Surface links which are coverings over the standard torus

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We introduce a new construction of a surface link in 4–space. We construct a surface link as a branched covering over the standard torus, which we call a torus-covering link. We show that a certain torus-covering T^2–link is equivalent to the split union of spun T^2–links and turned spun T^2–links. We show that a certain torus-covering T^2–link has a nonclassical link group. We give a certain class of ribbon torus-covering T^2–links. We present the quandle cocycle invariant of a certain torus-covering T^2–link obtained from a classical braid, by using the quandle cocycle invariants of the closure of the braid.

1 Introduction

A *surface link* is the image of a smooth embedding of a closed surface into the Euclidean 4–space $\mathbb{R}^4$. It is known (see Kamada [22; 24]) that any oriented surface link can be presented by the closure of a surface braid. Here, the closure of a surface braid is a surface link of the following form. Let S^2 be a standard 2–sphere in $\mathbb{R}^4$, ie the boundary of a standard 3–ball in $\mathbb{R}^3 \times \{0\}$. The *closure of a surface braid* is a surface link embedded in a tubular neighborhood $N(S^2)$ of S^2 in such a way that the projection of it to S^2 is a branched covering over S^2. We identify $N(S^2)$ with $I \times I \times S^2$, where I is an interval. For a surface link S of such a form, we consider the singular set of the image of S by the projection to $I \times S^2$, and the image of this singular set by the projection to S^2 forms a graph on S^2. An *m–chart on S^2* is such a graph with certain additional data. We can present the original surface link by its *m–chart on S^2* (see Kamada [23; 24]).

In this paper we introduce a "torus-covering link" as a new construction of a surface link, by considering a standard torus instead of a standard 2–sphere. Let T be a standard torus in $\mathbb{R}^4$, ie the boundary of a standard solid torus in $\mathbb{R}^3 \times \{0\}$. A *torus-covering link* is a surface link embedded in a tubular neighborhood $N(T)$ of T in such a way that the projection of it to T is a branched covering over T. For a surface link of such a form, we can define its *m–chart on T* in the same way as above. A torus-covering

link can be presented by an m–chart on T. The aim of this paper is to study various aspects of torus-covering links.

We introduce an equivalence relation, called the t–equivalence, among m–charts on T, and show that two torus-covering links are equivalent if their m–charts on T are t–equivalent (Theorem 2.5). A T^2–*link* is a surface link whose components are homeomorphic to tori. We show that a torus-covering T^2–link is determined from two commutative classical m–braids (Lemma 2.8), which we call *basis m–braids*, and we denote by $\mathcal{S}_m(a, b)$ the torus-covering T^2–link with basis m–braids a and b. A vertex of degree one (respectively six) of an m–chart is called a *black vertex* (respectively a *white vertex*). A torus-covering T^2–link is presented by an m–chart on T without black vertices (Lemma 2.7). We show that an m–chart on T with neither black nor white vertices presents the split union of spun T^2–links and turned spun T^2–links (Theorem 2.16).

The *link group* of a surface link or a classical link is the fundamental group of the link exterior. First we calculate the link group of $\mathcal{S}_m(a, b)$ (Proposition 3.1). It is known (see Livingston [26] and Boyle [8]) that a spun T^2–link or a turned spun T^2–link has a classical link group; thus the split union of spun T^2–links and turned spun T^2–links also has a classical link group. We will show that a certain 2–component torus-covering T^2–link has a nonclassical link group (Theorem 3.2). We show its knot version as well: a certain torus-covering T^2–knot has a nonclassical knot group (Theorem 3.7). As a corollary, we can see that the torus-covering T^2–link of Theorems 3.2 or 3.7 is not equivalent to the split union of spun T^2–links and turned spun T^2–links (Theorem 3.12).

An oriented surface link is called *ribbon* if it is the boundary of an immersed 3–manifold with "ribbon singularities" (see Yanagawa [32]). We give a certain class of ribbon torus-covering T^2–links (Theorem 4.1). As a corollary, we can see that the torus-covering T^2–link of Theorem 3.12 is ribbon (Corollary 4.2).

It is known (see Asami and Satoh [2]) that the quandle cocycle invariant of a twist spun 2–knot of a classical knot K can be presented by using the quandle cocycle invariants of a 1–tangle whose closure is K. From a similar viewpoint, we expect that an invariant of $\mathcal{S}_m(b, \Delta^{2n})$ can be presented by using invariants of an m–braid b, where Δ is a half-twist of a bundle of m parallel strands. In Theorem 5.2 we present the quandle cocycle invariant of $\mathcal{S}_m(b, \Delta^{2n})$, by using the quandle cocycle invariants of the closure of b. In Theorem 5.5, we calculate some concrete examples of Theorem 5.2. They give torus-covering T^2–knots whose triple point numbers are positive (Corollary 5.6).

The paper is organized as follows. In Section 2, we define a torus-covering link (Definition 2.3) and show Theorem 2.5. Further we study torus-covering T^2–links and

show Theorem 2.16. In Section 3, we study link groups of torus-covering T^2–links. We show Theorem 3.2 and Theorem 3.7. Further we show Theorem 3.12. In Section 4, we show Theorem 4.1. In Section 5, we calculate the quandle cocycle invariants and show Theorem 5.2. Further we show Theorem 5.5, by using Mochizuki's 3–cocycle.

2 Torus-covering links

A braided surface over a 2–disk was defined by Rudolph [30] and Kamada [24]. A surface braid is a braided surface with some boundary condition, and a notion of an m–chart on a 2–disk was introduced by Kamada [20; 24] to present a simple surface braid. Equivalent simple surface braids have distinct chart presentations. The notion of C–move equivalence between two m–charts on a 2–disk was introduced by Kamada [20; 23; 24] to give the equivalence class of an m–chart which represents the equivalence class of a simple surface braid. In this section, we modify the definitions to define a braided surface S over a closed surface Σ, an m–chart on Σ which presents S, and the notion of C–move equivalence between two m–charts on Σ. Using these terms, we define a torus-covering link, which is presented by an m–chart on the standard torus. We define t–equivalence between two m–charts, and show that the torus-covering links are equivalent if m–charts of them are t–equivalent (Theorem 2.5). Further we study torus-covering T^2–links. A torus-covering T^2–link is presented by an m–chart on T without black vertices (Lemma 2.7). We show Theorem 2.16: an m–chart on T with neither black nor white vertices presents the split union of spun T^2–links and turned spun T^2–links.

We work in the smooth category, and we assume that embeddings are locally flat. Let $D^2 = I \times I$, where $I = [0, 1]$. A *surface link* is the image of a smooth embedding of a closed surface into $\mathbb{R}^4$. Two surface links are said to be *equivalent* if one is taken to the other by an orientation-preserving self-diffeomorphism of $\mathbb{R}^4$.

Definition 2.1 A closed surface S embedded in $D^2 \times \Sigma$ is called a *braided surface over* Σ of degree m if $p_\Sigma|_S \colon S \to \Sigma$ is a branched covering map of degree m, where $p_\Sigma \colon D^2 \times \Sigma \to \Sigma$ is the projection to the second factor. A braided surface S is called *simple* if $\#(S \cap p_\Sigma^{-1}(x)) = m - 1$ or m for each $x \in \Sigma$. Take a base point x_0 of Σ. Two braided surfaces over Σ of degree m are *equivalent* if there is a fiber-preserving ambient isotopy of $D^2 \times \Sigma$ rel $p_\Sigma^{-1}(x_0)$ which carries one to the other.

When a simple braided surface S is given, we obtain a graph on Σ, as follows. Consider the singular set $\mathrm{Sing}(p_1(S))$ of the image of S by the projection p_1 to $I \times \Sigma$. Perturbing S if necessary, we can assume that $\mathrm{Sing}(p_1(S))$ consists of double

point curves, triple points, and branch points. Moreover we can assume that the singular set of the image of $\mathrm{Sing}(p_1(S))$ by the projection to Σ consists of a finite number of double points such that the preimages belong to double point curves of $\mathrm{Sing}(p_1(S))$. Thus the image of $\mathrm{Sing}(p_1(S))$ by the projection to Σ forms a finite graph Γ on Σ such that the degree of its vertex is either 1, 4 or 6. An edge of Γ corresponds to a double point curve, and a vertex of degree 1 (respectively 6) corresponds to a branch point (respectively a triple point).

For such a graph Γ obtained from a simple braided surface S, we give orientations and labels to the edges of Γ, as follows. Let us consider a path l in Σ such that $l \cap \Gamma$ is a point P of an edge e of Γ. Then $S \cap p_\Sigma^{-1}(l)$ is a classical m–braid with one crossing in $p_\Sigma^{-1}(l)$ such that P corresponds to the crossing of the m–braid. Let $\sigma_1, \sigma_2, \ldots, \sigma_{m-1}$ be the standard generators of the m–braid group B_m. Let σ_i^ϵ ($i \in \{1, 2, \ldots, m-1\}$, $\epsilon \in \{+1, -1\}$) be the presentation of $S \cap p_\Sigma^{-1}(l)$. Then label the edge e by i, and moreover give e an orientation such that the normal vector of l corresponds (respectively does not correspond) to the orientation of e if $\epsilon = +1$ (respectively -1). We call such an oriented and labeled graph an m–*chart of* S.

In general, we define an m–chart on Σ as follows.

Definition 2.2 Let m be a positive integer, and let Γ be a finite graph on Σ. Then Γ is called an m–*chart on* Σ if it satisfies the following conditions:

(i) Every edge is oriented and labeled by an element of $\{1, 2, \ldots, m-1\}$.

(ii) Every vertex has degree 1, 4 or 6.

(iii) The adjacent edges around each vertex are oriented and labeled as shown in Figure 1, where we depict a vertex of degree 1 by a black vertex, and a vertex of degree 6 by a white vertex.
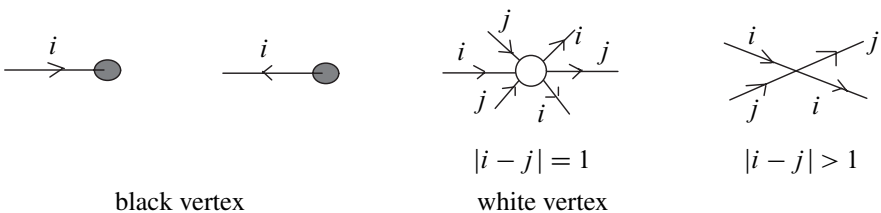

Figure 1: Vertices in an m–chart

When an m–chart Γ on Σ is given, we can reconstruct a simple braided surface S over Σ as follows. Let $N(\Gamma)$ be a neighborhood of Γ in Σ. Let us consider a trivial

braided surface $S = Q_m \times (\Sigma - N(\Gamma))$ over $\Sigma - N(\Gamma)$, where Q_m is a set of m interior points of D^2. We extend S over a neighborhood of each edge as follows. Identify a neighborhood of an edge e with $I \times I$ such that e is identified with $\{1/2\} \times I$. Let i be the label attached to e, and let $\epsilon = +1$ (respectively -1) if the orientation of e corresponds (respectively does not correspond) to the orientation of $\{0\} \times I$. Then let the braided surface S over the neighborhood of e be the braided surface which has a presentation $\sigma_i^\epsilon \times I$ and the image of the double point curve of $p_1(S)$ by the projection to Σ is e. Since Γ is as in Figure 1 around each vertex, S can be extended naturally over a neighborhood of each vertex. See Carter and Saito [14] and Kamada [21; 24] for more details. Thus we can construct a simple braided surface S over Σ such that the original m–chart Γ is an m–chart of S.

Two m–charts on Σ are *C–move equivalent* if they are related by a finite sequence of ambient isotopies of Σ rel $p_\Sigma^{-1}(x_0)$ and CI, CII, CIII–moves shown in Figure 2; see [24] for the complete set of CI–moves. It is shown as a minor modification of [20; 23; 24] that two simple braided surfaces over Σ of degree m are equivalent if and only if m–charts of them are C–move equivalent.
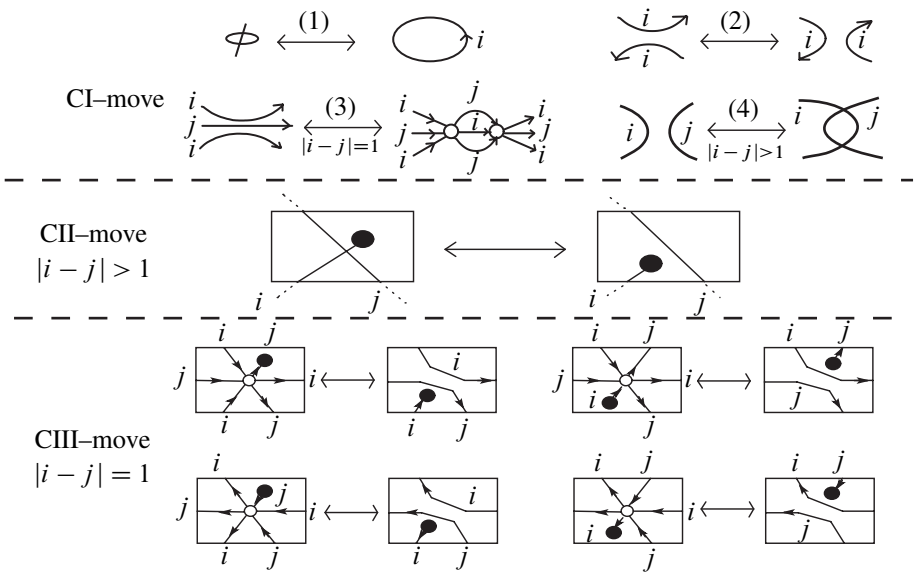

Figure 2: CI, CII, CIII–moves. For CI–moves, we give only several examples.

Now we define torus-covering links. Let T be the standard torus in $\mathbb{R}^4$, ie the boundary of the standard solid torus in $\mathbb{R}^3 \times \{0\}$. Let us fix a point x_0 of T, and take a meridian $\mathbf{m}$ and a longitude $\mathbf{l}$ of T with the base point x_0. A meridian is an oriented simple closed curve on T which bounds the 2–disk of the solid torus whose boundary

is T. A longitude is an oriented simple closed curve on T which is null-homologous in the complement of the solid torus in the three space $\mathbb{R}^3 \times \{0\}$. Let $N(T)$ be a tubular neighborhood of T in $\mathbb{R}^4$. Since T is the boundary of the standard solid torus in $\mathbb{R}^3 \times \{0\}$, the normal bundle of T in $\mathbb{R}^3 \times \{0\}$ is a trivial bundle. Let us identify it with $I \times T$. Then let us identify $N(T)$ with $I \times I \times T$, where the second I is an interval in the fourth axis of $\mathbb{R}^4$. From now on, we identify $N(T)$ with $D^2 \times T$.

Definition 2.3 A *torus-covering link* is a surface link in $\mathbb{R}^4$ presented by a simple braided surface over T, where we regard the braided surface as in $N(T) \subset \mathbb{R}^4$.

As we mentioned, for two m–charts on T, their presenting braided surfaces over T are equivalent if the m–charts are C–move equivalent. Hence it follows that for two m–charts on T, their presenting torus-covering links are equivalent if the m–charts are C–move equivalent. Since each component of a torus-covering link is a branched cover over a torus T, each component of a torus-covering link is of genus at least one. See Propositions 2.11, 2.12 and 2.13 for some examples of torus-covering links. Note that it is known (see Berstein and Edmonds [3; 4]) that any braided surface over T is approximated by a simple braided surface over T.

Regarding S^4 as the one-point compactification of $\mathbb{R}^4$, we regard a surface link as in S^4. Then $N(T) = D^2 \times T$ is embedded in S^4. Let $\mathbf{r} = \partial D^2 \times \{0\} \times \{0\}$ be a curve on $\partial N(T) = \partial D^2 \times T$. Put $E^4 = \mathrm{cl}(S^4 - N(T))$. Let r, m and l be the curves on ∂E^4, which are identified with $\mathbf{r}$, $\mathbf{m}$ and $\mathbf{l}$ under the natural identification map $i\colon \partial N(T) \to \partial E^4$. The curves $\mathbf{r}$, $\mathbf{m}$ and $\mathbf{l}$ represent a basis of $H_1(\partial E^4; \mathbb{Z})$. Let $f\colon \partial E^4 \to E^4$ be a diffeomorphism such that $f_*(\mathbf{r}\ \mathbf{m}\ \mathbf{l}) = (\mathbf{r}\ \mathbf{m}\ \mathbf{l})A^f$, where $A^f \in \mathrm{GL}(3, \mathbb{Z}) \cong \pi_0 \mathrm{Diffeo}(\partial E^4)$. It is known (see Montesinos [28]) that the map f can be extended to a self-diffeomorphism of E^4 if and only if $A^f \in H$, where:

$$H = \left\{ \begin{pmatrix} \pm 1 & 0 & 0 \\ * & \alpha & \gamma \\ * & \beta & \delta \end{pmatrix} \in \mathrm{GL}(3, \mathbb{Z}); \ \alpha + \beta + \gamma + \delta \equiv 0 \ (\mathrm{mod}\ 2) \right\}$$

Using this fact, we introduce an equivalence relation between two m–charts on T.

