cdots \otimes \Sigma A(\underline{\mathrm{f}}_s)) \to \Sigma A(\underline{\mathrm{g}}_1) \otimes \cdots \otimes \Sigma A(\underline{\mathrm{g}}_{r+s})$$

for each $\underline{\mathrm{g}}_1 \amalg \cdots \amalg \underline{\mathrm{g}}_{r+s} = \underline{\mathrm{e}} \amalg \underline{\mathrm{f}}$ given by a shuffle of the partitions $\underline{\mathrm{e}}_1 \amalg \cdots \amalg \underline{\mathrm{e}}_r = \underline{\mathrm{e}}$ and $\underline{\mathrm{f}}_1 \amalg \cdots \amalg \underline{\mathrm{f}}_s = \underline{\mathrm{f}}$.

4.2 The $\mathcal{K}$–structure of the iterated bar module: The generating symmetric module The purpose of the construction of this paragraph and of the next one is to prove that a composite $T^n \circ \mathsf{P}$, where P is a $\mathcal{K}$–operad, inherits a $\mathcal{K}$–structure. To begin with, we define a $\mathcal{K}(\underline{\mathrm{e}})$–diagram $T^n(\kappa) \subset T^n(\underline{\mathrm{e}})$ such that $T^n(\underline{\mathrm{e}}) = \operatorname{colim}_{\kappa \in \mathcal{K}(\underline{\mathrm{e}})} T^n(\kappa)$, for every finite set $\underline{\mathrm{e}}$.

Let $\kappa = (\mu, \sigma) \in \mathcal{K}(\underline{\mathrm{e}})$. Recall that T^n is defined as the iterated tensor coalgebra $T^n = (\mathrm{T}^c\ \Sigma)^n(\mathrm{I}) = \mathrm{T}^c(\Sigma T^{n-1})$. The submodule $T^n(\kappa) \subset T^n(\underline{\mathrm{e}})$ is defined by induction on n. For this purpose, we use the expansion of the tensor coalgebra

$$T^n(\underline{\mathrm{e}}) = \mathrm{T}^c(\Sigma T^{n-1})(\underline{\mathrm{e}}) = \bigoplus_{\substack{\underline{\mathrm{e}}_1 \amalg \cdots \amalg \underline{\mathrm{e}}_r = \underline{\mathrm{e}} \\ r \geq 1}} T^{n-1}(\underline{\mathrm{e}}_1) \otimes \cdots \otimes T^{n-1}(\underline{\mathrm{e}}_r)$$

(we omit marking suspensions to simplify the writing). A tensor $\xi = x_1 \otimes \cdots \otimes x_r \in T^{n-1}(\underline{\mathrm{e}}_1) \otimes \cdots \otimes T^{n-1}(\underline{\mathrm{e}}_r)$ belongs to the submodule $T^n(\kappa) \subset T^n$ when:

(1) each factor $x_i \in T^{n-1}$ satisfies $x_i \in T^n(\kappa|_{\underline{\mathrm{e}}_i})$, where $\kappa|_{\underline{\mathrm{e}}_i}$ is the restriction of the complete graph κ to the subset $\underline{\mathrm{e}}_i \subset \underline{\mathrm{e}}$;

(2) the indices $e, f \in \underline{\mathrm{e}}$ such that $\mu_{ef} < n-1$ belong to a same component $\underline{\mathrm{e}}_i$ of the partition $\underline{\mathrm{e}} = \underline{\mathrm{e}}_1 \amalg \cdots \amalg \underline{\mathrm{e}}_r$;

(3) for indices $e, f \in \underline{\mathrm{e}}$ in separate components $e \in \underline{\mathrm{e}}_i$, $f \in \underline{\mathrm{e}}_j$, $i \neq j$, and such that $\mu_{ef} = n-1$, the ordering $\sigma|_{ef}$ of the edge $\{e, f\}$ agrees with the order of $\{\underline{\mathrm{e}}_i, \underline{\mathrm{e}}_j\}$ in the decomposition $\underline{\mathrm{e}} = \underline{\mathrm{e}}_1 \amalg \cdots \amalg \underline{\mathrm{e}}_n$.

We clearly have $\kappa < \lambda \Rightarrow (T^n)(\kappa) \subset T^n(\lambda)$.

Note that every element $\xi \in T^n(\underline{\mathrm{e}})$ belongs to a component $T^n(\mu, \sigma)$ such that $\max(\mu_{ef}) < n$ for all e, f. Moreover, we have an equality $T^n(\underline{\mathrm{e}}) = T^n(\kappa)$, for κ sufficiently large, from which we deduce the identity $\operatorname{colim}_{\kappa \in \mathcal{K}(\underline{\mathrm{e}})} T^n(\kappa) = T^n(\underline{\mathrm{e}})$. The next observation follows from similar easy verifications:

4.3 Observation *Every element ξ of the generating collection $G^n(r) \subset T^n(r)$ belongs to a component $(T^n)(\mu, \sigma)$ such that $\max(\mu_{ef}) < n$ and $\sigma = \mathrm{id}$.*

If we have $\xi \in G^n \cap T^n(\kappa)$ for some $\kappa \in \mathcal{K}(r)$, then there is an element of the form (μ, id) such that $(\mu, \mathrm{id}) \leq \kappa$ and $\xi \in G^n \cap T^n(\mu, \mathrm{id})$.

The verification of this observation is straightforward from the inductive definition of $G^n(r) \subset T^n(r)$ in Proposition 2.8 and from the inductive definition of the $\mathcal{K}(r)$–diagrams $T^n(\kappa)$.

4.4 The $\mathcal{K}$–structure of the iterated bar module: The underlying free module Suppose now that P is a $\mathcal{K}$–operad. For each $m \in \mathbb{N}$, we have a $\mathcal{K}(m)$–diagram formed by the dg-modules $(T^n \circ \mathsf{P})(\kappa)$ spanned by formal composites $\xi(p_1, \ldots, p_r) \in T^n \circ \mathsf{P}(m)$, such that

$$\theta(\lambda_1, \ldots, \lambda_r) \leq \kappa,$$

where $\xi \in T^n(\theta)$ and $p_1 \in \mathsf{P}(\lambda_1), \ldots, p_r \in \mathsf{P}(\lambda_r)$. One can easily check that the objects $(T^n \circ \mathsf{P})(\kappa)$ inherits the structure of a right $\mathcal{K}$–module over P so that $\operatorname{colim}_{\kappa \in \mathcal{K}(r)}(T^n \circ \mathsf{P})(\kappa) = (T^n \circ \mathsf{P})(r)$. Hence, we obtain that the free right P–module $T^n \circ \mathsf{P}$ admits a $\mathcal{K}$–structure.

For the moment, we only need to apply the definition of the object $(T^n \circ \mathsf{P})(\kappa)$ to the commutative operad $\mathsf{P} = \mathsf{C}$, viewed as a constant $\mathcal{K}$–operad. In this particular case, we have:

4.5 Proposition *The module $(T^n \circ \mathsf{C})(\mu, \sigma)$ is spanned by composites $\xi(p_1, \ldots, p_r) \in (T^n \circ \mathsf{C})(\underline{\mathsf{e}})$, $\xi \in T^n(\kappa)$, $p_1 \in \mathsf{C}(\underline{\mathsf{e}}_1), \ldots, p_r \in \mathsf{C}(\underline{\mathsf{e}}_r)$, such that*

$$\kappa((0, \sigma|_{\underline{\mathsf{e}}_1}), \ldots, (0, \sigma|_{\underline{\mathsf{e}}_r})) \leq (\mu, \sigma),$$

where 0 represents the null collection $0_{ef} \equiv 0$ and $\sigma|_{\underline{\mathsf{e}}_i}$ refers to the restriction of the ordering σ to the subset $\underline{\mathsf{e}}_i$.

Proof By definition, the object $(T^n \circ \mathsf{C})(\mu, \sigma) \subset T^n \circ \mathsf{C}$ is spanned by formal composites $\xi(p_1, \ldots, p_r)$, $\xi \in T^n(\kappa)$, $p_1 \in \mathsf{C}(\lambda_1), \ldots, p_r \in \mathsf{C}(\lambda_r)$ such that $\kappa(\lambda_1, \ldots, \lambda_r) \leq (\mu, \sigma)$. By Observation 3.3, we have $\kappa(\lambda_1, \ldots, \lambda_r) \leq (\mu, \sigma)$ if and only if $\lambda_i \leq (\mu|_{\underline{\mathsf{e}}_i}, \sigma|_{\underline{\mathsf{e}}_i})$, for every $i = 1, \ldots, r$, and

$$\kappa((\mu|_{\underline{\mathsf{e}}_1}, \sigma|_{\underline{\mathsf{e}}_1}), \ldots, (\mu|_{\underline{\mathsf{e}}_r}, \sigma|_{\underline{\mathsf{e}}_r})) \leq (\mu, \sigma).$$

Since we clearly have

$$(0, \sigma|_{\underline{\mathsf{e}}_i}) \leq (\mu|_{\underline{\mathsf{e}}_i}, \sigma|_{\underline{\mathsf{e}}_i}), \quad \mathsf{C}(0, \sigma|_{\underline{\mathsf{e}}_i}) = \mathsf{C}(\mu|_{\underline{\mathsf{e}}_i}, \sigma|_{\underline{\mathsf{e}}_i}), \quad \text{for every } i,$$

and $$\kappa((0, \sigma|_{\underline{\mathsf{e}}_1}), \ldots, (0, \sigma|_{\underline{\mathsf{e}}_r})) \leq \kappa((\mu|_{\underline{\mathsf{e}}_1}, \sigma|_{\underline{\mathsf{e}}_1}), \ldots, (\mu|_{\underline{\mathsf{e}}_r}, \sigma|_{\underline{\mathsf{e}}_r})) \leq (\mu, \sigma),$$

we obtain that $\xi(p_1, \ldots, p_r)$ is a composite of the form of the Proposition. □

The crucial property which allows us to factor the twisting homomorphism of iterated bar modules to E_n–operads is given by the next Proposition:

4.6 Proposition *The twisting homomorphism $\partial_\gamma\colon T^n\circ\mathsf{C}\to T^n\circ\mathsf{C}$ of the iterated bar module B^n_{C} satisfies $\partial_\gamma(T^n(\kappa))\subset(T^n\circ\mathsf{C})(\kappa)$, for every $\kappa\in\mathcal{K}$.*

Proof Recall that ∂_γ is determined by the bar coderivations $\partial\colon \mathrm{T}^c(\Sigma A)\to\mathrm{T}^c(\Sigma A)$ which occur at each level of the composite $T^n\circ\mathsf{C}=(\mathrm{T}^c\,\Sigma\ldots\mathrm{T}^c\,\Sigma)(\mathsf{C})$. We prove the implication $\xi\in T^n(\kappa)\Rightarrow\partial_\gamma(\xi)\in(T^n\circ\mathsf{C})(\kappa)$ by induction on $n\geq 1$.

For $n=1$, the claim is checked by a quick inspection of the formula of Section 4.1 applied to $A=\mathsf{C}$.

Suppose that the assertion of the Lemma holds for $n-1$, where we assume $n>1$. Let $\xi=x_1\otimes\cdots\otimes x_r\in T^{n-1}(\underline{\mathrm{e}}_1)\otimes\cdots\otimes T^{n-1}(\underline{\mathrm{e}}_r)$ be a tensor such that $\xi\in T^n(\kappa)$, for some $\kappa\in\mathcal{K}(\underline{\mathrm{e}})$. By definition, we have

$$\begin{aligned}\partial_\gamma(\xi)=&\sum_{i=1}^{r-1}\pm x_1\otimes\cdots\otimes\mu(x_i,x_{i+1})\otimes\cdots\otimes x_r\\&+\sum_{i=1}^{r}\pm x_1\otimes\cdots\otimes\partial_\gamma(x_i)\otimes\cdots\otimes x_r,\end{aligned}$$

where $\mu(x_i,x_{i+1})$ refers to the shuffle product of x_i and x_{i+1} in $T^{n-1}=\mathrm{T}^c(\Sigma T^{n-2})$ and $\partial_\gamma(x_i)$ is determined by the bar differential $\partial_\gamma\colon T^{n-1}\circ\mathsf{C}\to T^{n-1}\circ\mathsf{C}$ coming from lower iterations of the bar complex.

By induction, we have $\partial_\gamma(x_i)\in(T^{n-1}\circ\mathsf{C})(\kappa|_{\underline{\mathrm{e}}_i})$, from which we deduce readily $x_1\otimes\cdots\otimes\partial_\gamma(x_i)\otimes\cdots\otimes x_r\in(T^n\circ\mathsf{C})(\kappa)$.

For indices $e,f\in\underline{\mathrm{e}}$ such that $e\in\underline{\mathrm{e}}_i$ and $f\in\underline{\mathrm{e}}_{i+1}$, we have necessarily $\mu_{ef}\geq n-1$ according to the definition of Section 4.2. This condition ensures that the shuffle product $\mu(x_i,x_{i+1})\in\mathrm{T}^c(\Sigma T^{n-2})(\underline{\mathrm{e}}_i\amalg\underline{\mathrm{e}}_{i+1})$ belongs to the cell

$$(T^{n-1})(\kappa|\,\underline{\mathrm{e}}_i\amalg\underline{\mathrm{e}}_{i+1})\subset T^{n-1}(\underline{\mathrm{e}}_i\amalg\underline{\mathrm{e}}_{i+1})=\mathrm{T}^c(\Sigma T^{n-2})(\underline{\mathrm{e}}_i\amalg\underline{\mathrm{e}}_{i+1}),$$

because:

- the product $\mu(x_i,x_{i+1})$ consists by definition of shuffles of tensors
$$x_i=y_1\otimes\cdots\otimes y_p\in T^{n-2}(\underline{\mathrm{g}}_1)\otimes\cdots\otimes T^{n-2}(\underline{\mathrm{g}}_p)$$
and $$x_{i+1}=z_1\otimes\cdots\otimes z_q\in T^{n-2}(\underline{\mathrm{h}}_1)\otimes\cdots\otimes T^{n-2}(\underline{\mathrm{h}}_q),$$
where $\underline{\mathrm{e}}_i=\underline{\mathrm{g}}_1\amalg\cdots\amalg\underline{\mathrm{g}}_p$ and $\underline{\mathrm{e}}_{i+1}=\underline{\mathrm{h}}_1\amalg\cdots\amalg\underline{\mathrm{h}}_q$;
- only pairs of the form (y_j,z_k) can be permuted in $\mu(x_i,x_{i+1})$, but the condition $\mu_{ef}\geq n-1$ for elements $e\in\underline{\mathrm{g}}_j\subset\underline{\mathrm{e}}_i$ and $f\in\underline{\mathrm{h}}_k\subset\underline{\mathrm{e}}_{i+1}$ ensures that the order between these pairs does not matter in the relation $\mu(x_i,x_{i+1})\in(T^{n-1}\circ\mathsf{C})(\kappa|\,\underline{\mathrm{e}}_i\cup\underline{\mathrm{e}}_{i+1})$.

From this observation, we readily conclude, by an easy inspection of conditions of Section 4.2, that the tensor product $x_1 \otimes \cdots \otimes \mu(x_i, x_{i+1}) \otimes \cdots \otimes x_r$ belongs to $(T^n \circ \mathsf{C})(\kappa)$. Therefore, we obtain finally $\xi \in T^n(\kappa) \Rightarrow \partial_\gamma(\xi) \in (T^n \circ \mathsf{C})(\kappa)$, for every $n \geq 1$. □

5 The restriction of iterated bar modules to E_n–operads

The goal of this section is to prove that the twisting homomorphism of the iterated bar module $B^n_{\mathsf{E}} = (T^n \circ \mathsf{E}, \partial_\epsilon)$ factors through $T^n \circ \mathsf{E}_n \subset T^n \circ \mathsf{E}$ when E is a $\mathcal{K}$–cellular E_∞–operads. From this observation, we conclude that the n–fold bar complex $\mathsf{B}^n\colon {}_{\mathsf{E}}\mathcal{E} \to \mathcal{E}$ admits an extension to the category of E_n–algebras ${}_{\mathsf{E}_n}\mathcal{E}$.

To begin with, we briefly review the structure of a $\mathcal{K}$–operad E, because we put additional assumptions on E in order to simplify our construction. The Barratt–Eccles operad still gives an example of a $\mathcal{K}$–cellular E_∞–operad which fulfils all our requirements.

5.1 Assumptions on $\mathcal{K}$–cellular E_∞–operads Recall (see Section 3.6) that a $\mathcal{K}$–cellular E_∞–operad consists of an E_∞–operad E equipped with a $\mathcal{K}$–structure satisfying suitable conditions (K1)–(K2) so that the colimit

$$\mathsf{E}_n(r) = \operatorname{colim}_{\kappa \in \mathcal{K}_n(r)} \mathsf{E}(\kappa),$$

where $$\mathcal{K}_n(r) = \big\{(\mu, \sigma) \in \mathcal{K}(r) \text{ such that } \max(\mu_{ij}) < n\big\} \subset \mathcal{K}(r),$$

forms an E_n–operad.

In the construction of Section 2, we assume that E is any E_∞–operad equipped with a section $\iota\colon \mathsf{C}(\underline{\mathsf{e}}) \to \mathsf{E}(\underline{\mathsf{e}})$ of the augmentation $\epsilon\colon \mathsf{E}(\underline{\mathsf{e}}) \to \mathsf{C}(\underline{\mathsf{e}})$ and with a contracting chain homotopy $\nu\colon \mathsf{E}(\underline{\mathsf{e}}) \to \mathsf{E}(\underline{\mathsf{e}})$ such that $\epsilon\iota = \mathrm{id}$, $\epsilon\nu = 0$ and $\delta(\nu) = \mathrm{id} - \iota\epsilon$, for every finite set $\underline{\mathsf{e}}$ equipped with an ordering $\underline{\mathsf{e}} = \{e_1 < \cdots < e_r\}$. From now on, we assume that E is a $\mathcal{K}$–cellular E_∞–operad in the sense of Section 3.6 together with:

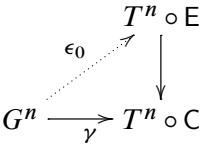
(E1) a section $\iota\colon \mathsf{C}(\underline{\mathsf{e}}) \to \mathsf{E}(\underline{\mathsf{e}})$ such that $\iota(\mathsf{C}(r)) \subset \mathsf{E}(0, \mathrm{id})$, for every $r \in \mathbb{N}$;

(E2) a chain-homotopy $\nu\colon \mathsf{E}(\underline{\mathsf{e}}) \to \mathsf{E}(\underline{\mathsf{e}})$ satisfying $\nu(\mathsf{E}(\kappa)) \subset \mathsf{E}(\kappa)$, for every $\kappa \in \mathcal{K}$ of the form $\kappa = (\mu, \mathrm{id})$, where we apply the definition of ι and ν to the standard ordinal $\underline{\mathsf{r}} = \{1 < \cdots < r\}$.

The section ι and the contracting chain-homotopy defined in Section 1.4 for the Barratt–Eccles operad satisfy these conditions (check definitions).

Under these assumptions (E1)–(E2), we obtain:

5.2 Lemma *The lifting of Theorem 2.9*



satisfies $\epsilon_0(G^n \cap T^n(\kappa)) \subset (T^n \circ \mathsf{E})(\kappa)$, *for every* $\kappa \in \mathcal{K}$ *of the form* $\kappa = (\mu, \mathrm{id})$.

Proof Recall that ϵ_0 is defined by a composite $\epsilon_0 = \widetilde{\tau} \cdot \gamma$.

By Proposition 4.6, we have $\partial_\gamma(T^n(\kappa)) \subset (T^n \circ \mathsf{C})(\kappa)$ for every $\kappa \in \mathcal{K}$. Thus, for an element of the form $\kappa = (\mu, \mathrm{id})$, the expansion of $\partial_\gamma(\xi)$, $\xi \in G^n \cap T^n(\kappa)$, consists by Proposition 4.5 of composites $x(p_1, \ldots, p_r) \in T^n \circ \mathsf{C}$, where $x \in (T^n)(\theta)$ and $p_1 \in \mathsf{C}(\underline{e}_1), \ldots, p_r \in \mathsf{C}(\underline{e}_r)$, for some θ such that $\theta((0, \mathrm{id}), \ldots, (0, \mathrm{id})) \leq \kappa$. By Σ_r–invariance of composites, we can moreover assume $x \in G^n \cap (T^n)(\theta)$.

