C^1–actions of Baumslag–Solitar groups on S^1

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Let $\mathrm{BS}(1,n)=\langle a,b \mid aba^{-1}=b^n\rangle$ be the solvable Baumslag–Solitar group, where $n\geq 2$. It is known that $\mathrm{BS}(1,n)$ is isomorphic to the group generated by the two affine maps of the line: $f_0(x)=x+1$ and $h_0(x)=nx$. The action on $S^1=\mathbb{R}\cup\infty$ generated by these two affine maps f_0 and h_0 is called the standard affine one. We prove that any faithful representation of $\mathrm{BS}(1,n)$ into $\mathrm{Diff}^1(S^1)$ is semiconjugated (up to a finite index subgroup) to the standard affine action.

37C85; 57S25, 37E10

1 Introduction

This paper is about the dynamics of the actions of the solvable Baumslag–Solitar group $\mathrm{BS}(1,n)=\langle a,b \mid aba^{-1}=b^n\rangle$, where $n\geq 2$, on the circle S^1.

It is known that $\mathrm{BS}(1,n)$ has many actions on S^1. The standard action on $S^1=\mathbb{R}\cup\infty$ is the action generated by the two affine maps $f_0(x)=x+1$ and $h_0(x)=nx$ (where $f_0\equiv b$ and $h_0\equiv a$).

Many people have studied actions of solvable groups on one-manifolds, for example Plante [9], Ghys [4], Navas [6], Farb and Franks [3], Moriyama [5] and Rebelo and Silva [10].

There are a number of results concerning $\mathrm{BS}(1,n)$–actions on S^1. Many of them are scattered in different articles and some of them are not so easy to find, so our aim is to present the state of the art for the case of the action of $\mathrm{BS}(1,n)$–group on S^1, in the case C^1.

It is known that any C^2 $\mathrm{BS}(1,n)$–action on S^1 admits a finite orbit. This fact was proven by Burslem and Wilkinson [1]. In fact they gave a classification (up to conjugacy) of representations ρ: $\mathrm{BS}(1,n)\to\mathrm{Diff}^r(S^1)$ with $r\geq 2$ or $r=\omega$.

These results are proved by using a dynamical approach. The dynamics of C^2 $\mathrm{BS}(1,n)$–actions on S^1 is now well understood, due to Navas' work [6] on solvable groups of

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circle diffeomorphisms. We will prove the same statement as in [1] but in the case that the action is C^1, that is, any C^1 $\mathrm{BS}(1,n)$–action on S^1 admits a finite orbit. That is:

Theorem 1 *Let $f, h\colon S^1 \to S^1$ be C^1 diffeomorphisms preserving orientation satisfying $h \circ f \circ h^{-1} = f^n$. If the $\mathrm{BS}(1,n)$–action on S^1, $\langle f, h\rangle$, is faithful then there exists $m \in \mathbb{N}$ such that f^{n-1} and h^m have a common fixed point.*

Our aim is giving a classification of faithful actions of $\mathrm{BS}(1,n)$ on S^1, in the case C^1.

So, our main result is the following:

Theorem 2 *Any faithful C^1 $\mathrm{BS}(1,n)$–action on S^1 preserving orientation, $\langle f, h\rangle$, is semiconjugated (up to a finite index subgroup $\langle f^{n-1}, h^m\rangle$) to the standard affine action.*

The proof uses in a crucial way (a slightly extended version of) an argument due to Cantwell and Conlon [2], and its use was suggested to us by Navas. It should be pointed out that, using this argument, Navas [7] has recently obtained a counterexample to the converse of Thurston Stability Theorem, and it is very likely that Cantwell and Conlon's argument will still reveal itself to be very fruitful for obtaining new obstructions to C^1 actions on 1–dimensional manifolds.

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2 Existence of a global finite orbit

As before, let $\langle f, h\rangle$ be a faithful action of $\mathrm{BS}(1,n)$ on S^1, where $h \circ f \circ h^{-1} = f^n$.

