

On the mapping space homotopy groups and the free loop space homology groups

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Let X be a Poincaré duality space, Y a space and $f: X \rightarrow Y$ a based map. We show that the rational homotopy group of the connected component of the space of maps from X to Y containing f is contained in the rational homology group of a space $L_f Y$ which is the pullback of f and the evaluation map from the free loop space LY to the space Y . As an application of the result, when X is a closed oriented manifold, we give a condition of a noncommutativity for the rational loop homology algebra $\mathbf{H}_*(L_f Y; \mathbb{Q})$ defined by Gruher and Salvatore which is the extension of the Chas–Sullivan loop homology algebra.

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1 Introduction

We assume that all topological spaces in this paper have a base point. Let M be a simply connected d -dimensional closed oriented manifold and LM the free loop space of M . We denote by $\text{aut}_1 M$ the path component of the monoid of the self-homotopy equivalences of M containing the identity map.

In [8], Félix and Thomas constructed the injective map from the rational homotopy group of $\text{aut}_1 M$ to the rational homology group of LM :

$$(1-1) \quad \pi_*(\text{aut}_1 M) \otimes \mathbb{Q} \longrightarrow H_{*-1+d}(LM; \mathbb{Q}).$$

Now recall that Jones [13] proved that $H^*(LM; \mathbf{k})$ is isomorphic as a vector space to the Hochschild homology of the singular cochain algebra $S^*(M; \mathbf{k})$ of M over a field \mathbf{k} :

$$H^*(LM; \mathbf{k}) \cong \text{HH}_*(S^*(M; \mathbf{k}); S^*(M; \mathbf{k})).$$

and the dual of the above isomorphism and the Poincaré duality of M yield an isomorphism of graded vector spaces $H_{*+d}(LM; \mathbf{k}) \cong \text{HH}^{-*}(S^*(M; \mathbf{k}); S^*(M; \mathbf{k}))$. We now note that the cochain algebra $S^*(M; \mathbb{Q})$ over \mathbb{Q} is weakly equivalent to a free commutative differential graded algebra over \mathbb{Q} , $(\Lambda V, d)$, called a Sullivan model for M ; see the end of Section 5, and so $H_{*+d}(LM; \mathbb{Q}) \cong \text{HH}^{-*}(\Lambda V; \Lambda V)$.

On the other hand, Block and Lazarev [1] and Lupton and Smith [14] constructed an isomorphism from the n -th rational homotopy groups of $\text{aut}_1 M$ to the $(-n)$ -th homology of the differential graded module of derivations of ΛV :

$$\pi_n(\text{aut}_1 M) \otimes \mathbb{Q} \xrightarrow{\cong} H^{-n}(\text{Der}^*(\Lambda V, \Lambda V)).$$

Also, we see that there is a map $J_1^*: H^*(\text{Der}^*(\Lambda V, \Lambda V)) \rightarrow \text{HH}^{*+1}(\Lambda V; \Lambda V)$; see Section 5 for a proper definition. The result of Félix and Thomas [8] also shows that a topological meaning of the map J_1^* is the map (1-1). That is, we get the following commutative square:

$$(1-2) \quad \begin{array}{ccc} H_{n-1+d}(LM; \mathbb{Q}) & \xrightarrow{\cong} & \text{HH}^{-n+1}(\Lambda V; \Lambda V) \\ \uparrow (1-1) & & \uparrow J_1^* \\ \pi_n(\text{aut}_1 M) \otimes \mathbb{Q} & \xrightarrow[\cong]{} & H^{-n}(\text{Der}^*(\Lambda V, \Lambda V)). \end{array}$$

The objective of this paper is to give a generalization of their works such as that mentioned below.

Let X and Y be simply connected spaces with homologies over \mathbf{k} of finite type and $f_1, f_2: X \rightarrow Y$ based maps. Here, the complex $S^*(X; \mathbf{k})$ is regarded as a $S^*(Y; \mathbf{k})$ -bimodule; that is a right and left $S^*(Y; \mathbf{k})$ -structure is via f_1^* and f_2^* , respectively. Denote by $P(Y; f_1, f_2)$ a pullback of the diagram

$$\begin{array}{ccc} P(Y; f_1, f_2) & \longrightarrow & \text{map}([0, 1], Y) \\ x \downarrow & & \downarrow (p_0, p_1) \\ X & \xrightarrow{(f_1, f_2)} & Y \times Y, \end{array}$$

where (p_0, p_1) is the map defined by $(p_0, p_1)(\varphi) = (\varphi(0), \varphi(1))$. Our first result is described as follows.

Theorem 1.1 *There is an isomorphism of \mathbf{k} -vector spaces*

$$\Theta_X: \text{HH}_*(S^*(Y; \mathbf{k}); S^*(X; \mathbf{k})) \xrightarrow{\cong} H^*(P(Y; f_1, f_2); \mathbf{k}).$$

In the proof, we use a cubical singular cochain complex instead of singular cochain algebra. In [4], Chen proved Theorem 1.1 in the case in which $\mathbf{k} = \mathbb{R}$. Our proof of the theorem is using ideas of Chen. As the relevant result of Theorem 1.1, we refer to the paper of Hess, Parent and Scott [12, Theorem 3.1]. They proved an integral version of the theorem, which also takes into account comultiplicative structure, that is,

Theorem 1.1 is a weaker assertion than their results. However, the important thing is that the isomorphism of **Theorem 1.1** is given by the map Θ_X described in **Section 4**.

Assume that X is a \mathbf{k} -Poincaré duality space of formal dimension d ; see **Section 5**. Let $\text{map}(X, Y; f)$ be the path component of the space of free maps from X to Y containing the based map $f: X \rightarrow Y$ and denote by $L_f Y$ the space $P(Y; f, f)$, especially. We consider the natural map

$$g: \Omega \text{map}(X, Y; f) \times X \longrightarrow L_f Y, \quad g(\gamma, x)(t) = \gamma(t)(x)$$

and the composite map for $n \geq 2$

$$\begin{aligned} \Gamma_1: \pi_n(\text{map}(X, Y; f)) \otimes \mathbf{k} &\xrightarrow{\cong} \pi_{n-1}(\Omega \text{map}(X, Y; f)) \otimes \mathbf{k} \\ &\xrightarrow{h} H_{n-1}(\Omega \text{map}(X, Y; f); \mathbf{k}) \xrightarrow{\Gamma} H_{n+d-1}(L_f Y; \mathbf{k}). \end{aligned}$$

Here ΩZ is the based loop space of Z , h is the Hurewicz map, Γ is the map defined by $\Gamma(a) = H(g)(a \otimes [X])$ and $[X] \in S_d(X; \mathbf{k})$ the fundamental class of X .

Let $\rho: (TV, d) \rightarrow S^*(Y; \mathbf{k})$ be a minimal free associative model for $S^*(Y; \mathbf{k})$ (see Halperin and Lemaire [111]) and $\text{Der}^*(TV, S^*(X; \mathbf{k}); f^* \circ \rho)$ the complex of $(f^* \circ \rho)$ -derivations; see **Section 5** for a proper definition. The next theorem is our main result of this paper.

Theorem 1.2 *If X is a \mathbf{k} -Poincaré duality space of formal dimension d , then, for any $n \geq 2$, there exists an isomorphism of \mathbf{k} -vector spaces Θ_X^* from $H_{*+d}(L_f Y; \mathbf{k})$ to $\text{HH}^*(TV; S^*(X; \mathbf{k}))$ and a \mathbf{k} -linear map Θ_1 from $\pi_n(\text{map}(X, Y; f)) \otimes \mathbf{k}$ to $H^{-n}(\text{Der}^*(TV, S^*(X; \mathbf{k}); f^* \circ \rho))$ such that the following square is commutative:*

$$\begin{array}{ccc} H_{n-1+d}(L_f Y; \mathbf{k}) & \xrightarrow[\cong]{\Theta_X^*} & \text{HH}^{-n+1}(TV; S^*(X; \mathbf{k})) \\ \Gamma_1 \uparrow & & \uparrow J_1^* \\ \pi_n(\text{map}(X, Y; f)) \otimes \mathbf{k} & \xrightarrow[\Theta_1]{} & H^{-n}(\text{Der}^*(TV, S^*(X; \mathbf{k}); f^* \circ \rho)). \end{array}$$

If $\mathbf{k} = \mathbb{Q}$, $X = Y$ and f is the identity map, then the diagram in **Theorem 1.2** coincides with the diagram (1-2); see the proof of Corollary 1.3. Thus **Theorem 1.2** is regarded as a generalization of [8]. We here note that, in general, the map Θ_1 in **Theorem 1.2** is not isomorphism. In the last paragraph, we use **Theorem 1.2** to deduce the following corollary.