We have by definition $\widetilde{\tau}(x(p_1, \ldots, p_r)) = x(\iota(p_1), \ldots, \iota(p_r))$ and our assumption on ι implies $\iota(p_i) \in \mathsf{E}(0, \mathrm{id})$. Therefore we obtain

$$\widetilde{\tau}(x(p_1, \ldots, p_r)) \in (T^n \circ \mathsf{C})(\theta((0, \mathrm{id}), \ldots, (0, \mathrm{id}))) \subset (T^n \circ \mathsf{C})(\kappa)$$

and the Lemma follows. □

5.3 Lemma *The homomorphism* $\epsilon_*\colon G^n \to T^n \circ \mathsf{E}$ *of Theorem 2.9 satisfies* $\epsilon_*(G^n \cap T^n(\kappa)) \subset (T^n \circ \mathsf{E})(\kappa)$, *for every element* $\kappa \in \mathcal{K}$ *of the form* $\kappa = (\mu, \mathrm{id})$.

The associated twisting homomorphism $\partial_\epsilon\colon T^n \circ \mathsf{E} \to T^n \circ \mathsf{E}$ *satisfies* $\partial_\epsilon((T^n \circ \mathsf{E})(\kappa)) \subset (T^n \circ \mathsf{E})(\kappa)$, *for every* $\kappa \in \mathcal{K}$.

Proof We have by definition $\epsilon_* = \sum_{m=0}^{\infty} \epsilon_m$, where $\epsilon_m = \sum_{p+q=m-1} \widetilde{\nu}(\epsilon_p \star \epsilon_q)$, for $m > 0$. We check that the assertions of the Lemma are satisfied by each term ϵ_m, $m \in \mathbb{N}$, of ϵ_*.

We have by definition $\epsilon_p \star \epsilon_q(\xi) = \partial_{\epsilon_p} \cdot \partial_{\epsilon_q}(\xi)$, where $\partial_{\epsilon_p}\colon T^n \circ \mathsf{E} \to T^n \circ \mathsf{E}$ is the homomorphism associated to ϵ_p. We assume by induction that $\partial_{\epsilon_p}((T^n \circ \mathsf{E})(\kappa)) \subset (T^n \circ \mathsf{E})(\kappa)$ for every element $\kappa \in \mathcal{K}$ and for every $p < m$.

Suppose κ is an element of the form $\kappa = (\mu, \mathrm{id})$. For $\xi \in G^n \cap T^n(\kappa)$, the expansion of $\partial_{\epsilon_p} \cdot \partial_{\epsilon_q}(\xi)$, consists by Observation 3.3 of composites $x(p_1, \ldots, p_r) \in T^n \circ \mathsf{E}$, where $x \in (T^n)(\theta)$ and $p_1 \in \mathsf{E}(\kappa|_{\underline{e}_1}), \ldots, p_r \in \mathsf{E}(\kappa|_{\underline{e}_r})$, for some $\theta \in \mathcal{K}(r)$ such that $\theta(\kappa|_{\underline{e}_1}, \ldots, \kappa|_{\underline{e}_r}) \leq \kappa$. By Σ_r–invariance of composites, we can moreover assume $x \in G^n \cap (T^n)(\theta)$.

We have by definition

$$\widetilde{\nu}(x(p_1,\ldots,p_r)) = \sum_{i=1}^{r} \pm x(\iota\epsilon(p_1),\ldots,\iota\epsilon(p_{i-1}),\nu(p_i),p_{i+1},\ldots,p_r).$$

Since $\iota(\mathsf{C}(r)) \subset \mathsf{E}(0,\mathrm{id}\,|_{\underline{e}_j})$, we have $\iota\epsilon(p_j) \in \mathsf{E}(0,\mathrm{id}) \subset \mathsf{E}(\kappa|_{\underline{e}_j})$, for $j = 1,\ldots,i-1$. By assumption, we have moreover $p_i \in \mathsf{E}(\kappa|_{\underline{e}_i}) \Rightarrow \nu(p_i) \in \mathsf{E}(\kappa|_{\underline{e}_i})$. Hence we obtain

$$\widetilde{\nu}(x(p_1,\ldots,p_r)) \in (T^n \circ \mathsf{E})(\theta(\kappa|_{\underline{e}_1},\ldots,\kappa|_{\underline{e}_r})) \subset (T^n \circ \mathsf{E})(\kappa),$$

from which we conclude $\xi \in G^n \cap T^n(\kappa) \Rightarrow \epsilon_m(\xi) \in (T^n \circ \mathsf{E})(\kappa)$.

The homomorphism $\partial_{\epsilon_m}\colon T^n \circ \mathsf{E} \to T^n \circ \mathsf{E}$ is defined on composites $x(p_1,\ldots,p_r) \in T^n \circ \mathsf{E}$ such that $x \in G^n(r)$ by $\partial_{\epsilon_m}(x(p_1,\ldots,p_r)) = \epsilon_m(x)(p_1,\ldots,p_r)$.

Suppose $x \in G^n \cap (T^n)(\theta)$ and $p_1 \in \mathsf{E}(\lambda_1),\ldots,p_r \in \mathsf{E}(\lambda_r)$. By Observation 4.3, we can assume that θ has the form $\theta = (\mu,\mathrm{id})$. We have then $\epsilon_m(x) \in (T^n \circ \mathsf{E})(\theta)$ and

$$\epsilon_m(x)(p_1,\ldots,p_r)) \in (T^n \circ \mathsf{E})(\theta(\lambda_1,\ldots,\lambda_r)).$$

From this assertion, we conclude that $\partial_{\epsilon_m}((T^n \circ \mathsf{E})(\kappa)) \subset (T^n \circ \mathsf{E})(\kappa)$ for every $\kappa \in \mathcal{K}$.

This verification completes the proof of the Lemma. □

5.4 Theorem *In the setting of Section 5.1, the twisting homomorphism $\partial_\epsilon\colon T^n \circ \mathsf{E} \to T^n \circ \mathsf{E}$ which arises from the definition of the iterated bar module $B^n_{\mathsf{E}} = (T^n \circ \mathsf{E}, \partial_\epsilon)$ in Sections 2.4–2.5 satisfies $\partial_\epsilon(T^n) \subset T^n \circ \mathsf{E}_n$ and admits a restriction to $T^n \circ \mathsf{E}_n \subset T^n \circ \mathsf{E}$.*

Thus we have a quasifree right E_n–module $B^n_{\mathsf{E}_n} = (T^n \circ \mathsf{E}_n, \partial_\epsilon)$ defined by the restriction of ∂_ϵ to $T^n \circ \mathsf{E}_n$ and this quasifree module satisfies the relation $B^n_{\mathsf{E}_n} \circ_{\mathsf{E}_n} \mathsf{E} \simeq B^n_{\mathsf{E}}$.

Proof Recall (see Section 4.2) that any element $\xi \in T^n$ belongs to a submodule $T^n(\kappa) \subset T^n$ such that $\kappa \in \mathcal{K}_n$ and $\mathsf{E}_n(r) = \mathrm{colim}_{\kappa \in \mathcal{K}_n(r)} \mathsf{E}(\kappa)$. Therefore Lemma 5.3 implies $\partial_\epsilon(\xi) \in T^n \circ \mathsf{E}_n$, for every $\xi \in T^n$, and $\partial_\epsilon(\xi(p_1,\ldots,p_r)) \in T^n \circ \mathsf{E}_n$ for every composite $\xi(p_1,\ldots,p_r)$ such that $p_1,\ldots,p_r \in \mathsf{E}_n$.

The relation $B^n_{\mathsf{E}_n} \circ_{\mathsf{E}_n} \mathsf{E} \simeq B^n_{\mathsf{E}}$ is an immediate consequence of the identity $(T^n \circ \mathsf{E}_n) \circ_{\mathsf{E}_n} \mathsf{E} \simeq T^n \circ \mathsf{E}$ for a free module. □

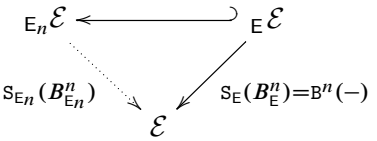
The quasifree right E_n–module $B^n_{\mathsf{E}_n} = (T^n \circ \mathsf{E}_n, \partial_\epsilon)$ determines a functor

$$\mathbb{S}_{\mathsf{E}_n}(B^n_{\mathsf{E}_n})\colon {}_{\mathsf{E}_n}\mathcal{E} \to \mathcal{E}$$

of the form

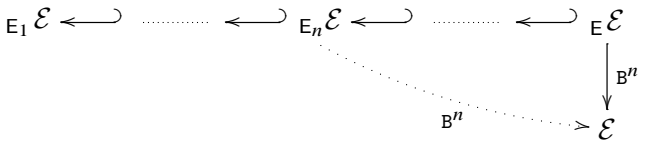
$$\mathbb{S}_{\mathsf{E}_n}(B^n_{\mathsf{E}_n}, A) = (\mathbb{S}_{\mathsf{E}_n}(T^n \circ \mathsf{E}_n, A), \partial_\epsilon) = ((\mathrm{T}^c\ \Sigma)^n(A), \partial_\epsilon),$$

for every E_n–algebra A. Thus we obtain that $\mathtt{S}_{\mathsf{E}_n}(B^n_{\mathsf{E}_n}, A)$ is a twisted dg-module defined by the same underlying functor as the n–fold tensor coalgebra $(\mathtt{T}^c\,\Sigma)^n(A)$ together with the twisting homomorphism yielded by the construction of Theorem 5.4 at the module level. The relation $B^n_{\mathsf{E}_n} \circ_{\mathsf{E}_n} \mathsf{E} \simeq B^n_{\mathsf{E}}$ implies that the diagram



commutes. To summarize:

5.5 Theorem *If the operad E fulfils the requirements of Section 5.1, then the n–fold bar complex $\mathtt{B}^n\colon {}_{\mathsf{E}}\mathcal{E} \to \mathcal{E}$ admits an extension to the category of E_n–algebras*



given by a construction of the form

$$\mathtt{B}^n(A) = ((\mathtt{T}^c\,\Sigma)^n(A), \partial_\epsilon),$$

where the twisting homomorphism ∂_ϵ, yielded by the result of Theorem 5.4, arises from a restriction of the twisting homomorphism given for E–algebras in Theorem 2.9. □

Iterated bar complexes and homology theories

The goal of this part is to prove that the n–fold desuspension of the n–fold bar complex $\Sigma^{-n}\,\mathtt{B}^n(A)$ determines the E_n–homology $\mathtt{H}^{\mathsf{E}_n}_*(A)$, for every $n \in \mathbb{N}$, including $n = \infty$. The infinite bar complex $\Sigma^{-\infty}\,\mathtt{B}^\infty(A)$ is just defined by the colimit of the complexes $\Sigma^{-n}\,\mathtt{B}^n(A)$ over the suspension morphisms $\sigma\colon \Sigma^{1-n}\,\mathtt{B}^{n-1}(A) \to \Sigma^{-n}\,\mathtt{B}^n(A)$.

The preliminary section (Section 6) is devoted to the computation of the homology of the iterated bar complex of usual commutative algebras, like trivial algebras and symmetric algebras. The next section (Section 7) is devoted to a short review of the definition of the homology theory $\mathtt{H}^{\mathsf{R}}_*(-)$ associated to an operad R. In the core section (Section 8), we determine the homology of the bar module $\Sigma^{-n}B^n_{\mathsf{E}_n}$ and we use the result to prove that $\Sigma^{-n}\,\mathtt{B}^n(A)$ determines the E_n–homology $\mathtt{H}^{\mathsf{E}_n}_*(A)$ in the case $n < \infty$. In the last section of this part (Section 9), we prove that $\Sigma^{-n}\,\mathtt{B}^n(A)$ determines the E_n–homology $\mathtt{H}^{\mathsf{E}_n}_*(A)$ in the case $n = \infty$ too.

To obtain our result, we use that the homology $\mathrm{H}^{\mathsf{R}}_*(-)$ can be represented by a generalized $\mathtt{Tor}$–functor $\mathtt{Tor}^{\mathsf{R}}_*(\mathsf{I}, -)$. The idea is to prove the relation

$$\mathrm{H}_*(\Sigma^{-n}\,\mathrm{B}^n(A)) = \mathtt{Tor}^{\mathsf{E}_n}_*(\mathsf{I}, A) = \mathrm{H}^{\mathsf{E}_n}_*(A)$$

by checking that the module $\Sigma^{-n}B^n_{\mathsf{E}_n}$, which represents the n–fold bar complex $\Sigma^{-n}\,\mathrm{B}^n(A)$, forms a cofibrant replacement of the composition unit I in the category of right E_n–modules.

Koszul complexes of operads In our arguments (Proposition 6.9 and Proposition 8.18), we make appear operadic Koszul complexes $\mathtt{K}(\mathsf{I}, \mathsf{P}, \mathsf{P})$ associated to the commutative and Lie operads $\mathsf{P} = \mathsf{C}, \mathsf{L}$. Recall simply that the Koszul complex $\mathtt{K}(\mathsf{I}, \mathsf{P}, \mathsf{P})$ associated to a Koszul operad P is an acyclic complex of right P–modules. Let $\mathtt{K}_{\mathsf{P}}(-) = \mathtt{S}_{\mathsf{P}}(\mathtt{K}(\mathsf{I}, \mathsf{P}, \mathsf{P}), -)$ be the functor on P–algebras determined by this right P–module $\mathtt{K}(\mathsf{I}, \mathsf{P}, \mathsf{P})$. In the case $\mathsf{P} = \mathsf{C}$, the functor $\mathtt{K}_{\mathsf{C}}\colon {}_{\mathsf{C}}\mathcal{E} \to \mathcal{E}$ is identified with the Harrison complex of commutative dg-algebras (see Section 6.8 for short recollections and [20, Section 6] for more detailed explanations). In the case $\mathsf{P} = \mathsf{L}$, the functor $\mathtt{K}_{\mathsf{L}}\colon {}_{\mathsf{L}}\mathcal{E} \to \mathcal{E}$ is identified with the usual Chevalley–Eilenberg complex of Lie dg-algebras (see [20, Section 6]). In this paper, we use the extension of these complexes to P–algebras in Σ_*–modules and, in order to identify the object $\mathtt{K}(\mathsf{I}, \mathsf{P}, \mathsf{P})$, we apply the relation

$$\mathtt{K}(\mathsf{I}, \mathsf{P}, \mathsf{P}) = \mathtt{S}_{\mathsf{P}}(\mathtt{K}(\mathsf{I}, \mathsf{P}, \mathsf{P}), \mathsf{P}) = \mathtt{K}_{\mathsf{P}}(\mathsf{P})$$

where the operad P is viewed as an algebra over itself in the category of Σ_*–modules (see explanations of Section 0.11).

Recall that we only deal with generalizations of modules of symmetric tensors. For us, the Chevalley–Eilenberg complex is defined by a symmetric algebra $\mathtt{S}(\Sigma G)$ on a suspension of Lie algebras G, as in the context of dg-modules over a field of characteristic zero, but we apply this construction to Σ_*–modules over any ring (see again Section 0.11 for more explanations). In Section 6.8, we apply a similar convention for the definition of the Harrison complex.

The operadic suspension of symmetric modules In this part, we use a functor $\Lambda\colon \mathcal{M} \to \mathcal{M}$ such that

$$\mathtt{S}(\Lambda M, \Sigma E) \simeq \Sigma\,\mathtt{S}(M, E),$$

for every Σ_*–module $M \in \mathcal{M}$. We call the Σ_*–module ΛM the operadic suspension of M. This Σ_*–module ΛM is defined in arity r by the tensor product

$$\Lambda M(r) = \Sigma^{1-r} M(r) \otimes \mathrm{sgn}_r,$$

where sgn_r refers to the signature representation of Σ_r (see [25]).

The operadic suspension of an operad $\Lambda\,\mathsf{P}$ inherits a natural operad structure and the suspension $\Sigma\colon E \mapsto \Sigma E$ induces an isomorphism from the category of P–algebras to the category of $\Lambda\,\mathsf{P}$–algebras (see [25] for details). Note that the operadic suspension $\Lambda\,\mathsf{P}$ has nothing to do with the suspension of the semimodel category of operads.

We have by definition $\Sigma\,\mathsf{P}(E) = \Lambda\,\mathsf{P}(\Sigma E)$ for a free P–algebra $A = \mathsf{P}(E)$. In the paper, we often use the equivalent relation $\mathsf{P}(\Sigma E) = \Sigma\Lambda^{-1}\,\mathsf{P}(E)$ which makes appear the operadic desuspension of P.

6 Prelude: Iterated bar complexes of usual commutative algebras

The homology of the iterated bar complexes of usual commutative algebras (exterior, polynomial, divided power, ...) is determined in [13] in the context of dg-modules. The purpose of this section is to review these classical homology calculations in the context of Σ_*–modules. To be specific, we determine the homology $\mathtt{H}_*(\mathtt{B}^n(A))$ of a free commutative algebra $A = \mathsf{C}(M)$ and of a trivial algebra $A = M$.

For our needs, we consider, throughout this section, a Σ_*–module M such that:

(M1) we have $M(0) = 0$ (thus M is connected as a Σ_*–module);

(M2) the differential of M is zero (in other words M defines a Σ_*–object in the category of graded modules);

(M3) and each component $M(r)$ is projective as a $\Bbbk$–module (but we do not assume that M is projective as a Σ_*–module).

6.1 Free commutative algebras in symmetric modules The free (nonunital) commutative algebra $\mathsf{C}(M)$ is identified with the (nonunital) symmetric algebra

$$\mathsf{C}(M) = \bigoplus_{r=1}^{\infty} (\mathsf{C}(r) \otimes M^{\otimes r})_{\Sigma_r} = \bigoplus_{r=1}^{\infty} (M^{\otimes r})_{\Sigma_r}.$$

The canonical morphism $\eta\colon M \to \mathsf{C}(M)$ identifies M with the summand $M \subset \bigoplus_{r=1}^{\infty}(M^{\otimes r})_{\Sigma_r}$ of $\mathsf{C}(M)$.

The element of $\mathsf{C}(M)$ represented by the tensor $x_1 \otimes \cdots \otimes x_r \in M^{\otimes r}$ is usually denoted by $x_1 \cdots x_r$ since this tensor represents the product of $x_1, \ldots, x_r \in M$ in $\mathsf{C}(M)$.

For a free commutative algebra $\mathsf{C}(M)$, the composite of the canonical morphism $\eta\colon M \to \mathsf{C}(M)$ with the suspension $\sigma\colon \Sigma\,\mathsf{C}(M) \to \mathtt{B}(\mathsf{C}(M))$ defines a natural morphism of Σ_*–modules $\sigma\colon \Sigma M \to \mathtt{B}(\mathsf{C}(M))$. We form the morphism of commutative algebras

$$\nabla\colon \mathsf{C}(\Sigma M) \to \mathtt{B}(\mathsf{C}(M))$$

such that $\nabla|_{\Sigma M} = \sigma$.

6.2 Proposition *The morphism* $\nabla\colon \mathsf{C}(\Sigma M) \to \mathtt{B}(\mathsf{C}(M))$ *defines a weak-equivalence of commutative algebras in* Σ_**–modules whenever the* Σ_**–module* M *satisfies the requirements (*M1*)–(*M3*).*

The result becomes much more complicated when $M(0) \neq 0$: the module $\mathtt{B}(\mathsf{C}(M))$ carries higher homological operations, like divided powers (see [13]), and the homology of $\mathtt{B}(\mathsf{C}(M))$ does not reduce to the free commutative algebra $\mathsf{C}(\Sigma M)$.

Proof We adapt classical arguments (given for instance in [13]).

Recall that a nonunital commutative algebra A is equivalent to a unital augmented commutative algebra such that $A_+ = \mathbf{1} \oplus A$. In this proof (and in this proof only), we use a free commutative algebra with unit $\mathsf{C}_+(M) = \mathbf{1} \oplus \mathsf{C}(M)$ and a unital version of the bar complex $B_+(A)$ such that $B_+(A) = \mathbf{1} \oplus \mathtt{B}(A)$.