Definition 2.4 We say that two m–charts on T are *t–equivalent* if they are related by a finite sequence of ambient isotopies of T, C–moves and a self-diffeomorphism of T given by an element of:

$$\left\{ \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \in \mathrm{GL}_+(2, \mathbb{Z}); \ \alpha + \beta + \gamma + \delta \equiv 0 \ (\mathrm{mod}\ 2) \right\}$$

Theorem 2.5 *Two torus-covering links in S^4 are equivalent if m–charts of them are t–equivalent.*

Proof Since C–move equivalent m–charts present equivalent torus-covering links, it suffices to show in the cases for an ambient isotopy of T and a self-diffeomorphism g of T of Definition 2.4.

An ambient isotopy of T induces a fiber-preserving ambient isotopy of $N(T)$ which relates the torus-covering links. This can be extended to the whole space by the Isotopy Extension Theorem (see Hirsch [18]). The terminal map of the resulting isotopy is an orientation-preserving diffeomorphism; thus the torus-covering links are equivalent.

Let Γ be an m–chart on T. Let S and S' be the torus-covering links presented by Γ and $g(\Gamma)$ respectively. Let g' be a self-diffeomorphism of $N(T)$ induced by g, ie $g' = \mathrm{id}_{D^2} \times g \colon N(T) \to N(T)$, where we regard $N(T)$ as $D^2 \times T$. Since $A^{g'}$ is in $\mathrm{GL}_+(3, \mathbb{Z})$, the map $g'|_{\partial N(T)}$ can be considered as an orientation-preserving self-diffeomorphism of ∂E^4. Since $A^{g'}$ is an element of H, $g'|_{\partial N(T)}$ can be extended to E^4, and hence to $S^4 = N(T) \cup_{\partial N(T)} E^4$. This is an orientation-preserving self-diffeomorphism of S^4 which maps S to S', and hence S and S' are equivalent in S^4. □

In particular, we have the following corollary. Let ρ, respectively τ, be a self-diffeomorphism of T given by:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{respectively} \quad \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

Corollary 2.6 *Two m–charts on T present equivalent torus-covering links if they are related by a finite sequence of ambient isotopies of T, C–moves, and moves as in Figure 3.*
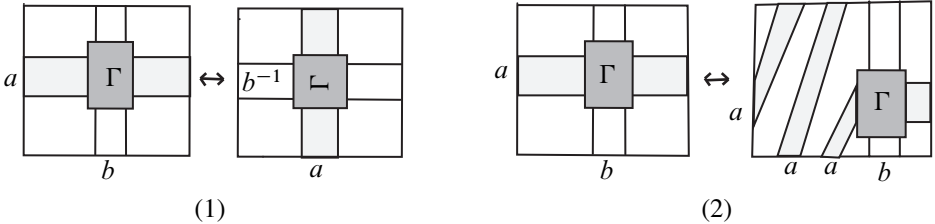

Figure 3: Equivalent moves of m–charts

Proof The moves as in Figure 3 (1) and (2) are related by ρ and τ^2 respectively. They give t–equivalence between two m–charts on T. Thus the conclusion follows from Theorem 2.5. □

Remark Teragaito [31] proved the same fact of the above corollary for the symmetry-spun version. The case of τ^2 for turned spun T^2–links was shown by Boyle [8], using the result of Gluck [16].

A *torus-covering T^2–link* is a torus-covering link whose components are homeomorphic to tori.

Lemma 2.7 *A torus-covering T^2–link S is presented by an m–chart without black vertices. Thus S is an unbranched cover over T.*

Proof Let Γ be an m–chart on T which presents S, and let $b(\Gamma)$ be the number of black vertices in Γ. Then S is a branched cover over T with $b(\Gamma)$ branch points, and the Euler characteristic of S is equal to $-b(\Gamma)$. Further, since the Euler characteristic of tori is equal to zero, we have $b(\Gamma) = 0$.   □

Let us consider a torus-covering T^2–link S. The intersections $S \cap p_T^{-1}(\mathbf{m})$ and $S \cap p_T^{-1}(\mathbf{l})$ are closures of classical braids. Cutting open the solid tori at the 2–disk $p_T^{-1}(x_0)$, we obtain a pair of classical braids. We call them *basis braids*.

Lemma 2.8   (1) *The basis braids of a torus-covering T^2–link are commutative.*

  (2) *For any commutative m–braids a and b, there exists a unique torus-covering T^2–link with basis braids a and b.*

For commutative m–braids a and b, we denote by $\mathcal{S}_m(a,b)$ the torus-covering T^2–link with basis m–braids a and b.

Proof (1) Let X_m be the configuration space of unordered m distinct points of a 2–disk D^2, ie the set of m–element subsets of D^2 such that each m–element subset consists of m distinct points. It is known (see Artin [1] and also Birman [5]) that $\pi_1(X_m) = B_m$. Since a torus-covering T^2–link is presented by an unbranched covering of degree m over T by Lemma 2.7, it is presented by a map $f \colon T \to X_m$. The induced map f_* gives a homomorphism $\mathbb{Z} \oplus \mathbb{Z} \cong \pi_1(T) \to \pi_1(X_m) = B_m$. Since the basis braids are the images of generators of $\mathbb{Z} \oplus \mathbb{Z}$ by this homomorphism, they are commutative.

(2) For any commutative m–braids a and b, let us consider a map $f \colon \mathbf{m} \cup \mathbf{l} \to X_m$ such that the closed paths $f|_{\mathbf{m}}$ and $f|_{\mathbf{l}}$ in X_m represent a and b respectively. Since a and b are commutative, $aba^{-1}b^{-1}$ is isotopic to the trivial braid, and it follows that the closed path l in X_m representing $aba^{-1}b^{-1}$ is null-homotopic. Hence we can take a 2–disk in X_m such that the boundary is l: thus f can be extended to a map

from T to X_m. Since $\pi_2(X_m) = 0$ (see Fadell and Neuwirth [15]), such an extension is unique (up to equivalence). This means that there exists a unique torus-covering T^2–link with basis braids a and b. □

By Corollary 2.6, we have the following corollary.

Corollary 2.9 *For commutative m–braids a and b, the following equivalence relations hold:*
$$\mathcal{S}_m(a, b) \sim \mathcal{S}_m(b^{-1}, a), \quad \mathcal{S}_m(a, b) \sim \mathcal{S}_m(a, a^2 b).$$

Proof Let Γ be an m–chart on T without black vertices and with basis braids a and b. By Lemma 2.7 and Lemma 2.8 (2), Γ presents $\mathcal{S}_m(a, b)$. By Corollary 2.6, the torus-covering T^2–links presented by Γ, $\rho(\Gamma)$ and $\tau^2(\Gamma)$ are equivalent. The basis braids of $\rho(\Gamma)$ are b^{-1} and a. Further, the basis braids of $\tau^2(\Gamma)$ are a and $a^2 b$; see Figure 3. Thus we have the required equivalent relations. □

Using this corollary, we can for example show the following proposition. We remark the result before the statement of Corollary 5.7. An oriented surface link S is *invertible* if S is equivalent to its orientation-reversed image $-S$.

Proposition 2.10 *For any integers p and n, $\mathcal{S}_4(\sigma_1 \sigma_2^p \sigma_3, \Delta^{2n})$ is invertible, where Δ is a half-twist of a bundle of 4 parallel strands.*

Proof Let us determine the basis braids of $-\mathcal{S}_m(a, b)$, as follows. Put $S = \mathcal{S}_m(a, b)$. Then $(-S) \cap p_T^{-1}(-\mathbf{m})$ is the closure of $-a = \bar{a}^{-1}$, and $(-S) \cap p_T^{-1}(-\mathbf{l})$ is the closure of $-b = \bar{b}^{-1}$. Here $-x$ ($x = \mathbf{m}, \mathbf{l}, a, b$) is the orientation-reversed image of x, and $\bar{a}$ (respectively $\bar{b}$) is the m–braid obtained from a (respectively b) by replacing σ_i with σ_i^{-1} for each standard generator σ_i of B_m. Hence $(-S) \cap p_T^{-1}(\mathbf{m})$ and $(-S) \cap p_T^{-1}(\mathbf{l})$ are the closures of $\bar{a}$ and $\bar{b}$ respectively; thus the basis braids of $-S$ are $\bar{a}$ and $\bar{b}$. From now on, put $S = \mathcal{S}_4(\sigma_1 \sigma_2^p \sigma_3, \Delta^{2n})$. By the above argument and Lemma 2.8 (2), $-S$ is equivalent to $\mathcal{S}_4(\sigma_1^{-1} \sigma_2^{-p} \sigma_3^{-1}, \Delta^{-2n})$. Applying the first equivalent relation of Corollary 2.9 twice, we have $\mathcal{S}_m(a, b) \sim \mathcal{S}_m(b^{-1}, a) \sim \mathcal{S}_m(a^{-1}, b^{-1})$. Thus $-S \sim \mathcal{S}_4(\sigma_3 \sigma_2^p \sigma_1, \Delta^{2n})$. Regarding the i–th string of the basis braids as the $(4-i)$–th string ($i = 1, \ldots, 4$), we can regard the basis braids as $\sigma_1 \sigma_2^p \sigma_3$ and Δ^{2n} respectively; thus $-S \sim \mathcal{S}_4(\sigma_1 \sigma_2^p \sigma_3, \Delta^{2n}) = S$. □

We show that some torus-covering T^2–links are equivalent to known T^2–links. Let b an m–braid, and let Q_m be the starting point set of b. Let us denote by $\hat{b}$ the closure of b.

Proposition 2.11 *The torus-covering T^2–link $\mathcal{S}_m(b,e)$ is equivalent to the spun T^2–link of $\hat{b}$.*

The 4–space $\mathbb{R}^4$ is constructed by rotating the upper half plane $\mathbb{R}^3_+ = \mathbb{R}^2 \times [0,\infty)$ in $\mathbb{R}^2 \times \mathbb{R}^2$ around the axis $\mathbb{R}^2 \times \{0\}$. This structure is called an *open book structure*. Let B^3 be a 3–ball in $\mathbb{R}^3_+$, and let us naturally identify the orbit of B^3 with $B^3 \times S^1$ in $\mathbb{R}^4$, where S^1 is a circle. Let $\pi\colon B^3 \times S^1 \to B^3$ be the projection. A surface link S in $B^3 \times S^1$ can be considered as $\bigcup_{t \in S^1} S_t \times \{t\}$, where $S_t = \pi(S \cap (B^3 \times \{t\})) \subset B^3$. We call the collection $\{S_t\}_{t \in S^1}$ the *motion picture* of S along S^1.

Let L be a classical link in B^3. The *spun T^2–link* of L is the surface link defined by the motion picture $S_t = L$ for $t \in S^1$ (see Livingston [26] and Boyle [7; 8]).

In an open book structure of $\mathbb{R}^4$, we naturally identify the orbit of B^3 with $B^3 \times S^1$. Let us consider un unknotted circle $S^1 \subset B^3$ with the base point x_0. The orbit of this S^1 is a standard torus. We identify it with T, by identifying $S^1 \times \{0\} \subset B^3 \times \{0\}$ with the meridian **m**, and $\{x_0\} \times S^1 \subset B^3 \times S^1$ with the longitude **l**. Further, we identify $N(T)$ with the orbit of the unknotted solid torus $N(\mathbf{m}) = \pi(p_T^{-1}(\mathbf{m}))$ in B^3. Put $D(x_0) = \pi(p_T^{-1}(x_0))$. Let us identify S^1 with $[0,1]/\sim$, where $0 \sim 1$. Let us assume that the closure $\hat{b}$ is in the solid torus $N(\mathbf{m})$ such that the identified corresponding ends are in $D(x_0)$. Further we regard a braided surface over T as in $N(T) \subset \mathbb{R}^4$.

Proof of Proposition 2.11 Let S be the surface defined by the motion picture $S_t = \hat{b}$ for $t \in S^1$. By definition, S is the spun T^2–link of $\hat{b}$. Since S is a braided surface over T with no branch points, S is a torus-covering T^2–link. Let us determine the basis braids. Since $\pi(S \cap p_T^{-1}(\mathbf{m})) = S_0 \cap N(\mathbf{m}) = \hat{b}$, one basis braid is b. By definition, we have $S \cap p_T^{-1}(\mathbf{l}) = \bigcup_{t \in [0,1]} (S_t \cap D(x_0)) \times \{t\}$. Since $S_t \cap D(x_0) = Q_m$ for any t, $S \cap p_T^{-1}(\mathbf{l})$ is the closure of the trivial m–braid $e = Q_m \times [0,1]$. Thus S is a torus-covering T^2–link with basis m–braids b and e, and it follows from Lemma 2.8 (2) that S is equivalent to $\mathcal{S}_m(b,e)$. $\qquad\square$

Let us identify the 3–ball B^3 with the unit ball in the xyz–space. Let us rotate a classical link L in B^3 around the z–axis once, and identify the resulting link with the original link. The orbit of L forms a surface link, called the *turned spun T^2–link* of L (see Livingston [26] and Boyle [8]).

Proposition 2.12 *The torus-covering T^2–link $\mathcal{S}_m(b,b)$ is equivalent to the turned spun T^2–link of $\hat{b}$.*

Proof We can assume that the solid torus $N(\mathbf{m}) \subset B^3$ is fixed as a set when we rotate it around the z–axis. Let $\{h_u\}_{u \in [0,1]}$ be an isotopy of B^3 which describes the rotation of the solid torus $N(\mathbf{m})$ around the z–axis once. Let S be the surface defined by the motion picture $S_t = \bigcup_{u \in [0,1]} h_t(\hat{b})$ for $t \in S^1$. By definition, S is the turned spun T^2–link of $\hat{b}$. By the same argument with the proof of Proposition 2.11, S is a torus-covering T^2–link with $S \cap p_T^{-1}(\mathbf{m}) = \hat{b}$. Regarding $N(\mathbf{m})$ as $D^2 \times S^1$ such that $D(x_0) = D^2 \times \{0\}$, we have $S_t \cap D(x_0) = h_t(\hat{b}) \cap (D^2 \times \{0\}) = p(\hat{b} \cap (D^2 \times \{t\})) \times \{0\}$ for $t \in [0,1]$, where $p \colon D^2 \times S^1 \to D^2$ is the projection; thus $S \cap p_T^{-1}(\mathbf{l}) = \bigcup_{t \in [0,1]} (S_t \cap D(x_0)) \times \{t\} = \hat{b}$. Thus the basis braids of S are b and b, and $S \sim \mathcal{S}_m(b, b)$ by Lemma 2.8 (2). $\qquad\square$

Let L be a classical link in B^3 such that rotating L around the z–axis by $2k\pi/n$ results in the original L as a set. Then let us rotate L around the z–axis by $2k\pi/n$, and identify the resulting link with the original link. The orbit of L forms a surface link, called a *symmetry-spun T^2–link* (see Teragaito [31]).

Let b^n be an m–braid in $D^2 \times [0,1]$ such that $b^n \cap (D^2 \times I_j) = b$, where $I_j = [(j-1)/n, \, j/n]$ ($j = 1, 2, \ldots, n$).

Proposition 2.13 *The torus-covering T^2–link $\mathcal{S}_m(b^n, b^k)$ is equivalent to a symmetry-spun T^2–link, which is constructed by turning $\widehat{b^n}$ by $2k\pi/n$ around the axis while spinning.*

By Teragaito [31, Theorem 8], the symmetry-spun T^2–link $\mathcal{S}_m(b^n, b^k)$ is equivalent to either $\mathcal{S}_m(b^r, e)$ or $\mathcal{S}_m(b^r, b^r)$, where $r = \gcd(n, k)$. This can be shown by Corollary 2.9, too.

Proof Let $\{h_u\}_{u \in [0,1]}$ be an isotopy of B^3 which describes the rotation of the solid torus $N(\mathbf{m})$ around the axis by $2k\pi/n$. Let S be the surface defined by the motion picture $S_t = \bigcup_{u \in [0,1]} h_t(\hat{b})$ for $t \in S^1$. By definition, S is the symmetry-spun T^2–link in question. By the same argument with the proof of Proposition 2.11, we can see that S is a torus-covering T^2–link with $S \cap p_T^{-1}(\mathbf{m}) = \widehat{b^n}$. Using the same notation with the proof of Proposition 2.12, we have $S_t \cap D(x_0) = h_t(\widehat{b^n}) \cap (D^2 \times \{0\}) = p(\widehat{b^n} \cap (D^2 \times \{kt/n\})) \times \{0\}$ for $t \in S^1$. So $S \cap p_T^{-1}(\mathbf{l}) = \bigcup_{t \in S^1} (S_t \cap D(x_0)) \times \{t\} = \widehat{b^k}$. Thus the basis braids of S are b^n and b^k, and $S \sim \mathcal{S}_m(b^n, b^k)$ by Lemma 2.8 (2). $\square$

Let us call an edge of an m–chart a *loop* if it is connected with no vertices. Let us consider an m–chart on T with no vertices.