The aim of this section is proving the existence of a common periodic point of f and h.

First, we prove that the rotation number of f is rational.

Proposition 2.1 *Let $f, h\colon S^1 \to S^1$ be homeomorphisms preserving orientation such that $h \circ f \circ h^{-1} = f^n$. Then there exists $l \in \mathbb{N}$ such that the rotation number of f, $\rho(f) = l/(n-1)$. Therefore f^{n-1} has a fixed point.*

Proof Since $h \circ f \circ h^{-1} = f^n$, we have that there exists $l \in \mathbb{Z}$ such that

$$n\rho(f) = \rho(f) + l,$$

then $\rho(f) = l/(n-1)$. □

We will prove that not only the rotation number of f is rational but the rotation number of h also is. For proving that, we need the following lemma.

In fact, its proof follows Cantwell and Conlon's one.

Proposition 2.2 (Modified Cantwell–Conlon) *Let f, h be C^1 diffeomorphisms preserving orientation on S^1 or* [0, 1] *and satisfying $h \circ f \circ h^{-1} = f^n$. Then, there is no interval $\mathcal{J}$ with the following properties:*

(1) *f fixes the endpoints of $\mathcal{J}$.*

(2) *f has no fixed points on the interior of $\mathcal{J}$.*

(3) *$\mathcal{J}$ is a wandering interval under h (ie $h^k(\mathcal{J}) \cap h^l(\mathcal{J}) = \varnothing$ whenever $k \neq l$).*

Proof Suppose there exists an interval $\mathcal{J}$ satisfying these three properties. We first note that since h preserves the set of fixed points of f, for any l, the interval $h^{-l}(\mathcal{J})$ has endpoints fixed by f, and no interior points fixed by f. Also, since the intervals $h^{-l}(\mathcal{J})$ are disjoint, we know that the length, $|h^{-l}(\mathcal{J})| \to 0$ as $l \to \infty$.

Fix $\epsilon > 0$ satisfying the condition $(1-\epsilon)^2 > \frac{3}{4}$.

Since $|h^{-l}(\mathcal{J})| \to 0$, and f fixes the endpoints of $h^{-l}(\mathcal{J})$ for all l, one can see that for all l sufficiently large, $f'(x) \geq 1-\epsilon$ for any $x \in h^{-l}(J)$. Furthermore, when $|h^{-l}(\mathcal{J})|$ is short enough, for $x, y \in h^{-l}(\mathcal{J})$, $h'(x)/h'(y) \geq 1-\epsilon$. We fix l and $J = h^{-l}(\mathcal{J})$ so that for all $s > 0$, if $x, y \in h^{-s}(J)$ then $h'(x)/h'(y) \geq 1-\epsilon$ and $f'(x) \geq 1-\epsilon$.

Let $x \in J$ and $I \subset J$ be the open arc between x and $f(x)$. It is easy to see that $\bigcup_{k\in\mathbb{Z}} f^k(I) \subset J$ and $f^k(I) \cap f^j(I) = \varnothing$ if $k \neq j$. Let us define the map $\Psi(\alpha_1, \alpha_2, \dots, \alpha_m)$ with $\alpha_i \in \{0,1\}$ as the map

$$(h^m \circ f^{\alpha_m} \circ h^{-m}) \circ \cdots \circ (h^2 \circ f^{\alpha_2} \circ h^{-2}) \circ (h \circ f^{\alpha_1} \circ h^{-1}).$$

Since $h^i \circ f^{\alpha_i} \circ h^{-i} = f^{n^i}$ if $\alpha_i = 1$ and $h^i \circ f^{\alpha_i} \circ h^{-i} = \mathrm{Id}$, otherwise; it follows that $\Psi(\alpha_1, \alpha_2, \dots, \alpha_m)(I) = f^{\beta}(I)$ where $\beta = \sum_{i|\alpha_i=1} n^i$.