Corollary 1.3 *If \mathbf{k} is \mathbb{Q} , then the map Γ_1 is injective.*

In [3], Chas and Sullivan constructed a product on $\mathbf{H}_*(LM) := H_{*+d}(LM)$ called the *loop product* and $\mathbf{H}_*(LM)$ is a commutative graded algebra. By Gruher and Salvatore [10], when X is a simply connected d -dimensional closed oriented manifold, we see that $\mathbf{H}_*(L_f Y)$ also has a graded algebra structure similar to the construction of loop products. As an application of the main result, we give a condition of a noncommutativity for $\mathbf{H}_*(L_f Y; \mathbb{Q})$ in rational cases. For details see Section 6.

The organization of this paper is as follows. In Section 2, we recall the Hochschild homology and cohomology. Section 3 gives a fundamental definition and facts on cubical singular chain complexes. Section 4 concentrates on the proof of Theorem 1.1. In Section 5, we prove the main result. Moreover, fundamental facts on rational homotopy theory and a proof of Corollary 1.3 are presented. Noncommutativity for $\mathbf{H}_*(L_f Y; \mathbb{Q})$ is described in Section 6.

2 Hochschild homology and cohomology

We begin with the definition of the Hochschild chain complex. Let (A, d) be a differential graded algebra over a field \mathbf{k} with augmentation $\varepsilon: A \rightarrow \mathbf{k}$ and $\bar{A} = \text{Ker } \varepsilon$ an augmentation ideal of A . Denote by $s\bar{A}$ the suspension of \bar{A} , that is $(s\bar{A})^n = \bar{A}^{n+1}$ and $T(s\bar{A})$ the tensor algebra on $s\bar{A}$. The *two-sided normalized bar construction* is the complex

$$\bar{\mathbf{B}}(A; A; A) = A \otimes T(s\bar{A}) \otimes A$$

with the differential $d_{\bar{\mathbf{B}}} = d_1 + d_2$ defined by

$$\begin{aligned} d_1(a[a_1|a_2|\cdots|a_k]b) &= d(a)[a_1|a_2|\cdots|a_k]b - \sum_{i=1}^k (-1)^{\varepsilon_i} a[a_1|a_2|\cdots|d(a_i)|\cdots|a_k]b \\ &\quad + (-1)^{\varepsilon_{k+1}} a[a_1|a_2|\cdots|a_k]d(b), \\ d_2(a[a_1|a_2|\cdots|a_k]b) &= (-1)^{|a|} a a_1[a_2|\cdots|a_k]b + \sum_{i=2}^k (-1)^{\varepsilon_i} a[a_1|\cdots|a_{i-1}a_i|\cdots|a_k]b \\ &\quad - (-1)^{\varepsilon_k} a[a_1|a_2|\cdots|a_{k-1}]a_k b. \end{aligned}$$

Here $\varepsilon_i = |a| + \sum_{j < i} |sa_j|$ and an element $a \otimes (sa_1 \otimes sa_2 \otimes \cdots \otimes sa_k) \otimes b$ in $\bar{\mathbf{B}}(A; A; A)$ is denoted by $a[a_1|a_2|\cdots|a_k]b$. We denote $\bar{\mathbf{B}}_n(A; A; A)$ by $A \otimes (s\bar{A})^{\otimes n} \otimes A$ for $n \geq 0$.

Let A^{op} be the opposite graded algebra of A and $A^e = A \otimes A^{\text{op}}$. Recall that any A -bimodule can be considered as a left (or right) A^e -module.

Lemma 2.1 [5, Lemma 4.3] *The left A^e -module map*

$$\varepsilon_A: \bar{\mathbf{B}}(A; A; A) \rightarrow A$$

defined by $\varepsilon_A([\])=1$ and $\varepsilon_A([a_1|a_2|\cdots|a_k])=0$ for $k > 0$ is a semifree resolution of A as a left A^e -module.

Let (M, d_M) be a differential graded A -bimodule, that is also a right A^e -module. The Hochschild chain complex of A with coefficient in M is the complex

$$C_*(A; M) = (M \otimes_{A^e} \bar{\mathbf{B}}(A; A; A), D = d_M \otimes 1 + 1 \otimes d_{\bar{\mathbf{B}}}).$$

The homology of $C_*(A; M)$ is denoted by $\text{HH}_*(A; M)$ called the Hochschild homology. Similarly, the Hochschild cochain complex of A with coefficient in M is the complex

$$C^*(A; M) = (\text{Hom}_{A^e}(\bar{\mathbf{B}}(A; A; A), M), D'),$$

where $D'(\varphi) = d_M \circ \varphi - (-1)^{|\varphi|} \varphi \circ d_{\bar{\mathbf{B}}}$ for $\varphi \in \text{Hom}_{A^e}(\bar{\mathbf{B}}(A; A; A), M)$ and the Hochschild cohomology is the homology of $C^*(A; M)$, written by $\text{HH}^*(A; M)$.

3 Cubical singular chain complex

Let $I^n = [0, 1]^n$ be the n times product of the closed unit interval, $[0, 1]$. An n -cube in a topological space Z is a continuous map $I^n \rightarrow Z$. An n -cube $\sigma: I^n \rightarrow Z$ is degenerate if there exist an integer i , $1 \leq i \leq n$, and an $(n-1)$ -cube $\sigma': I^{n-1} \rightarrow Z$ such that $\sigma(t_1, t_2, \dots, t_n) = \sigma'(t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n)$ for any $(t_1, t_2, \dots, t_n) \in I^n$. Note that all 0-cube are nondegenerate. We denote by $C_n(Z; \mathbf{k})$ the free \mathbf{k} -module generated by the set of all nondegenerate n -cubes in Z . We define the map

$$\lambda_i^\varepsilon: I^{n-1} \longrightarrow I^n; (t_1, t_2, \dots, t_{n-1}) \longmapsto (t_1, \dots, t_{i-1}, \varepsilon, t_i, \dots, t_{n-1})$$

for $\varepsilon = 0, 1$ and $1 \leq i \leq n$. Let $\partial = \sum_{i=1}^n (\lambda_i^{0*} - \lambda_i^{1*}): C_n(Z; \mathbf{k}) \rightarrow C_{n-1}(Z; \mathbf{k})$. Then ∂ is a well-defined differential of $C_*(Z; \mathbf{k})$ (see Massey [15, page 13]) and the chain complex $(C_*(Z; \mathbf{k}), \partial)$ is called the cubical singular chain complex of Z . The cubical singular cochain complex of Z over \mathbf{k} is the complex $C^n(Z; \mathbf{k}) = \text{Hom}_{\mathbf{k}}^{-n}(C_*(Z), \mathbf{k})$. The differential $d: C^{n-1}(Z; \mathbf{k}) \rightarrow C^n(Z; \mathbf{k})$ is defined by $d(\varphi) = \varphi \partial$ for $\varphi \in C^{n-1}(Z; \mathbf{k})$.

Remark 3.1 We see that the cubical singular chain complex $C_*(Z; \mathbf{k})$ is quasi-isomorphic to the singular chain complex $S_*(Z; \mathbf{k})$ by the method of acyclic models of Selick [18, Theorem 5.2.3’].

The Alexander–Whitney map and the Eilenberg–Zilber map are also defined in cubical singular chain complexes [15, pages 133, 137]. The Eilenberg–Zilber map

$$\text{EZ: } C_n(Z_1; \mathbf{k}) \otimes C_m(Z_2; \mathbf{k}) \longrightarrow C_{n+m}(Z_1 \times Z_2; \mathbf{k})$$

is defined by $\text{EZ}(\varphi \otimes \psi) = \varphi \times \psi$ where φ (resp. ψ) is an n (resp. m)–cube. The Alexander–Whitney map is defined as follows. Let J be any subset of $\{1, 2, \dots, n+m\}$ and J^c be the complementary subset of J . If $J = \{j_1, j_2, \dots, j_l\}$, then denote $\lambda_J^\varepsilon = \lambda_{j_1}^\varepsilon \lambda_{j_2}^\varepsilon \cdots \lambda_{j_l}^\varepsilon$. For any $(n+m)$ –cube $\sigma: I^{n+m} \rightarrow Z_1 \times Z_2$, we define a map $\text{AW: } C_{n+m}(Z_1 \times Z_2; \mathbf{k}) \rightarrow (C_*(Z_1; \mathbf{k}) \otimes C_*(Z_2; \mathbf{k}))_{n+m}$ by

$$\text{AW}(\sigma) = \sum_J (-1)^{\varepsilon(J)} (\text{pr}_1 \sigma \lambda_{J^c}^0) \otimes (\text{pr}_2 \sigma \lambda_J^1) \in (C_*(Z_1; \mathbf{k}) \otimes C_*(Z_2; \mathbf{k}))_{n+m}$$

where $\text{pr}_i: Z_1 \times Z_2 \rightarrow Z_i$ is the projection and $\varepsilon(J)$ is the cardinal number of the set $\{(i, j) \in J \times J^c \mid j < i\}$. We can see that EZ and AW are chain maps; see [15, pages 133, 138].

