Unexpected local minima in the width complexes for knots

ALEXANDER ZUPAN

In [9], Schultens defines the width complex for a knot in order to understand the different positions a knot can occupy in S^3 and the isotopies between these positions. She poses several questions about these width complexes; in particular, she asks whether the width complex for a knot can have local minima that are not global minima. In this paper, we find an embedding of the unknot 0_1 that is a local minimum but not a global minimum in the width complex for 0_1, resolving a question of Scharlemann. We use this embedding to exhibit for any knot K infinitely many distinct local minima that are not global minima of the width complex for K.

57M25, 57M27

1 Introduction

In [2], Gabai defines knot width and thin position as a measure of the complexity of various Morse functions on a given knot in S^3. One important aspect of thin position is that it yields an embedding of a given knot that is minimal with respect to certain types of isotopies. In [9], Schultens defines the width complex of a knot in order to better understand these isotopies and the various positions a given knot can occupy in S^3. Specifically, she asks the following two questions:

Question 12 Can the width complex of a knot have local minima that are not global minima?

Question 13 Is every vertex of the width complex of a knot connected to one of the global minima of this complex by a monotonically decreasing path?

Schultens also defines a similar width complex for 3–manifolds, and her Theorem 13 from [9] provides a positive answer to the 3–manifold version of Question 12, namely that there exist 3–manifolds whose width complexes contain local minima that are not global minima. On the other hand, combining the results of Bonahon and Otal [1], Scharlemann and Thompson [6] and Waldhausen [11], we see that if M is S^3 or a lens space, then the width complex of M has a unique minimum, corresponding to a minimal genus Heegaard splitting. Thus, it seems reasonable to expect that the

Published: 1 April 2011 DOI: 10.2140/agt.2011.11.1097

simplest knots might share this property. This is further suggested by Otal's proof that nonminimal bridge positions of the unknot and 2–bridge knots are stabilized [3] and Ozawa's recent proof of the same statement for torus knots [4].

Schultens compares Question 13 to one answered by Goeritz in 1934. Goeritz produced a nontrivial diagram of the unknot 0_1 such that any Reidemeister move increases the diagram's crossing number. As an analogue to Goeritz' result, Scharlemann poses the next question in his comprehensive treatment of thin position:

Question 3.5 [5] Suppose $K \subset S^3$ is the unknot. Is there an isotopy of K to thin position (ie a single minimum and maximum) via an isotopy during which the width is never increasing?

We provide an answer, finding a nontrivial embedding of 0_1 such that any isotopy must increase the width of the embedding. As a result, we give an affirmative answer to Schultens' first question, which shows that the answer to the second question must be no. In fact, we show the surprising and much stronger result that for every knot K, the width complex of K has infinitely many local minima that are not global minima.

2 Definitions

Let K be a knot in S^3, and fix a Morse function $h\colon S^3 \to \mathbb{R}$ such that h has exactly two critical points. We consider K to be an equivalence class, denoted $\mathcal{K}$, of the set of embeddings of S^1 into S^3 modulo ambient isotopy. In the usual definition of knot width, the embedding of K is fixed and the Morse function h is allowed to vary up to isotopy; however, this definition is equivalent with the one that follows. Let $k \in \mathcal{K}$ such that $h|_k$ is Morse, and let $c_0 < c_1 < \cdots < c_n$ be the critical levels of $h|_k$. Choose regular levels $c_0 < r_1 < c_1 < \cdots < r_n < c_n$, and define

$$w(k) = \sum_{i=1}^{n} |h^{-1}(r_i) \cap K|.$$

Now, let

$$w(K) = \min_{k \in \mathcal{K}} w(k).$$

The invariant $w(K)$ is called the width of K, and if $k \in \mathcal{K}$ satisfies $w(K) = w(k)$, we say that k is a thin position for K.