We consider the acyclic bar complex $\mathtt{B}(A_+, A, \mathbf{1})$ formed by the tensor products

$$\mathtt{B}(A_+, A, \mathbf{1}) = A_+ \otimes B_+(A) = \bigoplus_{d=0}^{\infty} A_+ \otimes \Sigma A^{\otimes d}$$

together with the differential such that

$$\partial(a_0 \otimes a_1 \otimes \cdots \otimes a_d) = \pm \sum_{i=0}^{d-1} a_0 \otimes \cdots \otimes \mu(a_i, a_{i+1}) \otimes \cdots \otimes a_d,$$

where $\mu\colon A \otimes A \to A$ refers to the product of A and its extension to A_+.

The tensor product of commutative algebras $\mathsf{C}_+(M) \otimes \mathsf{C}_+(\Sigma M)$ inherits a natural commutative algebra structure. Let $\partial\colon \mathsf{C}_+(M) \otimes \mathsf{C}_+(\Sigma M) \to \mathsf{C}_+(M) \otimes \mathsf{C}_+(\Sigma M)$ be the unique derivation of commutative algebras of degree -1 which vanishes on $\mathsf{C}_+(M)$ and extends the canonical homomorphism $\sigma\colon \Sigma M \to M$ on $\mathsf{C}_+(\Sigma M)$. We have $\partial^2 = 0$ so that the derivation ∂ provides the commutative algebra $\mathsf{C}_+(M) \otimes \mathsf{C}_+(\Sigma M)$ with a new dg-structure. The twisted object $K = (\mathsf{C}_+(M) \otimes \mathsf{C}_+(\Sigma M), \partial)$ is an analogue in Σ_*–modules of the usual Koszul complex.

For $r \in \mathbb{N}$ and any $N \in \mathcal{M}$, we set $\mathsf{C}_r(N) = (N^{\otimes r})_{\Sigma_r}$. The differential of the Koszul complex satisfies $\partial(\mathsf{C}_p(M) \otimes \mathsf{C}_q(\Sigma M)) \subset \mathsf{C}_{p+1}(M) \otimes \mathsf{C}_{q-1}(\Sigma M)$. We define (nonequivariant) maps $\nu\colon \mathsf{C}_p(M) \otimes \mathsf{C}_q(\Sigma M) \to \mathsf{C}_{p-1}(M) \otimes \mathsf{C}_{q+1}(\Sigma M)$ such that $\delta\nu + \nu\delta = \mathrm{id}$.

We have an identity

$$(\mathsf{C}_p(M)\otimes\mathsf{C}_q(\Sigma M))(r) = \bigoplus_{\underline{\mathrm{e}}_*\amalg\underline{\mathrm{f}}_*=\{1,\dots,r\}} (M(\underline{\mathrm{e}}_1)\otimes\cdots\otimes M(\underline{\mathrm{e}}_p))\otimes(\Sigma M(\underline{\mathrm{f}}_1)\otimes\cdots\otimes\Sigma M(\underline{\mathrm{f}}_q))/\equiv,$$

where the sum is divided out by the action of permutations $(u,v)\in\Sigma_p\times\Sigma_q$. For a tensor

$$\xi=(x_1\cdots x_p)\otimes(y_1\cdots y_q)\in(M(\underline{\mathrm{e}}_1)\otimes\cdots\otimes M(\underline{\mathrm{e}}_p))\otimes(\Sigma M(\underline{\mathrm{f}}_1)\otimes\cdots\otimes\Sigma M(\underline{\mathrm{f}}_q)),$$

we set

$$\nu(\xi)=\begin{cases}\pm(x_1\cdots\widehat{x_i}\cdots x_p)\otimes(x_i\cdot y_1\cdots y_q) & \text{if } 1\in\underline{\mathrm{e}}_i \text{ for some } i,\\ 0 & \text{otherwise.}\end{cases}$$

The relation $\delta\nu+\nu\delta=\mathrm{id}$ follows from an easy verification. The assumption $M(0)=0$ is used at this point, because our chain-homotopy does not work when $\{1,\dots,r\}$ is reduced to the empty set.

The tensor product $\mathsf{C}_+(M)\otimes\nabla$, where ∇ is the morphism of the Lemma, defines a morphism of commutative algebras

$$(\mathsf{C}_+(M)\otimes\mathsf{C}_+(\Sigma M),\partial)\xrightarrow{\mathsf{C}_+(M)\otimes\nabla}(\mathsf{C}_+(M)\otimes\mathtt{T}^c_+(\Sigma\,\mathsf{C}(M)),\partial) = \mathtt{B}(\mathsf{C}_+(M),\mathsf{C}(M),\mathbf{1}).$$

Both terms form quasifree resolutions of the unit object $\mathbf{1}$ in the category of left $\mathsf{C}_+(M)$–modules. By a standard result of homological algebra, we conclude that the morphism

$$\mathsf{C}_+(\Sigma M)\xrightarrow{\nabla}(\mathtt{T}^c_+(\Sigma\,\mathsf{C}(M)),\partial)=B_+(\mathsf{C}(M))$$

defines a weak-equivalence.

Use the natural splitting $A_+=\mathbf{1}\oplus A$ to get the result of the Proposition. □

Proposition 6.2 implies by an immediate induction:

6.3 Proposition *We have an isomorphism of commutative algebras in Σ_*–modules*

$$\nabla\colon\mathsf{C}(\Sigma^n M)\xrightarrow{\simeq}\mathtt{H}_*(\mathtt{B}^n(\mathsf{C}(M))),$$

for every $n\in\mathbb{N}$, whenever the Σ_*–module M satisfies the requirements (M1)–(M3). □

6.4 Hopf algebras Recall that the shuffle product commutes with the deconcatenation product of $\mathrm{T}^c(\Sigma M)$, so that $\mathrm{T}^c(\Sigma M)$ forms a commutative Hopf algebra (for any Σ_*–module M), and so does the bar complex $\mathrm{B}(A) = (\mathrm{T}^c(\Sigma A), \partial)$ when A is a commutative algebra (see Section 1.14).

The free commutative algebra $\mathrm{C}(\Sigma M)$ inherits a Hopf algebra structure as well. The diagonal $\Delta\colon \mathrm{C}(\Sigma M) \to \mathrm{C}(\Sigma M) \otimes \mathrm{C}(\Sigma M)$ of a product $x_1 \cdots x_r \in \mathrm{C}(\Sigma M)$ can be defined by the explicit formula

$$\Delta(x_1 \cdots x_r) = \sum_{\substack{p+q=r \\ p,q>0}} \left(\sum_{w \in \mathrm{shuffle}(p,q)} \pm (x_{w(1)} \cdots x_{w(p)}) \otimes (x_{w(p+1)} \cdots x_{w(p+q)}) \right),$$

where the inner sum ranges over the set of (p,q)–shuffles. Note that we assume $p, q > 0$ since we have removed units from the commutative algebra $\mathrm{C}(\Sigma M)$. The generators $\xi \in \Sigma M$ are primitive in the sense that $\Delta(\xi) = 0$ (since we do not consider unit).

The result of Proposition 6.2 can be improved to:

6.5 Proposition *The morphism of Proposition 6.2*

$$\nabla\colon \mathrm{C}(\Sigma M) \to \mathrm{B}(\mathrm{C}(M))$$

commutes with diagonals and yields an isomorphism

$$\nabla_*\colon \mathrm{C}(\Sigma M) \xrightarrow{\simeq} \mathrm{H}_*(\mathrm{B}(\mathrm{C}(M)))$$

in the category of Hopf algebras in Σ_*–modules.

Proof The image of an element $\xi \in \Sigma M$ under the suspension $\sigma\colon \Sigma M \to \mathrm{B}(\mathrm{C}(M))$ defines clearly a primitive element in $\mathrm{B}(\mathrm{C}(M))$. Hence ∇ preserves the diagonal of generators of the commutative algebra $\mathrm{C}(\Sigma M)$. We conclude readily that ∇ preserves the diagonal of any element of $\mathrm{C}(\Sigma M)$ by using the commutation relation between products and coproducts in Hopf algebras (without unit). □

The classical Milnor–Moore and Poincaré–Birkhoff–Witt theorems, which give the structure of cocommutative Hopf algebras, have a natural generalization in the context of Σ_*–modules (see [46]) and so do the dual statements which apply to commutative Hopf algebras. If we restrict ourselves to Σ_*–objects such that $M(0) = 0$, then the generalized Milnor–Moore and Poincaré–Birkhoff–Witt theorems hold over a ring, unlike the classical statements.

For the Hopf algebra $\mathrm{T}^c(\Sigma M)$, we obtain:

6.6 Fact *The Hopf algebra $\mathtt{T}^c(\Sigma M)$ is identified with the coenveloping coalgebra of the cofree Lie coalgebra $\mathtt{L}^c(\Sigma M)$.*

This assertion is a direct consequence of the adjoint definition of coenveloping coalgebras (see [21, Section 4.2] and [12, Chapitre II, Section 3] or [41, Sections 0.1–0.2] for the usual dual statement about enveloping algebras of free Lie algebras).

Then:

6.7 Fact (1) *The indecomposable quotient of $\mathtt{T}^c(\Sigma M)$ under the shuffle product is isomorphic to the cofree Lie coalgebra $\mathtt{L}^c(\Sigma M)$.*

(2) *The Hopf algebra $\mathtt{T}^c(\Sigma M)$ comes equipped with a filtration such that*

$$\mathrm{gr}_1\, \mathtt{T}^c(\Sigma M) \simeq \mathtt{L}^c(\Sigma M)$$

and we have an isomorphism of graded commutative Hopf algebras in Σ_*–modules

$$\mathtt{C}(\mathtt{L}^c(\Sigma M)) \simeq \mathrm{gr}_*\, \mathtt{T}^c(\Sigma M).$$

These assertions follow from the dual version of the generalized Milnor–Moore and Poincaré–Birkhoff–Witt theorems of [46]. In their usual form, these theorems are stated for Hopf algebras with units. Again, we can simply take augmentation ideals to obtain the objects required by our nonunital setting.

Note that the distribution relation $F(M \circ P) = F(M) \circ P$ holds for $F(-) = \mathtt{L}^c(\Sigma -)$ since the cofree Lie coalgebra $\mathtt{L}^c(\Sigma M)$ is identified with a quotient of $\mathtt{T}^c(\Sigma M)$. In fact, the functor $\mathtt{L}^c(\Sigma M)$ can be identified with a composite $\mathtt{L}^\vee \circ \Sigma M$, where $\mathtt{L}^\vee$ is the $\Bbbk$–dual of the Lie operad (see the digression and the discussion about coalgebra structures in [20, Sections 1.2.12–1.2.19]). From this observation we also deduce that $\mathtt{L}^c(\Sigma M)$ satisfies the Künneth isomorphism Theorem of Section 0.14.

6.8 The Harrison complex The identity $\mathtt{Indec}\, \mathtt{T}^c(\Sigma M) = \mathtt{L}^c(\Sigma M)$ implies that the indecomposable quotient of the bar complex of a commutative algebra is given by a twisted Σ_*–module of the form

$$\mathtt{Indec}\, \mathtt{B}(A) = (\mathtt{Indec}\, \mathtt{T}^c(\Sigma A), \partial) = (\mathtt{L}^c(\Sigma A), \partial).$$

The chain complex $(\mathtt{L}^c(\Sigma A), \partial)$ is a generalization, in the context of Σ_*–modules, of the standard Harrison complex with trivial coefficients (see [28]).

The next Proposition is classical for the standard Harrison complex of a free commutative algebra over a field of characteristic zero.

6.9 Proposition *The Harrison complex of the free commutative algebra $\mathtt{C}(M)$ is acyclic and we have $\mathtt{H}_*(\mathtt{L}^c(\Sigma\,\mathtt{C}(M)), \partial) = \Sigma M$ whenever M satisfies the requirements* (M1)–(M3).

This result does not hold for a usual commutative algebra in dg-modules if the ground ring is not a field or is a field of positive characteristic. To make the result hold in this setting, we really need the assumption $M(0) = 0$ and the Künneth isomorphism of Section 0.14.

Proof The complex $(\mathtt{L}^c(\Sigma\,\mathtt{C}(M)), \partial)$ can be identified with a composite Σ_*–module

$$(\mathtt{L}^c(\Sigma\,\mathtt{C}(M)), \partial) = \Sigma\,\mathtt{K}(\mathsf{I}, \mathsf{C}, \mathsf{C}) \circ M,$$

where $\mathtt{K}(\mathsf{I}, \mathsf{C}, \mathsf{C})$ is the Koszul complex of the commutative operad (see [20, Section 6.2, Section 6.6]). This complex $\mathtt{K}(\mathsf{I}, \mathsf{C}, \mathsf{C})$ is acyclic because the commutative operad is Koszul (see [20]). The results of [20, Section 2.3] imply moreover that the weak-equivalence $\mathtt{K}(\mathsf{I}, \mathsf{C}, \mathsf{C}) \xrightarrow{\sim} \mathsf{I}$ induces a weak-equivalence

$$\Sigma\,\mathtt{K}(\mathsf{I}, \mathsf{C}, \mathsf{C}) \circ M \xrightarrow{\sim} \Sigma\,\mathsf{I} \circ M = \Sigma M$$

under the assumptions (M1)–(M3). The Proposition follows. □

We now determine the homology of the iterated bar complexes $\mathtt{B}^n(M)$, where M is a Σ_*–module equipped with a trivial commutative algebra structure. For $n = 1$, the complex $\mathtt{B}(M)$ has a trivial differential. Therefore:

6.10 Fact *We have an identity of commutative algebras in Σ_*–module $\mathtt{B}(M) = \mathtt{T}^c(\Sigma M)$, where $\mathtt{T}^c(\Sigma M)$ is equipped with the shuffle product of tensors.*

Thus we study the bar complex of a commutative algebra of the form $\mathtt{T}^c(\Sigma M)$, $M \in \mathcal{M}$.

6.11 Proposition *The morphism $\sigma_*\colon \Sigma\,\mathtt{T}^c(\Sigma M) \to \mathtt{H}_*(\mathtt{B}(\mathtt{T}^c(\Sigma M)))$ induced by the suspension admits a factorization*

$$\begin{array}{ccc} \Sigma\,\mathtt{T}^c(\Sigma M) & \xrightarrow{\quad\sigma_*\quad} & \mathtt{H}_*(\mathtt{B}(\mathtt{T}^c(\Sigma M))) \\ & \searrow \quad \overset{\overline{\sigma}_*}{\nearrow} & \\ & \Sigma\,\mathtt{L}^c(\Sigma M) & \end{array}$$

and the morphism of commutative algebras in Σ_*–modules associated to $\overline{\sigma}_*$ defines an isomorphism

$$\nabla\colon \mathtt{C}(\Sigma\,\mathtt{L}^c(\Sigma M)) \xrightarrow{\simeq} \mathtt{H}_*(\mathtt{B}(\mathtt{T}^c(\Sigma M)))$$

whenever the requirements (M1)–(M3) *are fulfilled.*

Proof The first assertion of the Proposition follows from an observation of Section 1.9 and the identity $\mathtt{Indec}\,\mathtt{T}^c(\Sigma M) = \mathtt{L}^c(\Sigma M)$.

The filtration of the Hopf algebra $\mathtt{T}^c(\Sigma M)$ gives rise to a spectral sequence of commutative algebras $\mathtt{E}^r \Rightarrow \mathtt{H}_*(\mathtt{B}(\mathtt{T}^c(\Sigma M)))$ such that

$$(\mathtt{E}^0, \mathtt{d}^0) = \mathtt{B}(\mathrm{gr}_*\,\mathtt{T}^c(\Sigma M))).$$

Since $\mathrm{gr}_*\,\mathtt{T}^c(\Sigma M) \simeq \mathtt{C}(\mathtt{L}^c(\Sigma M))$, we have $\mathtt{E}^1 = \mathtt{C}(\Sigma\,\mathtt{L}^c(\Sigma M))$ by Proposition 6.2. By a straightforward inspection of the construction, we check that the morphism

$$\Sigma\,\mathtt{L}^c(\Sigma M) \to \mathtt{H}_*(\mathtt{B}(\mathtt{T}^c(\Sigma M)))$$

restricts to the isomorphism $\Sigma\,\mathtt{L}^c(\Sigma M) \xrightarrow{\simeq} \mathtt{E}^1_1$ on the E^1–term of the spectral sequence.

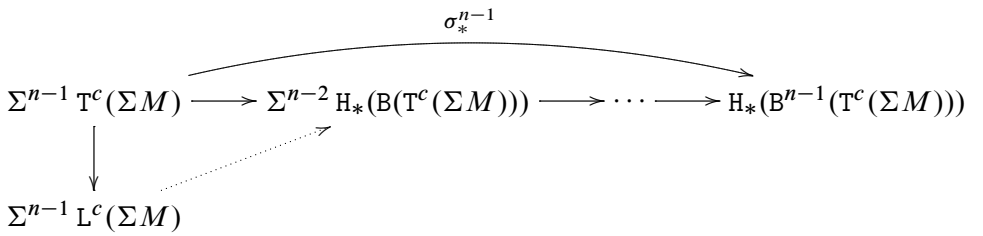
This observation implies that all differentials of the spectral sequence vanish since $\mathtt{E}^1 = \mathtt{C}(\Sigma\,\mathtt{L}^c(\Sigma M)) \Rightarrow \mathtt{H}_*(\mathtt{B}(\mathtt{T}^c(\Sigma M)))$ forms a spectral sequence of commutative algebras.

Hence the spectral sequence degenerates at the E^1–level and we conclude that our morphism $\Sigma\,\mathtt{L}^c(\Sigma M) \to \mathtt{H}_*(\mathtt{B}(\mathtt{T}^c(\Sigma M)))$ gives rise to an isomorphism of commutative algebras $\mathtt{C}(\Sigma\,\mathtt{L}^c(\Sigma M)) \xrightarrow{\simeq} \mathtt{H}_*(\mathtt{B}(\mathtt{T}^c(\Sigma M)))$. □

6.12 Proposition *For every $n > 1$, the natural morphism*

$$\overline{\sigma}^{n-1}_*\colon \Sigma^{n-1}\,\mathtt{L}^c(\Sigma M) \to \mathtt{H}_*(\mathtt{B}^{n-1}(\mathtt{T}^c(\Sigma M)))$$

which arises from the $(n-1)$–fold suspension



induces an isomorphism of commutative algebras in Σ_*–modules

$$\mathtt{C}(\Sigma^{n-1}\,\mathtt{L}^c(\Sigma M)) \xrightarrow{\simeq} \mathtt{H}_*(\mathtt{B}^{n-1}(\mathtt{T}^c(\Sigma M)))$$

whenever the requirements (M1)–(M3) *are fulfilled.*

Proof For any commutative dg-algebra A, we have a natural spectral sequence of commutative algebras $\mathtt{E}^r \Rightarrow \mathtt{H}_*(\mathtt{B}(A))$ such that $(\mathtt{E}^1, \mathtt{d}^1) = \mathtt{B}(\mathtt{H}_*(A))$. In the case $A =$

$\mathtt{B}^{n-2}(\mathtt{T}^c(\Sigma M))$, we have by induction $\mathtt{H}_*(A) \simeq \mathtt{C}(\Sigma^{n-2}\,\mathtt{L}^c(\Sigma M))$ and Proposition 6.2 implies $\mathtt{E}^2 \simeq \mathtt{C}(\Sigma^{n-1}\,\mathtt{L}^c(\Sigma M))$.

By a straightforward inspection of the construction, we check that the composite of the Proposition restricts to the isomorphism $\Sigma^{n-1}\,\mathtt{L}^c(\Sigma M) \xrightarrow{\simeq} \mathtt{E}^2_1$ on the E^2–term of the spectral sequence.