So for $(\alpha_1, \alpha_2, \dots, \alpha_m) \neq (\alpha'_1, \alpha'_2, \dots, \alpha'_m)$, the intersection $\Psi(\alpha_1, \alpha_2, \dots, \alpha_m)(I) \cap \Psi(\alpha'_1, \alpha'_2, \dots, \alpha'_m)(I)$ is empty. Hence,

$$\mathcal{I} = \bigcup_{\{0,1\}^m} \Psi(\alpha_1, \alpha_2, \dots, \alpha_m)(I)$$

is the union of 2^m disjoint arcs included in J.

We claim that the length of any arc $\Psi(\alpha_1,\alpha_2,\dots,\alpha_m)(I)$ may be bounded below by $|I|(\frac{3}{4})^m$:

Notice that

$$\Psi(\alpha_1,\alpha_2,\dots,\alpha_m) = h^k \circ f \circ h^{-l_1} \circ f \circ h^{-l_2} \circ f \circ \cdots \circ f \circ h^{-l_r}$$

where $l_1 + l_2 + \cdots + l_r = k$, $l_i > 0$ for $i = 1,\dots,r$. Hence,

$$\Psi'(\alpha_1,\alpha_2,\dots,\alpha_m)(u) = \prod_{i=1}^{r} f'(x_i) \prod_{j=1}^{k} \frac{h'(y_j)}{h'(w_j)},$$

where x_i, y_j and w_j are well defined points, $x_i \in \bigcup_{s\in\mathbb{N}} h^{-s}(J)$ and $y_j, w_j \in h^{-j}(J)$. Therefore, there exist points $\widehat{x_i}$ for $i = 1,..,r$, $\widehat{y_j}$ and $\widehat{w_j}$ for $j = 1,..,k$ such that the length of $\Psi(\alpha_1,\alpha_2,\dots,\alpha_m)(I)$,

$$|\Psi(\alpha_1,\alpha_2,\dots,\alpha_m)(I)| = |I| \prod_{i=1}^{r} |f'(\widehat{x_i})| \prod_{j=1}^{k} \frac{|h'(\widehat{y_j})|}{|h'(\widehat{w_j})|} \geq |I|(1-\epsilon)^{r+k}.$$

Since $r \leq k \leq m$ it follows that

$$|\Psi(\alpha_0,\alpha_1,\alpha_2,\dots,\alpha_m)(I)| \geq |I|(1-\epsilon)^{2m} \geq |I|(\tfrac{3}{4})^m.$$

Then, the length of $\mathcal{I} \geq 2^m|I|(\frac{3}{4})^m$ which tends to ∞ when $m \to \infty$. This is a contradiction with $\mathcal{I} \subset J$. □

Proposition 2.3 *Let f,h: $S^1 \to S^1$ be C^1 diffeomorphisms preserving orientation such that $h \circ f \circ h^{-1} = f^n$. If the* BS$(1,n)$*–action on S^1, $\langle f,h\rangle$, is faithful then $\rho(h)$, the rotation number of h, is rational.*

Proof We suppose that $\rho(h)$ is irrational, then we have two cases:

(1) h is conjugated to an irrational rotation. Note that the periodic points of f are preserved by h: let q be such that $f^k(q) = q$, then $f^{nk}(h(q)) = h(f^k(q)) = h(q)$. So $h(q)$ is a periodic point for f.

It follows that the periodic points of f are dense in S^1. This implies that there exists m such that $f^m = \mathrm{Id}$ contradicting that the action is faithful.

(2) The minimal set of h is a Cantor set, K. Notice that $K = \overline{\mathcal{O}_h(p)}$ where p is a fixed point of f^{n-1}. Then there exists an arc $J \subset K^c \subset S^1$, which endpoints are fixed by f^{n-1}, without f^{n-1}–fixed points in its interior satisfying that J is a wandering set under h. This is a contradiction with Proposition 2.2.

Therefore, the minimal set of h is not a Cantor set.