Proposition 2.14 *An m–chart on T with no vertices presents a spun T^2–link or a turned spun T^2–link.*

Proof By Lemma 2.7, an m–chart with no vertices presents a torus-covering T^2–link. Let us determine the basis braids, as follows. An m–chart with no vertices consists of a finite number of loops. By Lemma 2.15, we can assume that any loop does not bound a 2–disk in T. Then, by an ambient isotopy of T, we can make all the loops parallel and moreover geodesic. The m–chart is as in Figure 4; thus the basis braids
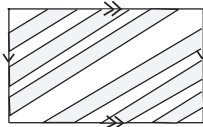

Figure 4

presented by Γ are b^n and b^k, for an m–braid b and integers n and k. Its presenting torus-covering T^2–link is equivalent to a symmetry-spun T^2–link by Proposition 2.13; thus it is equivalent to either a spun T^2–link or a turned spun T^2–link. □

Lemma 2.15 *An m–chart on T with no vertices is C–move equivalent to an m–chart such that each loop does not bound a 2–disk in T.*

Proof For a 2–disk D in T such that ∂D intersects an m–chart Γ transversely, if there are no black vertices in $\Gamma \cap D$, then by a CI–move we can redraw the m–chart within D as we like as long as it has no black vertices (see Kamada [24]). Hence, if there is a loop which bounds a 2–disk in T, then we can remove it by applying a CI–move around the neighborhood of the loop. □

Proposition 2.14 can be extended to an m–chart on T with neither black nor white vertices, as follows. The *split union* of two surface links S_1 and S_2 is a surface link presented by the union of the copies of S_1 and S_2 such that for a 3–sphere S^3 embedded in $\mathbb{R}^4$, S_1 is inside of S^3 and S_2 is outside. The 3–sphere S^3 is called a *separating 3–sphere*.

Theorem 2.16 *An m–chart on T with neither black nor white vertices presents either a spun T^2–link, a turned spun T^2–link or the split union of spun T^2–links and turned spun T^2–links.*

Let Γ be an m–chart with neither black nor white vertices. Then every vertex of Γ is of degree 4. Since the diagonal edges around a vertex of degree 4 have the same label and coherent orientation (see Figure 1), we can regard the union of connected edges of Γ with the label i as an oriented immersed circle with the label i. Let us call it just an *immersed circle*. Since the edges around a vertex of degree 4 have the

labels i and j with $|i - j| > 1$, we can regard Γ as consisting of immersed circles with transverse intersections such that each intersection is formed by two immersed circles with the labels i and j with $|i - j| > 1$.

Proof Let Γ be an m–chart with neither black nor white vertices. Let $i > 0$ be the minimum integer which does not appear as a label of Γ. Let $\Gamma_{<i}$ be a subgraph in T consisting of the edges of Γ with the labels smaller than i. Further, attach to each edge of this $\Gamma_{<i}$ the orientation and label induced from Γ. Since Γ can be regarded as consisting of immersed circles, so can $\Gamma_{<i}$; thus $\Gamma_{<i}$ presents a new m–chart on T with neither black nor white vertices. Similarly, let $\Gamma_{>i}$ be a subgraph in T consisting of the edges of Γ with the labels larger than i, with induced labels and orientations. Then $\Gamma_{>i}$ also presents a new m–chart on T with neither black nor white vertices. Since Γ has no edge with the label i, $\Gamma = \Gamma_{<i} \cup \Gamma_{<i}$. By Lemma 2.17, the torus-covering link presented by Γ is equivalent to the split union of two torus-covering links presented by $\Gamma_{<i}$ and $\Gamma_{>i}$.

If there is an immersed circle which bounds a 2–disk in T, then we can remove it by applying a CI–move by the same argument of Lemma 2.15. Thus, taking new i if necessary, we can assume that $\Gamma_{<i}$ satisfies the conditions of Lemma 2.18, ie (1) any immersed circle does not bound a 2–disk in T, and (2) there is at least one immersed circle with the label j for every label $j < i$. Thus $\Gamma_{<i}$ presents a spun T^2–link or a turned spun T^2–link by Lemma 2.18. Using induction for i, we can see that Γ presents a spun T^2–link, a turned spun T^2–link or the split union of spun T^2–links and turned spun T^2–links. □

Lemma 2.17 *The torus-covering link presented by $\Gamma_{<i} \cup \Gamma_{>i}$ is equivalent to the split union of two torus-covering links presented by $\Gamma_{<i}$ and $\Gamma_{>i}$.*

Proof Let us denote by S, S_1 and S_2 the torus-covering links presented by $\Gamma_{<i} \cup \Gamma_{>i}$, $\Gamma_{<i}$ and $\Gamma_{>i}$ respectively. Consider the open book decomposition of $\mathbb{R}^4$ as in Proposition 2.11. Let N_1 (respectively N_2) be a solid torus in $N(\mathbf{m})$ which contains the j–th starting point of the basis braids of S for every $j < i$ (respectively $j > i$). Since there are no edges of $\Gamma_{<i} \cup \Gamma_{>i}$ with the label i, we can assume that S is in the orbit $(N_1 \cup N_2) \times S^1$ such that S_1 and S_2 are contained in $N_1 \times S^1$ and $N_2 \times S^1$ respectively. Let us take a 2–disk in $\mathbb{R}^3_+$ as in Figure 5. The orbit of this 2–disk forms a separating 3–sphere. Thus S is the split union of S_1 and S_2. □

Lemma 2.18 *Let Γ be an m–chart on T with neither black nor white vertices satisfying (1) any immersed circle does not bound a 2–disk in T, and (2) there is at least one immersed circle with the label j for every label j. Then Γ presents either a spun T^2–link or a turned spun T^2–link.*

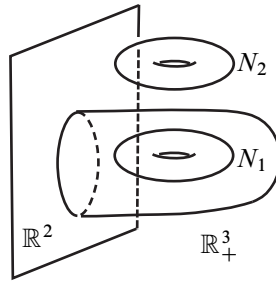Figure 5

Proof By the definition of an m–chart, there are no intersections between the immersed circles with the labels $j-1$ and j. Thus, together with (1), the immersed circles with the labels $j-1$ and j can be made parallel by an isotopy of T. Hence it follows from (2) that each oriented immersed circle goes n times around the meridian $\mathbf{m}$ and k times around the longitude $\mathbf{l}$, or $-n$ times around $\mathbf{m}$ and $--k$ times around $\mathbf{l}$, for some fixed integers n and k. Since there are only intersections of immersed circles with the labels i and j with $|i-j|>1$ as vertices of Γ, we can remove all the vertices by CI–moves of type (4) (see Figure 2). Then the m–chart Γ presents either a spun T^2–link or a turned spun T^2–link by Proposition 2.14. $\square$

We showed in Theorem 2.16 that an m–chart on T with neither black nor white vertices presents either a spun T^2–link, a turned spun T^2–link or the split union of spun T^2–links and turned spun T^2–links. We will show in Theorem 3.12 that there is a torus-covering T^2–link which is not equivalent to either a spun T^2–link, a turned spun T^2–link or the split union of spun T^2–links and turned spun T^2–links. Its presenting m–chart on T does not have black vertices (Lemma 2.7) but does have white vertices (Corollary 4.3 (2)).

3   Knot groups and link groups

From now on in this paper, we consider torus-covering T^2–links. By Lemma 2.8, the basis braids of a torus-covering T^2–link are commutative, and for any commutative m–braids a and b, there exists a unique torus-covering T^2–link with basis braids a and b. In this section, first we compute the link group of $\mathcal{S}_m(a,b)$ (Proposition 3.1). Using this proposition, we will show that a certain 2–component torus-covering T^2–link has a nonclassical link group (Theorem 3.2). We show its knot version as well: a certain torus-covering T^2–knot has a nonclassical knot group (Theorem 3.7). Further we show that the torus-covering T^2–link of Theorems 3.2 or 3.7 is not equivalent to

either a spun T^2–link, a turned spun T^2–link or the split union of spun T^2–links and turned spun T^2–links (Theorem 3.12).

Before the statement of Proposition 3.1, we will give the definition of Artin's automorphism (see Kamada [24]). Let b be an m–braid in a cylinder $D^2 \times [0,1]$, and let Q_m be the starting point set of b. Let $\{h_u\}_{u \in [0,1]}$ be an isotopy of D^2 rel ∂D^2 such that $\bigcup_{u \in [0,1]} h_u(Q_m) \times \{u\} = b$. Let $\mathcal{A}^b \colon (D^2, Q_m) \to (D^2, Q_m)$ be the terminal map h_1, and consider the induced map $\mathcal{A}^b_* \colon \pi_1(D^2 - Q_m) \to \pi_1(D^2 - Q_m)$. It is known (see Artin [1]) that $\mathcal{A}^b$ is uniquely determined from b. We call $\mathcal{A}^b_*$ *Artin's automorphism* associated with b. Note that $\pi_1(D^2 - Q_m)$ is naturally isomorphic to the free group F_m generated by the standard generators $x_1, x_2, \ldots, x_m$ of $\pi_1(D^2 - Q_m)$. By $\mathcal{A}^b_*$, the braid group B_m acts on $\pi_1(D^2 - Q_m)$. It is presented by

$$\mathcal{A}^{\sigma_i}_*(x_j) = \begin{cases} x_j x_{j+1} x_j^{-1} & \text{if } j = i, \\ x_{j-1} & \text{if } j = i+1, \\ x_j & \text{otherwise,} \end{cases}$$

and

$$\mathcal{A}^{\sigma_i^{-1}}_*(x_j) = \begin{cases} x_{j+1} & \text{if } j = i, \\ x_j^{-1} x_{j-1} x_j & \text{if } j = i+1, \\ x_j & \text{otherwise,} \end{cases}$$

where $i = 1, 2, \ldots, m-1$ and $j = 1, 2, \ldots, m$.

Proposition 3.1 *For commutative m–braids a and b, the link group of $\mathcal{S}_m(a,b)$ is presented by:*

$$\pi_1(\mathbb{R}^4 - \mathcal{S}_m(a,b)) = \langle x_1, \ldots, x_m \mid x_j = \mathcal{A}^a_*(x_j) = \mathcal{A}^b_*(x_j) \text{ for } j = 1, 2, \ldots, m \rangle$$

Proof Put $S = \mathcal{S}_m(a,b)$. Let Q_m be a set of m distinct interior points of D^2, and let q_0 be a point of ∂D^2. The space $N(T) - S$ is a fiber bundle over T with the fiber $D^2 - Q_m$ whose monodromy is given by $\mathcal{A}^a$ and $\mathcal{A}^b$. Let us take commutative $\mathcal{A}^a$ and $\mathcal{A}^b$. Then we have

$$N(T) - S \cong ((D^2 - Q_m) \times I \times I)/(x,0,u) \sim (\mathcal{A}^a(x),1,u), (x,u,0) \sim (\mathcal{A}^b(x),u,1)$$

$$= M \times I/(x,u,0) \sim (\mathcal{A}^b(x),u,1)$$

where $$M = ((D^2 - Q_m) \times I)/(x,0) \sim (\mathcal{A}^a(x),1)$$

and $x \in D^2 - Q_m$ and $u \in I$.

We compute $\pi_1(M)$, as follows. Since M is a mapping torus whose monodromy is given by $\mathcal{A}^a$, by van Kampen's theorem, we can see that $\pi_1(M)$ has a presentation

obtained from $\pi_1(D^2 - Q_m) * \mathbb{Z}$ by adding relations $s^{-1}xs = \mathcal{A}_*^a(x)$, where $x \in \pi_1(D^2 - Q_m)$, and s is the generator of $\mathbb{Z}$, which is represented by the loop $\{q_0\} \times S^1$ with the base point q_0. Since $\pi_1(D^2 - Q_m)$ is a free group generated by the standard generators $x_1, x_2, \ldots, x_m$ with the base point q_0, we have:

$$(3\text{-}1) \qquad \pi_1(M) = \langle \, x_1, x_2, \ldots, x_m, s \mid s^{-1}x_j s = \mathcal{A}_*^a(x_j) \text{ for } j = 1, 2, \ldots, m \, \rangle$$

We compute $\pi_1(N(T) - S)$, as follows. Since $N(T) - S = M \times I/(x, u, 0) \sim (\mathcal{A}^b(x), u, 1)$, where $x \in D^2 - Q_m$ and $u \in I$ with $(x, u) \in M$, $N(T) - S$ is a mapping torus whose monodromy is given by $\mathcal{A}^b \times \mathrm{id}$. Thus we can see that $\pi_1(N(T) - S)$ has a presentation obtained from $\pi_1(M) * \mathbb{Z}$ by adding relations $t^{-1}yt = (\mathcal{A}_*^b \times \mathrm{id}_*)(y)$, where $y \in \pi_1(M)$ and t is the generator of $\mathbb{Z}$. Hence together with (3-1), we can see that $\pi_1(N(T) - S)$ is presented by

$$(3\text{-}2) \quad \pi_1(N(T) - S) = \left\langle x_1, x_2, \ldots, x_m, s, t \, \middle| \, \begin{array}{l} s^{-1}x_j s = \mathcal{A}_*^a(x_j), \; t^{-1}x_j t = \mathcal{A}_*^b(x_j), \\ t^{-1}st = s, \text{ for } j = 1, 2, \ldots, m \end{array} \right\rangle$$

where s and t are represented by the loops **m** and **l** respectively.

We compute $\pi_1(S^4 - S)$, as follows. We have $S^4 - S = (N(T) - S) \cup_{\partial N(T)} E^4$. The fundamental group $\pi_1(N(T) - S)$ has the presentation (3-2). We obtain $\pi_1(E^4)$, as follows. Since $N(T)$ is a tubular neighborhood of T, and T is the boundary of the standard unknotted solid torus in $\mathbb{R}^3 \times \{0\}$, we can see that the fundamental group of $E^4 = \mathrm{cl}(S^4 - N(T))$ is the knot group of a trivial torus knot. Hence $\pi_1(E^4)$ is an infinite cyclic group, where the generator r is represented by the loop **r** (see Carter and Saito [14, Section 5.2]). Next we obtain $\pi_1(\partial E^4)$, as follows. Since $\partial E^4 = \partial N(T) = \partial D^2 \times T$ is a 3–dimensional torus $S^1 \times S^1 \times S^1$, $\pi_1(\partial N(T))$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, where the generators r', m' and l' are represented by the loops **r**, **m** and **l** respectively. Let $i_1 \colon \partial N(T) \to N(T) - S$ and $i_2 \colon \partial N(T) \to E^4$ be inclusion maps. Since $i_{1*}(r') = x_1 x_2 \cdots x_m$, $i_{1*}(m') = s$ and $i_{1*}(l') = t$ in $\pi_1(N(T) - S)$, and $i_{2*}(r') = r$, $i_{2*}(m') = 1$ and $i_{2*}(l') = 1$ in $\pi_1(E^4)$, by van Kampen's theorem $\pi_1(S^4 - S) = \pi_1((N(T) - S) \cup_{\partial N(T)} E^4)$ is presented by

$$\left\langle x_1, \ldots, x_m, s, t, r \, \middle| \, \begin{array}{l} s^{-1}x_j s = \mathcal{A}_*^a(x_j), \; t^{-1}x_j t = \mathcal{A}_*^b(x_j), \; t^{-1}st = s, \\ r = x_1 x_2 \cdots x_m, \; s = 1, \; t = 1, \text{ for } j = 1, 2, \ldots, m \end{array} \right\rangle$$

which is the required formula. $\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \square$

Theorem 3.2 *Put* $S_n = \mathcal{S}_4(\sigma_1\sigma_3, \Delta^{2n})$, *where* $\Delta = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1$ *(see Figure 6) and* n *is a positive integer. Then the link group of* S_n *is not a classical link group. Moreover,* S_n *and* S_m *are not equivalent for* $n \neq m$.
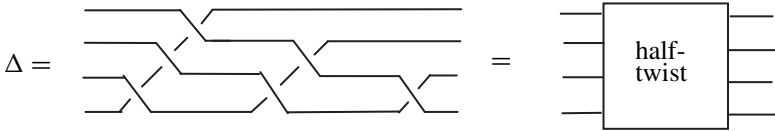
Figure 6: A half-twist Δ

Since Δ^{2n} is a central element of B_4 and the closure of $\sigma_1 \sigma_3$ is a classical link with two components, the torus-covering T^2–link S_n has two components. Each component of S_n is equivalent to $\mathcal{S}_2(\sigma_1, \sigma_1^{2n})$, which is equivalent to $\mathcal{S}_2(\sigma_1, e)$ by Corollary 2.9. By Proposition 2.11, it is equivalent to the spun T^2–knot of $\widehat{\sigma_1}$. Since $\widehat{\sigma_1}$ is a trivial knot, this is an unknotted T^2–knot. Thus each component of S_n is an unknotted T^2–knot.

Proof By Proposition 3.1, the link group G_n of S_n is computed as follows. Let $x_1, \ldots, x_4$ be the generators. The relations concerning the basis braid $\sigma_1 \sigma_3$ are $x_1 = x_2$ and $x_3 = x_4$. The other relations concerning the basis braid Δ^{2n} are:

$$x_1 = (x_1 x_2 x_3 x_4)^n \, x_1 \, (x_1 x_2 x_3 x_4)^{-n}$$
$$x_2 = (x_1 x_2 x_3 x_4)^n \, x_2 \, (x_1 x_2 x_3 x_4)^{-n}$$
$$x_3 = (x_1 x_2 x_3 x_4)^n \, x_3 \, (x_1 x_2 x_3 x_4)^{-n}$$
$$x_4 = (x_1 x_2 x_3 x_4)^n \, x_4 \, (x_1 x_2 x_3 x_4)^{-n}$$

Putting $a = x_1 = x_2$ and $b = x_3 = x_4$, we have:

$$G_n = \langle \, a, b \mid (a^2 b^2)^n b = b(a^2 b^2)^n, \ (a^2 b^2)^n a = a(a^2 b^2)^n \, \rangle$$

By Lemma 3.4, G_1 is not a classical link group. Let us consider the case for $n > 1$. For $n > 1$, let Z_n be the subgroup of G_n generated by $h_n = (a^2 b^2)^n$. By Lemma 3.5, Z_n is the center of G_n for $n > 1$. Further, Z_n is an infinite cyclic group; thus the center of G_n is nontrivial.