For our purposes it will be useful to split an embedding k into thick and thin levels. For $2 \le i \le n-1$, we say that a regular value r_i of $h|_k$ corresponds to a thick level $R_i = h^{-1}(r_i)$ if $|h^{-1}(r_i) \cap k| > |h^{-1}(r_{i-1}) \cap k|, |h^{-1}(r_{i+1}) \cap k|$. Likewise, r_i corresponds to a thin level $R_i = h^{-1}(r_i)$ if $|h^{-1}(r_i) \cap k| < |h^{-1}(r_{i-1}) \cap k|, |h^{-1}(r_{i+1}) \cap k|$. Let $a_0, \dots, a_m$ $(b_1, \dots, b_m)$ represent the regular values of $h\,|_k$ corresponding to thick (thin) levels $A_0, \dots, A_m$ $(B_1, \dots, B_m)$, where $a_0 < b_1 < a_1 < \dots < b_m < a_m$.

Note that $h^{-1}([b_i, a_i]) \cap k$ consists of vertical segments and arcs $\alpha_1, \dots, \alpha_l$, $l \geq 1$, where each α_j has exactly one minimum and is isotopic to an arc β_j in A_i. In this case, α_j cobounds a disk D with β_j such that D has no critical points with respect to h in its interior. We call D a *strict lower disk for k at A_i*. For any $r < c_0$, the lowest minimum of $h|_k$, we have that $h^{-1}([r, a_0]) \cap k$ consists of arcs $\alpha_1, \dots, \alpha_l$, which cobound pairwise disjoint strict lower disks with arcs $\beta_1, \dots, \beta_l$ contained in A_1.

Similarly, $h^{-1}([a_i, b_{i+1}]) \cap k$ consists of vertical segments and arcs $\alpha_1, \dots, \alpha_l$, $l \geq 1$, where each α_j has exactly one maximum and is isotopic to an arc β_j in A_i. Here α_j cobounds a disk E with β_j such that E has no critical points in its interior, and we call E a *strict upper disk for k at A_i*. For any $r > c_n$, the highest maximum of $h|_k$, we have that $h^{-1}([a_n, r]) \cap k$ consists of arcs $\alpha_1, \dots, \alpha_l$, which cobound pairwise disjoint strict upper disks with arcs $\beta_1, \dots, \beta_l$ contained in A_n.

Consider $k, k' \in K$ with corresponding thick/thin levels $A_0, B_1, A_1, \dots, B_l, A_l$ and $A'_0, B'_1, \dots, B'_{l'}, A'_{l'}$. We say that $k \sim k'$ if $l = l'$ and there is an isotopy of S^3 taking k to k', A_i to A'_i, and B_i to B'_i. In this case, we call this isotopy a *level isotopy*, and we have $w(k) = w(k')$, so that k and k' carry exactly the same information with respect to width and to upper and lower disks. Thus, from this point forward we will (under slight abuse of notation) let $\mathcal{K}$ denote the set of embeddings isotopic to K up to this equivalence.

3 The width complex of K

Now, we use the collection $\mathcal{K}$ and pairs of strict upper and lower disks to define the width complex of K, a directed graph Γ whose vertices correspond to elements of $\mathcal{K}$. We first make several definitions:

Definition 3.1 Suppose that $k \in \mathcal{K}$. If (D, E) is a pair of strict upper and lower disks for a thick level A_i such that $D \cap E$ is a single point in k, we say that A_i is *stabilized*. If $D \cap E = \varnothing$, we say that A_i is *weakly reducible*. In either case, we say that A_i is *reducible*. If A_i is not reducible, then A_i is *strongly irreducible*.

Elements of $k \in \mathcal{K}$ with reducible thick surfaces will be at the tail of directed edges in the width complex of K. If $k \in \mathcal{K}$ has a stabilized thick surface A_i, we can slide k along the pair (D, E) of upper and lower disks for A_i to cancel out a minimum and maximum, changing k to $k' \in \mathcal{K}$ such that $w(k') = w(k) - (2|A_i \cap k| - 2)$. As in [9], we call this a *Type I move*. If k has a weakly reducible thick surface A_i, we can again slide k along the pair (D, E) to move a minimum of k above a maximum of k. This changes k to $k' \in \mathcal{K}$ such that $w(k') = w(k) - 4$, and we call this a *Type II move*. In either case, we call (D, E) a *pair of reducing disks* at A_i and we make a directed edge from k to k' in Γ. The next theorem, Theorem 1 from [9], is important to our understanding of the width complex:

Theorem 3.2 *The width complex of a knot is connected.*

This theorem says that given $k, k' \in \mathcal{K}$, there is a series of level isotopies and Type I and Type II moves and their inverses taking k to k'. Schultens' width complex also contains higher dimensional cells, but we need only consider the one-skeleton of the complex in this context.