This observation implies again that all differentials of the spectral sequence vanish since $(\mathtt{E}^1, \mathtt{d}^1) = \mathtt{B}(\mathtt{H}_*(A)) \Rightarrow \mathtt{H}_*(\mathtt{B}(A))$ forms a spectral sequence of commutative algebras. The conclusion follows readily. □

7 Homology of algebras over operads and operadic Tor-functors

In this section, we review the definition of the homology theory $\mathtt{H}^{\mathsf{R}}_*$ associated to an operad R. Usually, the homology module $\mathtt{H}^{\mathsf{R}}_*$ is defined by a derived functor of indecomposables $L\,\mathtt{Indec}\colon \mathtt{Ho}({}_{\mathsf{R}}\mathcal{E}) \to \mathtt{Ho}(\mathcal{E})$. We apply theorems of [23, Section 15] to observe that $\mathtt{H}^{\mathsf{R}}_*$ is represented by a generalized $\mathtt{Tor}$–functor on R–modules $\mathtt{Tor}^{\mathsf{R}}_*(\mathsf{I}, -)$, where I is the composition unit of the category of Σ_*–modules. We use this representation in the next sections to prove that the n–fold bar complex determines the natural homology theory associated to E_n–operads.

7.1 Augmented operads The homology $\mathtt{H}^{\mathsf{R}}_*$ is defined for certain operads R equipped with an augmentation over the composition unit of Σ_*–modules I.

The unit relation gives an isomorphism $\mathsf{I} \circ \mathsf{I} \simeq \mathsf{I}$ which provides I with an obvious operad structure. The category of algebras ${}_{\mathsf{I}}\mathcal{E}$ associated to this operad is identified with the underlying category $\mathcal{E}$. Let R be an operad equipped with an augmentation $\epsilon\colon \mathsf{R} \to \mathsf{I}$. The restriction functor

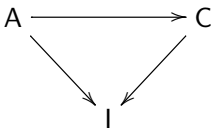
$$\mathcal{E} = {}_{\mathsf{I}}\mathcal{E} \xrightarrow{\epsilon^*} {}_{\mathsf{R}}\mathcal{E}$$

identifies an object $E \in \mathcal{E}$ with an R–algebra equipped with a trivial structure. The functor of indecomposables $\mathtt{Indec}\colon {}_{\mathsf{R}}\mathcal{E} \to \mathcal{E}$ represents the left adjoint of this category embedding $\mathcal{E} \hookrightarrow {}_{\mathsf{R}}\mathcal{E}$ and can be identified with the extension functor

$${}_{\mathsf{R}}\mathcal{E} \xrightarrow{\epsilon_!} {}_{\mathsf{I}}\mathcal{E} = \mathcal{E}$$

associated to the augmentation $\epsilon\colon \mathsf{R} \to \mathsf{I}$.

The associative operad A and the commutative operad C are canonically augmented over I in such a way that the diagram



commutes. In these examples $\mathsf{R} = \mathsf{A}, \mathsf{C}$, the indecomposable quotient of an R–algebra A is identified with the cokernel of the product $\mu\colon A \otimes A \to A$. Thus we retrieve the usual definition of Section 1.9.

The operads occurring in a nested sequence

$$\mathsf{E}_1 \to \cdots \to \mathsf{E}_n \to \cdots \to \operatorname{colim}_n \mathsf{E}_n = \mathsf{E}$$

inherit a canonical augmentation $\mathsf{E}_n \to \mathsf{I}$, since the E_∞–operad E is supposed to be augmented over C.

7.2 Homology of algebras over operads (standard definition) Recall (see Section 0.12) that the category of R–algebras ${}_{\mathsf{R}}\mathcal{E}$ inherits a semimodel structure if the operad R is Σ_*–cofibrant. The functors $\mathtt{Indec} = \epsilon_!\colon {}_{\mathsf{R}}\mathsf{E} \to \mathsf{E}$ and $\epsilon^*\colon \mathsf{E} \to {}_{\mathsf{R}}\mathsf{E}$ form a Quillen pair and as such determine a pair of adjoint derived functors

$$L\,\mathtt{Indec}\colon \mathtt{Ho}({}_{\mathsf{R}}\mathcal{E}) \rightleftarrows \mathtt{Ho}(\mathcal{E}) :\epsilon^*.$$

The derived functor of indecomposables maps an R–algebra A to the indecomposable quotient $\mathtt{Indec}\, Q_A$ of a cofibrant replacement $0 \rightarrowtail Q_A \xrightarrow{\sim} A$ in ${}_{\mathsf{R}}\mathcal{E}$. The homology of A is defined by $\mathtt{H}^{\mathsf{R}}_*(A) = \mathtt{H}_*(\mathtt{Indec}\, Q_A)$. (We refer to [23, Section 13, Section 16] for a comprehensive account on this background.)

7.3 Homology of algebras over operads and generalized Tor-functors We define the generalized $\mathtt{Tor}$–functor $\mathtt{Tor}^{\mathsf{R}}_*(M)\colon {}_{\mathsf{R}}\mathcal{E} \to \mathcal{E}$ associated to a right R–module M by the homology of the functor $\mathtt{S}_{\mathsf{R}}(P_M)\colon {}_{\mathsf{R}}\mathcal{E} \to \mathcal{E}$ associated to any cofibrant replacement $0 \rightarrowtail P_M \xrightarrow{\sim} M$ in $\mathcal{M}_{\mathsf{R}}$. Explicitly, we set

$$\mathtt{Tor}^{\mathsf{R}}_*(M, A) = \mathtt{H}_*(\mathtt{S}_{\mathsf{R}}(P_M, A)).$$

This definition makes sense for any $\mathcal{C}$–cofibrant operad R, for any $\mathcal{C}$–cofibrant right R–module M and for every $\mathcal{E}$–cofibrant R–algebra A: the assertions of [23, Theorems 15.1.A] imply that the homotopy type of $\mathtt{S}_{\mathsf{R}}(P_M, A)$ does not depend on the choice of the cofibrant replacement $0 \rightarrowtail P_M \xrightarrow{\sim} M$; moreover, the map $\mathtt{Tor}^{\mathsf{R}}_*\colon (M, A) \mapsto \mathtt{Tor}^{\mathsf{R}}_*(M, A)$ defines a bifunctor which satisfies reasonable homotopy invariance properties with respect to (M, A).

Pick a cofibrant replacement $0 \rightarrowtail Q_A \xrightarrow{\sim} A$ of A in $\mathcal{E}_{\mathsf{R}}$, assuming that the operad R is Σ_*–cofibrant. The weak-equivalences $P_M \xrightarrow{\sim} M$ and $Q_A \xrightarrow{\sim} A$ induce morphisms at the functor level and we have a commutative diagram

$$\begin{array}{ccc} \mathtt{S}_{\mathsf{R}}(P_M, Q_A) & \longrightarrow & \mathtt{S}_{\mathsf{R}}(P_M, A) \\ \downarrow & & \downarrow \\ \mathtt{S}_{\mathsf{R}}(M, Q_A) & \longrightarrow & \mathtt{S}_{\mathsf{R}}(M, A)\,. \end{array}$$

If M is also Σ_*–cofibrant and A is $\mathcal{E}$–cofibrant, then the left-hand vertical morphism and the upper horizontal morphism of the diagram are weak-equivalences (see [23, Theorems 15.1.A-15.2.A]). Thus, for a Σ_*–cofibrant module M, we have a natural isomorphism

$$\mathtt{Tor}^{\mathsf{R}}_*(M, A) = \mathtt{H}_*(\mathtt{S}_{\mathsf{R}}(P_M, A)) \simeq \mathtt{H}_*(\mathtt{S}_{\mathsf{R}}(M, Q_A)),$$

for every $\mathcal{E}$–cofibrant R–algebra A.

By [23, Theorem 7.2.2], the extension functor $\epsilon_!\colon {}_{\mathsf{R}}\mathsf{E} \to \mathsf{E}$ is identified with the functor $\epsilon_! = \mathtt{S}_{\mathsf{R}}(\mathsf{I})$, where we use the augmentation $\epsilon\colon \mathsf{R} \to \mathsf{I}$ to provide the composition unit I with a right R–module structure. Since the unit object I is obviously Σ_*–cofibrant, we have an identity

$$\mathtt{H}_*(\mathtt{S}_{\mathsf{R}}(P_{\mathsf{I}}, A)) = \mathtt{H}_*(\mathtt{S}_{\mathsf{R}}(M, Q_A)),$$

for any $\mathcal{E}$–cofibrant R–algebra A, from which we deduce the relation

$$\mathtt{Tor}^{\mathsf{R}}_*(\mathsf{I}, A) = \mathtt{H}^{\mathsf{R}}_*(A).$$

8 Iterated bar complexes and homology of algebras over E_n–operads

The goal of this section is to prove that the homology of the category of algebras over an E_n–operad E_n is determined by the n–fold bar complex $\mathtt{B}^n(A)$. For this purpose, we check that the n–fold bar module $B^n_{\mathsf{E}_n}$ defines a cofibrant replacement of I in the category of right E_n–modules and we apply the interpretation of $\mathtt{H}^{\mathsf{E}_n}_*$ in terms of operadic $\mathtt{Tor}$–functors.

The module $\Sigma^{-n} B^n_{\mathsf{E}_n}$ is cofibrant by construction (see Section 2.3). Our main task is to prove that it is acyclic.

We use a spectral sequence to reduce the problem to the acyclicity of a chain complex of the form $\mathtt{E}^1 = \mathtt{H}_*(\mathtt{B}^n(\mathsf{I})) \circ \mathtt{H}_*(\mathsf{E}_n)$, where $\mathtt{B}^n(\mathsf{I})$ is the iterated bar complex of the composition unit $\mathsf{I} \in \mathcal{M}$, viewed as a commutative algebra equipped with a trivial structure. We focus on the case $n > 1$. The homology operad $\mathtt{H}_*(\mathsf{E}_n)$ is determined in [15] and has a nice description as a composite of the commutative operad and a

desuspension of the Lie operad. This composite is usually called the Gerstenhaber operad and is denoted by G_n. The Gerstenhaber operad is Koszul (see [26; 36]) and this property gives the deep reason for the acyclicity of the iterated bar module $B^n_{\mathsf{E}_n}$. For technical reasons, we split the proof of the acyclicity of $\mathtt{E}^1 = \mathtt{H}_*(\mathtt{B}^n(\mathsf{I})) \circ \mathtt{H}_*(\mathsf{E}_n)$ in two steps and we rather use that the commutative operad and the Lie operad, which fit in the decomposition of the Gerstenhaber operad, are both Koszul operads.

8.1 The augmentation morphisms We already know that $B^n_{\mathsf{E}_n}$ is cofibrant. We also have a natural augmentation $\epsilon\colon \Sigma^{-n} B^n_{\mathsf{E}_n} \to \mathsf{I}$ which is defined as follows. For any commutative algebra A, the natural morphisms

$$\mathtt{T}^c(\Sigma A) = \bigoplus_{d=1}^{\infty} (\Sigma A)^{\otimes d} \to \Sigma A \to \Sigma\, \mathtt{Indec}\, A$$

give a morphism of commutative algebras $\epsilon\colon \mathtt{B}(A) \to \Sigma\, \mathtt{Indec}\, A$, where the object $\Sigma\, \mathtt{Indec}\, A$ is identified with a commutative algebra equipped with a trivial structure. For the composition unit I, viewed as a trivial commutative algebra in Σ_*–modules, we have $\mathtt{Indec}\, \mathsf{I} = \mathsf{I}$. Hence, in this case, we have a morphism of commutative algebras $\epsilon\colon \mathtt{B}(\mathsf{I}) \to \Sigma\, \mathsf{I}$. More generally, we have a morphism of commutative algebras $\epsilon\colon \mathtt{B}(\Sigma^{n-1}\, \mathsf{I}) \to \Sigma^n\, \mathsf{I}$, for any suspension $\Sigma^{n-1}\, \mathsf{I}$, and we can iterate the construction to obtain a morphism $\epsilon\colon \mathtt{B}^n(\mathsf{I}) \to \Sigma^n\, \mathsf{I}$ on the n–fold bar complex $\mathtt{B}^n(\mathsf{I})$.

The augmentation of an E_n–operad $\mathsf{E}_n \to \mathsf{I}$ defines a morphism of E_n–algebras in right E_n–modules and, by functoriality, gives rise to a morphism at the level of the n–fold bar complex. Note that we can view the composition unit I as a commutative algebra in right E_n–modules in the previous construction, so that the augmentation $\epsilon\colon \mathtt{B}^n(\mathsf{I}) \to \Sigma^n\, \mathsf{I}$ defines a morphism of right E_n–modules and not only of Σ_*–modules. The augmentation $\epsilon\colon \Sigma^{-n} B^n_{\mathsf{E}_n} \to \mathsf{I}$ is defined by a desuspension of the composite

$$B^n_{\mathsf{E}_n} = \mathtt{B}^n(\mathsf{E}_n) \to \mathtt{B}^n(\mathsf{I}) \to \Sigma^n\, \mathsf{I}.$$

It remains to determine the homology of $B^n_{\mathsf{E}_n}$ in order to prove that the morphism $\epsilon\colon B^n_{\mathsf{E}_n} \to \Sigma^n\, \mathsf{I}$ (or its desuspension) forms a weak-equivalence. In the remainder of this section, we focus on the case $n > 1$, because for $n = 1$ we immediately obtain:

8.2 Proposition *In the case $n = 1$, we have weak-equivalences $B_{\mathsf{E}_1} = \mathtt{B}(\mathsf{E}_1) \xrightarrow{\sim} \mathtt{B}(\mathsf{A}) \xrightarrow{\sim} \Sigma\, \mathsf{I}$.*

Proof Recall that the operad E_1 forms an A_∞–operad and is connected to the associative operad A by a weak-equivalence $\epsilon\colon \mathsf{E}_1 \xrightarrow{\sim} \mathsf{A}$. This augmentation induces a weak-equivalence at the bar complex level:

$$B_{\mathsf{E}_1} = \mathtt{B}(\mathsf{E}_1) \xrightarrow{\sim} \mathtt{B}(\mathsf{A}).$$

The bar complex $\mathtt{B}(\mathsf{A})$ is identified with the suspension of the Koszul complex $\mathtt{K}(\mathsf{I}, \mathsf{A}, \mathsf{A})$ of the associative operad A (see [20, Section 5.2] and [27]). This complex is acyclic, because the associative operad is an instance of a Koszul operad (see [27]). □

The spectral sequence, used to reduce the calculation of $\mathtt{H}_*(B^n_{\mathsf{E}_n})$ for $n > 1$, arises from the definition of $B^n_{\mathsf{E}_n}$ as a quasifree module over the operad E_n.

8.3 The natural spectral sequence of a quasifree module Let $M = (K \circ \mathsf{R}, \partial_\alpha)$ be a quasifree module over an operad R such that $\mathsf{R}(0) = 0$. Let $\mathtt{F}_0\, K \subset \cdots \subset \mathtt{F}_s\, K \subset \cdots \subset K$ be the filtration of K formed by the Σ_*–modules such that

$$\mathtt{F}_s\, K(r) = \begin{cases} K(r) & \text{if } s \leq r, \\ 0 & \text{otherwise.} \end{cases}$$

Claim *If $\mathsf{R}(0) = 0$, then the twisting homomorphism $\partial_\alpha\colon K \circ \mathsf{R} \to K \circ \mathsf{R}$ satisfies $\partial_\alpha(\mathtt{F}_s\, K \circ \mathsf{R}) \subset \mathtt{F}_s\, K \circ \mathsf{R}$, for every $s \in \mathbb{N}$.*

Proof Observe that $\xi \in K \circ \mathsf{R}(r) \Rightarrow \xi \in \mathtt{F}_r\, K \circ \mathsf{R}(r)$. Indeed, for a composite $y(q_1, \ldots, q_s) \in K \circ \mathsf{R}(r)$, where $y \in K(s)$ and $q_1 \in \mathsf{R}(n_1), \ldots, q_s \in \mathsf{R}(n_s)$, we have $n_1 + \cdots + n_s = r$. Since $\mathsf{R}(0) = 0$, we have necessarily $n_i > 0$, for $i = 1, \ldots, s$, from which we deduce $s \leq n_1 + \cdots + n_s = r$. Hence, we obtain $y(q_1, \ldots, q_s) \in \mathtt{F}_r\, K \circ \mathsf{R}(r)$.

For a generating element $x \in K(r)$, we have $\partial_\alpha(x) \in (K \circ \mathsf{R})(r) \Rightarrow \partial_\alpha(x) \in (\mathtt{F}_r\, K \circ \mathsf{R})(r)$. For a composite $x(p_1, \ldots, p_r) \in K \circ \mathsf{R}$, we still have $\partial_\alpha(x(p_1, \ldots, p_r)) \in \mathtt{F}_r\, K \circ \mathsf{R}$ since $\partial_\alpha(x(p_1, \ldots, p_r)) = \partial_\alpha(x)(p_1, \ldots, p_r)$. Therefore we conclude $\xi \in \mathtt{F}_s\, K \circ \mathsf{R} \Rightarrow \partial_\alpha(\xi) \in \mathtt{F}_s\, K \circ \mathsf{R}$. □

The relation $\partial_\alpha(\mathtt{F}_s\, K \circ \mathsf{R}) \subset \mathtt{F}_s\, K \circ \mathsf{R}$ implies that the quasifree module $M = (K \circ \mathsf{R}, \partial_\alpha)$ has a filtration by submodules such that

$$\mathtt{F}_s\, M = (\mathtt{F}_s\, K \circ \mathsf{R}, \partial_\alpha).$$

The spectral sequence $\mathtt{E}^r(M) \Rightarrow \mathtt{H}_*(M)$ associated to a quasifree module M is the spectral sequence defined by this filtration $\mathtt{F}_0\, M \subset \cdots \subset \mathtt{F}_s\, M \subset \cdots \subset M$.

We have the easy observation:

8.4 Observation *Each term of the spectral sequence $\mathtt{E}^r(M)$ inherits a natural right $\mathtt{H}_*(\mathsf{R})$–action. This right $\mathtt{H}_*(\mathsf{R})$–action is preserved by the differential $\mathtt{d}^r\colon \mathtt{E}^r(M) \to \mathtt{E}^r(M)$ so that $\mathtt{E}^r(M) \Rightarrow \mathtt{H}_*(M)$ defines a spectral sequence of right $\mathtt{H}_*(\mathsf{R})$–modules.*

8.5 Functoriality of the spectral sequence Let $\phi\colon M \to N$ be any morphism between quasifree modules $M = (K \circ \mathsf{R}, \partial)$ and $N = (L \circ \mathsf{R}, \partial)$.

For a generating element $x \in K(r)$, we have $\phi(x) \in K \circ \mathsf{R}(r) \Rightarrow \phi(x) \in \mathtt{F}_r\, K \circ \mathsf{R}(r)$. From this assertion we immediately deduce that ϕ preserves the filtrations of the spectral sequence of Section 8.3. Thus we obtain that $\phi\colon M \to N$ induces a morphism of spectral sequences $\mathtt{E}^r(\phi)\colon \mathtt{E}^r(M) \to \mathtt{E}^r(N)$.

8.6 The E^0–term of the spectral sequence We adopt the notation $\mathtt{E}^0_s = \mathtt{F}_s \,/\, \mathtt{F}_{s-1}$ for the subquotient of any filtration. We have an obvious isomorphism $\mathtt{E}^0_s(M) = (\mathtt{E}^0_s(K) \circ \mathsf{R}, \partial_\alpha)$, where

$$\mathtt{E}^0_s(K)(r) = \begin{cases} K(s) & \text{if } r = s, \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence, the projection $\mathtt{F}_s\, M \to \mathtt{E}^0_s\, M$ has an obvious section $\mathtt{E}^0_s\, M \to \mathtt{F}_s\, M$ and we have a natural isomorphism $\mathtt{E}^0(M) \simeq K \circ \mathsf{R}$.