We will show that G_n ($n > 1$) is not a classical link group. Since the torus-covering T^2–link S_n consists of two components, we show that G_n is not a classical 2–component link group, as follows. It is known (see Burde and Murasugi [9]) that if the center of the group of a classical 2–component link L is nontrivial, then the link group of L is isomorphic to one of the groups of type (a), (b) or (c) as follows:

(a) $Z \times \mathbb{Z}$,

(b) $(Z \times \mathbb{Z}) *_Z \mathbb{Z}$,

(c) $(Z \times \mathbb{Z}) *_Z (\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z})$

where $\mathbb{Z}$ is an infinite cyclic group, and $Z = \langle h \rangle$ is a "special" infinite cyclic group which is the center of the link group of types (b) and (c). In type (b), the amalgamation concerning the last factor $\mathbb{Z} = \langle q \rangle$ is given by $h = q^\alpha$ for an integer $\alpha > 1$. In type (c), the last factor $\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$ is the group of the torus (α, β)–knot, ie $\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z} = \langle\, x, y \mid x^\alpha = y^\beta \,\rangle$ for coprime positive integers α and β, and the amalgamation is given by $h = x^\alpha = y^\beta$.

Since an infinite cyclic group Z_n is the center of G_n by Lemma 3.5, it suffices to show that G_n $(n > 1)$ is neither of type (b) nor (c). Further, in these cases, the center Z equals Z_n and $h = h_n$.

Case (b) If G_n is of type (b), then $G_n = (Z_n \times \mathbb{Z}) *_{Z_n} \mathbb{Z} = (\langle h_n \rangle \times \langle k \rangle) *_{Z_n} \langle q \rangle$, where the amalgamation is given by $h_n = q^\alpha$ for an integer $\alpha > 1$. Put $h'_n = f(h_n)$ and $q' = f(q)$ for a natural epimorphism

$$(3\text{-}3) \qquad\qquad f\colon G_n \to \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} = \langle a' \rangle * \langle b' \rangle$$

where $a' = f(a)$ and $b' = f(b)$, which are the basis. Since $h_n = (a^2 b^2)^n$, we see that $h'_n = 1$. Since $h_n = q^\alpha$, it follows that $q'^\alpha = 1$. If $q' = 1$, then $f(G_n) = \langle f(k) \rangle$, which is generated by at most one generator. However, $f(G_n)$ is generated by two generators a' and b'. Hence q' is nontrivial. Since $\alpha > 1$ and a nontrivial element of $f(G_n) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ has order 2 or ∞ by Lemma 3.6, q' has order 2; thus $\alpha = 2$. Consider the abelianization map

$$(3\text{-}4) \qquad\qquad \phi\colon G_n \to G_n/[G_n, G_n] = \mathbb{Z} \times \mathbb{Z}$$

and put $\bar{a} = \phi(a)$ and $\bar{b} = \phi(b)$, which are the basis. Since $h_n = q^2$ and $\phi(h_n) = \bar{a}^{2n}\bar{b}^{2n}$, it follows that $\phi(q) = \bar{a}^n \bar{b}^n$. Consider the abelianization map

$$\phi'\colon f(G_n) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

and put $\bar{a}' = \phi'(a')$ and $\bar{b}' = \phi'(b')$, which are the basis. Since $\phi(q) = \bar{a}^n \bar{b}^n$, it follows that $\phi'(q') = 1$ if n is even, and $\phi'(q') = \bar{a}'\bar{b}'$ if n is odd; thus $\phi'(q') = 1$ or $\bar{a}'\bar{b}'$. However, $\phi'(q') = \bar{a}'$ or $\bar{b}'$, as follows. Since $h_n = q^2$, it follows that $q'^2 = 1$; thus $q' = \xi^{-1}a'\xi$ or $\xi^{-1}b'\xi$ for some $\xi \in f(G_n)$ by Lemma 3.6. Thus $\phi'(q') = \bar{a}'$ or $\bar{b}'$. This is a contradiction.

Case (c) If G_n is of type (c), then $G_n = (Z_n \times \mathbb{Z}) *_{Z_n} (\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z})$, where $\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z} = \langle\, x, y \mid x^\alpha = y^\beta \,\rangle$ for coprime positive integers α and β, and the amalgamation is given by $h_n = x^\alpha = y^\beta$. Since $h_n = (a^2 b^2)^n$, we see that $h'_n = 1$, where $h'_n = f(h_n)$. Since $h_n = x^\alpha = y^\beta$, it follows that $x'^\alpha = y'^\beta = 1$, where $x' = f(x)$ and $y' = f(y)$. If $x' = 1$ and $y' = 1$, then $f(\langle\, x, y \mid x^\alpha = y^\beta \,\rangle) = 1$ and it follows that $f(G_n)$ is generated by at most one generator. However, $f(G_n)$ is generated by two generators a' and b'. Hence we can assume that x' is nontrivial. Since any element of $f(G_n) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ has

order 2 or ∞ by Lemma 3.6, it follows that $\alpha = 2$. Since $h_n = x^2$ and $\phi(h_n) = \bar{a}^{2n}\bar{b}^{2n}$, it follows that $\phi(x) = \bar{a}^n\bar{b}^n$, and hence $\phi'(x') = 1$ or $\bar{a}'\bar{b}'$ by the same argument as in Case (b). However, since $h_n = x^2$, it follows that $x'^2 = 1$, and hence $\phi'(x') = \bar{a}'$ or $\bar{b}'$ by the same argument as in Case (b). This is a contradiction. Thus G_n is not a classical link group.

Now we will show that S_n and S_m are not equivalent for $n \neq m$. The center Z_1 of G_1 is a free abelian group of rank 2 (see the proof of Lemma 3.4), while, for $n > 1$, the center Z_n of G_n is an infinite cyclic group by Lemma 3.5; thus it suffices to show in the case when $n, m > 1$. The abelianization of G_n/Z_n for $n > 1$ is $\mathbb{Z} \times \mathbb{Z}/2n\mathbb{Z}$; thus G_n is not isomorphic to G_m for $n \neq m$, and hence $S_n \not\sim S_m$ for $n \neq m$. $\qquad\square$

A *2–link* is a surface link whose components are homeomorphic to 2–spheres. It is known (see Hillman [17, Chapter 3, Corollary 2]) that if the center of a μ–component 2–link group with $\mu > 1$ is nontrivial, then the center must be a torsion group. Hence we have a corollary.

Corollary 3.3 *The link group of the 2–component torus-covering T^2–link of Theorem 3.2 is not a 2–component 2–link group.*

Proof For any $n > 0$, the center Z_n of G_n is nontrivial and torsion free by Lemma 3.4 and Lemma 3.5; thus the conclusion follows from [17, Chapter 3, Corollary 2]. $\qquad\square$

Lemma 3.4 *The group G_1 of Theorem 3.2 is not a classical link group.*

Proof Let Z_1 be the subgroup of G_1 generated by $\{a^2, b^2\}$. We will show that Z_1 is the center of G_1, as follows. Let N be a normal subgroup of G_1. If the center of G_1/N is trivial, then N contains the center of G_1. Since Z_1 consists of central elements, Z_1 is a normal subgroup of G_1 such that Z_1 is contained in the center of G_1. Hence it suffices to show that the center of the quotient group G_1/Z_1 is trivial. Since $G_1/Z_1 = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, the center of G_1/Z_1 is trivial; thus Z_1 is the center of G_1. Let us take the abelianization map ϕ given by (3-4). Since $\phi(a^2) = \bar{a}^2$ and $\phi(b^2) = \bar{b}^2$, the center Z_1 is a free abelian group of rank 2; thus Z_1 is generated by two generators. Hence it follows from Burde and Murasugi [9] that if G_1 is a classical link group, then it is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ (type (a)): G_1 is commutative. However, since the image of G_1 by the natural epimorphism f given by (3-3) is a noncommutative group $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, G_1 is not commutative. This is a contradiction. Thus G_1 is not a classical link group. $\qquad\square$

Lemma 3.5 *Let us consider the group G_n of [Theorem 3.2](#). For $n > 1$, let Z_n be the subgroup of G_n generated by $h_n = (a^2b^2)^n$. Then Z_n is the center of G_n. Moreover Z_n is an infinite cyclic group.*

Proof By the same argument as in the proof of [Lemma 3.4](#), in order to show that Z_n is the center of G_n, it suffices to show that the center of the quotient group G_n/Z_n is trivial. We see that $G_n/Z_n = \langle a, b \mid (a^2b^2)^n = 1 \rangle$, which is an amalgamated product $\langle a \rangle *_U \langle b, x \mid x^n = 1 \rangle$, where $U = \langle a^2 \rangle = \langle xb^{-2} \rangle = \mathbb{Z}$ and the amalgamation is given by $a^2 = xb^{-2}$. Put $H_1 = \langle a \rangle$ and $H_2 = \langle b, x \mid x^n = 1 \rangle$. We can take $\{1, a\}$ as a set of right-handed coset representatives of U in H_1.

Let h be a central element of $G_n/Z_n = H_1 *_U H_2$. By Neumann [29] or Bogopolski [6, Theorem 11.3, page 73], h is uniquely written as $h = ua^\delta c_1 a c_2 \cdots a c_t a^\epsilon$, where $u \in U$ and $c_1, \ldots, c_t$ are nontrivial elements of a set of right-handed coset representatives of U in H_2 and $\delta, \epsilon \in \{0, 1\}$, which is called a *normal form*. Since $ah = ha$, it follows that $aua^\delta c_1 a c_2 \cdots a c_t a^\epsilon = ua^\delta c_1 a c_2 \cdots a c_t a^\epsilon a$. Since $ua = au$ in the amalgamated product $H_1 *_U H_2$, it follows that $ua^\delta a c_1 a c_2 \cdots a c_t a^\epsilon = ua^\delta c_1 a c_2 \cdots a c_t a a^\epsilon$, and hence $ac_1 a c_2 \cdots a c_t = c_1 a c_2 \cdots a c_t a$ as elements in $H_1 *_U H_2$.

If $t > 0$, then $ac_1 a c_2 \cdots a c_t$ and $c_1 a c_2 \cdots a c_t a$ are in distinct normal forms, which is a contradiction. Hence $t = 0$ and $h = ua^\delta = a^k$ for an integer k. Since $hb = bh$, $a^k b = ba^k$. If $k = 1$, then $ab = ba$. In this case, if b is not in U, then we can take b as a nontrivial right-handed coset representative of U in H_2. It follows that then ab and ba are in distinct normal forms, which is a contradiction. If $k = 2l + 1$ (respectively $k = 2l$) for a nonzero integer l, then $a^k b = uab$ and $ba^k = ca$ (respectively $a^k b = ub$ and $ba^k = c$), where in both cases $u = a^{2l} \in U$ and $c = b(xb^{-2})^l$. In these cases, if neither b nor c is in U and we can take b and c as distinct right-handed coset representatives of U in H_2, then $a^k b$ and ba^k have distinct normal forms uab and ca (respectively ub and c), which is a contradiction. Then it follows that $k = 0$ and hence $h = 1$; thus the center of G_n/Z_n is trivial.

It remains to show that neither b nor $c = b(xb^{-2})^l$ ($l \neq 0$) is in U and we can take b and c as distinct right-handed coset representatives of U in H_2. The group $H_2 = \langle b, x \mid x^n = 1 \rangle$ is the free product of $\langle b \rangle$ and $\langle x \mid x^n = 1 \rangle$. By [29; 6, Theorem 11.3, page 73] again, every element of H_2 has a normal form $b_1^\delta x_1 b_2 x_2 \cdots b_t x_t^\epsilon$, where $b_1, b_2, \ldots, b_t$ (respectively $x_1, x_2, \ldots, x_t$) are nontrivial elements of $\langle b \rangle$ (respectively $\langle x \mid x^n = 1 \rangle$) and $\delta, \epsilon \in \{0, 1\}$. Let us determine the normal forms of b and c. Put $l_0 = |l|$, a positive integer. We can see that b has a normal form b, and $c = b(xb^{-2})^l$ has a normal form $b(xb^{-2})^{l_0}$ (respectively $b^3 x^{-1}(b^2 x^{-1})^{l_0-1}$) if $l > 0$ (respectively $l < 0$). Further, an element of $U = \langle xb^{-2} \rangle$ in H_2 has a normal form 1, $(xb^{-2})^{m_0}$ or $(b^2 x^{-1})^{m_0}$, where m_0 is a positive integer. Hence, by the

uniqueness of normal forms, we can see that neither b nor c is in U. Similarly, an element of Ub has a normal form b, $(xb^{-2})^{m_0-1}xb^{-1}$ or $(b^2x^{-1})^{m_0}b$, and an element of $Uc = Ub(xb^{-2})^l$ has a normal form $b(xb^{-2})^{l_0}$, $b^3x^{-1}(b^2x^{-1})^{l_0-1}$, $(xb^{-2})^{m_0-1}xb^{-1}(xb^{-2})^{l_0}$, $(xb^{-2})^{m_0-1}xbx^{-1}(b^2x^{-1})^{l_0-1}$, $(b^2x^{-1})^{m_0}b(xb^{-2})^{l_0}$ or $(b^2x^{-1})^{m_0}b^3x^{-1}(b^2x^{-1})^{l_0-1}$. By the uniqueness of normal forms, we can see that $Ub \neq Uc$. Thus neither b nor c is in U and we can take b and c as distinct right-handed coset representatives of U in H_2, and it follows that the center of G_n/Z_n is trivial. Therefore Z_n is the center of G_n.

Let us take the abelianization map ϕ given by (3-4). Since the image $\phi(h_n)$ is $\overline{a}^{2n}\overline{b}^{2n}$, the center Z_n ($n > 1$) is an infinite cyclic group. $\qquad\square$

Lemma 3.6 Let us consider the group $\mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}$, the free product of $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/q\mathbb{Z}$, where p, q are positive integers greater than one. Then the order of a nontrivial element $\mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}$ is either ∞, a divisor of p or a divisor of q. Further, if the order is finite, then the element can be written as a conjugate of an element of the same order in $\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Z}/q\mathbb{Z}$.

Proof Let z be a nontrivial element of $\mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}$. By Neumann [29] or Bogopolski [6, Theorem 11.3, page 73], z has a normal form

(3-5) $$x_1 y_1 x_2 y_2 \cdots x_t y_t$$

(3-6) $$x_1 y_1 x_2 y_2 \cdots x_{t-1} y_{t-1} x_t$$

(3-7) $$y_1 x_2 y_2 \cdots x_t y_t$$

(3-8) $$y_1 x_2 y_2 \cdots x_{t-1} y_{t-1} x_t$$

where t is an integer with $t > 1$ in (3-8) and $t > 0$ otherwise, and $x_1, x_2, \ldots, x_t$ (respectively $y_1, y_2, \ldots, y_t$) are nontrivial elements of $\mathbb{Z}/p\mathbb{Z}$ (respectively $\mathbb{Z}/q\mathbb{Z}$).

In cases (3-5) and (3-8), z^l has a normal form which is not 1 for any positive integer l. Thus the order of z is infinite.

In cases (3-6) and (3-7), we show the lemma using induction for t, as follows. If $t = 1$, then $z = x_1$ (respectively y_1) for (3-6) (respectively (3-7)) and the order of z is a divisor of p (respectively q). Now let us assume that if $t < s$, then the order of any element z with the normal form (3-6) or (3-7) is infinite or a divisor of p or q. Let us consider z with the normal form (3-6) with $t = s$. If $x_1 x_t = 1$, then $x_1^{-1} z x_1$ has a normal form (3-7) with $t = s - 1$; thus, by the assumption, the statement of the lemma holds. If $x_1 x_t \neq 1$, then z^l has a nontrivial normal form for any positive integer l; thus the order of z is infinite. For z with the normal form (3-7), we can apply the same argument. Further we can see that if the order $\text{ord}(z)$ of z is finite, ie a divisor of p

or q, then z can be written as $\xi^{-1}x\xi$, where $\xi \in \mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}$, and x is an element of order $\operatorname{ord}(z)$ in $\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Z}/q\mathbb{Z}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We can consider the knot version of Theorem 3.2.

Theorem 3.7 *Put $S_n = \mathcal{S}_4(\sigma_1\sigma_3, \Delta^{2n+1})$, where n is a positive integer. Then the knot group of S_n is not a classical knot group. Moreover, S_n and S_m are not equivalent for $n \neq m$.*

The torus-covering T^2–knot $S_0 = \mathcal{S}_4(\sigma_1\sigma_3, \Delta)$ is unknotted (see Corollary 4.4).