In the rest of this section, we recall the map called the *integration map* or the *slant product*. Let $\sigma \in C_q(Z_1; \mathbf{k})$, then define a map $\int_\sigma: C^{n+q}(Z_1 \times Z_2; \mathbf{k}) \rightarrow C^n(Z_2; \mathbf{k})$ by $(\int_\sigma(x))(\varphi) = x(\sigma \times \varphi)$ for any $\varphi \in C_n(Z_2; \mathbf{k})$. The equality

$$(3-1) \quad d\left(\int_\sigma x\right) = (-1)^q \left(\int_\sigma dx - \int_{\partial\sigma} x\right)$$

is easily seen as follows:

$$\begin{aligned} \left(\int_\sigma dx\right)(\varphi) &= dx(\sigma \times \varphi) = x(\partial\sigma \times \varphi) + (-1)^q x(\sigma \times \partial\varphi) \\ &= \left(\int_{\partial\sigma} x\right)(\varphi) + (-1)^q d\left(\int_\sigma x\right)(\varphi). \end{aligned}$$

We note that the Equation (3-1) is a particular version of Stokes’ theorem.

4 Proof of Theorem 1.1

In this section, we denote $C^*(-; \mathbf{k})$ by $C^*(-)$ for convenience. We begin recalling the $C^*(Y)$ –bimodule structure on $C^*(X)$ defined for $\nu \in C^*(X)$ and $\omega, \omega' \in C^*(Y)$ by

$$\omega' \cdot \nu \cdot \omega = f_2^*(\omega') \nu f_1^*(\omega).$$

Let $\Delta^n = \{(t_1, t_2, \dots, t_n) \in \mathbb{R}^n \mid 0 \leq t_1 \leq t_2 \leq \dots \leq t_n \leq 1\}$ be the standard n –simplex and $\kappa_n: I^n \rightarrow \Delta^n$ be a nondegenerate cubical chain defined by

$$\kappa_n(t_1, t_2, \dots, t_n) = (x_1, x_2, \dots, x_n), \quad x_i = 1 - t_1 t_2 \cdots t_i.$$

We now consider a map $\alpha_k: \Delta^k \times P(Y; f_1, f_2) \longrightarrow X \times Y^{\times k}$ defined by

$$\alpha_k((t_1, t_2, \dots, t_k), \gamma) = (\chi(\gamma), \gamma(t_1), \gamma(t_2), \dots, \gamma(t_k)).$$

Then we obtain the following composition map Θ_X^n

$$\begin{aligned} \Theta_X^n: C^*(X) \otimes_{C^*(Y)^e} \bar{\mathbf{B}}_n(C^*(Y); C^*(Y); C^*(Y)) &\xrightarrow{s_n} C^*(X) \otimes C^*(Y)^{\otimes n} \\ &\xrightarrow{\text{AW}} C^*(X \times Y^{\times n}) \xrightarrow{\int_{\kappa_n} \alpha_n^*} C^*(P(Y; f_1, f_2)), \end{aligned}$$

where $s_n(v \otimes \omega[|\omega_1| |\omega_2| \dots |\omega_n|] \omega') = (-1)^\varrho \omega' v \omega \otimes \omega_1 \otimes \omega_2 \otimes \dots \otimes \omega_n$,

$$\varrho = |\omega'|(|v| + |\omega| + \sum_{j=0}^n |s\omega_j|) + \sum_{j=0}^{n-1} \sum_{i=1}^j (|v| + |s\omega_i|),$$

and put $\Theta_X = \sum_{n \geq 0} \Theta_X^n: C_*(C^*(Y); C^*(X)) \rightarrow C^*(P(Y; f_1, f_2))$. Essentially, the map Θ_X is the map in Félix, Oprea and Tanré [7, Theorem 9.64] which is defined using iterated integrals.

Lemma 4.1 *The map Θ_X is a chain map.*

Proof The Equation (3-1) enables us to give

$$\begin{aligned} d\Theta_X^n &= (-1)^n \int_{\kappa_n} d\alpha_n^* \text{AW } s_n - (-1)^n \int_{\partial\kappa_n} \alpha_n^* \text{AW } s_n \\ &= (-1)^n \int_{\kappa_n} \alpha_n^* \text{AW } ds_n + (-1)^{n+1} \int_{\kappa_{n-1}} \alpha_{n-1}^* \text{AW } \delta s_n, \end{aligned}$$

where the map $\delta: C^*(X) \otimes C^*(Y)^{\otimes n} \rightarrow C^*(X) \otimes C^*(Y)^{\otimes(n-1)}$ is defined by

$$\begin{aligned} \delta(v \otimes \omega_1 \otimes \dots \otimes \omega_n) &= v f_1^*(\omega_1) \otimes \omega_2 \otimes \dots \otimes \omega_n \\ &\quad + \sum_{i=2}^n (-1)^{i-1} v \otimes \omega_1 \otimes \dots \otimes \omega_{i-1} \omega_i \otimes \dots \otimes \omega_n \\ &\quad + (-1)^{|\omega_n|(|v| + \sum_{i=1}^{n-1} |\omega_i|) + n} f_2^*(\omega_n) v \otimes \omega_1 \otimes \dots \otimes \omega_{n-1}. \end{aligned}$$

A straightforward calculation shows $(-1)^n ds_n = s_n(d \otimes 1 + 1 \otimes d_1)$ and $(-1)^{n-1} \delta s_n = s_{n-1}(1 \otimes d_2)$. We hence have $d\Theta_X = \Theta_X D$. □

Let $\rho: (TV, d) \rightarrow C^*(Y)$ be a minimal free associative model for $C^*(Y)$ [11], that is, TV is a tensor algebra over \mathbf{k} , ρ is a quasi-isomorphism of differential graded algebras,

$V = \{V^p\}_{p \geq 2}$, each V^p is finite dimensional and d is decomposable; $d(V) \subset T^{\geq 2}V$. Since the map

$$C_*(\rho; C^*(X)): C_*(TV; C^*(X)) \rightarrow C_*(C^*(Y); C^*(X))$$

is a quasi-isomorphism by [5, Proposition 2.4], it is only necessary to show that the composition map $\Theta_X \circ C_*(\rho; C^*(X))$ is a quasi-isomorphism to prove Theorem 1.1. We put $\bar{\Theta}_X = \Theta_X \circ C_*(\rho; C^*(X))$.

We begin with the Hochschild chain complex $C_*(TV; C^*(X))$ and filter it by

$$F^p = C^{\geq p}(X) \otimes_{(TV)^e} \bar{\mathbf{B}}(TV; TV; TV).$$

Lemma 4.2 *The spectral sequence associated the above filtration, denote by (E_r, d_r) , satisfy $E_2^{p,q} \cong H^p(X) \otimes \text{HH}_q(TV; \mathbf{k})$ as \mathbf{k} -vector spaces.*

Proof Recall that there is an isomorphism as follows:

$$\begin{aligned} E_0^{p,q} &= \frac{(F^p)^{p+q}}{(F^{p+1})^{p+q}} \cong C^p(X) \otimes (\mathbf{k} \otimes_{(TV)^e} \bar{\mathbf{B}}(TV, TV, TV)^q) \\ &= C^p(X) \otimes C_q(TV; \mathbf{k}), \end{aligned}$$

$$\nu \otimes \omega[\omega_1 | \cdots | \omega_n] \omega' \mapsto (-1)^{|\omega'|(|\nu| + |\omega| + \sum_i |s\omega_i|)} \omega' \nu \otimes (1 \otimes [\omega_1 | \cdots | \omega_n]).$$

If the degrees of ω or ω' are not zero, then $\nu \otimes \omega[\omega_1 | \cdots | \omega_n] \omega'$ is zero in $E_0^{p,q}$. It follows that the above correspondence is one-to-one. The differential $d_0: E_0^{p,q} \rightarrow E_0^{p,q+1}$ is the induced map of the differential of the Hochschild chain complex $C_*(TV; C^*(X))$. Since $\overline{TV}^0 = 0$, we have $d_0 = 1 \otimes D$ where D is the differential of $C_*(TV; \mathbf{k})$ and so $E_1^{p,q} \cong C^p(X) \otimes \text{HH}_q(TV; \mathbf{k})$. The differential d_1 is defined by

$$\begin{aligned} d_1: E_1^{p,q} = H^{p+q}(F^p/F^{p+1}) &\xrightarrow{\partial^*} H^{p+q+1}(F^{p+1}) \\ &\xrightarrow{\pi_*} H^{p+q+1}(F^{p+1}/F^{p+2}) = E_1^{p+1,q} \end{aligned}$$

where ∂^* is the connecting homomorphism and $\pi: F^{p+1} \rightarrow F^{p+1}/F^{p+2}$ is the quotient map. For any $\sum \sigma \otimes [\omega_1 | \omega_2 | \cdots | \omega_k] \in H^{p+q}(F^p/F^{p+1})$, we have

$$d_1 \left(\sum \sigma \otimes [\omega_1 | \omega_2 | \cdots | \omega_k] \right) = \sum d(\sigma) \otimes [\omega_1 | \omega_2 | \cdots | \omega_k]$$

since $\overline{TV}^0 = 0$ and $\overline{TV}^1 = 0$. Therefore, we conclude that $d_1 = \partial \otimes 1$ and it means that $E_2^{p,q} \cong H^p(X) \otimes \text{HH}_q(TV; \mathbf{k})$. □