We apply the spectral sequence $\mathtt{E}^0(M) \Rightarrow \mathtt{H}_*(M)$ to the iterated bar module $B^n_{\mathsf{E}_n} = (T^n \circ \mathsf{E}_n, \partial_\epsilon)$. In one argument (Lemma 8.10), we also use the spectral sequence of the m–fold bar complex

$$B^m_{\mathsf{E}_n} = B^m_{\mathsf{E}_m} \circ_{\mathsf{E}_m} \mathsf{E}_n = (T^m \circ \mathsf{E}_n, \partial_\epsilon),$$

for $m < n$, and the morphism of spectral sequences $\mathtt{E}^r(\sigma)\colon \mathtt{E}^r(\Sigma^{-m} B^m_{\mathsf{E}_n}) \to \mathtt{E}^r(\Sigma^{-n} B^n_{\mathsf{E}_n})$ induced by the composite suspension

$$\Sigma^{-m} B^m_{\mathsf{E}_n} \xrightarrow{\sigma} \Sigma^{-m-1} B^{m+1}_{\mathsf{E}_n} \xrightarrow{\sigma} \dots \xrightarrow{\sigma} \Sigma^{-n} B^n_{\mathsf{E}_n}.$$

We go back to the inductive definition of the twisting homomorphism $\partial_\epsilon\colon T^n \circ \mathsf{E}_n \to T^n \circ \mathsf{E}_n$ in order to determine the differentials $\mathtt{d}^r$ of $\mathtt{E}^r(B^n_{\mathsf{E}_n})$ for $r = 0, 1$. In the case $r = 0$, we obtain the following result:

8.7 Proposition *The chain complex $(\mathtt{E}^0(B^n_{\mathsf{E}_n}), \mathtt{d}^0)$ is isomorphic to the composite $\mathtt{B}^n(\mathsf{I}) \circ \mathsf{E}_n$, where $\mathtt{B}^n(\mathsf{I}) = ((\mathtt{T}^c\, \Sigma)^n(\mathsf{I}), \partial)$ is the n–fold bar complex of the composition unit of Σ_*–modules, viewed as a trivial commutative algebra in Σ_*–modules.*

Proof The twisting homomorphism of the iterated bar complex $\mathtt{B}^n(\mathsf{C})$ has a splitting $\partial_\gamma = \partial^0_\gamma + \partial^1_\gamma$ such that:

- the component $\partial^1_\gamma\colon (\mathtt{T}^c\, \Sigma)^n(\mathsf{I}) \circ \mathsf{C} \to (\mathtt{T}^c\, \Sigma)^n(\mathsf{I}) \circ \mathsf{C}$ is yielded by the bar differential of the first factor of the composite

$$\mathtt{B}^n(\mathsf{C}) = \underset{n}{\mathtt{B}} \circ \cdots \circ \underset{2}{\mathtt{B}} \circ \underset{1}{\mathtt{B}}(\mathsf{C})\,;$$

- the component $\partial^0_\gamma\colon (\mathtt{T}^c\, \Sigma)^n(\mathsf{I}) \circ \mathsf{C} \to (\mathtt{T}^c\, \Sigma)^n(\mathsf{I}) \circ \mathsf{C}$ is yielded by the bar differential of factors $2, \dots, n$ of $\mathtt{B}^n(\mathsf{C})$.

The component ∂^0_γ is identified with the homomorphism of free right C–modules

$$\partial^0 \circ \mathsf{C}\colon (\mathsf{T}^c\,\Sigma)^n(\mathsf{I}) \circ \mathsf{C} \to (\mathsf{T}^c\,\Sigma)^n(\mathsf{I}) \circ \mathsf{C},$$

where $\partial^0\colon (\mathsf{T}^c\,\Sigma)^n(\mathsf{I}) \to (\mathsf{T}^c\,\Sigma)^n(\mathsf{I})$ is the differential of the iterated bar complex $\mathsf{B}^n(\mathsf{I})$.

The twisting homomorphism $\partial_\gamma\colon \mathsf{T}^c(\Sigma\,\mathsf{I}) \circ \mathsf{C} \to \mathsf{T}^c(\Sigma\,\mathsf{I}) \circ \mathsf{C}$ of the bar complex $\mathsf{B}(\mathsf{C}) = (\mathsf{T}^c(\Sigma\,\mathsf{I}) \circ \mathsf{C}, \partial_\gamma)$ clearly satisfies $\partial_\gamma(\mathsf{F}_s\,\mathsf{T}^c(\Sigma\,\mathsf{I})) \subset \mathsf{F}_{s-1}\,\mathsf{T}^c(\Sigma\,\mathsf{I}) \circ \mathsf{C}$. This relation implies that the induced differential on $\mathsf{B}^n(\mathsf{C})$ satisfies

$$\partial_\gamma(\mathsf{F}_s(\mathsf{T}^c\,\Sigma)^n(\mathsf{I}) \circ \mathsf{C}) \subset \mathsf{F}_{s-1}(\mathsf{T}^c\,\Sigma)^n(\mathsf{I}) \circ \mathsf{C}$$

and vanishes in $\mathsf{E}^0_s(B^n_{\mathsf{C}})$.

Recall that the homomorphism $\epsilon_*\colon G^n \to T^n \circ \mathsf{E}_n$ which determines the twisting homomorphism of $B^n_{\mathsf{E}_n}$ is a sum $\epsilon = \sum_{m=0}^{\infty} \epsilon_m$, whose terms ϵ_m are determined inductively by formulas of the form

$$\epsilon_0(\xi) = \widetilde{\iota}\gamma(\xi) \quad \text{and} \quad \epsilon_m(\xi) = \sum_{p+q=m-1} \widetilde{\nu}\partial_{\epsilon_p}\epsilon_q(\xi),$$

for any generating element $\xi \in G^n(r)$. The homomorphisms $\widetilde{\iota}$ and $\widetilde{\nu}$ defined in Section 2.4 clearly satisfy $\widetilde{\iota}(\mathsf{F}_s\,B^n_{\mathsf{E}_n}) \subset \mathsf{F}_s\,B^n_{\mathsf{E}_n}$ and $\widetilde{\nu}(\mathsf{F}_s\,B^n_{\mathsf{E}_n}) \subset \mathsf{F}_s\,B^n_{\mathsf{E}_n}$. From this observation, we deduce the relation

$$\epsilon_0(\xi) \equiv \widetilde{\iota}\partial^0(\xi) \mod \mathsf{F}_{r-1}\,B^n_{\mathsf{E}_n}$$

and we easily conclude that the splitting $\mathsf{E}^r(B^n_{\mathsf{E}_n}) \simeq T^n \circ \mathsf{E}_n$ identifies the class of $\mathsf{d}^0(\xi) = \epsilon_*(\xi) \mod \mathsf{F}_{r-1}\,B^n_{\mathsf{E}_n}$ with $\partial^0(\xi) \in (\mathsf{T}^c\,\Sigma)^n(\mathsf{I})$.

For a composite $\xi(p_1, \ldots, p_r) \in T^n \circ \mathsf{E}_n$, where $\xi \in G^n(\underline{\mathrm{e}})$, we have

$$\mathsf{d}^0(\xi(p_1, \ldots, p_r)) = \partial^0(\xi)(p_1, \ldots, p_r) + \sum_{i=1}^{r} \pm\xi(p_1, \ldots, \delta(p_i), \ldots, p_r),$$

where δ refers to the internal differential of E_n. Hence, we conclude that the differential $\mathsf{d}^0\colon \mathsf{E}^0(B^n_{\mathsf{E}_n}) \to \mathsf{E}^0(B^n_{\mathsf{E}_n})$ is identified with the natural differential of the composite

$$\mathsf{B}^n(\mathsf{I}) \circ \mathsf{E}_n = ((\mathsf{T}^c\,\Sigma)^n(\mathsf{I}), \partial^0) \circ \mathsf{E}_n\,.$$

The Proposition follows. □

8.8 The homology of E_n–operads and the Gerstenhaber operad We study the homology of the factors of the composite $(\mathtt{E}^0, \mathtt{d}^0) = \mathtt{B}^n(\mathsf{I}) \circ \mathsf{E}_n$ to determine the E^1–term of the spectral sequence $\mathtt{E}^1(B^n_{\mathsf{E}_n}) \Rightarrow \mathtt{H}_*(B^n_{\mathsf{E}_n})$. We still focus on the case $n > 1$.

The homology of $\mathtt{B}^n(\mathsf{I})$ is given by the result of Proposition 6.12.

The results of [15] imply that the homology of E_n, $n > 1$, is isomorphic to the Gerstenhaber operad G_n. The structure of a G_n–algebra consists of a commutative algebra A equipped with a Lie bracket $\lambda_{n-1}\colon A \otimes A \to A$ of degree $n-1$ which satisfies a distribution relation with respect to the product $\mu\colon A \otimes A \to A$. This description of the structure of a G_n–algebra reflects a definition of G_n by generators and relations (see [25; 36]).

The element $\mu \in \mathsf{G}_n$ which represents the commutative product of G_n–algebras generates a suboperad of G_n isomorphic to the commutative operad C. The element $\lambda_{n-1} \in \mathsf{G}_n$ which represents the Lie product of G_n–algebras generates a suboperad of G_n isomorphic to the $(n-1)$–desuspension $\Lambda^{1-n}\mathsf{L}$ of the Lie operad L. The embeddings $\mathsf{C} \hookrightarrow \mathsf{G}_n$ and $\Lambda^{1-n}\mathsf{L} \hookrightarrow \mathsf{G}_n$ assemble to an isomorphism $\mathsf{C} \circ \Lambda^{1-n}\mathsf{L} \simeq \mathsf{G}_n$.

8.9 Proposition *For $n > 1$, we have an isomorphism*

$$\mathtt{E}^1(B^n_{\mathsf{E}_n}) \simeq \mathsf{C}(\Sigma^{n-1}\mathsf{L}^c(\Sigma\,\mathsf{I})) \circ \mathsf{G}_n \simeq \mathsf{C}(\Sigma^{n-1}\mathsf{L}^c(\Sigma\,\mathsf{G}_n)).$$

Proof In Section 0.14, we recall that the Künneth morphism $\mathtt{H}_*(M) \circ \mathtt{H}_*(N) \to \mathtt{H}_*(M \circ N)$ is an isomorphism for any composite $M \circ N$ such that the Σ_*–modules M, N and their homology $\mathtt{H}_*(M), \mathtt{H}_*(N)$ consist of projective $\Bbbk$–modules, under the connectedness assumption $N(0) = 0$.

The E_n–operad E_n is Σ_*–cofibrant (and hence $\Bbbk$–projective) by assumption. The iterated bar complex $\mathtt{B}^n(\mathsf{I})$ is Σ_*–cofibrant (and hence $\Bbbk$–projective as well) by the result of Proposition 2.8.

We have $\mathtt{H}_*(\mathtt{B}^n(\mathsf{I})) = \mathsf{C}(\Sigma^{n-1}\mathsf{L}^c(\Sigma\,\mathsf{I}))$ by Proposition 6.12 and we just recall that $\mathtt{H}_*(\mathsf{E}_n) = \mathsf{G}_n = \mathsf{C} \circ \Lambda^{1-n}\mathsf{L}$. The Lie operad L consists of free $\Bbbk$–modules (see for instance [12, Chapitre II, Section 2] or [41, Corollary 0.10]) and so does the commutative operad C. The result of [20, Lemma 1.3.9] shows that a composite $M \circ N$, where M, N are $\Bbbk$–projective Σ_*–modules, is still $\Bbbk$–projective under the connectedness assumption $N(0) = 0$. Thus, the Σ_*–modules $\mathtt{H}_*(\mathtt{B}^n(\mathsf{I}))$ and $\mathtt{H}_*(\mathsf{E}_n)$ are both $\Bbbk$–projective.

From these observations, we conclude that the Künneth morphism $\mathtt{H}_*(M) \circ \mathtt{H}_*(N) \to \mathtt{H}_*(M \circ N)$ yields, in the case $M = \mathtt{B}^n(\mathsf{I})$ and $N = \mathsf{E}_n$, an isomorphism

$$\mathsf{C}(\Sigma^{n-1}\mathsf{L}^c(\Sigma\,\mathsf{I})) \circ \mathsf{G}_n \simeq \mathtt{H}_*(\mathtt{B}^n(\mathsf{I})) \circ \mathtt{H}_*(\mathsf{E}_n) \xrightarrow{\simeq} \mathtt{H}_*(\mathtt{E}^0(B^n_{\mathsf{E}_n}), \mathtt{d}^0).$$

The Proposition follows. □

The differential of certain particular elements of $\mathtt{E}^1(B^n_{\mathsf{E}_n})$ can easily be determined:

8.10 Lemma *The restriction of the differential* $\mathtt{d}^1\colon \mathtt{E}^1 \to \mathtt{E}^1$ *to the summand*

$$\Sigma^{n-1}\,\mathtt{L}^c(\Sigma\,\mathsf{G}_n) \subset \mathtt{C}(\Sigma^{n-1}\,\mathtt{L}^c(\Sigma\,\mathsf{G}_n))$$

is identified with the differential of the Harrison complex $(\mathtt{L}^c(\Sigma\,\mathsf{G}_n), \partial)$.

Proof In Section 8.3, we observe that the suspension

$$\mathtt{B}(\mathsf{E}_n) \xrightarrow{\sigma} \mathtt{B}^n(\mathsf{E}_n) = B^n_{\mathsf{E}_n}$$

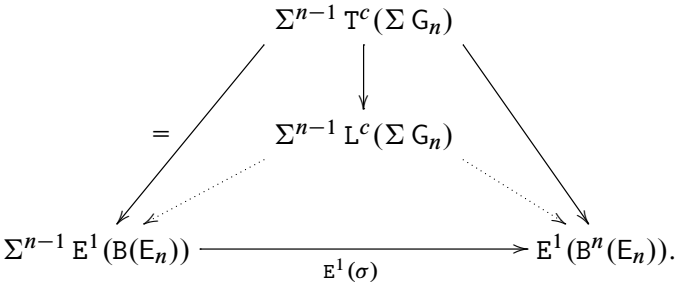
determines a morphism of spectral sequences

$$\mathtt{E}^r(\sigma)\colon \mathtt{E}^r(\mathtt{B}(\mathsf{E}_n)) \to \mathtt{E}^r(\mathtt{B}^n(\mathsf{E}_n)).$$

For the spectral sequence $\mathtt{E}^r(\mathtt{B}(\mathsf{E}_n)) \Rightarrow \mathtt{H}_*(\mathtt{B}(\mathsf{E}_n))$, we also have $(\mathtt{E}^0, \mathtt{d}^0) = \mathtt{T}^c(\Sigma\,\mathsf{E}_n)$ since the differential of the 1–fold bar complex decreases filtrations (see proof of Proposition 8.7). Hence, we obtain

$$\mathtt{E}^1(\mathtt{B}(\mathsf{E}_n)) = \mathtt{T}^c(\Sigma\,\mathtt{H}_*(\mathsf{E}_n)) = \mathtt{T}^c(\Sigma\,\mathsf{G}_n).$$

By immediate inspection of constructions, we see that the morphism $\Sigma^{n-1}\,\mathtt{L}^c(\Sigma\,\mathsf{G}_n) \to \mathtt{E}^1(B^n_{\mathsf{E}_n})$ fits in a commutative diagram



Thus we are reduced to determine the bar differential of the representative $\xi = p_1 \otimes \cdots \otimes p_s \in \mathtt{T}^c(\Sigma\,\mathsf{E}_n)$ of an element of $\mathtt{T}^c(\Sigma\,\mathsf{G}_n)$.

The definition of the differential of $\mathtt{B}(\mathsf{E}_n)$ gives immediately

$$\begin{aligned}\partial(p_1 \otimes \cdots \otimes p_s) &= \sum_{t=2}^{s}\Bigg(\sum_{i=1}^{s-t+1} \pm p_1 \otimes \cdots \otimes \mu_t(p_i, \ldots, p_{i+t-1}) \otimes \cdots \otimes p_s\Bigg)\\ &\equiv \sum_{i=1}^{s-1} \pm p_1 \otimes \cdots \otimes \mu_2(p_i, p_{i+1}) \otimes \cdots \otimes p_s \mod \mathtt{F}_{s-2}\,\mathtt{B}(\mathsf{E}_n),\end{aligned}$$

where $\mu_2 \in \mathsf{E}_2(2)$ is a representative of the product $\mu \in \mathsf{G}_2(2)$. Hence we obtain that $\mathrm{d}^1(p_1 \otimes \cdots \otimes p_s)$ is given by the differential of the bar complex $\mathtt{B}(\mathsf{G}_n)$. The conclusion follows. □

8.11 Lemma *For an element of the form*

$$\xi = x(e_1) \cdot x(e_2) \in \mathsf{C}(\Sigma^{n-1}\, \mathtt{L}^c(\Sigma\, \mathsf{I}))(\{e_1, e_2\}),$$

where x represents the canonical generator of $\Sigma\, \mathsf{I}(1) = \mathtt{L}^c(\Sigma\, \mathsf{I})(1)$, we have

$$\mathrm{d}^1(\xi) = \lambda_{n-1}(e_1, e_2) \in \mathsf{G}_n(\{e_1, e_2\}),$$

where λ_{n-1} is the operation of $\mathsf{G}_n(2)$ which represents the Lie bracket of G_n–algebras.

Proof In this proof, it is convenient to adopt the notation $\otimes_m$ to refer to a tensor product in the m–th factor of the composite

$$(\mathtt{T}^c\, \Sigma)^n(\mathsf{I}) = \underset{n}{\mathtt{T}^{\mathrm{c}}} \circ \cdots \circ \underset{1}{\mathtt{T}^{\mathrm{c}}}(\mathsf{I}).$$

Observe that

$$(\mathtt{T}^c\, \Sigma)^n(\mathsf{I})(\{e_1, e_2\}) = \bigoplus_{m=1}^{n} \Big\{ \Bbbk x(e_1) \otimes_m x(e_2) \oplus \Bbbk x(e_2) \otimes_m x(e_1) \Big\},$$

where x refers to a generator of $\mathsf{I}(1) = \Bbbk$. The element ξ is represented by the product of $x(e_1)$ and $x(e_2)$ in $\mathtt{B}^n(\mathsf{E}_n)$:

$$x(e_1) \cdot x(e_2) = x(e_1) \otimes_n x(e_2) + \pm x(e_2) \otimes_n x(e_1) \in \mathtt{T}^c(\Sigma(\mathtt{T}^c\, \Sigma)^{n-1}(\mathsf{I})).$$

(1) First, we prove inductively that the differential in the iterated bar complex $\mathtt{B}^n(\mathsf{E}_n)$ of an element of the form $x(e_1) \otimes_m x(e_2)$ is given by a sum

$$\partial_\epsilon(x(e_1) \otimes_m x(e_2)) = \partial^0(x(e_1), x(e_2)) + \upsilon_m(e_1, e_2)$$

where $\partial^0(x(e_1), x(e_2)) = x(e_1) \otimes_{m-1} x(e_2) + \pm x(e_1) \otimes_{m-1} x(e_2) \in (\mathtt{T}^c\, \Sigma)^{m-1}(\mathsf{I})(\{e_1, e_2\})$

is the shuffle product of $x(e_1)$ and $x(e_2)$ in $(\mathtt{T}^c\, \Sigma)^{m-1}(\mathsf{I})$ and

$$\upsilon_m(e_1, e_2) \in \mathsf{E}_n(\{e_1, e_2\}) \subset (\mathtt{T}^c\, \Sigma)^n(\mathsf{E}_n)(\{e_1, e_2\})$$

is a representative of the $\cup_m$–product.

Recall that the operad E is supposed to be equipped with a chain-homotopy $\nu\colon \mathsf{E} \to \mathsf{E}$ so that $\delta(\nu) = \mathrm{id} - \iota\epsilon$ for some fixed section $\iota\colon \mathsf{C} \to \mathsf{E}$ of the augmentation $\epsilon\colon \mathsf{E} \to \mathsf{C}$. The $\cup_m$–products υ_m are defined inductively by $\upsilon_0 = \iota\mu$, where $\mu \in \mathsf{C}(2)$ represents

the product of commutative algebras, and $\upsilon_m = \nu(\upsilon_{m-1} + \pm\tau\upsilon_{m-1})$, where $\tau \in \Sigma_2$ denotes the transposition of $(1,2)$. Our assumptions on E_∞–operads ensure that $\upsilon_m \in \mathsf{E}_n(2)$ for $m < n$. The cocycle $\upsilon_{n-1} + \pm\tau\upsilon_{n-1} \in \mathsf{E}_n(2)$ defines a representative of the operation $\lambda_{n-1} \in \mathsf{G}_n(2)$.