Proof By Proposition 3.1, the knot group G_n of S_n is computed as follows. Let $x_1, \ldots, x_4$ be the generators. Then the relations concerning the basis braid $\sigma_1\sigma_3$ are $x_1 = x_2$ and $x_3 = x_4$. The other relations concerning the basis braid Δ^{2n+1} are:

$$x_1 = (x_1x_2x_3x_4)^n\, x_1x_2x_3x_4x_3^{-1}x_2^{-1}x_1^{-1}\, (x_1x_2x_3x_4)^{-n}$$
$$x_2 = (x_1x_2x_3x_4)^n\, x_1x_2x_3x_2^{-1}x_1^{-1}\, (x_1x_2x_3x_4)^{-n}$$
$$x_3 = (x_1x_2x_3x_4)^n\, x_1x_2x_1^{-1}\, (x_1x_2x_3x_4)^{-n}$$
$$x_4 = (x_1x_2x_3x_4)^n\, x_1\, (x_1x_2x_3x_4)^{-n}$$

Putting $a = x_1 = x_2$ and $b = x_3 = x_4$, we have:

$$G_n = \langle\, a, b \mid b(a^2b^2)^n = (a^2b^2)^na,\ a(a^2b^2)^{n+1} = (a^2b^2)^{n+1}b \,\rangle$$

Let us assume that G_n is a classical knot group. Let Z_n be the subgroup of G_n generated by $h_n = (a^2b^2)^{2n+1}$, which is a central element. By Lemma 3.8, Z_n is the center of G_n. Further, Z_n is an infinite cyclic group. It is known (see Burde and Zieschang [10]) that if the center of a classical knot group is nontrivial, then the knot is a torus knot. Hence, by the assumption, G_n is isomorphic to a torus knot group. Let $G_{p,q}$ be the (p,q)–torus knot group isomorphic to G_n, where p and q are coprime positive integers. Let $Z_{p,q}$ be the center of $G_{p,q}$. Then $G_{p,q} = \langle\, x, y \mid x^p = y^q \,\rangle$ and $Z_{p,q}$ is generated by $h = x^p = y^q$. Put $G'_{p,q} = G_{p,q}/Z_{p,q}$, which is $\langle\, x, y \mid x^p = y^q = 1 \,\rangle = \mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}$. The abelianization of $G'_{p,q}$ is isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$.

Consider the quotient group $G'_n = G_n/Z_n$. By (3-9) in the proof of Lemma 3.8, $G'_n = \langle\, a, x \mid x^{2n+1} = (a^2x^n)^2 = 1 \,\rangle$. The abelianization of G'_n is presented by $\langle\, a, x \mid x^{2n+1} = (a^2x^n)^2 = 1, ax = xa \,\rangle$, which equals $\langle\, a \mid a^{4(2n+1)} \,\rangle = \mathbb{Z}/4(2n+1)\mathbb{Z}$. Since $G_{p,q}$ and G_n are isomorphic, so are the abelianizations of $G'_{p,q}$ and G'_n. Hence, comparing the order of the groups we see that $pq = 4(2n+1)$. Since G'_n has an element of order $2n+1$ by Lemma 3.9, and the order of a nontrivial torsion element

of $G'_{p,q}$ is a divisor of p or q by Lemma 3.6, it follows that $2n + 1$ is a divisor of p or q. Hence we can determine coprime positive integers p and q by $p = 4$ and $q = 2n + 1$.

For any element z of order 2 in $G'_{4,2n+1} = \mathbb{Z}/4\mathbb{Z} * \mathbb{Z}/(2n+1)\mathbb{Z}$, z can be written as $z = z'^2$ for some element z' of order 4 by Lemma 3.10. Since $y = a^2 x^n \in G'_n$ is of order 2 by Lemma 3.9, and G'_n and $G'_{4,2n+1}$ are isomorphic, there is an element $y' \in G'_n$ with $y = y'^2$, and hence $G'_n = \langle a, x, y' \mid x^{2n+1} = 1, \ y'^4 = 1, \ a^2 x^n = y'^2 \rangle$. Let N_w be the normal subgroup of $G'_{4,2n+1}$ generated by an element w of order $2n + 1$. The quotient group $G'_{4,2n+1}/N_w$ does not depend on the choice of w and $G'_{4,2n+1}/N_w = \mathbb{Z}/4\mathbb{Z}$ by Lemma 3.11. We will denote it by $G'_{4,2n+1}/N$. Let N_x be the normal subgroup of G'_n generated by x. Since x has order $2n + 1$ by Lemma 3.9, G'_n/N_x is isomorphic to $G'_{4,2n+1}/N = \mathbb{Z}/4\mathbb{Z}$: G'_n/N_x is abelian. Adding the relation $x = 1$ to the presentation of G'_n, we see that $G'_n/N_x = \langle a, y' \mid y'^4 = 1, \ a^2 = y'^2 \rangle$. Since there is a natural epimorphism f from G'_n/N_x onto $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ with the basis $f(a)$ and $f(y')$, G'_n/N_x is not abelian. This is a contradiction. Thus G_n is not a classical knot group. Since the abelianization of G'_n is $\mathbb{Z}/4(2n+1)\mathbb{Z}$, it follows that $S_n \not\sim S_m$ for $n \neq m$. $\square$

Lemma 3.8 *Let us consider the group G_n of Theorem 3.7. For $n > 0$, let Z_n be the subgroup of G_n generated by $h_n = (a^2 b^2)^{2n+1}$. Then Z_n is the center of G_n. Moreover Z_n is an infinite cyclic group.*

Proof Since h_n is a central element, by the same argument as in the proof of Lemma 3.4, in order to show that Z_n is the center of G_n, it suffices to show that the center of $G'_n = G_n/Z_n$ is trivial. The quotient group $G'_n = G_n/Z_n$ is presented by $\langle a, b, x \mid x = a^2 b^2, \ bx^n = x^n a, \ ax^{n+1} = x^{n+1}b, \ x^{2n+1} = 1 \rangle$. By eliminating b by $b = x^n a x^{-n}$, we have

$$(3\text{-}9) \qquad\qquad G'_n = \langle a, x \mid x^{2n+1} = (a^2 x^n)^2 = 1 \rangle$$

which is an amalgamated product $\langle a \rangle *_U \langle x, y \mid x^{2n+1} = 1, \ y^2 = 1 \rangle$, where $U = \langle a^2 \rangle = \langle yx^{-n} \rangle = \mathbb{Z}$ and the amalgamation is given by $a^2 = yx^{-n}$. We can show that the center of G'_n is trivial by the following argument similar to the proof of Lemma 3.5, as follows. Put $H_1 = \langle a \rangle$ and $H_2 = \langle x, y \mid x^{2n+1} = 1, \ y^2 = 1 \rangle$. Note that H_2 is a free product of $\langle x \mid x^{2n+1} = 1 \rangle = \mathbb{Z}/(2n+1)\mathbb{Z}$ and $\langle y \mid y^2 = 1 \rangle = \mathbb{Z}/2\mathbb{Z}$. By Neumann [29] or Bogopolski [6, page 73, Theorem 11.3], any element of $G'_n = H_1 *_U H_2$ has a normal form $u a^\delta c_1 a c_2 \cdots a c_t a^\epsilon$, where $u \in U$ and $c_1, \ldots, c_t$ are nontrivial elements of a set of right handed coset representatives of U in H_2 and $\delta, \epsilon \in \{0, 1\}$.

Let h be a central element of G'_n. By the same argument as in the proof of Lemma 3.5, by using normal forms, we see that $h = a^k$ for an integer k. Since $hx = xh$, $a^k x = xa^k$.

If $k = 2l + 1$ (respectively $2l$) for a nonzero integer l, then $a^k x = uax$ and $xa^k = ca$ (respectively $a^k x = ux$ and $xa^k = c$), where $u = a^{2l} \in U$ and $c = x(yx^{-n})^l$ in both cases. If neither x nor c is in U and we can choose x and c as distinct right-handed coset representatives of U in H_2, then in both cases $a^k x$ and xa^k have distinct normal forms, which is a contradiction. Then $k = 0$, and it follows that $h = 1$.

It remains to show that neither x nor $c = x(yx^{-n})^l$ ($l \neq 0$) is in U and we can choose x and c as distinct right-handed coset representatives of U in H_2. By [29; 6, Theorem 11.3, page 73=, any element of the free product $H_2 = \langle x \mid x^{2n+1} = 1 \rangle * \langle y \mid y^2 = 1 \rangle$ has a normal form $x_1^\delta y x_2 y \cdots x_t y^\epsilon$, where $x_1, x_2, \ldots, x_t$ are nontrivial elements of $\langle x \mid x^{2n+1} = 1 \rangle$ and $\delta, \epsilon \in \{0, 1\}$. Let us determine the normal forms of x and c. Put $l_0 = |l|$, a positive integer. We can see that x has a normal form x, and $c = x(yx^{-n})^l$ has a normal form $x(yx^{-n})^{l_0}$ (respectively $x^{n+1} y(x^n y)^{l_0-1}$) if $l > 0$ (respectively $l < 0$). Further, an element of $U = \langle yx^{-n} \rangle$ has a normal form either 1, $(yx^{-n})^{m_0}$ or $(x^n y)^{m_0}$, where m_0 is a positive integer. Hence, by the uniqueness of normal forms, neither x nor c is in U. Similarly, if $n = 1$ (respectively $n > 1$), then an element of Ux has a normal form either x, $(yx^{-1})^{m_0-1} y$ or $(xy)^{m_0} x$ (respectively x, $(yx^{-n})^{m_0-1} yx^{-n+1}$ or $(x^n y)^{m_0} x$). Hence in both cases c is not an element of Ux. Thus neither x nor c is in U and we can choose x and c as distinct right-handed coset representatives of U in H_2, and it follows that the center of G_n' is trivial. Thus Z_n is the center. Considering the abelianization map of G_n, we see that Z_n is an infinite cyclic group. $\qquad \square$

Lemma 3.9 *The element x (respectively $y = a^2 x^n$) of G_n' of Theorem 3.7 (see (3-9)) is of order $2n + 1$ (respectively 2).*

Proof By Lemma 3.8, G_n' is an amalgamated product. Seeing the normal forms of the powers of x and y, we can show that the order of x is $2n + 1$ and the order of y is 2. $\qquad \square$

Lemma 3.10 *For any element z of order 2 in $\mathbb{Z}/4\mathbb{Z} * \mathbb{Z}/(2n+1)\mathbb{Z}$, z can be written as $z = z'^2$ for some element z' of order 4.*

Proof The order of z is 2. It is a divisor of 4, and it is not a divisor of $2n + 1$. Thus it follows from Lemma 3.6 that $z = \xi^{-1} u^2 \xi$, where $\xi \in \mathbb{Z}/4\mathbb{Z} * \mathbb{Z}/(2n+1)\mathbb{Z}$ and u is a generator of $\mathbb{Z}/4\mathbb{Z}$. Thus z can be written as $z = z'^2$, where $z' = \xi^{-1} u \xi$. Since the order of u is 4, so is the order of z'. $\qquad \square$

Lemma 3.11 *Put $G_{4, 2n+1}' = \mathbb{Z}/4\mathbb{Z} * \mathbb{Z}/(2n+1)\mathbb{Z}$, and let N_w be a normal subgroup of $G_{4, 2n+1}'$ generated by an element w of order $2n + 1$. Then $G_{4, 2n+1}'/N_w$ does not depend on the choice of w, and $G_{4, 2n+1}'/N_w = \mathbb{Z}/4\mathbb{Z}$.*

Proof Since w has order $2n+1$, $w = \xi^{-1}v^k\xi$ by Lemma 3.6, where $\xi \in \mathbb{Z}/4\mathbb{Z} *$ $\mathbb{Z}/(2n+1)\mathbb{Z}$ and v is a generator of $\mathbb{Z}/(2n+1)\mathbb{Z}$, and k is an integer such that v^k has order $2n+1$. Put $X = \{\overline{kl} \mid l = 1, 2, \ldots, 2n\}$, where $\overline{kl} = kl$ mod $2n+1$. If $\overline{kl} = \overline{kl'}$ for $l \neq l'$ $(0 < l, l' < 2n+1)$, then $k|l - l'| \equiv 0$ mod $2n+1$, and it follows that the order of w is a divisor of $|l - l'| < 2n$. Then the order is smaller than $2n+1$, which is a contradiction. Hence, if $l \neq l'$ $(0 < l, l' < 2n+1)$, then $\overline{kl} \neq \overline{kl'}$. Since v^k has order $2n+1$, $v^{kl} \neq 1$ for $0 < l < 2n+1$; thus $\overline{0} \notin X$. Thus $X = \{\overline{1}, \overline{2}, \ldots, \overline{2n}\}$, and hence X contains $\overline{1}$. Hence $\overline{kl_0} = \overline{1}$ for some integer l_0, and it follows that $w^{l_0} = \xi^{-1}v^{kl_0}\xi = \xi^{-1}v\xi$. Since $v = \xi w^{l_0}\xi^{-1}$, we have $v \in N_w$. Since N_w contains the generator v of N_v, we have $N_v \subset N_w$. Similarly, since $w = \xi^{-1}v^k\xi$, we have $N_w \subset N_v$. Hence $N_v = N_w$. Thus $G'_{4,\,2n+1}/N_w = G'_{4,\,2n+1}/N_v = \mathbb{Z}/4\mathbb{Z}$. $\square$

Using the results of Theorem 3.2 and Theorem 3.7, we have the following theorem.

Theorem 3.12 *For an integer $l > 1$, $\mathcal{S}_4(\sigma_1\sigma_3, \Delta^l)$ is not equivalent to either a spun T^2–link, a turned spun T^2–link or the split union of spun T^2–links and turned spun T^2–links.*

Proof Since the link group of the spun T^2–link or the turned spun T^2–link of a classical link L is isomorphic to the link group of L (see Livingston [26] and Boyle [8]), it is classical. Thus the link group of the split union of spun T^2–links and turned spun T^2–links is also classical. However, the link group of $\mathcal{S}_4(\sigma_1\sigma_3, \Delta^l)$ $(l > 1)$ is not classical by Theorem 3.2 and Theorem 3.7. Thus we have the conclusion. $\square$

4 Ribbon torus-covering T^2–links

In this section we show Theorem 4.1: for certain basis mn–braids, $\mathcal{S}_{mn}(a, b)$ is ribbon. As a corollary, we can see that the torus-covering T^2–link of Theorem 3.12 is ribbon (Corollary 4.2).

Let M be a disjoint union of a finite number of handlebodies. The image of M into $\mathbb{R}^4$ by an immersion ϕ is called a *3–ribbon* (see Yanagawa [32]) if the singularity set consists of *ribbon singularities*, ie the self-intersection of $\phi(M)$ consists of a finite number of mutually disjoint 2–disks, and for each 2–disk D, the preimage $\phi^{-1}(D)$ consists of a pair of 2–disks D', D'' such that $D' \cap D'' = \varnothing$, $D' \subset \text{Int}\,M$ and $\partial D'' = D'' \cap \partial M$. An oriented surface link is *ribbon* if it bounds a 3–ribbon (see [32]).

For an m–braid b, we denote by $b^{(n)}$ the n–parallel of b, ie $b^{(n)}$ is the mn–braid obtained from b by replacing each string of b with its n parallel copies; see Figure 7. For n–braids $b_1, b_2, \ldots, b_m$, we denote by $b_1 \circ b_2 \circ \cdots \circ b_m$ the mn–braid depicted in Figure 8.
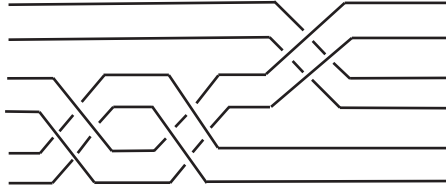
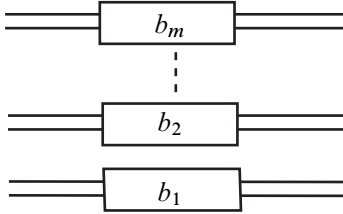Figure 7: The 2–parallel $(\sigma_1^2\sigma_2^{-1})^{(2)}$ of the 3–braid $\sigma_1^2\sigma_2^{-1}$

Figure 8: The mn–braid $b_1 \circ b_2 \circ \cdots \circ b_m$

Theorem 4.1 Let α be a classical n–braid whose closure $\hat{\alpha}$ is a trivial knot. Let a and b be mn–braids given by

$$a = \overbrace{\alpha \circ \alpha \circ \cdots \circ \alpha}^{m}$$
$$b = b'^{(n)} \cdot (\alpha^{l_1} \circ \alpha^{l_2} \circ \cdots \circ \alpha^{l_m})$$

where b' is an m–braid and l_j is an integer $(j = 1, 2, \ldots, m)$; note that a and b are commutative. Then $\mathcal{S}_{mn}(a, b)$ is ribbon.

Proof Let the braid word presentation of b' be $b' = \sigma_{i_1}^{\epsilon_1} \cdot \sigma_{i_2}^{\epsilon_2} \cdots \sigma_{i_\nu}^{\epsilon_\nu}$, where $i_k \in \{1, 2, \ldots, m-1\}$ and $\epsilon_k \in \{+1, -1\}$ for $k = 1, 2, \ldots, \nu$. Let $\pi \colon \mathbb{R}_+^3 \times S^1 \to \mathbb{R}_+^3$ be the projection. Let us take a solid torus $N(\mathbf{m}) \subset \mathbb{R}_+^3$ and a disk $D(x_0) \subset N(\mathbf{m})$ as in the proof of Proposition 2.11. Let us take the closure $\hat{a}$ of a in $N(\mathbf{m})$ as shown in Figure 9, where we take the m parallel copies of $\hat{\alpha}$ in such a position that the identified corresponding ends are in $D(x_0)$.