We next recall the Serre spectral sequence associated to the fibration $\chi: P(Y; f_1, f_2) \rightarrow X$. For any nondegenerate p -cube $\sigma: I^p \rightarrow X$, a $(q+p)$ -cube $\bar{\sigma}: I^q \times I^p \rightarrow P(Y; f_1, f_2)$ is a *fibred q -cube over σ* if the diagram

$$\begin{array}{ccc} I^q \times I^p & \xrightarrow{\bar{\sigma}} & P(Y; f_1, f_2) \\ \text{pr}_2 \downarrow & & \downarrow \chi \\ I^p & \xrightarrow{\sigma} & X \end{array}$$

is commutative. Denote by \tilde{F}_p the subcomplex of $C_*(P(Y; f_1, f_2))$ generated by nondegenerate cubes fibred by some $\sigma \in C_{\leq p}(X)$ and put

$$(4-1) \quad \tilde{F}^p = \{\varphi \in C^*(P(Y; f_1, f_2)) \mid \varphi|_{\tilde{F}_{p-1}} = 0\}.$$

Then, we get a spectral sequence, written by $(\tilde{E}_r, \tilde{d}_r)$, associated to the filtration which is called the Serre spectral sequence.

Proposition 4.3 [19, Chapter II 8, Proposition 6] *There is an isomorphism of \mathbf{k} -vector space*

$$\tilde{E}_2^{p,q} \cong H^p(X) \otimes H^q(\Omega Y).$$

Lemma 4.4 *The map $\bar{\Theta}_X$ is filtration preserving. Moreover, the morphism of spectral sequences induced by $\bar{\Theta}_X$ is of the form*

$$1 \otimes H(\bar{\Theta}_{\text{pt}})^\pm: E_2^{p,n-p} \longrightarrow \tilde{E}_2^{p,n-p}$$

at the 2-terms. Here, pt is the one point space and the map $H(\bar{\Theta}_{\text{pt}})^\pm$ from $\text{HH}_*(TV; \mathbf{k})$ to $H^*(\Omega Y)$ is defined by

$$H(\bar{\Theta}_{\text{pt}})^\pm([\omega_1 | \omega_2 | \cdots | \omega_k]) = (-1)^{p(k+n-p)} H(\bar{\Theta}_{\text{pt}})([\omega_1 | \omega_2 | \cdots | \omega_k]).$$

Proof Given $v \otimes [\omega_1 | \omega_2 | \cdots | \omega_k] \in F^p$ and n -cube $\bar{\sigma}: I^n \rightarrow P(Y; f_1, f_2)$ in \tilde{F}_{p-1} where $n = |v| + \sum_i |\omega_i| - k$. By the definition of \tilde{F}_{p-1} , there exists a nondegenerate m -cube σ ($m < n$) such that the following square commutes:

$$\begin{array}{ccc} I^{n-m} \times I^m & \xrightarrow{\bar{\sigma}} & P(Y; f_1, f_2) \\ \text{pr}_2 \downarrow & & \downarrow \chi \\ I^m & \xrightarrow{\sigma} & X. \end{array}$$

Then, we have

$$\bar{\Theta}_X(v \otimes [\omega_1 | \omega_2 | \cdots | \omega_k])(\bar{\sigma}) = (v \otimes \rho(\omega_1) \otimes \rho(\omega_2) \otimes \cdots \otimes \rho(\omega_k)) \text{AW}(\alpha_k(\kappa_k \times \bar{\sigma})).$$

It is only necessary to show $AW(\alpha_k(\kappa_k \times \bar{\sigma})) = 0$ in $C_{|v|}(X) \otimes \bigotimes_i C_{|\omega_i|}(Y)$. We may write

$$AW(\alpha_k(\kappa_k \times \bar{\sigma})) = \sum \pm \psi \otimes \psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_k$$

where ψ is a $|v|$ -cube in X and ψ_i is a $|\omega_i|$ -cube in Y . By the definition of the Alexander–Whitney map, there is a subset $J = \{j_1 < j_2 < \cdots < j_{|v|}\}$ of $\{1, 2, \dots, n+k\}$ such that the diagram is commutative:

$$\begin{array}{ccc} I^k \times I^n & \xrightarrow{\kappa_k \times \bar{\sigma}} & \Delta^k \times P(Y; f_1, f_2) & \xrightarrow{\alpha_k} & X \times Y^{\times k} \\ \lambda_{J^c}^0 \uparrow & & & & \downarrow \text{pr}_1 \\ I^{|v|} & \xrightarrow{\psi} & & & X. \end{array}$$

We put $\lambda_{J^c}^0(t) = ((u_1, u_2, \dots, u_k), (u_{k+1}, u_{k+2}, \dots, u_{k+n})) \in I^k \times I^n$ for any $t \in I^{|v|}$. If $(u_1, u_2, \dots, u_k) \neq 0$ for some t , we see that ψ is a degenerate cube by the commutativity of the above diagram. Hence, $\psi = 0$ in $C_{|v|}(X)$. If $(u_1, u_2, \dots, u_k) = 0$ for any t , that is the composition map

$$j: I^{|v|} \xrightarrow{\lambda_{J^c}^0} I^k \times I^n \xrightarrow{\text{pr}_2} I^n$$

is the inclusion, then we see the commutativity of the diagram

$$(4-2) \quad \begin{array}{ccccccc} & & & \psi & & & \\ & & & \curvearrowright & & & \\ I^{|v|} & \xrightarrow{j} & I^{n-m} \times I^m & \xrightarrow{\text{pr}_2} & I^m & \xrightarrow{\sigma} & X. \end{array}$$

Since $m \leq p - 1 < p \leq |v|$, ψ is a degenerate cube. Therefore, we conclude that $AW(\alpha_k(\kappa_k \times \bar{\sigma})) = 0$. This finishes a proof of the first assertion.

Recall that $E_0^{p,q} \cong C^p(X) \otimes C_q(TV, \mathbf{k})$ and $\tilde{E}_0^{p,n-p} \cong \text{Hom}_{\mathbf{k}}((\tilde{F}_p)^n / (\tilde{F}_{p-1})^n, \mathbf{k})$. We consider the case $|v| = p$ and $\bar{\sigma} \in (\tilde{F}_p)^n / (\tilde{F}_{p-1})^n$, that means the σ is a p -cube. Then, the diagram (4-2) shows that $\psi = \sigma$. Therefore we have

$$AW(\alpha_k(\kappa_k \times \bar{\sigma})) = \sigma \otimes (-1)^{p(k+n-p)} AW(\alpha_k(\kappa_k \times \bar{\sigma}|_{I^{n-p}})).$$

in $C_{|v|}(X) \otimes \bigotimes_i C_{|\omega_i|}(Y)$ where the sign $(-1)^{p(k+n-p)}$ is appeared by the Alexander–Whitney map and so

$$\bar{\Theta}_X(v \otimes [\omega_1|\omega_2|\cdots|\omega_k])(\bar{\sigma}) = v(\sigma) \otimes (-1)^{p(k+n-p)} \bar{\Theta}_{\text{pt}}([\omega_1|\omega_2|\cdots|\omega_k])(\bar{\sigma}|_{I^{n-p}}).$$

The equality shows the second assertion. □

Before proving [Theorem 1.1](#), we recall the following theorem.

Theorem 4.5 (McCleary [16, Theorem 3.26]) *Let E_r and \tilde{E}_r be first quadrant spectral sequences of cohomological type over a field \mathbf{k} and $\phi_r: E_r \rightarrow \tilde{E}_r$ a morphism of spectral sequences such that $E_2^{p,q} = E_2^{p,0} \otimes E_2^{0,q}$, $\tilde{E}_2^{p,q} = \tilde{E}_2^{p,0} \otimes \tilde{E}_2^{0,q}$ and $\phi_2^{p,q} = \phi_2^{p,0} \otimes \phi_2^{0,q}$. Then any two of the following conditions imply the third:*

- (1) $\phi_2^{p,0}: E_2^{p,0} \rightarrow \tilde{E}_2^{p,0}$ is an isomorphism for all p .
- (2) $\phi_2^{0,q}: E_2^{0,q} \rightarrow \tilde{E}_2^{0,q}$ is an isomorphism for all q .
- (3) $\phi_\infty^{p,q}: E_\infty^{p,q} \rightarrow \tilde{E}_\infty^{p,q}$ is an isomorphism for all p, q .