Definition 3.3 We call $k \in \mathcal{K}$ a *local minimum* of the width complex if there are no directed edges leaving k in Γ.

The position k is called a local minimum because any isotopy that changes k to $k' \in \mathcal{K}$ must increase $w(K)$. Let $\widehat{\mathcal{K}} \subset \mathcal{K}$ denote the set of local minima of the width complex of K. It is clear that any thin position k for K must come from $\widehat{\mathcal{K}}$; otherwise there is an isotopy decreasing $w(k)$. We also have the following, the proof of which is clear from the definition of the width complex:

Lemma 3.4 *An element $k \in \mathcal{K}$ is in $\widehat{\mathcal{K}}$ if and only if every thick level of k is strongly irreducible.*

Using the definitions of this section, we can reformulate Schultens' questions as follows:

Question 12 Is there a knot K with $k \in \widehat{\mathcal{K}}$ such that $w(k) > w(K)$?

Question 13 Given $k \in \mathcal{K}$, is there a directed path in Γ starting at k and ending at a thin position for K?

Explicitly, a directed path is a sequence of vertices $k = k_0, k_1, \ldots, k_n$ such that there is a directed edge from k_i to k_{i+1} for each $i < n$.

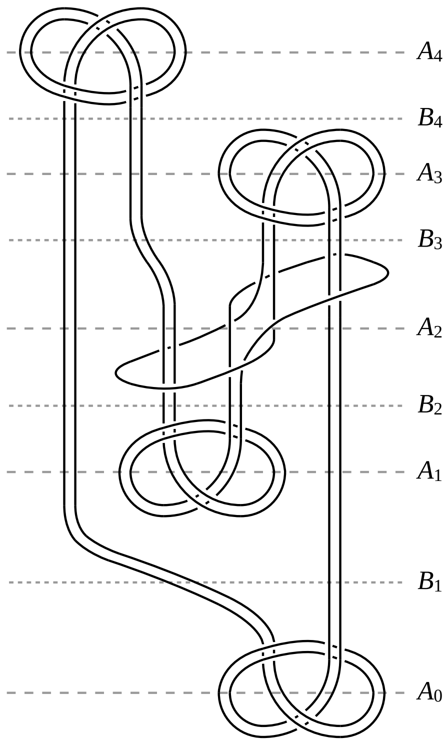

Figure 1: A troublesome embedding k of the unknot, shown with thick/thin levels

4 A local minimum in the width complex of the unknot

Let K be the unknot in S^3, and let $k \in \mathcal{K}$ be the position of the unknot depicted in Figure 1, where h is the standard height projection onto a vertical axis.

We will label the thick/thin levels of k as $A_0, B_1, A_1, B_2, A_2, B_3, A_3, B_4, A_4$, as shown. First, we need several results about bridge position.

Definition 4.1 For any knot K with embedding k, the bridge number of k, $b(k)$, is defined to be the number of maxima in k with respect to h, and the bridge number of K, $b(K)$, is the minimum of $b(k)$ over $k \in \mathcal{K}$. We call k a *minimal bridge position* for K if $b(k) = b(K)$ and k has exactly one thick level, called a bridge sphere for k.

Schubert shows in [7] that the bridge number of any (p,q)–cable of a 2–bridge knot is $2q$ (this was later reproved by Schultens in [8]), and we demonstrate in [12] that any thin position is a minimal bridge position for such a knot. In this case, the bridge sphere must be strongly irreducible. We will use this fact in the following:

Theorem 4.2 *The pictured embedding k of the unknot is a local minimum in the width complex.*

Proof By Lemma 3.4, it suffices to show that every thick surface of k is strongly irreducible. Observe that the regions between consecutive thin surfaces around thick surfaces A_0, A_1, A_3, A_4 are identical except for the extra vertical segments passing through A_1 and A_3. Thus, we need only show that A_0 and A_2 are strongly irreducible.