We consider a surface link S determined by the motion picture $S_t = \pi(S \cap (\mathbb{R}_+^3 \times \{t\}))$ along S^1, which is the orbit of the isotopy from $\hat{a}$ to $\hat{a}$, given by the composition of the following (1) and (2).

(1) Concerning $b'^{(n)}$, let us take the isotopy from $\hat{a}$ to $\hat{a}$ as follows. For each $(\sigma_{i_k}^{\epsilon_k})^{(n)}$, we consider the isotopy shown in Figure 10 if $\epsilon_k = +1$, and its inverse if $\epsilon_k = -1$. Further, we consider the composition of them for all k.
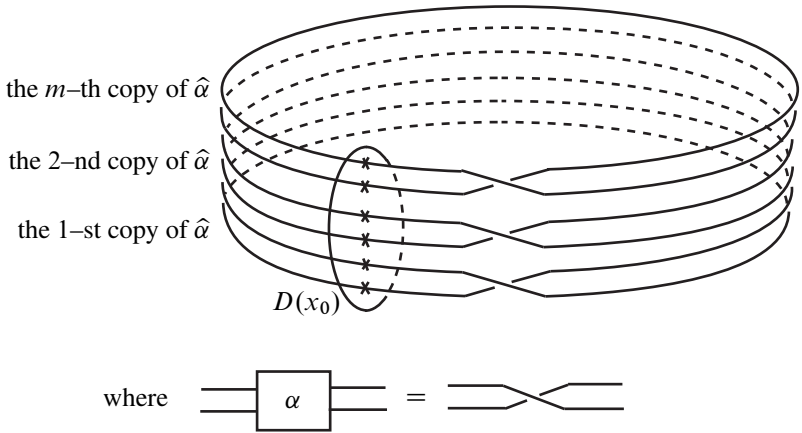
Figure 9: The closure $\hat{a}$ of a

the (i_k+1)–st copy of $\hat{\alpha}$

the i_k–th copy of $\hat{\alpha}$

Figure 10: We consider this isotopy (1), concerning $(\sigma_{i_k}^{\epsilon_k})^{(n)}$, if $\epsilon_k = +1$.

(2)   Concerning $\alpha^{l_1} \circ \alpha^{l_2} \circ \cdots \circ \alpha^{l_m}$, let us take the isotopy from $\hat{a}$ to $\hat{a}$ which turns the j–th copy of $\hat{\alpha}$ l_j times as shown in Figure 11, for each $j = 1, 2, \ldots, m$.

Since each isotopy is from $\hat{a}$ to $\hat{a}$, $\{S_t\}$, and hence S, is well-defined.
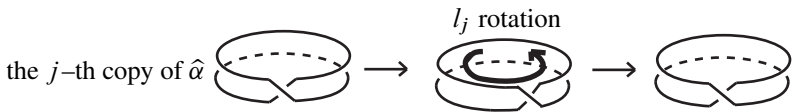
Figure 11: The isotopy (2), concerning $\alpha^{l_1} \circ \alpha^{l_2} \circ \cdots \circ \alpha^{l_m}$

Next we show that S is equivalent to $\mathcal{S}_{mn}(a, b)$, as follows. It suffices to see that the orbit of $S_t \cap D(x_0)$ forms b. The orbit of $S_t \cap D(x_0)$ by the isotopy (1) is as in Figure 12; thus it forms $(\sigma_{i_k}^{\epsilon_k})^{(n)}$. Since the isotopy (2) turns the j–th copy of $\hat{\alpha}$ l_j times ($j = 1, 2, \ldots, m$), by the similar argument to the proof of Proposition 2.13, we can see that the orbit of $S_t \cap D(x_0)$ by this isotopy forms $\alpha^{l_1} \circ \alpha^{l_2} \circ \cdots \circ \alpha^{l_m}$. Thus, the orbit of $S_t \cap D(x_0)$ as a whole forms $(\sigma_{i_1}^{\epsilon_1} \sigma_{i_2}^{\epsilon_2} \cdots \sigma_{i_\nu}^{\epsilon_\nu})^{(n)} \cdot (\alpha^{l_1} \circ \alpha^{l_2} \circ \cdots \circ \alpha^{l_m})$, which is b. Thus S is equivalent to $\mathcal{S}_{mn}(a, b)$ by Lemma 2.8 (2).

(the (i_k+1)–st $\hat{\alpha}$) $\cap D(x_0)$
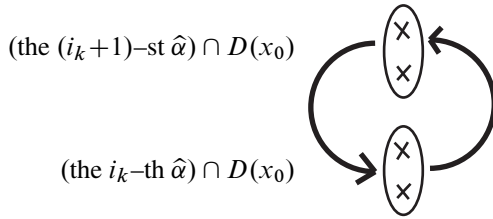
(the i_k–th $\hat{\alpha}$) $\cap D(x_0)$

Figure 12: The orbit of $S_t \cap D(x_0)$ by the isotopy (1) concerning $(\sigma_{i_k}^{\epsilon_k})^{(n)}$, if $\epsilon_k = +1$

Now let us construct an immersed 3–manifold M such that $\partial M = S$, which is determined by $M_t = \pi(M \cap (\mathbb{R}^3_+ \times \{t\}))$ as follows. Since $\hat{\alpha}$ is a trivial knot, we can take a disk bounded by $\hat{\alpha}$ as shown in Figure 13. For each S_t, let M_t be the union of such disks bounded by S_t. As the union of M_t, we naturally obtain an immersed 3–manifold M such that $\partial M = S$.
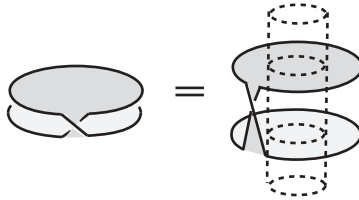
Figure 13: The disk bounded by $\hat{\alpha}$

In order to show that $\mathcal{S}_{mn}(a, b)$ is ribbon, it is sufficient to show that M is a 3–ribbon, ie M has only ribbon singularities. Since M has no singularity in the motion picture of the isotopy (2), it is sufficient to show that the singularity in the motion picture of the isotopy (1) consists of ribbon singularities. Let us consider the singularity of M in the motion picture of the isotopy (1) shown in Figure 10. This singularity is of the form of the singularity of the motion picture shown in Figure 14, and hence this singularity set is the disk itself. Therefore M has only ribbon singularities. □

Corollary 4.2 *For any integer l, $\mathcal{S}_4(\sigma_1\sigma_3, \Delta^l)$ is ribbon.*

Proof Put $a = \sigma_1\sigma_3$ and $b = \Delta^l$. Let α be a 2–braid σ_1. Then $\hat{\alpha}$ is a trivial knot, and $a = \alpha \circ \alpha$. By definition, $\sigma_1^{(2)} = \sigma_2\sigma_1\sigma_3\sigma_2$. Since $\Delta = (\sigma_2\sigma_1\sigma_3\sigma_2) \cdot \sigma_1\sigma_3 = \sigma_1^{(2)} \cdot \sigma_1\sigma_3$, and $\sigma_i \cdot \sigma_1^{(2)} = \sigma_1^{(2)} \cdot \sigma_j$ for $\{i, j\} = \{1, 3\}$, together with the fact that σ_1 and σ_3 commute, it follows that Δ^l can be written as $\Delta^l = \sigma_1^{l\,(2)} \cdot \sigma_1^l\sigma_3^l$; thus $b = b'^{(2)} \cdot (\alpha^l \circ \alpha^l)$, where $b' = \sigma_1^l$. Thus the basis braids a and b have the required presentations of Theorem 4.1, and hence $\mathcal{S}_4(\sigma_1\sigma_3, \Delta^l)$ is ribbon by the theorem. □
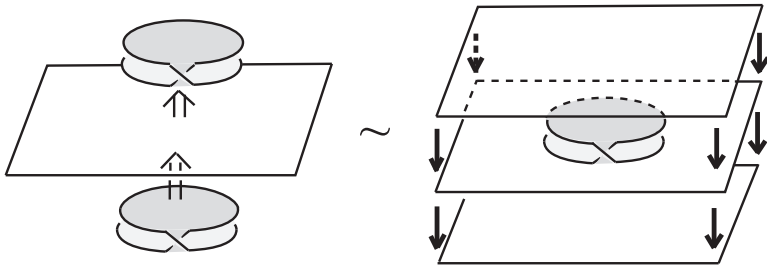
Figure 14: The motion picture of an upward move of the disk (the left picture) is equivalent to the motion picture of a downward move of a horizontal plane (the right picture)

So the torus-covering T^2–link of Theorem 3.12 is ribbon. Together with Theorem 3.12, this indicates the following corollary.

Corollary 4.3 *For an integer $l > 1$, $\mathcal{S}_4(\sigma_1\sigma_3, \Delta^l)$ has the following properties: (1) it can be presented by an m–chart on S^2 without white vertices, however (2) any m–chart on T presenting it has at least one white vertex.*

Proof Put $S = \mathcal{S}_4(\sigma_1\sigma_3, \Delta^l)$.

(1) Any ribbon surface link is presented by an m–chart on the standard 2–sphere S^2 without white vertices (see Kamada [20; 24]). Since S is ribbon by Corollary 4.2, it is presented by an m–chart on S^2 without white vertices.

(2) By Lemma 2.7 and Theorem 2.16, if an m–chart on T presenting a torus-covering T^2–link does not have a white vertex, then it presents either a spun T^2–link, a turned spun T^2–link or the split union of spun T^2–links and turned spun T^2–links. Since S is not equivalent to such a surface link by Theorem 3.12, it cannot be presented by an m–chart on T without white vertices. □

Concerning $\mathcal{S}_4(\sigma_1\sigma_3, \Delta)$, we have the following corollary, by using the 3–ribbon constructed in the proof of Theorem 4.1.

Corollary 4.4 *The torus-covering T^2–knot $\mathcal{S}_4(\sigma_1\sigma_3, \Delta)$ is unknotted.*

A 1–*handle* attaching to a surface link S is a 3–ball h embedded in $\mathbb{R}^4$ such that $S \cap h$ is a pair of 2–disks in ∂h. The closure (as a set) of $(S \cup \partial h) - (S \cap h)$ is a surface link. We call it the surface link obtained from S by a 1–*handle surgery* along a 1–handle h. A 2–*handle* attaching to S is a 3–ball h embedded in $\mathbb{R}^4$ such that $S \cap h$ is an annulus in ∂h. The closure (as a set) of $(S \cup \partial h) - (S \cap h)$ is a surface link. We call it the surface link obtained from S by a 2–*handle surgery* along a 2–handle h. The inverse operation of a 1–handle surgery is a 2–handle surgery, and vice versa.

Proof By Corollary 4.2, $\mathcal{S}_4(\sigma_1\sigma_3, \Delta)$ is equivalent to $S = \partial M$ in the proof of Theorem 4.1. We use the notation of the theorem. By Corollary 4.2, α is the 2–braid σ_1, and the basis braids are the 4–braids given by $\alpha \circ \alpha$ and $(\sigma_1)^{(2)} \cdot (\alpha \circ \alpha)$. Since $\partial M_0 = S_0$ is the closure of $\alpha \circ \alpha$, it consists of two components; thus the part of M of the motion picture of the isotopy (1) consists of two connected components. Let us denote by h one of the components containing the first copy of $\hat{\alpha}$ in ∂M_0. Since h is an embedded 3–ball such that $S \cap h$ is an annulus in ∂h (see Figure 10), it is a 2–handle on S. Let $M' = \text{cl}(M - h)$, and put $S' = \partial M'$. Then S' is the surface link obtained from S by a 2–handle surgery along h. Since a 2–handle surgery is the inverse operation of a 1–handle surgery, h is a 1–handle on S', and S is obtained from S' by a 1–handle surgery along h. Since the singularity set of M is contained in h (see the proof of Theorem 4.1), M' is an embedded 3–ball with no singularity; thus S' is an unknotted sphere. It is known (see Boyle [7, Corollary 5]) that if a surface knot is unknotted, then the result of a 1–handle surgery for any 1–handle is also unknotted. Thus S, hence $\mathcal{S}_4(\sigma_1\sigma_3, \Delta)$, is unknotted. □

5 Quandle cocycle invariants

It is known (see Asami and Satoh [2]) that the quandle cocycle invariant of a twist-spun 2–knot of a classical knot K can be presented by using the quandle cocycle invariants of a 1–tangle whose closure is K. In this section we present the quandle cocycle invariant of $\mathcal{S}_m(b, \Delta^{2n})$ for an m–braid b (Theorem 5.2), by using the quandle cocycle invariants of the closure of b. Here Δ is a half-twist of a bundle of m parallel strands. In Theorem 5.5, we calculate some concrete examples.

This section is organized as follows. In Section 5.1, we review the quandle cocycle invariants and the shadow cocycle invariants. Further, we give a certain 2–cocycle, which is determined from a 3–cocycle. Using these terms, we give the statement of Theorem 5.2. In Section 5.2, we study triple points of $\mathcal{S}_m(b, \Delta^{2n})$, and prove Theorem 5.2. In Section 5.3, we show Theorem 5.5, using a dihedral quandle and Mochizuki's 3–cocycle.

5.1 Quandle cocycle invariant of $\mathcal{S}_m(b, \Delta^{2n})$

A *quandle* (see Joyce [19]) is a set Q with a binary operation $*$ satisfying the following conditions:

  (i)   For any $x \in Q$, $x * x = x$.
  (ii) For any $x, y \in Q$, there exists a unique $z \in Q$ such that $x = z * y$.
  (iii) For any $x, y, z \in Q$, $(x * y) * z = (x * z) * (y * z)$.

From now on, assume that Q is a finite quandle, ie a quandle consisting of finitely many elements.

For an oriented classical link L or an oriented surface link S, let us denote by D a diagram of L or S, ie the image of L or S by a generic projection to $\mathbb{R}^2$ or $\mathbb{R}^3$. In order to indicate crossing information of the diagram, we break the under-arc or the under-sheet into two pieces missing the over-arc or the over-sheet. Then the diagram is presented by a disjoint union of arcs, or compact surfaces which are called *broken sheets*. Let $B(D)$ be the set of such arcs or broken sheets. A Q–*coloring* for a diagram D of L or S is a map $C: B(D) \to Q$ as in Figure 15. The image by C is called the *color*.
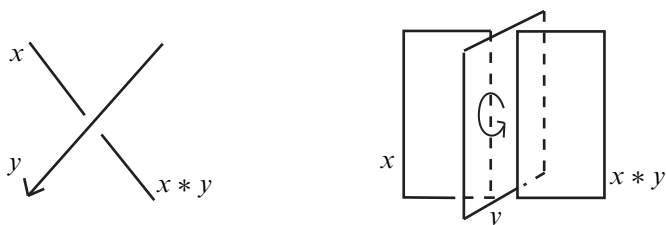
Figure 15: A Q–coloring C, where x, y and $x * y$ are the colors of arcs or broken sheets given by C

Let G be an abelian group. A *2–cocycle* with the coefficient group G is a map $f: Q^2 \to G$ satisfying

$$f(s, u) + f(s * u, t * u) = f(s, t) + f(s * t, u)$$

and
$$f(s, s) = 0$$

for any $s, t, u \in Q$. A 3–*cocycle* is a map $f: Q^3 \to G$ satisfying

$$f(s, t, u) + f(s * u, t * u, v) + f(s, u, v) = f(s * t, u, v)$$
$$+ f(s, t, v) + f(s * v, t * v, u * v),$$
$$f(s, s, t) = 0 \quad \text{and} \quad f(s, t, t) = 0$$

for any $s, t, u, v \in Q$.

For a Q–coloring C for a diagram D of a classical link L or a surface link S, we briefly review the quandle cocycle invariant as follows (for details see Carter, Jelsovsky, Kamada, Langford and Saito [11]), where G is written multiplicatively. For the case

of a classical link, at each crossing r of the diagram D, the *weight* $W_f(r;C)$ at r for a 2–cocycle f is given as in Figure 16. Put

$$\Phi_f(L;C) = \prod_{r\in X_2(D)} W_f(r;C)$$

where $X_2(D)$ is the set of the crossings of D. For the case of a surface link, at each triple point t of the diagram D, the *weight* $W_f(t;C)$ at t for a 3–cocycle f is given as in Figure 17 (see [11, Sections 10 and 11] and also Carter and Saito [14, Proposition 4.43 (3)]). Put

$$\Phi_f(S;C) = \prod_{t\in X_3(D)} W_f(t;C)$$

where $X_3(D)$ is the set of the triple points of D. It is known [11] that $\Phi_f(L;C)$ or
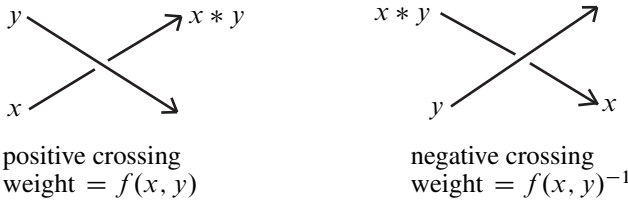
Figure 16: The weight at a crossing, where x, y and $x*y$ are the colors of arcs by C, and f is a 2–cocycle
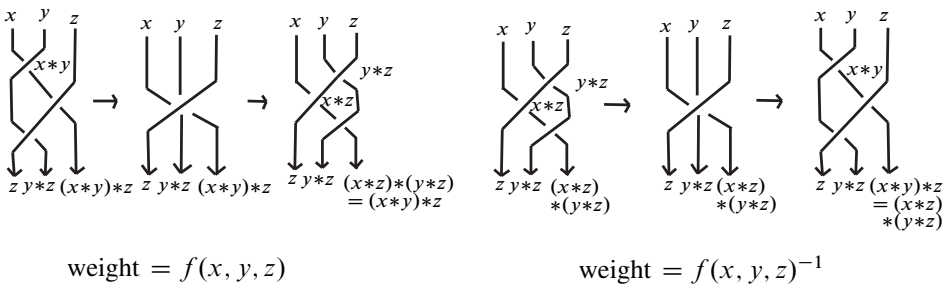
Figure 17: The weight at a triple point, where the triple point is presented by a motion picture around it, and x, y, z, etc are the colors by C, and f is a 3–cocycle

$\Phi_f(S;C)$ is an invariant of L or S. We call it the *quandle cocycle invariant* of L or S associated with a Q–coloring C (see [11]). Since $B(D)$ is a finite set, so is

the set of Q–colorings for D. Let $\mathrm{Col}_Q(D)$ be the set of all the Q–colorings. Then define $\Phi_f(L)$ or $\Phi_f(S)$ by

$$\Phi_f(X) = \sum_{C \in \mathrm{Col}_Q(D)} \Phi_f(X;C) \in \mathbb{Z}[G]$$

where $X = L$ or S, and f is a 2–cocycle (respectively 3–cocycle) if $X = L$ (respectively $X = S$). We call $\Phi_f(X)$ the *quandle cocycle invariant* of X associated with f [11].