Proof of Theorem 1.1 Since the both spectral sequences E_r and \tilde{E}_r are strong convergent, by [16, Theorem 3.9], it is only enough to show that $H(\bar{\Theta}_{pt})^\pm$ is an isomorphism to prove the theorem. We consider the following pullback diagram

$$\begin{array}{ccc}
 P(Y; 1_Y, c_*) & \longrightarrow & \text{map}(I, Y) \\
 x \downarrow & & \downarrow (p_0, p_1) \\
 Y & \xrightarrow{(1_Y, c_*)} & Y \times Y,
 \end{array}$$

where $c_*: Y \rightarrow Y$ is the constant map to the base point. The space $P(Y; 1_Y, c_*)$ is contractible, we see that $H^*(P(Y; 1_Y, c_*)) \cong \mathbf{k}$. On the other hand, when the $C^*(Y)$ -bimodule structure on $C^*(Y)$ is defined by $\omega' \cdot \nu \cdot \omega = c_*^*(\omega')\nu\omega$, the Hochschild homology $\text{HH}_*(C^*(Y); C^*(Y))$ is \mathbf{k} . In effect, we now note that any element $\nu \otimes \omega[\omega_1|\omega_2|\dots|\omega_k]\omega'$ in $C_*(C^*(Y); C^*(Y))$ is zero if $|\omega'| > 0$ since $c_*^*(\omega') = 0$ and so assume that $|\omega'| = 0$, that is $c_*^*(\omega') \in \mathbf{k}$. Define a map

$$h: C_*(C^*(Y); C^*(Y)) \rightarrow C_*(C^*(Y); C^*(Y))$$

$$\text{by } h(\nu \otimes \omega[\omega_1|\omega_2|\dots|\omega_k]\omega') = \begin{cases} 0 & |\nu| = |\omega| = 0, \\ 1 \otimes 1[c_*^*(\omega')\nu\omega|\omega_1|\omega_2|\dots|\omega_k]1 & \text{otherwise.} \end{cases}$$

An easy calculation gives us the equation $Dh + hD = 1$, where D is the differential of $C_*(C^*(Y); C^*(Y))$. Hence, we have $\text{HH}_*(C^*(Y); C^*(Y)) \cong \mathbf{k}$ and, by Theorem 4.5, the map $H(\bar{\Theta}_{pt})^\pm$ is an isomorphism. □

5 Main result

Let (A, d) and (M, d) be differential graded algebras and $\xi: A \rightarrow M$ a differential graded algebra map. We here recall the complex of ξ -derivations from A to M , $\text{Der}^*(A, M; \xi)$. An element θ in $\text{Der}^n(A, M; \xi)$ is a \mathbf{k} -linear map of degree n

with $\theta(xy) = \theta(x)\xi(y) + (-1)^{|x|}\xi(x)\theta(y)$. The differentials $\delta: \text{Der}^*(A, M; \xi) \rightarrow \text{Der}^{*+1}(A, M; \xi)$ send θ to $d\theta - (-1)^{|\theta|}\theta d$. Then we have the natural map

$$J_1: \text{Der}^n(A, M; \xi) \longrightarrow C^{n+1}(A; M)$$

$$J_1(\theta)([\omega_1|\omega_2|\cdots|\omega_k]) = \begin{cases} (-1)^{|\theta|}\theta(\omega_1) & k = 1, \\ 0 & k \geq 2, k = 0, \end{cases}$$

and it is readily seen that J_1 is a cochain map of degree 1, that is, $J_1 D = -\delta J_1$.

Suppose that X is a \mathbf{k} -Poincaré duality space of formal dimension d ; that is, the space X is equipped with a fundamental class $[X] \in H_d(X)$ such that the cap product

$$-\cap[X]: H^*(X) \longrightarrow H_{d-*}(X)$$

is an isomorphism. We also denote by $[X] \in C_d(X)$ the representative element of $[X] \in H_d(X)$. By dualizing Theorem 1.1, we obtain the isomorphism of \mathbf{k} -vector space

$$\Phi_X: H_*(L_f Y) \xrightarrow{\cong} H^*(L_f Y)^\vee \xrightarrow{H(\bar{\Theta}_X)^\vee} \text{HH}_*(TV; C^*(X))^\vee,$$

where $(-)^\vee = \text{Hom}_{\mathbf{k}}(-, \mathbf{k})$ is the graded dual space. Let $\varepsilon: C_*(X) \rightarrow C^*(X)^\vee$ be the evaluation map; $\varepsilon(\sigma)(\omega) = (-1)^{|\sigma|}\omega(\sigma)$ for $\sigma \in C_*(X)$ and $\omega \in C^*(X)$. We here remark that the evaluation map ε is not a chain map by the definition of the differentials of $C^*(X)$. However, ε induces the map $H(C_*(X)) \rightarrow H(C^*(X)^\vee)$ in homology and the induced map is an isomorphism.

For simplicity we denote by $\bar{\mathbf{B}}(C^*(X))$ the two-sided normalized bar construction $\bar{\mathbf{B}}(C^*(X); C^*(X); C^*(X))$. In [9], Félix, Thomas and Vigué-Poirrier proved that the map of $C^*(X)$ -bimodules with degree $-d$

$$\theta_{\varepsilon[X]}: \bar{\mathbf{B}}(C^*(X)) \longrightarrow C^*(X)^\vee$$

defined by $\theta_{\varepsilon[X]}(\omega[\]\omega') = \varepsilon(\omega\omega' \cap [X])$ and $\theta_{\varepsilon[X]}(\omega[\omega_1|\omega_2|\cdots|\omega_k]\omega') = 0$ for $k > 0$ is a quasi-isomorphism [9, Theorem 12]. Here the $C^*(X)$ -bimodule structure on $C^*(X)^\vee$ is defined by

$$(\omega_1 \cdot \varphi \cdot \omega_2)(\omega) = (-1)^{|\omega_1||\varphi|}\varphi(\omega_1\omega_2\omega)$$

for $\omega, \omega_i \in C^*(X)$ and $\varphi \in C^*(X)^\vee$. Therefore, by [5, Proposition 2.4], we have the isomorphism

$$\Psi_X: \text{HH}^*(TV; C^*(X)) \xrightarrow{\text{HH}(TV; \varepsilon_{C^*(X)})^{-1}} \text{HH}^*(TV; \bar{\mathbf{B}}(C^*(X)))$$

$$\xrightarrow{\text{HH}(TV; \theta_{\varepsilon[X]})} \text{HH}^{*-d}(TV; C^*(X)^\vee)$$

$$\xrightarrow{\iota_*} \text{HH}_{-*+d}(TV; C^*(X))^\vee$$

where ι_* the induced map of the isomorphism of complexes

$$\iota: \text{Hom}_{(TV)^e}(\overline{\mathbf{B}}(TV), C^*(X)^\vee) \longrightarrow \text{Hom}(C^*(X) \otimes_{(TV)^e} \overline{\mathbf{B}}(TV), \mathbf{k})$$

defined by $\iota(\varphi)(\omega \otimes \sigma) = (-1)^{|\sigma||\omega|} \varphi(\sigma)(\omega)$ for $\sigma \in \overline{\mathbf{B}}(TV)$ and $\omega \in C^*(X)$.

Now we define the map for any $n \geq 2$

$$\Theta_1: \pi_n(\text{map}(X, Y; f)) \otimes \mathbf{k} \longrightarrow H^{-n}(\text{Der}^*(TV, C^*(X); f^* \circ \rho))$$

by $\Theta_1(\alpha)(x) = (-1)^{n|x|} \int_{[S^n]} C^*(\overline{\alpha})\rho(x)$ for any $\alpha \in \pi_n(\text{map}(X, Y; f)) \otimes \mathbf{k}$ and $x \in TV$, where $\overline{\alpha}: S^n \times X \rightarrow Y$ is the adjoint of α and $[S^n] \in C_n(S^n)$ be the fundamental class defined by

$$[S^n]: I^n \rightarrow I^n / \partial I^n \cong S^n, \quad (t_1, t_2, \dots, t_n) \mapsto [1 - t_1, 1 - t_2, \dots, 1 - t_n].$$

A straightforward calculation shows that $\Theta_1(\alpha)$ is a $(f^* \circ \rho)$ -derivation. If two maps $\overline{\alpha}$ and $\overline{\beta}: S^n \times X \rightarrow Y$ are homotopic, then we have $\int_{[S^n]} C^*(\overline{\alpha})\rho - \int_{[S^n]} C^*(\overline{\beta})\rho = \delta(\int_{[S^n]} \int_{\text{id}_I} C^*(H)\rho)$ where $\text{id}_I \in C_1(I)$ is the identity map and $H: I \times S^n \times X \rightarrow Y$ is a homotopy from $\overline{\alpha}$ to $\overline{\beta}$. Hence, Θ_1 is a well-defined map. In addition, the map Θ_1 is a homomorphism. Indeed, for any α and β in $\pi_n(\text{map}(X, Y; f)) \otimes \mathbf{k}$, the adjoint of the sum $\alpha + \beta \in \pi_n(\text{map}(X, Y; f)) \otimes \mathbf{k}$ is the composite map

$$S^n \times X \xrightarrow{\mu' \times 1} (S^n \vee S^n) \times X \xrightarrow{(\overline{\alpha}|\overline{\beta})} Y$$

where $\mu': S^n \rightarrow S^n \vee S^n$ is the pinching map and $(\overline{\alpha}|\overline{\beta})$ is a map defined by $(\overline{\alpha}|\overline{\beta})((u, *), x) = \overline{\alpha}(u, x)$ and $(\overline{\alpha}|\overline{\beta})((*), (u), x) = \overline{\beta}(u, x)$ for $u \in S^n$ and $x \in X$. Then, we see that the following diagram is commutative:

$$\begin{array}{ccc} (C_*(S^n) \otimes C_*(X)) & \xrightarrow{(C_*(i_1) + C_*(i_2)) \otimes 1} & C_*(S^n \vee S^n) \otimes C_*(X) \\ \downarrow C_*(\overline{\alpha}) + C_*(\overline{\beta}) & & \downarrow \text{EZ} \\ C_*(Y) & \xleftarrow{C_*(\overline{\alpha}|\overline{\beta})} & C_*((S^n \vee S^n) \times X), \end{array}$$

where i_1 and $i_2: S^n \rightarrow S^n \vee S^n$ are the inclusions on the first and second factors respectively. A commutativity of the diagram shows that $C^*(\overline{\alpha}|\overline{\beta}) = C^*(\overline{\alpha}) + C^*(\overline{\beta})$ and hence the map Θ_1 is a homomorphism.