By equivariance, we can assume $e_1 < e_2$. Observe that $x(e_1) \otimes_m x(e_2)$ belongs to the generating Σ_*–module of $\mathtt{B}^n(\mathsf{E}_n)$. By definition of the twisting homomorphism ∂_ϵ, the differential of $x(e_1) \otimes_m x(e_2)$ in $\mathtt{B}^n(\mathsf{E}_n)$ has an expansion of the form

$$\partial_\epsilon(x(e_1) \otimes_m x(e_2)) = \sum_{r=0}^{\infty} \epsilon_r(x(e_1) \otimes_m x(e_2)).$$

For $m = 1$, the definition of Section 2.4 returns

$$\epsilon_0(x(e_1) \otimes_1 x(e_2)) = \iota\mu(e_1, e_2) \in \mathsf{E}_n(\{e_1 < e_2\}).$$

For $m > 1$, we obtain

$$\epsilon_0(x(e_1) \otimes_m x(e_2)) = x(e_1) \otimes_{m-1} x(e_2) + \pm x(e_2) \otimes_{m-1} x(e_1),$$

the shuffle product of $x(e_1)$ and $x(e_2)$ in $(\mathrm{T}^c\,\Sigma)^{m-1}(\mathsf{I})$.

From the definition of Section 2.4, we see by an easy induction on m and r that the term $\epsilon_r(x(e_1) \otimes_m x(e_2))$ vanishes for $0 < r < m-1$.

For $r = m-1$, we obtain

$$\begin{aligned}\epsilon_{m-1}(x(e_1) \otimes_m x(e_2)) &= \widetilde{\nu} \cdot \partial_{\epsilon_{m-2}} \cdot \partial_{\epsilon_0}(x(e_1) \otimes_m x(e_2)) \\ &= \widetilde{\nu}\partial_{\epsilon_{m-2}}(x(e_1) \otimes_{m-1} x(e_2) + \pm x(e_2) \otimes_{m-1} x(e_1)),\end{aligned}$$

where $\widetilde{\nu}$ arises from a natural extension of the chain-homotopy ν. This identity gives by an immediate induction

$$\epsilon_{m-1}(x(e_1) \otimes_m x(e_2)) = \upsilon_m(e_1, e_2) \in \mathsf{E}_n(\{e_1 < e_2\}),$$

and the terms $\epsilon_r(x(e_1) \otimes_m x(e_2))$ are trivial for $r > m-1$.

(2) From the result

$$\partial_\epsilon(x(e_1) \otimes_n x(e_2)) = x(e_1) \otimes_{n-1} x(e_2) + x(e_2) \otimes_{n-1} x(e_1) + \upsilon_n(e_1, e_2),$$

we deduce

$$\begin{aligned}\partial_\epsilon(x(e_1) \cdot x(e_2)) &= \partial_\epsilon(x(e_1) \otimes_n x(e_2)) + \pm\partial_\epsilon(x(e_2) \otimes_n x(e_1)) \\ &= (x(e_1) \otimes_{n-1} x(e_2) + \pm \pm x(e_1) \otimes_{n-1} x(e_2)) \\ &\quad + (\pm x(e_2) \otimes_{n-1} x(e_1) + \pm x(e_2) \otimes_{n-1} x(e_1)) \\ &\quad + (\upsilon_n(e_1, e_2) + \pm\upsilon_n(e_2, e_1)).\end{aligned}$$

In our verification, we have not specified any sign, but the coherence of dg-algebra ensures that:

- the terms $x(e_1) \otimes_{n-1} x(e_2)$ cancel each other in the expansion of $\partial_\epsilon(x(e_1) \cdot x(e_2))$, and so do the terms $x(e_2) \otimes_{n-1} x(e_1)$;
- the terms $\upsilon_n(e_1, e_2)$ and $\upsilon_n(e_2, e_1)$ assemble to the Lie bracket $\lambda_{n-1}(e_1, e_2) = \upsilon_n(e_1, e_2) + \pm\upsilon_n(e_2, e_1)$.

Thus we obtain

$$\partial_\epsilon(x(e_1) \cdot x(e_2)) = \lambda_{n-1}(e_1, e_2).$$

The conclusion about the differential of $x(e_1) \cdot x(e_2)$ in $\mathtt{E}^1(B^n_{\mathsf{E}_n})$ follows. □

We use the coalgebra structure of the iterated bar complexes to determine the E^2–term of the spectral sequence $\mathtt{E}^1(B^n_{\mathsf{E}_n}) \Rightarrow \mathtt{H}_*(B^n_{\mathsf{E}_n})$ from the partial calculations of Lemmas 8.10–8.11.

Claim *The filtration of Section 8.3 satisfies the relation*

$$\Delta(\mathtt{F}_r\, B^n_{\mathsf{E}_n}) \subset \sum_{s+t=r} \mathtt{F}_s\, B^n_{\mathsf{E}_n} \otimes \mathtt{F}_t\, B^n_{\mathsf{E}_n}$$

with respect to the coproduct of $B^n_{\mathsf{E}_n}$.

Proof Indeed, the isomorphism $(\mathtt{T}^c\,\Sigma)^n(M) \simeq (\mathtt{T}^c\,\Sigma)^n(\mathsf{I}) \circ M = T^n \circ M$ defines an isomorphism of coalgebras in the sense that the diagonal of $(\mathtt{T}^c\,\Sigma)^n(M)$ corresponds to the composite of the morphism

$$\Delta \circ M\colon T^n \circ M \to (T^n \otimes T^n) \circ M$$

induced by the diagonal of $T^n = (\mathtt{T}^c\,\Sigma)^n(\mathsf{I})$ with the distribution isomorphism

$$(T^n \otimes T^n) \circ M \simeq T^n \circ M \otimes T^n \circ M.$$

The diagonal of T^n satisfies $\Delta(T^n(r)) \subset \bigoplus_{s+t=r} \Sigma_r \otimes_{\Sigma_s \times \Sigma_t} T^n(s) \otimes T^n(t)$ by definition of the tensor product of Σ_*–modules. Hence, we obtain $\Delta(\mathtt{F}_r\, T^n) \subset \sum_{s+t=r} \mathtt{F}_s\, T^n \otimes \mathtt{F}_t\, T^n$, from which we deduce $\Delta(\mathtt{F}_r\, T^n \circ \mathsf{E}_n) \subset \sum_{s+t=r} \mathtt{F}_s\, T^n \circ \mathsf{E}_n \otimes \mathtt{F}_t\, T^n \circ \mathsf{E}_n$ and the claim follows. □

This observation implies that the spectral sequence of Section 8.3 defines a spectral sequence of coalgebras $\mathtt{E}^r(B^n_{\mathsf{E}_n}) \Rightarrow \mathtt{H}_*(B^n_{\mathsf{E}_n})$.

Recall that any commutative algebra $\mathtt{C}(M)$ is equipped with a natural Hopf algebra structure so that M is primitive in $\mathtt{C}(M)$. The result of Proposition 8.9 can be improved to:

8.12 Lemma *The isomorphism of Proposition 8.9*

$$\mathrm{E}^1(B^n_{\mathsf{E}_n}) \simeq \mathsf{C}(\Sigma^{n-1}\,\mathrm{L}^c(\Sigma\,\mathsf{G}_n))$$

defines an isomorphism of coalgebras.

Proof Let $M = \Sigma^{n-1}\,\mathrm{L}^c(\Sigma\,\mathsf{I})$. The distribution isomorphism

$$\mathsf{C}(M \circ \mathsf{G}_n) \simeq \mathsf{C}(M) \circ \mathsf{G}_n$$

maps the natural diagonal of the commutative algebra to the composite

$$\mathsf{C}(M) \circ \mathsf{G}_n \xrightarrow{\Delta \circ \mathsf{G}_n} (\mathsf{C}(M) \otimes \mathsf{C}(M)) \circ \mathsf{G}_n \simeq \mathsf{C}(M) \circ \mathsf{G}_n \otimes \mathsf{C}(M) \circ \mathsf{G}_n,$$

where Δ refers to the diagonal of $\mathsf{C}(M)$.

The isomorphism $\mathrm{H}_*(\mathsf{B}^n(\mathsf{I})) \simeq \mathsf{C}(\Sigma^{n-1}\,\mathrm{L}^c(\Sigma\,\mathsf{I}))$ is an isomorphism of coalgebras by Proposition 6.5 and the Künneth morphism $\mathrm{H}_*(T^n) \circ \mathrm{H}_*(\mathsf{E}_n) \to \mathrm{H}_*(T^n \circ \mathsf{E}_n)$ forms clearly a morphism of coalgebras with respect to the coalgebra structure yielded by T^n. The conclusion follows immediately. □

8.13 Coderivations on cofree coalgebras Recall (see Section 6.4) that the diagonal of a monomial $\xi = x_1 \cdots x_r \in \mathsf{C}(M)$ in the free (nonunitary) commutative algebra $\mathsf{C}(M)$ is defined by the formula:

$$\Delta(x_1 \cdots x_r) = \sum_{\substack{p+q=r \\ p,q>0}} \Big(\sum_{w \in \mathrm{shuffle}(p,q)} \pm (x_{w(1)} \cdots x_{w(p)}) \otimes (x_{w(p+1)} \cdots x_{w(p+q)}) \Big),$$

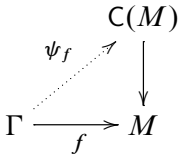
where the inner sum ranges over the set of (p,q)–shuffles in Σ_r. Recall again that $\mathsf{C}(M)$ refers to a free commutative algebra without unit. Therefore the expansion of $\Delta(x_1 \cdots x_r)$ runs over pairs (p,q) such that $p,q > 0$. Note that the (p,q)–shuffle $(x_{w(1)} \cdots x_{w(p)}) \otimes (x_{w(p+1)} \cdots x_{w(p+q)})$ includes a permutation of the inputs $\underline{\mathrm{e}}_i$ of the elements $x_i \in M(\underline{\mathrm{e}}_i)$.

Let $\mathsf{C}_r(M) = (M^{\otimes r})_{\Sigma_r}$. The n–fold diagonal $\Delta^n \colon \mathsf{C}(M) \to \mathsf{C}(M)^{\otimes n}$ is defined inductively by $\Delta^n = \Delta^{n-1} \otimes \mathrm{id} \cdot \Delta$ (as in the unital setting since the coassociativity relation does not change). The composite of the n–fold diagonal Δ^n with the canonical projection $\mathsf{C}(M) \to \mathsf{C}_1(M) = M$ vanishes over the summands $\mathsf{C}_s(M)$, $s \neq r$, and is identified with the trace morphism on the summand $\mathsf{C}_r(M)$. Recall that the trace morphism Tr_{Σ_r} is the morphism

$$(M^{\otimes r})_{\Sigma_r} \xrightarrow{\mathrm{Tr}_{\Sigma_r}} (M^{\otimes r})^{\Sigma_r} \subset M^{\otimes r}$$

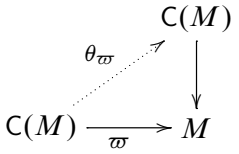
defined by the sum of all tensor permutations $w_* \colon M^{\otimes r} \to M^{\otimes r}$, $w \in \Sigma_r$.

For a Σ_*–module M such that $M(0) = 0$ the trace Tr_{Σ_*} is an isomorphism (see [19, Section 1.1] or [20, Sections 1.2.12–1.2.19]) and this observation implies that $\mathsf{C}(M) = \bigoplus_{n=1}(M^{\otimes r})_{\Sigma_r}$ satisfies the universal property of a cofree coalgebra (without counit): any morphism $f\colon \Gamma \to M$, where Γ is a coalgebra in Σ_*–modules, admits a unique lifting



such that ψ_f is a morphism of coalgebras.

Similarly, any homomorphism $\varpi\colon \mathsf{C}(M) \to M$ has a unique lifting



such that θ_ϖ is a coderivation. This coderivation is given on products $x_1 \cdots x_r \in \mathsf{C}_r(M)$ by the formula

$$\theta_\varpi(x_1 \cdots x_r) = \sum_{\substack{p+q=r \\ p,q>0}} \left\{ \sum_{w \in \mathrm{shuffle}(p,q)} \pm x_{w(1)} \cdots x_{w(p)} \cdot \varpi(x_{w(p+1)} \cdots x_{w(p+q)}) \right\}.$$

Note that ϖ is recovered from the associated coderivation θ_ϖ by the composite of $\theta_\varpi\colon \mathsf{C}(M) \to \mathsf{C}(M)$ with the canonical projection $\epsilon\colon \mathsf{C}(M) \to M$.

Lemma 8.12 implies that the differential $\mathtt{d}^1$ of the spectral sequence $\mathtt{E}^1(B^n_{\mathsf{E}_n}) \Rightarrow \mathtt{H}_*(B^n_{\mathsf{E}_n})$ is given by a formula of this form.

8.14 The natural spectral sequence of a cofree coalgebra The cofree coalgebra $\mathsf{C}(M)$ admits a canonical filtration defined by

$$\mathtt{F}_s\,\mathsf{C}(M) = \bigoplus_{r \leq s} \mathsf{C}_r(M).$$

Let $\Gamma = (\mathsf{C}(M), \partial)$ be a quasicofree coalgebra so that the twisting homomorphism $\partial\colon \mathsf{C}(M) \to \mathsf{C}(M)$ is a coderivation. Thus we have $\partial = \theta_\varpi$, for a certain homomorphism $\varpi\colon \mathsf{C}(M) \to M$.

The formula of Section 8.13 implies that $\theta_\varpi(\mathtt{F}_s\,\mathsf{C}(M)) \subset \mathtt{F}_s\,\mathsf{C}(M)$. From this observation, we deduce that the quasicofree coalgebra $\Gamma = (\mathsf{C}(M), \partial)$ has a filtration by

twisted dg-modules such that

$$\mathrm{F}_s\,\Gamma = (\mathrm{F}_s\,\mathrm{C}(M), \partial).$$

Let $\mathrm{D}^r(\Gamma) \Rightarrow \mathrm{H}_*(\Gamma)$ be the spectral sequence defined by this filtration.

Note that $\Delta(\mathrm{F}_r\,\mathrm{C}(M)) \subset \sum_{p+q=r} \mathrm{F}_p\,\mathrm{C}(M) \otimes \mathrm{F}_q\,\mathrm{C}(M)$. This relation implies that the spectral sequence $\mathrm{D}^r(\Gamma) \Rightarrow \mathrm{H}_*(\Gamma)$ forms a spectral sequence of coalgebras.

We have an obvious isomorphism $\mathrm{D}^0(\Gamma) \simeq \mathrm{C}(M)$.

Recall that ϖ is determined from $\partial = \theta_\varpi$ by the composite

$$\mathrm{C}(M) \xrightarrow{\partial} \mathrm{C}(M) \xrightarrow{\epsilon} M,$$

where ϵ refers to the canonical projection of $\mathrm{C}(M)$ onto the summand $M \subset \mathrm{C}(M)$. Let $\varpi_r\colon \mathrm{C}_{r+1}(M) \to M$ be the restriction of $\varpi\colon \mathrm{C}(M) \to M$ to the summand $\mathrm{C}_{r+1}(M) \subset \mathrm{C}(M)$, for $r \in \mathbb{N}$. Let $\partial^r = \theta_{\varpi_r}$ be the coderivation associated to this homomorphism. We clearly have $\varpi = \sum_r \varpi_r \Rightarrow \partial = \sum_{r=0}^{\infty} \partial^r$.

The formula of Section 8.13 implies moreover $\partial^r(\mathrm{F}_s\,\mathrm{C}(M)) = \theta_{\varpi_r}(\mathrm{F}_s\,\mathrm{C}(M)) \subset \mathrm{F}_{s-r}\,\mathrm{C}(M)$. As a consequence, the differential d^0 of the spectral sequence $\mathrm{D}^r(\Gamma)$ is identified with the coderivation $\partial^0 = \theta_{\varpi_0}\colon \mathrm{C}(M) \to \mathrm{C}(M)$. Note that the assumption $\varpi_0(M) \subset M$ implies that $\partial^0 = \theta_{\varpi_0}$ defines a differential on M and the quasicofree coalgebra $(\mathrm{D}^0, \mathrm{d}^0) = (\mathrm{C}(M), \partial^0)$ is isomorphic to the cofree coalgebra $(\mathrm{D}^0, \mathrm{d}^0) = \mathrm{C}(M, \partial^0)$ cogenerated by the twisted Σ_*–object (M, ∂^0).

We apply the spectral sequence of cofree coalgebras to the quasicofree coalgebra

$$\mathrm{E}^1(B^n_{\mathsf{E}_n}) = \mathrm{C}(\Sigma^{n-1}\,\mathrm{L}^c(\Sigma\,\mathsf{G}_n)).$$

We have in this case:

8.15 Lemma *The D^0–term of the spectral sequence $\mathrm{D}^r(\mathrm{E}^1(B^n_{\mathsf{E}_n}))$ is identified with the cofree coalgebra on the $(n-1)$–fold suspension of the Harrison complex $(\mathrm{L}^c(\Sigma\,\mathsf{G}_n), \partial)$.*

Proof By observations of Section 8.14, the differential $\mathrm{d}^0\colon \mathrm{D}^0 \to \mathrm{D}^0$ is the coderivation of E^1 associated to the component

$$\begin{array}{ccc} \mathrm{C}(\Sigma^{n-1}\,\mathrm{L}^c(\Sigma\,\mathsf{G}_n)) & \xrightarrow{\mathrm{d}^1} & \mathrm{C}(\Sigma^{n-1}\,\mathrm{L}^c(\Sigma\,\mathsf{G}_n)) \\ \uparrow & & \downarrow \\ \Sigma^{n-1}\,\mathrm{L}^c(\Sigma\,\mathsf{G}_n) & \xrightarrow{\partial^0} & \Sigma^{n-1}\,\mathrm{L}^c(\Sigma\,\mathsf{G}_n) \end{array}$$

of the differential of E^1. In Lemma 8.10, we observe that this component of d^1 is identified with the Harrison differential. The conclusion follows. □

From this Lemma, we deduce:

8.16 Proposition *We have* $\mathtt{D}^1(\mathtt{E}^1(B^n_{\mathsf{E}_n})) = \mathtt{C}(\Sigma^n \Lambda^{1-n} \mathtt{L})$.

Proof The composite $\mathtt{G}_n = \mathtt{C} \circ \Lambda^{1-n} \mathtt{L}$ is identified with the free commutative algebra $\mathtt{G}_n = \mathtt{C}(\Lambda^{1-n} \mathtt{L})$. Thus we have $\mathtt{H}_*(\mathtt{L}^c(\Sigma\, \mathtt{G}_n), \partial) = \Lambda^{1-n} \mathtt{L}$ by Proposition 6.9.

Since the Σ_*–modules $\mathtt{L}^c(\Sigma\, \mathtt{G}_n)$ and $\Lambda^{1-n} \mathtt{L}$ are both $\mathbb{k}$–projective and connected, the Künneth morphism

$$\mathtt{C} \circ \mathtt{H}_*(\mathtt{L}^c(\Sigma\, \mathtt{G}_n), \partial) \to \mathtt{H}_*(\mathtt{C} \circ (\mathtt{L}^c(\Sigma\, \mathtt{G}_n), \partial))$$

is an isomorphism (according to Section 0.14). Equivalently, we have an isomorphism

$$\mathtt{C}(\mathtt{H}_*(\mathtt{L}^c(\Sigma\, \mathtt{G}_n), \partial)) \xrightarrow{\simeq} \mathtt{H}_*(\mathtt{C}(\mathtt{L}^c(\Sigma\, \mathtt{G}_n), \partial)) = \mathtt{H}_*(\mathtt{D}^0(\Gamma), \mathtt{d}^0),$$

from which we conclude $\mathtt{D}^1(\Gamma) = \mathtt{C}(\Sigma^n \Lambda^{1-n} \mathtt{L})$. □

For the next stage of the spectral sequence, we obtain:

8.17 Lemma *For* $\Gamma = \mathtt{E}^1(B^n_{\mathsf{E}_n})$, *the chain complex* $(\mathtt{D}^1(\Gamma), \mathtt{d}^1(\Gamma))$ *is identified with the Chevalley–Eilenberg complex of the free Lie algebra* $\mathtt{L}(\Sigma^{n-1} \mathsf{I})$.