Next we define a shadow coloring for a classical link. For a diagram of a classical link with a given Q–coloring, its shadow color is determined from the color of the unbounded region (see Carter, Kamada and Saito [13]), which we will call the *base color*.

Let C be a Q–coloring for a diagram D of a classical link L. A *shadow coloring of D extending C with the base color $x \in Q$* is a map $C_x^*: B^*(D) \to Q$, where $B^*(D)$ is the union of $B(D)$ and the set of regions of $\mathbb{R}^2$ separated by the immersed strings of the diagram D, satisfying the following conditions:

  (i) C_x^* restricted to $B(D)$ is coincident with C.

 (ii)   The color of the regions are as in Figure 18.

(iii)   The color of the unbounded region is x.

By [13], C_x^* exists uniquely for given C and x. For a 3–cocycle f, let us define the weight at a positive (respectively negative) crossing r by $W_f^*(r;C,x) = f(w,y,z)$ (respectively $f(w,y,z)^{-1}$) in G, where y, z and w are the colors shown in Figure 18. Put:

$$\Psi_f^*(L;C,x) = \prod_{r \in X_2(D)} W_f^*(r;C,x)$$

It is known [13] that $\Psi_f^*(L;C,x)$ is an invariant of L. We will call $\Psi_f^*(L;C,x)$ the *shadow cocycle invariant* of L associated with the Q–coloring C and the base color x (see [13]).

Let $\mathcal{R}_y: Q \to Q$ be a map defined by $\mathcal{R}_y(x) = x * y$ for $x, y \in Q$. Further, let $\mathcal{R}_\varnothing = \mathrm{id}_Q$. We will denote $\mathcal{R}_{y_l} \circ \cdots \circ \mathcal{R}_{y_2} \circ \mathcal{R}_{y_1}$ by $\mathcal{R}_{(y_1, y_2, \ldots, y_l)}$ for $(y_1, y_2, \ldots, y_l) \in Q^l$. For quandles Q and Q', a map $\phi: Q \to Q'$ is called a *quandle homomorphism* if $\phi(x * y) = \phi(x) * \phi(y)$ for any $x, y \in Q$. By the condition (iii) of a quandle, for any $\mathbf{y} = (y_1, y_2, \ldots, y_l)$, $\mathcal{R}_\mathbf{y}$ is a quandle homomorphism.
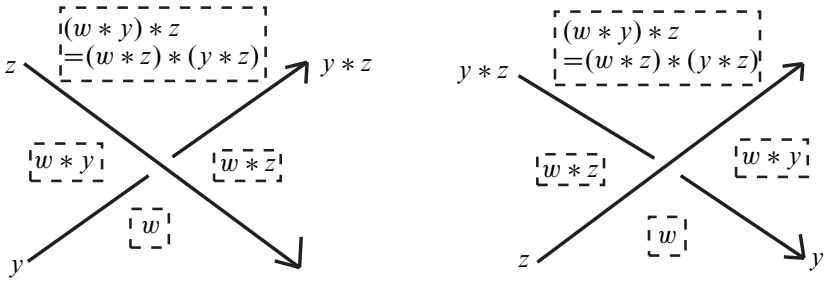
Figure 18: A shadow coloring

For a G–valued 3–cocycle f and $\mathbf{y} = (y_1, \ldots, y_l) \in Q^l$, let $\widehat{f}_{\mathbf{y}} \colon Q^2 \to G$ be the map defined by:

$$(5\text{-}1) \qquad \widehat{f}_{\mathbf{y}}(s, t) = \sum_{j=1}^{l} f(\mathcal{R}_{(y_1,\ldots,y_{j-1})}(s), \mathcal{R}_{(y_1,\ldots,y_{j-1})}(t), y_j)$$

Lemma 5.1 *Assume that $\mathcal{R}_{\mathbf{y}} = \mathrm{id}_Q$. Then the map $\widehat{f}_{\mathbf{y}}$ is a 2–cocycle.*

Proof We show that $\widehat{f}_{\mathbf{y}}$ satisfies the condition of a 2–cocycle, ie

$$\widehat{f}_{\mathbf{y}}(s, u) + \widehat{f}_{\mathbf{y}}(s * u, t * u) = \widehat{f}_{\mathbf{y}}(s, t) + \widehat{f}_{\mathbf{y}}(s * t, u),$$

as follows. Put:

$$F = \widehat{f}_{\mathbf{y}}(s, u) + \widehat{f}_{\mathbf{y}}(s * u, t * u) - \widehat{f}_{\mathbf{y}}(s, t) - \widehat{f}_{\mathbf{y}}(s * t, u)$$

Since f is a 3–cocycle, f satisfies:

$$f(s, u, v) + f(s * u, t * u, v) - f(s, t, v) - f(s * t, u, v)$$
$$= f(s * v, t * v, u * v) - f(s, t, u)$$

As we mentioned, $\mathcal{R}_{(y_1,\ldots,y_{j-1})}$ is a quandle homomorphism; thus we have:

$$F = \sum_{j=1}^{l} \big(f(\mathcal{R}_{(y_1,\ldots,y_{j-1})}(s) * y_j, \mathcal{R}_{(y_1,\ldots,y_{j-1})}(t) * y_j, \mathcal{R}_{(y_1,\ldots,y_{j-1})}(u) * y_j)$$
$$- f(\mathcal{R}_{(y_1,\ldots,y_{j-1})}(s), \mathcal{R}_{(y_1,\ldots,y_{j-1})}(t), \mathcal{R}_{(y_1,\ldots,y_{j-1})}(u)) \big)$$

Since $\mathcal{R}_{(y_1,\ldots,y_{j-1})}(s) * y_j = \mathcal{R}_{(y_1,\ldots,y_j)}(s)$, we have

$$F = f(\mathcal{R}_{\mathbf{y}}(s), \mathcal{R}_{\mathbf{y}}(t), \mathcal{R}_{\mathbf{y}}(u)) - f(\mathcal{R}_{\varnothing}(s), \mathcal{R}_{\varnothing}(t), \mathcal{R}_{\varnothing}(u))$$

which is zero from $\mathcal{R}_{\varnothing} = \mathrm{id}$ and the assumption $\mathcal{R}_{\mathbf{y}} = \mathrm{id}$. Thus $\widehat{f}_{\mathbf{y}}$ is a 2–cocycle. $\square$

For $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{x}' = (x_1', \ldots, x_m')$, let us denote $(x_1, \ldots, x_m, x_1', \ldots, x_m')$ by $\mathbf{x}\mathbf{x}'$.

Theorem 5.2 *For a given Q–coloring of $\hat{b}$, let x_i $(i = 1, 2, \ldots, m)$ be the color of the i–th initial arc of the m–braid b. Put $\mathbf{x} = (x_1, \ldots, x_m)$, and put $\mathbf{y} = \mathbf{x}_1 \mathbf{x}_2 \cdots \mathbf{x}_n$, where $\mathbf{x}_1 = \mathbf{x}$ and $\mathbf{x}_j = \mathcal{R}_{\mathbf{x}_{j-1}}(\mathbf{x}_{j-1})$ $(j > 1)$. Assume that for any $C \in \mathrm{Col}_Q(\hat{b})$, $\mathcal{R}_\mathbf{y} = \mathrm{id}_Q$. Then the quandle cocycle invariant of $\mathcal{S}_m(b, \Delta^{2n})$ associated with a 3–cocycle f is presented by*

$$\Phi_f(\mathcal{S}_m(b, \Delta^{2n})) = \sum_{C \in \mathrm{Col}_Q(\hat{b})} \Phi_{\hat{f}_\mathbf{y}}(\hat{b}; C) \cdot \prod_{i=1}^{m} \prod_{j=1}^{n} \Psi_f^*(\hat{b}; \mathcal{R}_\mathbf{x}^{j-1}(C), \mathcal{R}_\mathbf{x}^{j-1}(x_i))^{-1}$$

where $\Phi_{\hat{f}_\mathbf{y}}(\hat{b}; C)$ is the quandle cocycle invariant of $\hat{b}$, and $\Psi_f^*(\hat{b}; C, x)$ is the shadow cocycle invariant of $\hat{b}$. Here $\mathbf{x}$ is determined from C and b, and $\hat{f}_\mathbf{y}$ is the 2–cocycle determined from f and $\mathbf{y}$ by (5-1).

5.2 Proof of Theorem 5.2

5.2.1 Triple points of $\mathcal{S}_m(b, \Delta^{2n})$ Regarding the tubular neighborhood $N(T)$ of T as $I \times I \times T$, we take for the surface diagram D of $\mathcal{S}_m(b, \Delta^{2n})$ the image of the braided surface by the projection to $I \times T \subset \mathbb{R}^3 \times \{0\}$. Cutting $N(T)$ by $p_T^{-1}(\mathbf{m} \cup \mathbf{l})$, we can see that $\mathcal{S}_m(b, \Delta^{2n})$ is described by a braided surface over a 2–disk presenting $b \cdot \Delta^{2n} \cdot b^{-1} \cdot \Delta^{-2n} \to e$; thus $b \cdot \Delta^{2n} \to \Delta^{2n} \cdot b$, where we use the same notation c for a diagram of a classical braid c. Thus the triple points of D appear when we slide b along Δ^{2n}, ie when we transform $b \cdot \Delta^{2n}$ to $\Delta^{2n} \cdot b$ fixing the diagram of Δ^{2n}. Each triple point appears when a Reidemeister move of type III occurs. Since the braid Δ^2 is isotopic relative the boundary to the form as in Figure 19, $\mathcal{S}_m(b, \Delta^{2n})$ is equivalent to the form such that the basis braid Δ^{2n} is n powers of Δ^2 as in Figure 19. Since equivalent surface links have the same quandle cocycle invariant, we can assume that the diagram of Δ^2 is as in Figure 19. Sliding b along Δ^{2n} is equal to sliding b along Δ^2 n times. When we slide b through the j–th Δ^2 $(j = 1, 2, \ldots, n)$, a crossing r of b slides over m arcs, and then under m arcs of Δ^2 (see Figure 20). Each time when r slides over or under an arc, a Reidemeister move of type III occurs; thus a triple point of D appears. Let $t_1^{j,+}(r), \ldots, t_m^{j,+}(r), t_1^{j,-}(r), \ldots, t_m^{j,-}(r)$ be the triple points which appear in this order.

For a given 3–cocycle f and a Q–coloring C, we have the following lemma. Before a crossing r slides over an arc, around r there are three strings. Two strings form r, and they separate the other string into three arcs. According to the orientation, let us call the first arc of the three arcs the *initial arc over which r slides*. For the colors x and y as in Figure 16, we call the pair (x, y) the *color of the crossing r* by C (see Carter, Jelsovsky, Kamada, Langford and Saito [11]).
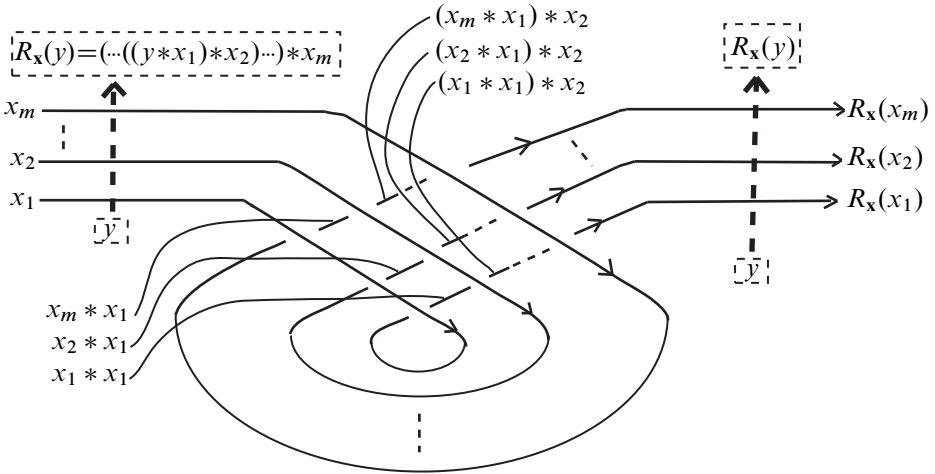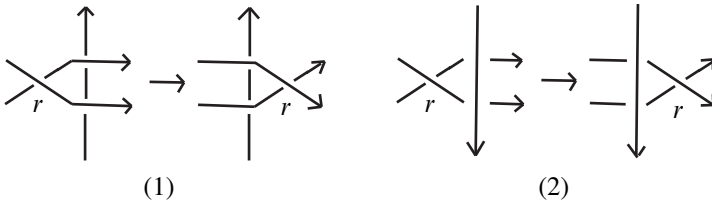
Figure 19: The braid Δ^2

Figure 20: (1) A crossing r slides over an arc, and (2) r slides under an arc

Lemma 5.3 *The weight of $t_i^{j,+}(r)$ is $f(z, x, y)^{-\epsilon}$, and the weight of $t_i^{j,-}(r)$ is $f(x, y, z)^{\epsilon}$, where z is the color of the initial arc over or under which r slides when it forms the triple point, and (x, y) is the color of r before sliding over or under the arc, and $\epsilon = +1$ (respectively -1) if r is a positive (respectively negative) crossing.*

Proof Put $t_+ = t_i^{j,+}(r)$. If r is a positive crossing, then the motion picture around t_+ is as in Figure 21. Thus, for this case, the weight of t_+ is $f(z, x, y)^{-1}$; see Figure 17. If r is a negative crossing, then around t_+ is as in Figure 22, which is equivalent to the right figure of Figure 23. Thus the weight of t_+ is $f(z, x, y)$; see Figure 17. The weight of $t_i^{j,-}(r)$ is obtained likewise.   □

5.2.2 Proof of Theorem 5.2 We take the surface diagram D of $\mathcal{S}_m(b, \Delta^{2n})$ as in Section 5.2.1. First we show that $\mathrm{Col}_Q(D)$ and $\mathrm{Col}_Q(\widehat{b})$ has one-to-one correspondence, as follows. Here, we have the assumption that for any $C \in \mathrm{Col}_Q(\widehat{b})$, $\mathcal{R}_\mathbf{y} = \mathrm{id}$. For a given Q–coloring C of D, by restricting C to the diagram of the closure of
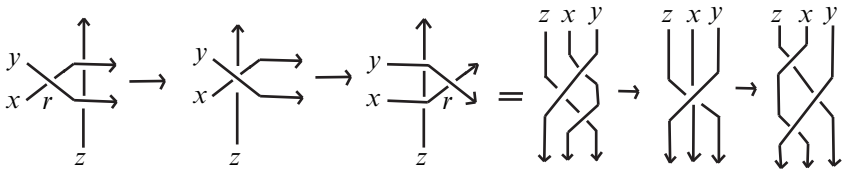
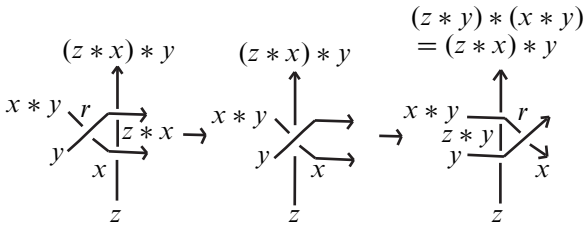Figure 21: Around t_+ if r is a positive crossing

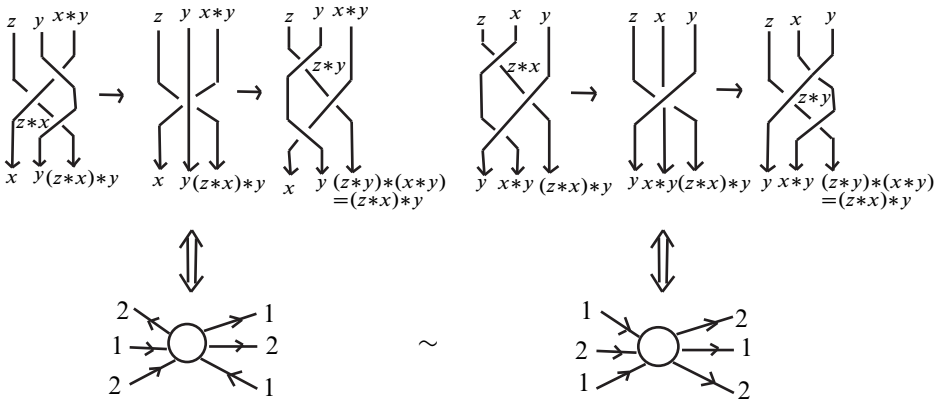Figure 22: Around t_+ if r is a negative crossing

Figure 23: The motion picture of Figure 22 and its presenting white vertex (the left figure) are equivalent to those of the right figure

the basis braid b, we have a Q–coloring of $\hat{b}$. Conversely, let us consider a given $C \in \mathrm{Col}_Q(\hat{b})$. Let us give the other basis braid Δ^{2n} a Q–coloring such that the colors of the initial arcs are $\mathbf{x}$. Since the color of the i–th initial arc of the j–th Δ^2 is the i–th element of $\mathbf{x}_j$ by Lemma 5.4, it follows that $\mathcal{R}_\mathbf{y}(\mathbf{x})$ are the colors of the terminal arcs of Δ^{2n}. Since $\mathcal{R}_\mathbf{y} = \mathrm{id}$, C can be extended uniquely to the diagram of the closure of Δ^{2n}; thus to the closures of the basis braids of $\mathcal{S}_m(b, \Delta^{2n})$. Since $\mathcal{S}_m(b, \Delta^{2n})$ is

determined from the basis braids by Lemma 2.8 (2), C can be extended uniquely to the surface diagram D.