Proof of Theorem 1.2 We consider the following diagram:

$$(5-1) \quad \begin{array}{ccc} H_{n-1+d}(L_f Y; \mathbf{k}) & \xrightarrow{\Phi_X} & \mathrm{HH}_{n-1+d}(TV; C^*(X)) \vee \xleftarrow{\Psi_X} \mathrm{HH}^{-n+1}(TV; C^*(X)) \\ \Gamma_1 \uparrow & & \uparrow J_1^* \\ \pi_n(\mathrm{map}(X, Y; f)) \otimes \mathbf{k} & \xrightarrow{\Theta_1} & H^{-n}(\mathrm{Der}^*(TV, C^*(X); f^* \circ \rho)). \end{array}$$

Given $\alpha \in \pi_n(\mathrm{map}(X, Y; f)) \otimes \mathbf{k}$. By the definition of Γ_1 ,

$$\Gamma_1(\alpha) = H_{n-1+d}(g(\alpha' \times 1_X))(EZ([S^{n-1}] \otimes [X]))$$

where $\alpha': S^{n-1} \rightarrow \Omega \mathrm{map}(X, Y; f)$ is the adjoint map of α , and denote

$$\gamma_1(\alpha) = C_{n-1+d}(g(\alpha' \times 1_X))(EZ([S^{n-1}] \otimes [X])) \in C_{n-1+d}(L_f Y)$$

by the representative element of $\Gamma_1(\alpha)$. For any element $v \otimes [\omega_1 | \omega_2 | \cdots | \omega_k]$ in $C_{n-1+d}(TV; C^*(X))$, we have

$$\begin{aligned} (\Phi_X \Gamma_1)(\alpha)(v \otimes [\omega_1 | \omega_2 | \cdots | \omega_k]) \\ = \pm((v \otimes \rho \omega_1 \otimes \rho \omega_2 \otimes \cdots \otimes \rho \omega_k) \circ \mathrm{AW} \circ \alpha_{k*})(EZ(\kappa_k \otimes \gamma_1(\alpha))). \end{aligned}$$

We may write

$$(\mathrm{AW} \circ \alpha_{k*})(EZ(\kappa_k \otimes \gamma_1(\alpha))) = \sum \pm \psi \otimes \psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_k$$

where ψ is a $|v|$ -cube in X and ψ_i is a $|\omega_i|$ -cube in Y . If $k \geq 2$, it is readily seen that some ψ_i are degenerate, that is $(\Phi_X \Gamma_1)(\alpha)(v \otimes [\omega_1 | \omega_2 | \cdots | \omega_k]) = 0$. If $k = 1$, we see that

$$\alpha_{1*}(EZ(\kappa_1 \otimes \gamma_1(\alpha))) = \sum n_x \tau_x.$$

Here, we may write $[X] = \sum_{[X]} n_x x \in C_d(X)$ some $n_x \in \mathbf{k}$ and $x: I^d \rightarrow X$. Then, $(n+d)$ -cube, τ_x is the compositions

$$\tau_x: I \times I^{n-1} \times I^d \xrightarrow{\kappa_1 \times [S^{n-1}] \times x} \Delta^1 \times S^{n-1} \times X \xrightarrow{\alpha_1(1 \times g(\alpha' \times 1))} X \times Y.$$

Then,
$$\mathrm{AW}(\tau_x) = \sum_{\substack{J \subset \{1, 2, \dots, n+d\}, \\ \#J = |v|}} (-1)^{\varepsilon(J)} \tau_{x1} \lambda_{Jc}^0 \otimes \tau_{x2} \lambda_J^\varepsilon$$

where $\#J$ is the cardinal number of J and τ_{xi} is the composition of the projection pr_i and τ_x . If there is $i \in J$ such that $i < n$, then $\tau_{x1} \lambda_{Jc}^0$ is degenerate since

$\tau_{x1}: I \times I^{n-1} \times I^d \rightarrow X$ depends only on I^d . Hence,

$$AW(\tau_x) = \sum_{\substack{J \subset \{1, 2, \dots, n+d\}, \\ \#J=|v|, \min J \geq n}} (-1)^{\varepsilon(J)} \tau_{x1} \lambda_{J^c}^0 \otimes \tau_{x2} \lambda_J^\varepsilon$$

and so

$$\begin{aligned} & (\Phi_X \Gamma_1)(\alpha)(v \otimes [\omega_1]) \\ &= (-1)^{|v|} (v \otimes \rho(\omega_1)) AW\left(\sum_{[X]} n_x \tau_x\right) \\ &= (-1)^{|v|+|v||\omega_1|} \sum_{[X]} \sum_{\substack{J \subset \{1, 2, \dots, n+d\}, \\ \#J=|v|, \min J \geq n}} (-1)^{\varepsilon(J)} n_x (v(\tau_{x1} \lambda_{J^c}^0)) (\rho(\omega_1)(\tau_{x2} \lambda_J^\varepsilon)). \end{aligned}$$

On the other hand, $(\Psi_X J_1^* \Theta_1)(\alpha)(v \otimes [\omega_1 | \omega_2 | \dots | \omega_k]) = 0$ for $k \geq 2$ and $k = 0$ by the definition of J_1 , and

$$\begin{aligned} & (\Psi_X J_1^* \Theta_1)(\alpha)(v \otimes [\omega_1]) \\ &= (-1)^{|v||s\omega_1|} \varepsilon(J_1^* \Theta_1(\alpha)([\omega_1]) \cap [X])(v) \\ &= (-1)^{|v||s\omega_1|+|v|} v(J_1^* \Theta_1(\alpha)([\omega_1]) \cap [X]) \\ &= (-1)^{|v||s\omega_1|+|v|+n} v(\Theta_1(\alpha)(\omega_1) \cap [X]) \\ &= (-1)^{|v||s\omega_1|+|v|+n+n|\omega_1|} v\left(\int_{[S^n]} C^*(\bar{\alpha}) \rho(\omega_1) \cap [X]\right) \\ &= (-1)^{|v||s\omega_1|+|v|+n+n|\omega_1|+|v|(d-|v|)} \\ &\quad \times \sum_{[X]} \sum_{\substack{J \subset \{1, 2, \dots, d\}, \\ \#J=|v|}} (-1)^{\varepsilon(J)} n_x (v(x \lambda_{J^c}^0)) \left(\int_{[S^n]} C^*(\bar{\alpha}) \rho(\omega_1)(x \lambda_J^1)\right) \\ &= (-1)^{|v||s\omega_1|+nd+(n+d)|v|} \\ &\quad \times \sum_{[X]} \sum_{\substack{J \subset \{1, 2, \dots, n+d\}, \\ J=|v|, \min J \geq n}} (-1)^{\varepsilon(J)+n|v|} n_x (v(\tau_{x1} \lambda_{J^c}^0)) (\rho(\omega_1)(\bar{\alpha}([S^n] \times x) \lambda_J^1)). \end{aligned}$$

Since $\rho(\omega_1)(\tau_{x2} \lambda_J^1) = \rho(\omega_1)(\bar{\alpha}([S^n] \times x) \lambda_J^1)$, we have

$$(\Phi_X \Gamma_1)(\alpha)(v \otimes [\omega_1]) = (-1)^{nd+d|v|} (\Psi_X J_1^* \Theta_1)(\alpha)(v \otimes [\omega_1]).$$