Proof Note that $\Sigma^{n-1} \Lambda^{1-n} \mathtt{L} = \mathtt{L}(\Sigma^{n-1} \mathsf{I})$ by definition of operadic suspensions.

By observations of Section 8.14, the component

$$\begin{array}{ccc} \mathtt{C}(\Sigma^{n-1}\, \mathtt{L}^c(\Sigma\, \mathtt{G}_n)) & \xrightarrow{\partial} & \mathtt{C}(\Sigma^{n-1}\, \mathtt{L}^c(\Sigma\, \mathtt{G}_n)) \\ \uparrow & & \downarrow \\ \mathtt{C}_{r+1}(\Sigma^{n-1}\, \mathtt{L}^c(\Sigma\, \mathtt{G}_n)) & \xrightarrow[\varpi_{r+1}]{} & \Sigma^{n-1}\, \mathtt{L}^c(\Sigma\, \mathtt{G}_n) \end{array}$$

of the differential of $\mathtt{E}^1$ determines a coderivation ∂^r such that $\partial^r(\mathtt{F}_s\, \mathtt{E}^1) \subset \mathtt{F}_{s-r}\, \mathtt{E}^1$. Thus, to determine the differential $\mathtt{d}^1\colon \mathtt{D}^1 \to \mathtt{D}^1$, we have to determine the restriction of $\mathtt{d}^1\colon \mathtt{E}^1 \to \mathtt{E}^1$ to the submodule

$$\mathtt{D}^1_2 = \mathtt{C}_2(\Sigma^n \Lambda^{1-n} \mathtt{L}) \subset \mathtt{C}_2(\Sigma^{n-1}\, \mathtt{L}^c(\Sigma\, \mathtt{G}_n)).$$

In Lemma 8.11, we prove that the differential in $\mathtt{E}^1$ of an element of the form

$$x(e_1) \cdot x(e_2) \in \mathtt{C}_2(\Sigma^n \mathsf{I})(\{e_1, e_2\})$$

is given by $\mathtt{d}^1(x(e_1) \cdot x(e_2)) = \lambda_{n-1}(e_1, e_2)$. Since $\mathtt{d}^1\colon \mathtt{E}^1 \to \mathtt{E}^1$ is a differential of right $\mathtt{G}_n$–modules, we have $\mathtt{d}^1(p_1 \cdot p_2) = \lambda_{n-1}(p_1, p_2)$ for any product of elements $p_1, p_2 \in \mathtt{L}(\Sigma^{n-1} \mathsf{I})$.

Hence, we have $\varpi_2(p_1 \cdot p_2) = \lambda(p_1, p_2)$ and the coderivation associated to ϖ_2 has an expansion of the form

$$\partial^1(p_1 \cdots p_r) = \sum \pm p_1 \cdots \widehat{p_i} \cdots \widehat{p_j} \cdots \lambda_{n-1}(p_i, p_j).$$

The terms on the right-hand side belong to

$$\mathtt{D}^1 = \mathtt{C}_r(\Sigma^n \Lambda^{1-n} \mathtt{L}) \subset \mathtt{C}_2(\Sigma^{n-1} \mathtt{L}^c(\Sigma\, \mathtt{G}_n)).$$

Thus the expansion of $\mathtt{d}^1\colon \mathtt{D}^1 \to \mathtt{D}^1$ is given by the same formula, which is also identified with the expression of the Chevalley–Eilenberg differential. The Lemma follows. □

Then:

8.18 Proposition *We have* $\mathtt{D}^2(\mathtt{E}^1(B^n_{\mathtt{E}_n})) = \Sigma^n \mathtt{I}$.

Proof We have the distribution relation

$$(\mathtt{C}(\Sigma\, \mathtt{L}(\Sigma^{n-1} \mathtt{I})), \partial) = (\mathtt{C}(\Sigma\, \mathtt{L}(\mathtt{I})), \partial) \circ (\Sigma^{n-1} \mathtt{I}),$$

and the Chevalley–Eilenberg complex $(\mathtt{C}(\Sigma\, \mathtt{L}(\Sigma^{n-1} \mathtt{I})), \partial)$ is identified with a composite

$$(\mathtt{C}(\Sigma\, \mathtt{L}(\Sigma^{n-1} \mathtt{I})), \partial) = \Sigma\, \mathtt{K}(\mathtt{I}, \mathtt{L}, \mathtt{L}) \circ (\Sigma^{n-1} \mathtt{I})$$

where $\mathtt{K}(\mathtt{I}, \mathtt{L}, \mathtt{L})$ is the Koszul complex of the Lie operad (see [20, Sections 6.3–6.4]). This complex $\mathtt{K}(\mathtt{I}, \mathtt{L}, \mathtt{L})$ is acyclic because the Lie operad is Koszul (see [20]). Therefore, we obtain

$$\mathtt{D}^2 = \mathtt{H}_*(\mathtt{C}(\Sigma\, \mathtt{L}(\Sigma^{n-1} \mathtt{I})), \partial) = \Sigma^n \mathtt{I}.$$ □

This result implies immediately:

8.19 Proposition *The spectral sequence* $\mathtt{D}^r(\mathtt{E}^1, \mathtt{d}^1) \Rightarrow \mathtt{H}_*(\mathtt{E}^1, \mathtt{d}^1)$ *degenerates at* D^2 *and gives* $\mathtt{H}_*(\mathtt{E}^1, \mathtt{d}^1) = \Sigma^n \mathtt{I}$. □

Thus, we have

$$\mathtt{E}^2(B^n_{\mathtt{E}_n}) = \mathtt{H}_*(\mathtt{E}^1, \mathtt{d}^1) = \Sigma^n \mathtt{I},$$

from which we obtain:

8.20 Proposition *The spectral sequence* $\mathtt{E}^r(B^n_{\mathtt{E}_n}) \Rightarrow \mathtt{H}_*(B^n_{\mathtt{E}_n})$ *degenerates at the* E^2*–stage and returns* $\mathtt{H}_*(B^n_{\mathtt{E}_n}) = \Sigma^n \mathtt{I}$. □

Finally, we conclude:

8.21 Theorem *The augmentation $\epsilon\colon B^n_{\mathsf{E}_n} \to \Sigma^n \mathsf{I}$ defines a weak-equivalence of right E_n–modules and the bar module $B^n_{\mathsf{E}_n}$ produced by Theorem 5.4 defines, after desuspension, a cofibrant replacement of the composition unit I in the category of right E_n–modules.* □

As we explain in Section 7.3, this Theorem implies:

8.22 Theorem *Let E be a $\mathcal{K}$–cellular E_∞–operad satisfying all requirements of Section 5.1. We consider the E_n–operad E_n deduced from the $\mathcal{K}$–cellular structure of E and the n–fold bar complex $\mathtt{B}^n(-)\colon {}_{\mathsf{E}_n}\mathcal{E} \to \mathcal{E}$ of Theorem 5.4. We have*

$$\mathtt{H}_*(\Sigma^{-n}\,\mathtt{B}^n(A)) = \mathtt{Tor}^{\mathsf{E}_n}_*(\mathsf{I}, A) = \mathtt{H}^{\mathsf{E}_n}_*(A)$$

for every E_n–algebra $A \in {}_{\mathsf{E}_n}\mathcal{E}$ which forms a cofibrant object in the underlying category $\mathcal{E}$. □

9 Infinite bar complexes

The goal of this section is to extend the result of Theorem 8.22 to the case $n = \infty$. First of all, we have to define an infinite bar complex $\Sigma^{-\infty}\,\mathtt{B}^\infty(A)$ on the category of algebras over an E_∞–operad E.

9.1 The infinite bar complex and the infinite bar module Throughout this section, we use the letter R to refer either to the commutative operad $\mathsf{R} = \mathsf{C}$ or to an E_∞–operad $\mathsf{R} = \mathsf{E}$. In our construction, we observe that the iterated bar modules B^n_{R} are connected by suspension morphisms so that the diagram

$$\begin{array}{ccccccccc}
\Sigma^{-1}B_{\mathsf{E}} & \xrightarrow{\sigma_{\mathsf{E}}} & \cdots\cdots & \xrightarrow{\sigma_{\mathsf{E}}} & \Sigma^{1-n}B^{n-1}_{\mathsf{E}} & \xrightarrow{\sigma_{\mathsf{E}}} & \Sigma^{-n}B^n_{\mathsf{E}} & \xrightarrow{\sigma_{\mathsf{E}}} & \cdots\cdots \\
\sim\downarrow & & & & \sim\downarrow & & \sim\downarrow & & \\
\Sigma^{-1}B_{\mathsf{C}} & \xrightarrow{\sigma_{\mathsf{C}}} & \cdots\cdots & \xrightarrow{\sigma_{\mathsf{C}}} & \Sigma^{1-n}B^{n-1}_{\mathsf{C}} & \xrightarrow{\sigma_{\mathsf{C}}} & \Sigma^{-n}B^n_{\mathsf{C}} & \xrightarrow{\sigma_{\mathsf{C}}} & \cdots\cdots
\end{array}$$

commutes. For short, we may write $\sigma = \sigma_{\mathsf{R}}$. These suspension morphisms induce natural transformations at the functor level

$$\Sigma^{-1}\,\mathtt{B}(A) \xrightarrow{\sigma} \cdots \xrightarrow{\sigma} \Sigma^{1-n}\,\mathtt{B}^{n-1}(A) \xrightarrow{\sigma} \Sigma^{-n}\,\mathtt{B}^n(A) \xrightarrow{\sigma} \cdots.$$

Set
$$\Sigma^{-\infty}B^\infty_{\mathsf{R}} = \operatorname{colim}_n \Sigma^{-n}B^n_{\mathsf{R}}$$

and form the infinite bar complex

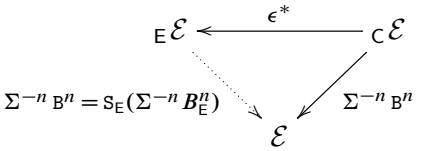
$$\Sigma^{-\infty}\,\mathtt{B}^\infty(A) = \operatorname{colim}_n \Sigma^{-n}\,\mathtt{B}^n(A).$$

Since the functor $(M, A) \mapsto \mathtt{S}_{\mathsf{R}}(M, A)$ commutes with colimits in M, we have the relation

$$\Sigma^{-\infty}\,\mathtt{B}^{\infty}(A) = \mathtt{S}_{\mathsf{R}}(\Sigma^{-\infty} B^{\infty}_{\mathsf{R}}, A)$$

for every R–algebra A.

The relationship $\Sigma^{-n} B^{n}_{\mathsf{E}} \circ_{\mathsf{E}} \mathsf{C} \simeq \Sigma^{-n} B^{n}_{\mathsf{C}}$ extends to $n = \infty$, because the suspension morphisms satisfy the coherence relation $\sigma_{\mathsf{E}} \circ_{\mathsf{E}} \mathsf{C} = \sigma_{\mathsf{C}}$, and the diagram



is still commutative at $n = \infty$.

We prove in Proposition 2.13 that the suspension morphisms are all cofibrations in the category of right R–modules. As a corollary:

9.2 Proposition *The infinite bar modules $\Sigma^{-\infty} B^{\infty}_{\mathsf{R}}$ form cofibrant objects in the category of right R–modules for $\mathsf{R} = \mathsf{E}, \mathsf{C}$.*

The morphisms $\epsilon\colon \Sigma^{-n} B^{n}_{\mathsf{E}} \xrightarrow{\sim} \Sigma^{-n} B^{n}_{\mathsf{C}}$, $n \in \mathbb{N}$, yield a weak-equivalence at $n = \infty$. □

We adapt the arguments of Section 8 to prove that the infinite bar complex determines the homology of E–algebras. We prove that the infinite bar modules $\Sigma^{-\infty} B^{\infty}_{\mathsf{R}}$, where $\mathsf{R} = \mathsf{E}, \mathsf{C}$, define a cofibrant replacement of I in the category of right R–modules to obtain the relation $\mathtt{Tor}^{\mathsf{R}}_{*}(\mathsf{I}, A) = \mathtt{H}_{*}(\mathtt{S}_{\mathsf{R}}(\Sigma^{-\infty} B^{\infty}_{\mathsf{R}}, A)$. We can address the cases $\mathsf{R} = \mathsf{C}$ and $\mathsf{R} = \mathsf{E}$ in parallel. In fact, Proposition 9.2 implies that $\Sigma^{-\infty} B^{\infty}_{\mathsf{E}}$ forms a cofibrant replacement of $\Sigma^{-\infty} B^{\infty}_{\mathsf{C}}$ in the category of right E–modules. Therefore it is sufficient to gain the result at the level of the commutative operad C. This situation contrasts with the case of finite iterations of the bar construction, addressed in Section 8, where we can not avoid the study of extended bar modules $B^{n}_{\mathsf{E}_n}$.

The right R–module $\Sigma^{-\infty} B^{\infty}_{\mathsf{R}}$ comes equipped with a natural augmentation $\epsilon\colon \Sigma^{-\infty} B^{\infty}_{\mathsf{R}} \to \mathsf{I}$ yielded by the morphisms of Section 8.1:

$$\Sigma^{-n} B^{n}_{\mathsf{R}} = \Sigma^{-n}\,\mathtt{B}^{n}(\mathsf{R}) \to \Sigma^{-n}\,\mathtt{B}^{n}(\mathsf{I}) \to \mathsf{I}.$$

We already observed that $\Sigma^{-\infty} B^{\infty}_{\mathsf{R}}$ forms a cofibrant object. It remains to check:

9.3 Lemma *We have $\mathtt{H}_{*}(\Sigma^{-\infty} B^{\infty}_{\mathsf{R}}) = \mathsf{I}$.*

Proof We have $\mathrm{H}_*(\Sigma^{-\infty} B^{\infty}_{\mathsf{E}}) = \mathrm{H}_*(\Sigma^{-\infty} B^{\infty}_{\mathsf{C}})$ by Proposition 9.2. Therefore we are reduced to prove the vanishing of $\mathrm{H}_*(\Sigma^{-\infty} B^{\infty}_{\mathsf{R}}) = \operatorname{colim}_n \mathrm{H}_*(\Sigma^{-n} B^{n}_{\mathsf{R}})$ for the commutative operad $\mathsf{R} = \mathsf{C}$.

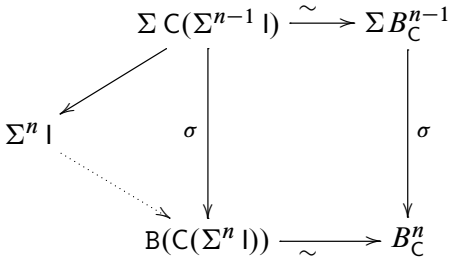
By Proposition 6.2, we have weak-equivalences

$$\mathsf{C}(\Sigma^n \mathsf{I}) \xrightarrow{\sim} \mathrm{B}(\mathsf{C}(\Sigma^{n-1} \mathsf{I})) \xrightarrow{\sim} \cdots \xrightarrow{\sim} \mathrm{B}^n(\mathsf{C}(\mathsf{I})) = B^n_{\mathsf{C}}.$$

In Section 1.9, we observe that the morphism

$$\sigma_*\colon \Sigma A \to \mathrm{H}_*(\mathrm{B}(A))$$

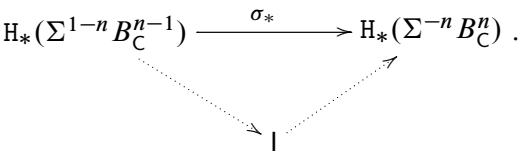
induced by the suspension factors through the indecomposables $\mathtt{Indec}\, A$, for every commutative algebra A. In the case $A = \mathsf{C}(\Sigma^{n-1}\mathsf{I})$, we have $\mathtt{Indec}\, A = \Sigma^{n-1}\mathsf{I}$ and we obtain a commutative diagram



from which we deduce that the morphism

$$\mathrm{H}_*(\Sigma^{1-n} B^{n-1}_{\mathsf{C}}) \xrightarrow{\sigma_*} \mathrm{H}_*(\Sigma^{-n} B^{n}_{\mathsf{C}})$$

admits a factorization



Hence we conclude

$$\mathrm{H}_*(\Sigma^{-\infty} B^{\infty}_{\mathsf{E}}) = \mathrm{H}_*(\Sigma^{-\infty} B^{\infty}_{\mathsf{C}}) = \operatorname{colim}_n \mathrm{H}_*(\Sigma^{-n} B^{n}_{\mathsf{C}}) = \mathsf{I}\,. \qquad \square$$

To summarize:

9.4 Proposition *The morphism $\epsilon\colon \Sigma^{-\infty} B^{\infty}_{\mathsf{R}} \to \mathsf{I}$ defines a weak-equivalence of right R–modules, so that $\Sigma^{-\infty} B^{\infty}_{\mathsf{R}}$ forms a cofibrant replacement of the composition unit I in the category of right R–modules for $\mathsf{R} = \mathsf{E}, \mathsf{C}$.*

Hence, we have

$$\mathtt{H}_*(\Sigma^{-\infty}\,\mathtt{B}^{\infty}(A)) = \mathtt{H}_*(\mathtt{S}_{\mathsf{R}}(\Sigma^{-\infty}B^{\infty}_{\mathsf{R}}, A)) = \mathtt{Tor}^{\mathsf{R}}_*(\mathsf{I}, A),$$

for $\mathsf{R} = \mathsf{E}, \mathsf{C}$, and from the identity $\mathtt{H}^{\mathsf{E}}_*(A) = \mathtt{Tor}^{\mathsf{E}}_*(\mathsf{I}, A)$ we conclude:

9.5 Theorem *For an E–algebra $A \in {}_{\mathsf{E}}\mathcal{E}$, we have the identity*

$$\mathtt{H}_*(\Sigma^{-\infty}\,\mathtt{B}^{\infty}(A)) = \mathtt{H}^{\mathsf{E}}_*(A)$$

as long as A forms a cofibrant object in the underlying category $\mathcal{E}$. □

Recall that the identity $\mathtt{H}^{\mathsf{R}}_*(A) = \mathtt{Tor}^{\mathsf{R}}_*(\mathsf{I}, A)$ makes sense for a Σ_*–cofibrant operad only, though the $\mathtt{Tor}$–functor $\mathtt{Tor}^{\mathsf{R}}_*(\mathsf{I}, A)$ can be defined as long as the operad R is $\mathcal{C}$–cofibrant. In the case of a commutative algebra, we have a restriction relation

$$\mathtt{S}_{\mathsf{E}}(\Sigma^{-\infty}B^{\infty}_{\mathsf{E}}, \epsilon^* A) \simeq \mathtt{S}_{\mathsf{C}}(\Sigma^{-\infty}B^{\infty}_{\mathsf{E}} \circ_{\mathsf{E}} \mathsf{C}, A) \simeq \mathtt{S}_{\mathsf{C}}(\Sigma^{-\infty}B^{\infty}_{\mathsf{C}}, A)$$

from which we deduce

$$\mathtt{H}^{\mathsf{E}}_*(\epsilon^* A) \simeq \mathtt{Tor}^{\mathsf{E}}_*(\mathsf{I}, \epsilon^* A) \simeq \mathtt{Tor}^{\mathsf{C}}_*(\mathsf{I}, A)$$

(we prefer to mark the restriction functor $\epsilon^*\colon {}_{\mathsf{C}}\mathcal{E} \to {}_{\mathsf{E}}\mathcal{E}$ in these formulas).

9.6 Remarks: Relationship with Γ–homology of commutative algebras In the case of an E_∞–operad E, the homology theory $\mathtt{H}^{\mathsf{E}}_*(A)$, defined abstractly in Section 7.2, represents the Γ–homology of A over $\Bbbk$ with trivial coefficients $\Bbbk$. The usual notation for this homology theory is $H\Gamma_*(A|\Bbbk, \Bbbk) = \mathtt{H}^{\mathsf{E}}_*(A)$.

The article [43] gives another chain complex $C\Xi_*(A|R, E)$ (rather denoted there by $\Xi_*(A|R, E)$) which determines the Γ–homology $H\Gamma_*(A|R, E)$ in the case where A is a commutative algebra over another commutative algebra R, and for any coefficient E in the category of A–modules. The author of [43] deals with unital commutative algebras over R, but a unital commutative algebra A_+ can be replaced by a quotient $A = A_+/R$ to give a normalized chain complex $N\Xi_*(A|R, E)$ equivalent to $C\Xi_*(A_+|R, E)$.