Claim 1 A_0 is strongly irreducible. Suppose not. Then there is a pair of reducing disks (D, E) at A_0. Let b_1 denote the regular value corresponding to the thin level B_1. If we restrict our attention to $k^* = k \cap h^{-1}(-\infty, b_1]$, we can easily see that by adding two arcs to the four intersection points of k with B_1, we can complete k^* to a $(p,2)$–cable of the trefoil for some p, whose thin position is bridge position by the discussion above, and such that A_0 becomes a bridge sphere. Thus, the pair (D, E) of reducing disks at A_0 is also a pair of reducing disks at the bridge sphere A_0 of the trefoil's cable, a contradiction to the fact that this cable is in thin position. We conclude that A_0 and thus A_1, A_3, and A_4 are strongly irreducible.

Claim 2 A_2 is strongly irreducible. Let b_2 and b_3 be the regular values corresponding to B_2 and B_3, respectively. Then $\mathcal{A}_2 = h^{-1}([b_2, b_3])$ is homeomorphic to $S^2 \times I$, and $k' = k \cap \mathcal{A}_2$ has exactly one maximum contained in an arc κ_1 and exactly one minimum contained in an arc κ_2 properly embedded in $\mathcal{A}_2$. Note that $\mathcal{A}_2$ intersects six additional vertical segments, two of which extend from B_1 to B_4, call these γ_1 and γ_1', two of which extend from B_2 to B_4, call these γ_2 and γ_2', and two of which extend from B_1 to B_3, call these γ_3 and γ_3'.

As above, we will suppose A_2 is reducible and add extra arcs along the endpoints of components of k' to derive a contradiction. If A_2 is reducible, there is a pair of reducing disks (D, E) for k at A_2, where D contains the maximum of κ_1 and E contains the minimum of κ_2. Note that $\kappa_1 \cap \kappa_2 = \varnothing$, implying $D \cap E = \varnothing$. Thus by extending D down to B_2 and E up to B_3, we can find disjoint disks D' and E' such that $\partial D' = \kappa_1 \cup \delta$ for some level arc $\delta \subset B_2$ and $\partial E' = \kappa_2 \cup \eta$ for some level arc $\eta \subset B_3$. Note further that each pair of vertical arcs γ_i and γ_i' cobounds a rectangle R_i with level arcs $\iota_i \subset B_2$ and $\iota_i' \subset B_3$. After isotopy, we may assume that R_1, R_2, R_3, D', and E' are pairwise disjoint.

Now, as pictured in Figure 2, we add four arcs to $k' \cap B_2$ and four arcs to $k' \cap B_3$ to get a link k'', which is an unlinked square knot (the connected sum of two trefoils, one left-hand and one right-handed) and two unknots. In addition, we may attach the arcs so that two arcs of the square knot component cobound a rectangle R with ι_i and δ and two other arcs of this component cobound a rectangle R' with ι_i' and η, where these

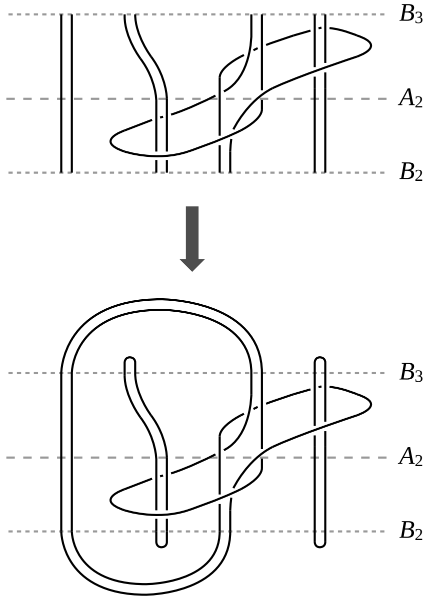

Figure 2: The tangle k' is shown at top, and the link k'' is shown at bottom. If A_2 is reducible, the square knot component of k'' is isotopic to the unknot.