Now we show the required formula, as follows. By definition:

$$\Phi_f(\mathcal{S}_m(b, \Delta^{2n}); C) = \prod_{r \in X_2(\widehat{b})} \prod_{i=1}^{m} \prod_{j=1}^{n} \left(W_f(t_i^{j,+}(r); C) \cdot W_f(t_i^{j,-}(r); C) \right)$$

First we calculate

$$\prod_{r \in X_2(\widehat{b})} \prod_{i=1}^{m} \prod_{j=1}^{n} W_f(t_i^{j,+}(r); C),$$

as follows. Since the color of the i–th initial arcs of the j–th Δ^2 is $\mathcal{R}_{\mathbf{x}}^{j-1}(x_i)$ ($i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$) by Lemma 5.4, it follows that the Q–coloring of b before sliding the j–th Δ^2 is $\mathcal{R}_{\mathbf{x}}^{j-1}(C)$. The color of a crossing $r \in X_2(\widehat{b})$ does not change when r slides over an arc. When r forms the triple point $t_i^{j,+}(r)$, the color of the initial arc over which r slides is the color w depicted in Figure 18, determined from a shadow coloring extending $\mathcal{R}_{\mathbf{x}}^{j-1}(C)$ with the base color which is the color of the $(m+1-i)$–th initial arc of the j–th Δ^2, ie with the base color $\mathcal{R}_{\mathbf{x}}^{j-1}(x_{m+1-i})$ by Lemma 5.4; see Figure 24. Thus

$$W_f(t_i^{j,+}(r); C) = W_f^*(r; \mathcal{R}_{\mathbf{x}}^{j-1}(C), \mathcal{R}_{\mathbf{x}}^{j-1}(x_{m+1-i}))^{-1}$$

by Lemma 5.3, so:

$$\prod_{r \in X_2(\widehat{b})} \prod_{i=1}^{m} W_f(t_i^{j,+}(r); C) = \prod_{i=1}^{m} \Psi_f^*(\widehat{b}; \mathcal{R}_{\mathbf{x}}^{j-1}(C), \mathcal{R}_{\mathbf{x}}^{j-1}(x_i))^{-1}$$

Hence:

$$\prod_{r \in X_2(\widehat{b})} \prod_{i=1}^{m} \prod_{j=1}^{n} W_f(t_i^{j,+}(r); C) = \prod_{i=1}^{m} \prod_{j=1}^{n} \Psi_f^*(\widehat{b}; \mathcal{R}_{\mathbf{x}}^{j-1}(C), \mathcal{R}_{\mathbf{x}}^{j-1}(x_i))^{-1}$$

Next we calculate

$$\prod_{r \in X_2(\widehat{b})} \prod_{i=1}^{m} \prod_{j=1}^{n} W_f(t_i^{j,-}(r); C),$$

as follows. For each crossing $r \in X_2(\widehat{b})$, the color (x, y) of r by C changes to $(x * z, y * z)$, ie the color by $\mathcal{R}_z(C)$, when r slides under an arc as in Figure 20 (2), where z is the color of the arc under which r slides. Let us denote the k–th element of $\mathbf{y} = \mathbf{x}_1 \cdots \mathbf{x}_n$ by y_k ($k = 1, 2, \ldots, mn$). When r slides under the i–th initial arc of the
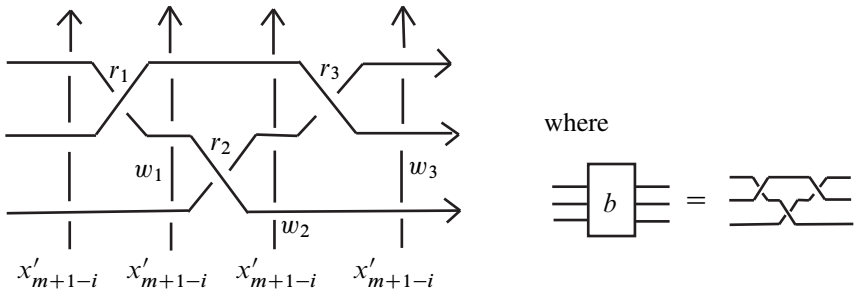
Figure 24: The color w_k of the initial arc over which a crossing r_k slides, when it forms $t_i^{j,+}(r_k)$, where x'_{m+1-i} is the color of the $(m+1-i)$–th initial arc of the j–th Δ^2

j–th Δ^2, r has slid under the arcs from the first initial arc of the first Δ^2 to the $(i-1)$–th initial arc of the j–th Δ^2, whose colors are presented by $(y_1, y_2, \ldots, y_{m(j-1)+i-1})$ by Lemma 5.4. Thus, when r forms the triple point $t_i^{j,-}(r)$, the color of r before sliding under an arc is the color by $\mathcal{R}_{(y_1,\ldots,y_{m(j-1)+i-1})}(C)$. The arc under which r slides is the i–th initial arc of the j–th Δ^2; thus its color is the i–th element of $\mathbf{x}_j$, ie $y_{m(j-1)+i}$, by Lemma 5.4. Hence it follows from Lemma 5.3 that $W_f(t_i^{j,-}(r); C) = f(\mathcal{R}_{(y_1,\ldots,y_{k-1})}(x), \mathcal{R}_{(y_1,\ldots,y_{k-1})}(y), y_k)^\epsilon$, where $k = m(j-1)+i$ and (x, y) is the color of r by C, and $\epsilon = +1$ (respectively -1) if r is a positive (respectively negative) crossing; thus

$$\prod_{i=1}^{m}\prod_{j=1}^{n} W_f(t_i^{j,-}(r); C) = W_{\widehat{f_\mathbf{y}}}(r; C),$$

and we have:

$$\prod_{r \in X_2(\widehat{b})} \prod_{i=1}^{m}\prod_{j=1}^{n} W_f(t_i^{j,-}(r); C) = \Phi_{\widehat{f_\mathbf{y}}}(\widehat{b}; C)$$

Hence we have the required formula. $\qquad\square$

Lemma 5.4 *In the situation of Theorem 5.2, the color of the i–th initial arc of the j–th Δ^2 is the i–th element of $\mathbf{x}_j$. Further, $\mathbf{x}_j = \mathcal{R}_\mathbf{x}^{j-1}(\mathbf{x})$.*

Proof The color of the i–th initial arc of the j–th Δ^2 is the i–th element of $\mathbf{x}$ (respectively $\mathcal{R}_{\mathbf{x}_{j-1}}(\mathbf{x}_{j-1})$) if $j = 1$ (respectively $j > 1$), where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ (see Figure 19); thus it is the i–th element of $\mathbf{x}_j$. Since there exists a unique shadow coloring for a given Q–coloring and a base color (see Carter, Kamada and Saito [13]), $\mathcal{R}_{\mathcal{R}_\mathbf{x}(\mathbf{x})}(y) = \mathcal{R}_\mathbf{x}(y)$ for any $y \in Q$; see Figure 19. Thus $\mathbf{x}_j = \mathcal{R}_\mathbf{x}(\mathbf{x}_{j-1}) = \mathcal{R}_\mathbf{x}^{j-1}(\mathbf{x})$. $\qquad\square$

5.3 Concrete calculations

The *dihedral quandle* of order p, denoted by R_p, is the set $\{0, 1, \ldots, p-1\}$ with the binary operation $x * y = 2y - x \pmod{p}$. Mochizuki [27] showed that for any odd prime p, the 3–cocycles for R_p with the coefficient group $\mathbb{Z}/p\mathbb{Z}$ forms a group isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Its generator is reduced (see Asami and Satoh [2]) to a map given by:

$$\theta_p(s, t, u) = v^{(s-t)((2u-t)^p + t^p - 2u^p)/p} \in \langle\, v \mid v^p = 1 \,\rangle = \mathbb{Z}/p\mathbb{Z}$$

We call θ_p *Mochizuki's 3–cocycle*. We identify the group ring $\mathbb{Z}[\mathbb{Z}/p\mathbb{Z}]$ with the Laurent polynomial ring $\mathbb{Z}[v, v^{-1}]/(v^p - 1)$.

Theorem 5.5 *For an odd prime p, we have:*

$$\Phi_{\theta_p}(\mathcal{S}_4(\sigma_1 \sigma_2^p \sigma_3, \Delta^{2n})) = p \sum_{i=0}^{p-1} v^{4ni^2} \in \mathbb{Z}[v, v^{-1}]/(v^p = 1)$$

The *triple point number* of a surface link S is the minimum number of triple points among all possible diagrams of S. By definition, the quandle cocycle invariant of a surface link with the triple point number zero has an integer value. Thus we have the following corollary.

Corollary 5.6 *If n is not divisible by p, the triple point number of $\mathcal{S}_4(\sigma_1 \sigma_2^p \sigma_3, \Delta^{2n})$ is positive.*

Proof of Theorem 5.5 Let us give a R_p–coloring for the diagram of the basis 4–braid $b = \sigma_1 \sigma_2^p \sigma_3$. By the definition of a R_p–coloring, we have $x_1 = x_2$ and $x_3 = x_4$. We will denote the colors by x and y respectively; we have $\mathbf{x} = (x, x, y, y)$. By a direct calculation, we can see that

$$(z * w) * w = z$$

for any $z, w \in R_p$; thus $\mathcal{R}_{(w,w)} = \mathrm{id}$. Thus it follows that $\mathcal{R}_{\mathbf{x}} = \mathcal{R}_{(y,y)} \circ \mathcal{R}_{(x,x)} = \mathrm{id}$. Hence, by Theorem 5.2:

$$(5\text{-}2) \quad \Phi_{\theta_p}(\mathcal{S}_4(\sigma_1 \sigma_2^p \sigma_3, \Delta^{2n})) = \sum_{C \in \mathrm{Col}_O(\hat{b})} (\Phi_{\hat{\theta}_{px}}(\hat{b}; C))^n \cdot (\Psi_{\theta_p}^*(\hat{b}; C, x) \cdot \Psi_{\theta_p}^*(\hat{b}; C, y))^{-2n}$$

where
$$\hat{\theta}_{p\mathbf{x}}(s, t) = \prod_{i=1}^{4} \theta_p(\mathcal{R}_{(x_1, \ldots, x_{i-1})}(s), \mathcal{R}_{(x_1, \ldots, x_{i-1})}(t), x_i)$$

We calculate $\Phi_{\widehat{\theta}_{px}}(\widehat{b}; C)$, as follows. Since $x_1 = x_2 = x$ and $x_3 = x_4 = y$, we can see that $\mathcal{R}_{(x_1,x_2)} = \mathrm{id}$ and $\mathcal{R}_{(x_1,x_2,x_3)} = \mathcal{R}_y$. Thus:

$$\widehat{\theta}_{px}(s,t) = \theta_p(s,t,x) \cdot \theta_p(s*x, t*x, x) \cdot \theta_p(s,t,y) \cdot \theta_p(s*y, t*y, y) \in \mathbb{Z}/p\mathbb{Z}$$

By a direct calculation (see Asami and Satoh [2]), we can see that θ_p satisfies

$$\theta_p(s*u, t*u, u) = \theta_p(s,t,u)^{-1}$$

for any $s, t, u \in R_p$. Thus $\widehat{\theta}_{px} = 1$, and hence $\Phi_{\widehat{\theta}_{px}}(\widehat{b}; C) = 1$ for any C.

We calculate $\Psi^*_{\theta_p}(\widehat{b}; C, x) \cdot \Psi^*_{\theta_p}(\widehat{b}; C, y)$, as follows. In [2], they calculated that $\Psi^*_{\theta_p}(\widehat{b}; C, x) = v^{-(x-y)^2}$, using the diagram of the right figure of Figure 25. Since
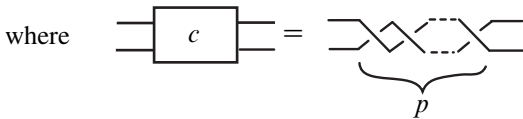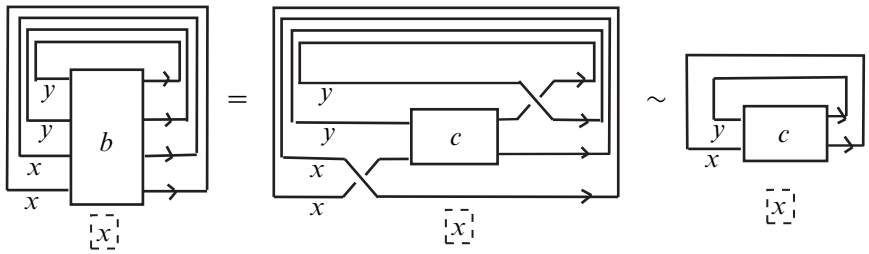

Figure 25: The shadow coloring for $\widehat{b}$ with the base color x

the diagram of $\widehat{b}$ with the R_p–coloring C with the base color y is transformed as in Figure 26 by Reidemeister moves, $\Psi^*_{\theta_p}(\widehat{b}; C, y) = v^{-(y-x*y)^2}$, which equals $v^{-(y-(2y-x))^2} = v^{-(x-y)^2}$. Thus $\Psi^*_{\theta_p}(\widehat{b}; C, x) \cdot \Psi^*_{\theta_p}(\widehat{b}; C, y) = v^{-2(x-y)^2}$.
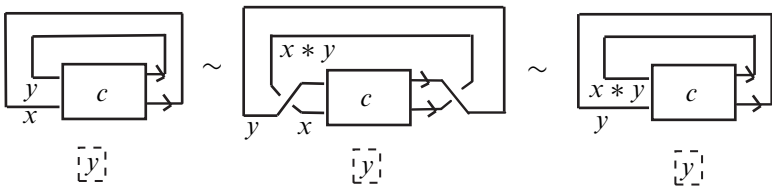

Figure 26: The shadow coloring for $\widehat{b}$ with the base color y

Hence, by (5-2):

$$\Phi_{\theta_p}(\mathcal{S}_4(\sigma_1\sigma_2^p\sigma_3, \Delta^{2n})) = \sum_{x,y \in R_p} v^{4n(x-y)^2}$$

$$= p\sum_{i=0}^{p-1} v^{4ni^2} \in \mathbb{Z}[v, v^{-1}]/(v^p=1) \qquad \square$$

The quandle cocycle invariant $\Phi_{\theta_p}(\mathcal{S}_4(\sigma_1\sigma_2^p\sigma_3, \Delta^{2n}))$ has the same value with that of the orientation-reversed mirror image of $4n$–twist spun $(2, p)$–torus knot $\tau^{4n}T_p$ (see Asami and Satoh [2]). An oriented surface link S is *invertible* if S is equivalent to its orientation-reversed image $-S$, and $(-)$–*amphicheiral* if S is equivalent to its orientation-reversed mirror image $-S^*$. The $4n$–twist spun 2–knot $\tau^{4n}T_p$ is equivalent to its mirror image $\tau^{4n}T_p^*$ (see Litherland [25]), and in [2] they showed the following fact: for an odd prime p with $p \equiv 3 \pmod 4$, if n is not divisible by p, then $\Phi_{\theta_p}(\tau^{4n}T_p) \neq \Phi_{\theta_p}(-\tau^{4n}T_p^*)$ (see also Carter, Jelsovsky, Kamada and Saito [12]). This means that under the above conditions $\tau^{4n}T_p$ is not invertible. Though $\mathcal{S}_4(\sigma_1\sigma_2^p\sigma_3, \Delta^{2n})$ is invertible for any p and n (see Proposition 2.10), comparing the quandle cocycle invariants, we have the following corollary.

Corollary 5.7 *For an odd prime p with $p \equiv 3 \pmod 4$, if n is not divisible by p, then $\mathcal{S}_4(\sigma_1\sigma_2^p\sigma_3, \Delta^{2n})$ is not $(-)$–amphicheiral.*

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