In the case $E = R = \Bbbk$, the normalized chain complex $N\Xi_*(A|\Bbbk, \Bbbk)$ can be identified with the functor $N\Xi_*(A|\Bbbk, \Bbbk) = \mathtt{S}_{\mathsf{C}}(N\Xi_{\mathsf{C}}, A)$ associated to a particular cofibrant replacement of I in the category of right C–modules. This cofibrant replacement $N\Xi_{\mathsf{C}}$ is defined over the ring $\Bbbk = \mathbb{Z}$.

Observe that any category of dg-modules over a ring R forms a symmetric monoidal category $\mathcal{E} = \mathcal{C}_R$ over the base category $\mathcal{C} = \mathcal{C}_{\Bbbk}$ of dg-modules over $\Bbbk$. An augmented

algebra over R is equivalent to a (nonunital nonaugmented) commutative algebra in $\mathcal{C}_R$ and the extended functor $\mathtt{S}_{\mathsf{C}}(N\Xi_{\mathsf{C}})\colon \mathcal{C}_R \to \mathcal{C}_R$ satisfies

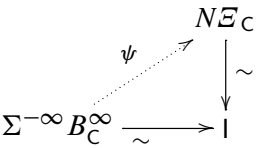
$$N\Xi_*(A|R, R) = \mathtt{S}_{\mathsf{C}}(N\Xi_{\mathsf{C}}, A),$$

for any $A \in {}_{\mathsf{C}}\mathcal{C}_R$.

The chain complex of [43] inherits a grading from $N\Xi_{\mathsf{C}}$ and an internal grading from A, so that $N\Xi_*(A|R, R)$ forms naturally a bigraded object. So does the infinite bar complex of commutative algebras. The morphism

$$\underbrace{\mathtt{S}_{\mathsf{C}}(\Sigma^{-\infty}B^{\infty}_{\mathsf{C}}, A)}_{=\,\Sigma^{-\infty}\mathtt{B}^{\infty}(A)} \xrightarrow{\mathtt{S}_{\mathsf{C}}(\psi, A)} \underbrace{\mathtt{S}_{\mathsf{C}}(N\Xi_{\mathsf{C}}, A)}_{N\Xi_*(A|\Bbbk, \Bbbk)}$$

associated to a lifting



preserves clearly bigradings. Hence, in the context of commutative algebras over a ring R, our Theorem gives a natural isomorphism of bigraded objects

$$\mathtt{H}_{*,*}(\Sigma^{-\infty}\mathtt{B}^{\infty}(A)) = H\Gamma_{*,*}(A|R, R)$$

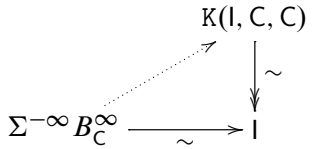
for every commutative algebra A which is cofibrant (or simply flat) in the underlying category of dg-modules over R.

9.7 Remarks: Relationship with Koszul duality and applications to the Lie operad The Koszul duality of operads gives another quasifree complex $\mathtt{K}(\mathsf{I}, \mathsf{C}, \mathsf{C})$ together with an acyclic fibration $\epsilon\colon \mathtt{K}(\mathsf{I}, \mathsf{C}, \mathsf{C}) \xrightarrow{\sim} \mathsf{I}$. In the proof of Proposition 6.9, we already recalled that this Koszul complex is identified with a desuspension of the Harrison complex of the commutative operad C, viewed as a commutative algebra in the category of right modules over itself. Equivalently, the Koszul complex $\mathtt{K}(\mathsf{I}, \mathsf{C}, \mathsf{C})$ is a quasifree module such that

$$\mathtt{K}(\mathsf{I}, \mathsf{C}, \mathsf{C}) = (\Sigma^{-1}\mathsf{L}^{c}(\Sigma\,\mathsf{C}), \partial) = (\Lambda^{-1}\mathsf{L}^{\vee} \circ \mathsf{C}, \partial),$$

where $\mathsf{L}^{\vee}$ refers to the $\Bbbk$–dual of the Lie operad L. This quasifree module is not cofibrant since the Lie operad does not form a cofibrant Σ_*–module.

Nevertheless, we can pick a lifting in the diagram



to obtain a weak-equivalence $\kappa\colon \Sigma^{-\infty} B^{\infty}_{\mathsf{C}} \xrightarrow{\sim} \mathtt{K}(\mathsf{I}, \mathsf{C}, \mathsf{C})$. By [20, Theorem 2.1.14] or [23, Theorem 15.3.A], the quasifree structure is sufficient to ensure that the morphism $\kappa \circ_{\mathsf{C}} \mathsf{I}$ is still a weak-equivalence. Thus, since $\Sigma^{-\infty} B^{\infty}_{\mathsf{C}} \circ_{\mathsf{C}} \mathsf{I} = \Sigma^{-\infty}\, \mathtt{B}^{\infty}(\mathsf{I})$ and $\mathtt{K}(\mathsf{I}, \mathsf{C}, \mathsf{C}) \circ_{\mathsf{C}} \mathsf{I} = \Lambda^{-1}\, \mathsf{L}^{\vee}$, we have a weak-equivalence of Σ_*–modules

$$\overline{\kappa}\colon \Sigma^{-\infty}\, \mathtt{B}^{\infty}(\mathsf{I}) \xrightarrow{\sim} \Lambda^{-1}\, \mathsf{L}^{\vee}.$$

Note that $\Sigma^{-\infty}\, \mathtt{B}^{\infty}(\mathsf{I})$ forms a cofibrant Σ_*–module. Thus the object $\Sigma^{-\infty}\, \mathtt{B}^{\infty}(\mathsf{I})$ defines a cofibrant replacement of the $\Bbbk$–dual of the Lie operad $\Lambda^{-1}\, \mathsf{L}^{\vee}$ in the category of Σ_*–modules. From this observation, we deduce an identity

$$\mathtt{H}_*(\mathtt{S}(\Sigma^{-\infty}\, \mathtt{B}^{\infty}(\mathsf{I}), \Bbbk)) = \bigoplus_{r=0}^{\infty} \mathtt{Tor}_*^{\Sigma_r}(\Lambda^{-1}\, \mathsf{L}^{\vee}(r), \Bbbk),$$

where $\mathtt{S}(\Sigma^{-\infty}\, \mathtt{B}^{\infty}(\mathsf{I}), \Bbbk)$ denotes the image of the free module of rank 1 under the symmetric tensor functor associated to $\Sigma^{-\infty}\, \mathtt{B}^{\infty}(\mathsf{I})$. The object $\mathtt{S}(\Sigma^{-\infty}\, \mathtt{B}^{\infty}(\mathsf{I}), \Bbbk)$ is identified with the iterated bar complex $\Sigma^{-\infty}\, \mathtt{B}^{\infty}(A)$ of a trivial algebra $A = \Bbbk e$ (which represents the nonunital algebra associated to a unital exterior algebra in one generator), since we have the restriction relation

$$\mathtt{S}(\Sigma^{-\infty}\, \mathtt{B}^{\infty}(\mathsf{I}), \Bbbk) = \mathtt{S}(\Sigma^{-\infty} B^{\infty}_{\mathsf{C}} \circ_{\mathsf{C}} \mathsf{I}, \Bbbk) = \mathtt{S}_{\mathsf{C}}(\Sigma^{-\infty} B^{\infty}_{\mathsf{C}}, \Bbbk e) = \Sigma^{-\infty}\, \mathtt{B}^{\infty}(\Bbbk e).$$

The arguments of [13] give the homology of the infinite bar complex $\Sigma^{-\infty}\, \mathtt{B}^{\infty}(A)$ for this particular commutative algebra, for every ground ring $\Bbbk$. Thus our result relates the calculation of $\mathtt{Tor}_*^{\Sigma_r}(\Lambda^{-1}\, \mathsf{L}^{\vee}(r), \Bbbk)$ to classical homological computations. Such $\mathtt{Tor}$–functors are determined by other methods in [2; 3] in the case $\Bbbk = \mathbb{F}_p$ (see also [10] for another approach to this computation).

Afterword: Applications to the cohomology of iterated loop spaces

The goal of this concluding part is to explain the applications of our main theorems to the cohomology of iterated loop spaces.

Let $\bar{N}^*(X)$ denote to the reduced normalized cochain complex of a simplicial set X. By [29] (see also [9; 40]), the cochain complex $\bar{N}^*(X)$ inherits an action of an E_∞–operad $\mathtt{E}$ so that the map $N^*\colon X \mapsto \bar{N}^*(X)$ defines a functor from the category of simplicial sets $\mathcal{S}$ to the category of $\mathtt{E}$–algebras in dg-modules ${}_{\mathtt{E}}\mathcal{C}$. Moreover, all functorial actions of an E_∞–operad $\mathtt{E}$ on the functor $\bar{N}^*\colon \mathcal{S} \to \mathcal{C}$ are homotopy equivalent. In [9], we prove by an explicit construction that the Barratt–Eccles operad is an instance of an E_∞–operad which acts on cochain complexes.

In [24], we prove that the n–fold bar complex $\mathtt{B}^n(\bar{N}^*(X))$ of a cochain complex $\bar{N}^*(X)$ determines the cohomology of the n–fold loop space of X under reasonable finiteness and completeness assumptions. Thus, with the new results of the present article, we obtain (in the case $\Bbbk = \mathbb{F}_p$):

Theorem *Let X be a pointed simplicial set whose cohomology modules $\mathtt{H}^*(X, \mathbb{F}_p)$ are degreewise finitely generated. Let $\bar{N}^*(X)$ be the reduced cochain complex of X with coefficients in $\Bbbk = \mathbb{F}_p$.*

Let $\mathtt{E}$ be any E_∞–operad which, like the Barratt–Eccles operad, acts on cochain complexes of spaces, admits a $\mathcal{K}$–cellular structure, and fulfils the requirements of Section 5.1, so that the conclusions of Theorem 5.5 and Theorem 8.22 hold for this operad. Then we have identities

$$\mathtt{H}_*^{\mathtt{E}_n}(\bar{N}^*(X)) = \mathtt{H}_*(\Sigma^{-n}\,\mathtt{B}^n(\bar{N}^*(X))) = \operatorname{colim}_s \bar{H}^*(\Sigma^n \Omega^n R_s X),$$

where $R_s X$ refers to the Bousfield–Kan tower of X. □

A similar result can be stated in the case $\Bbbk = \mathbb{Z}$ or in the case $\Bbbk = \mathbb{Q}$ (assuming in this case that X is n–connected).

The explicit construction of the n–fold bar complex implies the existence of a spectral sequence

$$\mathtt{B}^n(\mathtt{H}^*(X)) \Rightarrow \mathtt{H}^*(\mathtt{B}^n(X)).$$

We refer to [44; 45] for another definition of a similar spectral sequence converging to $\mathtt{H}_*(\Omega^n X)$. We conjecture that our spectral sequence is isomorphic (from the E^2–stage and up to $\Bbbk$–duality) to the $\mathtt{H}_*(-, \Bbbk)$–version of the spectral sequence of [1], defined from Goodwillie's approximations of the functor $\Sigma^\infty \operatorname{Map}(S^n, X)_+$. We prove in [17] that E_n–operads are (up to operadic suspension) self-dual in the sense of Koszul duality of operads. We deduce from this result another representation of the homology theory $\mathtt{H}_*^{\mathtt{E}_n}(A)$ which relates the n–fold bar complex $\mathtt{B}^n(A)$ to the $\Bbbk$–dual of the E_n–operad $\mathtt{E}_n$. We conjecture that this relationship reflects the occurrence of the little n–cubes operad in [1].

The reference [13] gives the homology of the iterated bar complexes of many usual commutative algebras, like exterior algebras, polynomial algebras, divided power algebras, abelian group algebras. These results could be used to determine E^2–terms in the spectral sequence $\mathtt{B}^n(\mathtt{H}^*(X)) \Rightarrow \mathtt{H}^*(\mathtt{B}^n(X))$. Note that the calculations of [13] are performed over $\mathbb{Z}$.

References

[1] **S T Ahearn**, **N J Kuhn**, *Product and other fine structure in polynomial resolutions of mapping spaces*, Algebr. Geom. Topol. 2 (2002) 591–647 MR1917068

[2] **G Arone**, **M Kankaanrinta**, *The homology of certain subgroups of the symmetric group with coefficients in $\mathcal{L}ie(n)$*, J. Pure Appl. Algebra 127 (1998) 1–14 MR1609516

[3] **G Arone**, **M Mahowald**, *The Goodwillie tower of the identity functor and the unstable periodic homotopy of spheres*, Invent. Math. 135 (1999) 743–788 MR1669268

[4] **M G Barratt**, **P J Eccles**, *Γ^+–structures. I. A free group functor for stable homotopy theory*, Topology 13 (1974) 25–45 MR0348737

[5] **M Basterra**, **M A Mandell**, *Homology of E_n ring spectra and iterated THH* arXiv:1007.5315

[6] **M A Batanin**, *Symmetrisation of n–operads and compactification of real configuration spaces*, Adv. Math. 211 (2007) 684–725 MR2323542

[7] **M A Batanin**, *The Eckmann–Hilton argument and higher operads*, Adv. Math. 217 (2008) 334–385 MR2365200

[8] **C Berger**, *Opérades cellulaires et espaces de lacets itérés*, Ann. Inst. Fourier (Grenoble) 46 (1996) 1125–1157 MR1415960

[9] **C Berger**, **B Fresse**, *Combinatorial operad actions on cochains*, Math. Proc. Cambridge Philos. Soc. 137 (2004) 135–174 MR2075046

[10] **S Betley**, **J Słomińska**, *New approach to the groups $H_*(\Sigma_n, \mathrm{Lie}_n)$ by the homology theory of the category of functors*, J. Pure Appl. Algebra 161 (2001) 31–43 MR1834077

[11] **J M Boardman**, **R M Vogt**, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Math. 347, Springer, Berlin (1973) MR0420609

[12] **N Bourbaki**, *Éléments de mathématique. Fasc. XXXVII. Groupes et algèbres de Lie. Chapitre II: Algèbres de Lie libres. Chapitre III: Groupes de Lie*, Act. Sci. et Ind. 1349, Hermann, Paris (1972) MR0573068

[13] **H Cartan**, **J C Moore**, **R Thom**, **J-P Serre**, *Algèbres d'Eilenberg–Mac Lane et homotopie*, 2ème éd edition, Séminaire Henri Cartan de l'Ecole Normale Supérieure, 1954/1955, Secrétariat math. (1956) MR0087935

[14] **D Chataur**, **M Livernet**, *Adem–Cartan operads*, Comm. Algebra 33 (2005) 4337–4360 MR2184004

[15] **F R Cohen**, **T J Lada**, **J P May**, *The homology of iterated loop spaces*, Lecture Notes in Math. 533, Springer, Berlin (1976) MR0436146

[16] **B Fresse**, *Koszul duality complexes for the cohomology of iterated loop spaces of spheres* `arXiv:1001.4650`

[17] **B Fresse**, *Koszul duality of E_n–operads*, to appear in Selecta Math. `arXiv:0904.3123`

[18] **B Fresse**, *La catégorie des arbres élagués de Batanin est de Koszul* `arXiv:0909.5447`

[19] **B Fresse**, *On the homotopy of simplicial algebras over an operad*, Trans. Amer. Math. Soc. 352 (2000) 4113–4141 MR1665330

[20] **B Fresse**, *Koszul duality of operads and homology of partition posets*, from: "Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K–theory", (P Goerss, S Priddy, editors), Contemp. Math. 346, Amer. Math. Soc. (2004) 115–215 MR2066499

[21] **B Fresse**, *Théorie des opérades de Koszul et homologie des algèbres de Poisson*, Ann. Math. Blaise Pascal 13 (2006) 237–312 MR2275449

[22] **B Fresse**, *Iterated bar complexes and the poset of pruned trees*, Preprint (2008) Available at `http://math.univ-lille1.fr/~fresse/IteratedBarAppendix.html`

[23] **B Fresse**, *Modules over operads and functors*, Lecture Notes in Math. 1967, Springer, Berlin (2009) MR2494775

[24] **B Fresse**, *The bar complex of an E–infinity algebra*, Adv. Math. 223 (2010) 2049–2096 MR2601008

[25] **E Getzler**, **J Jones**, *Operads, homotopy algebra and iterated integrals for double loop spaces* `arXiv:hep-th/9403055`

[26] **G Ginot**, *Homologie et modèle minimal des algèbres de Gerstenhaber*, Ann. Math. Blaise Pascal 11 (2004) 95–127 MR2077240

[27] **V Ginzburg**, **M Kapranov**, *Koszul duality for operads*, Duke Math. J. 76 (1994) 203–272 MR1301191

[28] **D K Harrison**, *Commutative algebras and cohomology*, Trans. Amer. Math. Soc. 104 (1962) 191–204 MR0142607

[29] **V A Hinich**, **V V Schechtman**, *On homotopy limit of homotopy algebras*, from: "K–theory, arithmetic and geometry (Moscow, 1984–1986)", (Y I Manin, editor), Lecture Notes in Math. 1289, Springer, Berlin (1987) 240–264 MR923138

[30] **M Hovey**, *Model categories*, Math. Surveys and Monogr. 63, Amer. Math. Soc. (1999) MR1650134

[31] **M Kontsevich**, *Operads and motives in deformation quantization*, Lett. Math. Phys. 48 (1999) 35–72 MR1718044 Moshé Flato (1937–1998)

[32] **M Livernet**, **B Richter**, *An interpretation of E_n–homology as functor homology*, to appear in Math. Z. arXiv:0907.1283

[33] **J-L Loday**, *Generalized bialgebras and triples of operads*, Astérisque 320, Soc. Math. France (2008) MR2504663

[34] **S Mac Lane**, *Homology*, Grund. der math. Wissenschaften 114, Academic Press, New York (1963) MR0156879

[35] **M A Mandell**, *Topological André–Quillen cohomology and E_∞ André–Quillen cohomology*, Adv. Math. 177 (2003) 227–279 MR1990939

[36] **M Markl**, *Distributive laws and Koszulness*, Ann. Inst. Fourier (Grenoble) 46 (1996) 307–323 MR1393517

[37] **M Markl**, *Models for operads*, Comm. Algebra 24 (1996) 1471–1500 MR1380606

[38] **M Markl**, **S Shnider**, **J Stasheff**, *Operads in algebra, topology and physics*, Math. Surveys and Monogr. 96, Amer. Math. Soc. (2002) MR1898414

[39] **J P May**, *The geometry of iterated loop spaces*, Lectures Notes in Math. 271, Springer, Berlin (1972) MR0420610

[40] **J E McClure**, **J H Smith**, *Multivariable cochain operations and little n–cubes*, J. Amer. Math. Soc. 16 (2003) 681–704 MR1969208

[41] **C Reutenauer**, *Free Lie algebras*, London Math. Soc. Monogr., New Series 7, Oxford Science Publ., The Clarendon Press, Oxford Univ. Press, New York (1993) MR1231799

[42] **B Richter**, *A lower bound for coherences on the Brown–Peterson spectrum*, Algebr. Geom. Topol. 6 (2006) 287–308 MR2199461

[43] **A Robinson**, *Gamma homology, Lie representations and E_∞ multiplications*, Invent. Math. 152 (2003) 331–348 MR1974890

[44] **V A Smirnov**, *On the chain complex of an iterated loop space*, Izv. Akad. Nauk SSSR Ser. Mat. 53 (1989) 1108–1119, 1135–1136 MR1024456

[45] **V A Smirnov**, *The homology of iterated loop spaces*, Forum Math. 14 (2002) 345–381 MR1899289 With an appendix by F Sergeraert

[46] **C R Stover**, *The equivalence of certain categories of twisted Lie and Hopf algebras over a commutative ring*, J. Pure Appl. Algebra 86 (1993) 289–326 MR1218107

UMR CNRS 8524, UFR de Mathématiques, Université Lille 1 - Sciences et Technologies
Cité Scientifique - Bâtiment M2, 59655 Villeneuve d'Ascq Cedex, France

Benoit.Fresse@math.univ-lille1.fr

http://math.univ-lille1.fr/~fresse/

Received: 6 December 2010