rectangles are disjoint from $\mathrm{int}(R_1)$, $\mathrm{int}(D')$, and $\mathrm{int}(E')$. But this implies that the square knot component of k'' bounds a disk $D' \cup R \cup R_1 \cup R' \cup E'$, a contradiction. We conclude that A_2 is strongly irreducible, completing the proof. □

5 Local minima in the width complex of an arbitrary knot

Suppose that k_1 and k_2 are embeddings representing local minima in the width complexes of knots K_1 and K_2. Then we can find an embedding k of $K_1 \# K_2$ by connecting the highest maximum of k_1 to the lowest minimum of k_2. Observe that this creates a new thin surface but does not interfere with the reducibility of the thick surfaces of k_1 and k_2. Thus, every thick surface of k is strongly irreducible, and by Lemma 3.4, k represents a local minimum in the width complex of $K_1 \# K_2$. For instance, consider the embedding of the figure eight knot 4_1 shown in Figure 3. Note that minimal bridge position is thin position for 4_1 by Thompson [10]. Here we have taken k_1 to be minimal bridge position of the figure eight knot and k_2 to be the unknot embedding shown above, creating a new embedding k of 4_1. Since every thick sphere is strongly irreducible, this embedding corresponds to a local minimum in the knot's width complex. This suggests the following:

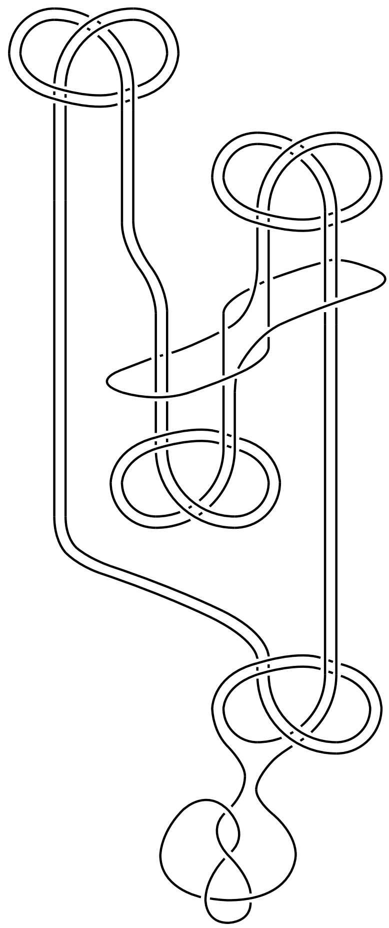

Figure 3: A local minimum in the width complex of the figure eight knot

Corollary 5.1 *The width complex of every knot contains infinitely many local minima.*

Proof Let K be an arbitrary knot, with embedding k representing a local minimum in the width complex of K. For any such k, we exhibit another local minimum k' of the width complex of K with $w(k') > w(k)$, showing that there are infinitely many such embeddings. Let K_0 denote the unknot, and let k_0 be the embedding representing the local minimum of the width complex in Theorem 4.2. Since $K \# K_0 = K$, we can attach k to k_0 by connecting the highest maximum of k to the lowest minimum of k_0 to get a new embedding k' of K with $w(k') > w(k)$. By the above argument, every thick sphere of k' is strongly irreducible, so k' is another local minimum in the width complex. □

Acknowledgements I would like to thank Makoto Ozawa for pointing out the connection between Theorem 4.2 and Scharlemann's Question 3.5 and the reviewer for helping to make the proof of Theorem 4.2 more clear. In addition, I am sincerely grateful to Maggy Tomova, Charlie Frohman and Ying-Qing Wu for their insights and their support.

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Department of Mathematics, University of Iowa
14 MacLean Hall, Iowa City IA 52242, USA

`alexander-zupan@uiowa.edu`

`http://math.uiowa.edu/~azupan`

Received: 20 September 2010 Revised: 15 January 2011