If d is even, then the diagram (5-1) is commutative. We consider the case that d is odd. When we define Ψ_X , we replace $\theta_{\varepsilon[X]}$ with the map of degree $-d$, $\tilde{\theta}_{\varepsilon[X]}: \mathbf{B}(C^*(X)) \rightarrow C^*(X)^\vee$ defined by $\tilde{\theta}_{\varepsilon[X]}(\omega[\]\omega') = (-1)^{|\omega\omega'|} \varepsilon(\omega\omega' \cap [X])$ and $\tilde{\theta}_{\varepsilon[X]}(\omega[\omega_1 | \omega_2 | \dots | \omega_k]\omega') = 0$ for $k > 0$. Also $\tilde{\theta}_{\varepsilon[X]}$ is a quasi-isomorphism and

similar calculation described above enable us to get the equation

$$(\Phi_X \Gamma_1)(\alpha)(v \otimes [\omega_1]) = (-1)^{nd+d+(d+1)|v|}(\Psi_X J_1^* \Theta_1)(\alpha)(v \otimes [\omega_1]).$$

That is, the diagram (5-1) is commutative up to sign, completing the proof. □

We here recall a *minimal Sullivan model* for a simply connected space X with finite type. It is a free commutative differential graded algebra over \mathbb{Q} of the form $(\Lambda V, d)$ with $V = \bigoplus_{i \geq 2} V^i$ where each V^i is of finite dimension and d is decomposable; that is, $d(V) \subset \Lambda^{\geq 2} V$. Moreover, $(\Lambda V, d)$ is equipped with a quasi-isomorphism $(\Lambda V, d) \xrightarrow{\cong} A_{PL}(X)$ to the commutative differential graded algebra $A_{PL}(X)$ of differential polynomial forms on X [6, Section 12]. Observe that, as algebras, $H^*(\Lambda V, d) \cong H^*(A_{PL}(X)) \cong H^*(X; \mathbb{Q})$. Let $f: X \rightarrow Y$ be a map between spaces of finite type. Then there exists a commutative differential graded algebra map \tilde{f} from a minimal Sullivan model $(\Lambda V_Y, d)$ for Y to a minimal Sullivan model $(\Lambda V_X, d)$ for X which makes the diagram

$$\begin{array}{ccc} A_{PL}(Y) & \xrightarrow{A_{PL}(f)} & A_{PL}(X) \\ \cong \uparrow & & \uparrow \cong \\ \Lambda V_Y & \xrightarrow{\tilde{f}} & \Lambda V_X \end{array}$$

commutative up to homotopy. We call \tilde{f} a *Sullivan model* for f .

Proposition 5.1 *Let ΛV_X and ΛV_Y be a minimal Sullivan model for X and Y , respectively, and \tilde{f} a Sullivan model for f . Then, the cochain map*

$$J_1: \text{Der}^*(\Lambda V_Y, \Lambda V_X; \tilde{f}) \longrightarrow C^{*+1}(\Lambda V_Y; \Lambda V_X)$$

is injective in homology.

For giving a proof of Proposition 5.1, we introduce a semifree resolution of ΛV_Y as a left $\Lambda V_Y \otimes \Lambda V_Y$ -module that is different from the two-sided bar resolution and give some lemmas. We consider the commutative differential graded algebra $\Lambda V_Y \otimes \Lambda V_Y \otimes \Lambda(sV_Y)$ with the differential d defined by

$$\begin{aligned} d(v \otimes 1 \otimes \bar{1}) &= dv \otimes 1 \otimes \bar{1}, & d(1 \otimes v \otimes \bar{1}) &= 1 \otimes dv \otimes \bar{1}, \\ d(1 \otimes 1 \otimes sv) &= (v \otimes 1 - 1 \otimes v) \otimes \bar{1} - \sum_{i=1}^{\infty} \frac{(sd)^i}{i!} (v \otimes 1 \otimes \bar{1}). \end{aligned}$$

Here $\bar{1}$ is the unit of $\Lambda(sV_Y)$, and s is the unique degree -1 derivation of the algebra $\Lambda V_Y \otimes \Lambda V_Y \otimes \Lambda(sV_Y)$ defined by

$$s(v \otimes 1 \otimes \bar{1}) = 1 \otimes 1 \otimes sv = s(1 \otimes v \otimes \bar{1}), \quad s(1 \otimes 1 \otimes sv) = 0.$$

By [6, Section 15 Example 1], the map

$$\mu \cdot \bar{\varepsilon}: \Lambda V_Y \otimes \Lambda V_Y \otimes \Lambda(sV_Y) \longrightarrow \Lambda V_Y$$

is a semifree resolution of ΛV_Y as a left $\Lambda V_Y \otimes \Lambda V_Y$ -module, where μ is the product of ΛV_Y and $\bar{\varepsilon}$ is the canonical augmentation of $\Lambda(sV_Y)$. Since the map $\varepsilon_{\Lambda V_Y}: \bar{\mathbf{B}}(\Lambda V_Y, \Lambda V_Y, \Lambda V_Y) \rightarrow \Lambda V_Y$ is a surjective quasi-isomorphism, by [6, Proposition 14.6], there exists a differential graded algebra map ϕ such that the following diagram is commutative:

$$\begin{array}{ccc} \Lambda V_Y \otimes \Lambda V_Y & \hookrightarrow & \bar{\mathbf{B}}(\Lambda V_Y, \Lambda V_Y, \Lambda V_Y) \\ \downarrow & \nearrow \phi & \downarrow \varepsilon_{\Lambda V_Y} \\ \Lambda V_Y \otimes \Lambda V_Y \otimes \Lambda(sV_Y) & \xrightarrow{\mu \cdot \bar{\varepsilon}} & \Lambda V_Y. \end{array}$$

A commutativity of the diagram shows that the map ϕ is a quasi-isomorphism. We now recall a construction of ϕ . For any basis element $v \in V_Y$, we put $\phi(v \otimes 1 \otimes \bar{1}) = v [] 1$ and $\phi(1 \otimes v \otimes \bar{1}) = 1 [] v$. By induction on degree of V_Y , we construct $\phi(1 \otimes 1 \otimes sv)$. For any $v' \in V$ such that $dv' = 0$, we defined $\phi(1 \otimes 1 \otimes sv') = 1 [v'] 1$. Assume that ϕ is defined in $\Lambda V_Y \otimes \Lambda V_Y \otimes \Lambda(sV_Y^{\leq |v|})$ for some basis element $v \in V$, that is, $\phi d(1 \otimes 1 \otimes sv)$ is also defined. Since $\varepsilon_{\Lambda V_Y}$ is a quasi-isomorphism, the equation

$$\varepsilon_{\Lambda V_Y} \phi d(1 \otimes 1 \otimes sv) = (\mu \cdot \bar{\varepsilon}) d(1 \otimes 1 \otimes sv) = 0 = \varepsilon_{\Lambda V_Y} d(1 [v] 1)$$

shows that there is $\beta \in \bar{\mathbf{B}}(\Lambda V_Y, \Lambda V_Y, \Lambda V_Y)$ such that $\phi d(1 \otimes 1 \otimes sv) - d(1 [v] 1) = d\beta$. Then, we put $\phi(1 \otimes 1 \otimes sv) = 1 [v] 1 + \beta$. The above construction of ϕ and the differential of $\Lambda V_Y \otimes \Lambda V_Y \otimes \Lambda(sV_Y)$ establishes that $d\beta$ has no term of the form $x [] x'$, that is, β does not have terms of the form $x [\omega] x'$. So we have the following lemma.

Lemma 5.2 *There is a quasi-isomorphism*

$$\phi: \Lambda V_Y \otimes \Lambda V_Y \otimes \Lambda(sV_Y) \rightarrow \bar{\mathbf{B}}(\Lambda V_Y, \Lambda V_Y, \Lambda V_Y)$$

of $\Lambda V_Y \otimes \Lambda V_Y$ -modules such that the following diagram is commutative

$$\begin{array}{ccc} \Lambda V_Y \otimes \Lambda V_Y \otimes \Lambda(sV_Y) & \xrightarrow{\phi} & \bar{\mathbf{B}}(\Lambda V_Y, \Lambda V_Y, \Lambda V_Y) \\ \varepsilon \cdot \varepsilon \cdot \text{pr} \downarrow & & \downarrow \varepsilon \cdot \text{pr}' \cdot \varepsilon \\ sV_Y & \hookrightarrow & s\Lambda V_Y, \end{array}$$

where $\varepsilon: \Lambda V_Y \rightarrow \mathbb{Q}$ is the canonical augmentation and $\text{pr}: \Lambda(sV_Y) \rightarrow sV_Y$ and $\text{pr}': T(s\Lambda V_Y) \rightarrow s\Lambda V$ are the canonical projections.