Hence, we have proved that $\rho(h)$ is rational. □

Proof of Theorem 1 Let m such that $\rho(h^m)=0$. Let q be a fixed point for f^{n-1} (we have already seen that there exists a fixed point for f^{n-1}), then $\{h^{lm}(q)\}_{l\in\mathbb{N}}$ is included in the set of fixed points of f^{n-1} since h preserves the fixed points of f^{n-1}.

Let u be an accumulation point of $\{h^{lm}(q)\}$. It follows from continuity of f that u is a fixed point for f^{n-1} and since the nonwandering set, $\Omega(h^m)$, is included in the set of fixed points of h^m then u is also a fixed point for h^m. □

3 Semiconjugation to the standard affine action

Recall that the standard affine action on $S^1=\mathbb{R}\cup\infty$ is the action generated by the two affine maps $f_0(x)=x+1$ and $h_0(x)=nx$.

Following results or ideas of Cantwell and Conlon, Navas and Rivas, from now on we will prove that any faithful $C^1-\mathrm{BS}(1,n)$ action on S^1 is semiconjugated (up to the finite index subgroup $\langle f^{n-1},h^m\rangle$) to the standard affine action.

Remark 3.1 Since the C^1 diffeomorphisms f^{n-1} and h^m have a common fixed point, we can study $\langle f^{n-1},h^m\rangle$, a C^1 action of $\mathrm{BS}(1,(n-1)n^m)$ on the interval [0, 1] instead of S^1.

Due to a classical result that appears, for example in *Groups acting on the circle* [4] by Ghys , it is known that for a countable infinite group G, G is left orderable if and only if G acts faithfully on the real line by orientation preserving homeomorphisms. Let us note that Ghys proved that G acts faithfully on the real line constructing the *dynamical realization of a left ordering*. For this construction he fixed an enumeration $\{g_i\}$ of G. He defined an order preserving map $t\colon G\to\mathbb{R}$, in such a way that $g(t(g_i))=t(gg_i)$. This action was extended to the closure of $t(G)$, and later to the whole real line. As Rivas noted [11, Remark 4.4], and it was proved by Navas [8] that the dynamical realization associated to different enumerations of G (but the same ordering) are topologically conjugated.

It was proven by Rivas [11] that the set of left orderings of $\mathrm{BS}(1,n)$ is made up of four Conradian orderings and an uncountable set of non-Conradian left orderings. Each one of these infinitely many non-Conradian orderings can be realized as an induced ordering that comes from an affine action on $\mathrm{BS}(1,n)$. Moreover, in the proof of this result it was shown that the dynamical realization of any non-Conradian ordering is semiconjugated to the standard affine action. (In fact, Rivas proved this result for $\mathrm{BS}(1,2)$ but the proof for $\mathrm{BS}(1,n)$ is the same).

As Ghys' and Rivas' results are "topological", they hold in an interval instead of the real line.

We will call an "exotic" action to the $\mathrm{BS}(1,n)$ one that is induced by a Conradian ordering.

Let $\langle\langle f\rangle\rangle$ be the largest abelian subgroup containing f. For any "exotic" action in the interval [0, 1] (that is induced by one of the four Conradian orderings), $\langle\langle f\rangle\rangle$ is a convex subgroup (in the sense of ordering; see the proof of case 2 of Proposition 4.2 of [11]).

In fact, in this proof it was shown that there exist two f–fixed points c and d in $(0,1)$ such that f has no fixed points in (c,d) and $h(c,d)$ is disjoint of (c,d).

It can be proven that $h^k(c,d)\cap(c,d)=\varnothing$ for any k therefore (c,d) is a wandering interval under h. As it was proven in Proposition 2.2 it is not possible when f and h are C^1 diffeomorphisms.

It follows that any C^1 BS–action on an interval I is semiconjugated to the standard affine action. By Remark 3.1 the same holds for C^1 BS–action on S^1.

So, we have proved Theorem 2.

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