Consider the canonical isomorphism

$$\zeta: \text{Hom}_{\Lambda V_Y \otimes \Lambda V_Y}(\Lambda V_Y \otimes \Lambda V_Y \otimes \Lambda(sV_Y), \Lambda V_X) \longrightarrow \text{Hom}_{\mathbb{Q}}(\Lambda(sV_Y), \Lambda V_X).$$

and define $\bar{D} = \zeta D \zeta^{-1}$, where D is the differential of

$$\text{Hom}_{\Lambda V_Y \otimes \Lambda V_Y}(\Lambda V_Y \otimes \Lambda V_Y \otimes \Lambda(sV_Y), \Lambda V_X).$$

Then, for $\psi \in \text{Hom}_{\mathbb{Q}}(\Lambda(sV_Y), \Lambda V_X)$ and $sv_1sv_2 \cdots sv_p \in \Lambda(sV_Y)$,

$$\begin{aligned} \bar{D}(\psi)(sv_1sv_2 \cdots sv_p) &= d\psi(sv_1sv_2 \cdots sv_p) \\ &+ (-1)^{|\psi|} \sum_{i=1}^p \sum_{v_i} \sum_{k=1}^p \pm \omega_{i_1} \cdots \omega_{i_{k-1}} \omega_{i_{k+1}} \cdots \omega_{i_p} \psi(sv_1 \cdots sv_{i-1} s\omega_{i_k} sv_{i+1} \cdots sv_p), \end{aligned}$$

where $dv_i = \sum_{v_i} \omega_{i_1} \omega_{i_2} \cdots \omega_{i_p}$ and the sign \pm is the Koszul sign convention. In fact, for example $p = 1$ and $v = v_1 \in V$ with $dv = \sum_v \omega_1 \cdots \omega_p$,

$$\begin{aligned} \bar{D}(\psi)(sv) &= d\zeta^{-1}(\psi)(1 \otimes 1 \otimes sv) - (-1)^{|\psi|} \zeta^{-1}(\psi) d(1 \otimes 1 \otimes sv) \\ &= d\psi(sv) + (-1)^{|\psi|} \zeta^{-1}(\psi) \left(\sum_{i=1}^{\infty} \frac{(sd)^i}{i!} (v \otimes 1 \otimes \bar{1}) \right) \\ &= d\psi(sv) + (-1)^{|\psi|} \sum_v \sum_{j=1}^p \pm \omega_1 \cdots \omega_{j-1} \omega_{j+1} \cdots \omega_p \psi(s\omega_j) \\ &\quad + \zeta^{-1}(\psi) \left(\sum_{i=2}^{\infty} \frac{(sd)^i}{i!} (v \otimes 1 \otimes \bar{1}) \right). \end{aligned}$$

An induction on the degree of v gives that $\zeta^{-1}(\psi)((sd)^2(v \otimes 1 \otimes \bar{1})) = 0$. Therefore, we see that $\text{Hom}_{\mathbb{Q}}(\Lambda(sV_Y), \Lambda V_X)$ decomposes into a direct sum of complexes

$$(5-2) \quad (\text{Hom}_{\mathbb{Q}}(\Lambda(sV_Y), \Lambda V_X), \bar{D}) = \bigoplus_{p \geq 0} (\text{Hom}_{\mathbb{Q}}(\Lambda^p(sV_Y), \Lambda V_X), \bar{D}).$$

Note that the decomposition is a Hochschild cohomology version of Vigué’s work [20].

Proof of Proposition 5.1 By Lemma 5.2, the following diagram of complexes is commutative:

$$\begin{array}{ccc} C^*(\Lambda V_Y, \Lambda V_X) & \xrightarrow{\zeta \phi^*} & \text{Hom}_{\mathbb{Q}}^*(\Lambda(sV_Y), \Lambda V_X) \\ J_1 \uparrow & & \uparrow \\ \text{Der}^{*-1}(\Lambda V_Y, \Lambda V_X; \tilde{f}) & \xrightarrow{\zeta_1} & \text{Hom}_{\mathbb{Q}}^*(sV_Y, \Lambda V_X), \end{array}$$

where ζ_1 is the canonical degree 1 isomorphism of complexes defined by $\zeta_1(\theta)(sv) = (-1)^{|\theta|} \theta(v)$ for $\theta \in \text{Der}^{*-1}(\Lambda V_Y, \Lambda V_X; \tilde{f})$ and $v \in V_Y$. Therefore, the decomposition (5-2) shows that J_1 is injective in homology. □

Before proving **Corollary 1.3**, we recall the definition of the isomorphism

$$\Phi: \pi_n(\text{map}(X, Y; f)) \otimes \mathbb{Q} \rightarrow H^{-n}(\text{Der}^*(\Lambda V_Y, \Lambda V_X; \tilde{f}))$$

defined by [1; 14]. Let $\alpha \in \pi_n(\text{map}(X, Y; f)) \otimes \mathbb{Q}$ and $g: S^n \times X \rightarrow Y$ be the adjoint of α . Denote by $\tilde{g}: \Lambda V_X \rightarrow \Lambda V_{S^n} \otimes \Lambda V_Y$ a Sullivan model for g . Since S^n is formal, there is a quasi-isomorphism $\phi: \Lambda V_{S^n} \rightarrow (H^*(S^n; \mathbb{Q}), 0)$ and, for any $v \in \Lambda V$, we may write

$$(\phi \otimes 1)\tilde{g}(v) = 1 \otimes \tilde{f}(v) + e_n \otimes v'.$$

Then we put $\Phi(\alpha)(v) = v'$.

Proof of Corollary 1.3 By the definition of Θ_1 and Φ , we have the following commutative diagram:

$$\begin{array}{ccc} \text{HH}^{-n+1}(TV; C^*(X)) & \xrightarrow{\cong} & \text{HH}^{-n+1}(\Lambda V_Y; \Lambda V_X) \\ J_1^* \uparrow & & \uparrow J_1^* \\ H^{-n} \text{Der}^*(TV, C^*(X); f^* \circ \rho) & & H^{-n}(\text{Der}^*(\Lambda V_Y, \Lambda V_X; \tilde{f})) \\ \Theta_1 \swarrow & & \searrow \Phi \\ & \pi_n(\text{map}(X, Y; f)) \otimes \mathbb{Q}, & \end{array}$$

where the isomorphism at the top of the above diagram is the map induced by chains of natural quasi-isomorphisms [6, Corollary 10.10]

$$\begin{aligned} TV &\xrightarrow{\cong} C^*(Y) \xrightarrow{\cong} \dots \xleftarrow{\cong} A_{\text{PL}}(Y) \xleftarrow{\cong} \Lambda V_Y, \\ C^*(X) &\xrightarrow{\cong} \dots \xleftarrow{\cong} A_{\text{PL}}(X) \xleftarrow{\cong} \Lambda V_X. \end{aligned}$$

Since Φ is an isomorphism, the commutativity of (5-1) and **Proposition 5.1** show the assertion. □

6 Noncommutativity for $\mathbf{H}_*(L_f Y; \mathbb{Q})$

We retain the notation described in the section above. Let X be a simply connected d -dimensional closed oriented manifold, Y a simply connected space with finite type and $f: X \rightarrow Y$ a based space. We see that the shifted homology $\mathbf{H}_*(L_f Y)$ has a graded algebra structure by Gruher and Salvatore [10]. As an application for the main result, we have the following proposition.

Proposition 6.1 *If the rational homotopy group $\pi_{\geq 2}(\text{map}(X, Y; f)) \otimes \mathbb{Q}$ has a non-trivial Whitehead product, then $\mathbf{H}_*(L_f Y; \mathbb{Q})$ is a noncommutative graded algebra.*

Proof By [21, Chapter X, Theorem (7.10)], $\pi_{\geq 2}(\text{map}(X, Y; f)) \otimes \mathbb{Q}$ has a nontrivial Whitehead product if and only if there is a nontrivial Samelson product on $\pi_{\geq 1}(\Omega \text{map}(X, Y; f)) \otimes \mathbb{Q}$. We denote $\langle \beta_1, \beta_2 \rangle$ by the nontrivial Samelson product for some β_1 and β_2 . Then, by [21, Chapter X, Theorem (6.3)], we have the equality $h(\langle \beta_1, \beta_2 \rangle) = h(\beta_1)h(\beta_2) - (-1)^{|\beta_1||\beta_2|}h(\beta_2)h(\beta_1)$, where h is the Hurewicz map. We note that a graded algebra structure on $H_*(\Omega \text{map}(X, Y; f); \mathbb{Q})$ is determined by the H-space structure on $\Omega \text{map}(X, Y; f)$. Since the map $g: \Omega \text{map}(X, Y; f) \times X \rightarrow L_f Y$ is a morphism of fiberwise monoids from the projection $\Omega \text{map}(X, Y; f) \times X \rightarrow X$ to the map $\chi: L_f Y \rightarrow X$, by [10, Theorem 4.1 (ii)], the map $\Gamma: H_*(\Omega \text{map}(X, Y; f); \mathbb{Q}) \rightarrow \mathbf{H}_*(L_f Y; \mathbb{Q})$ stated in Section 1 is an algebra map. Therefore, we see that

$$\Gamma_1(\langle \beta_1, \beta_2 \rangle) = \Gamma_1(\beta_1)\Gamma_1(\beta_2) - (-1)^{|\beta_1||\beta_2|}\Gamma_1(\beta_2)\Gamma_1(\beta_1)$$

and Corollary 1.3 shows that $\Gamma_1(\beta_1)\Gamma_1(\beta_2) \neq (-1)^{|\beta_1||\beta_2|}\Gamma_1(\beta_2)\Gamma_1(\beta_1)$. □

In the rest of this section, we give an example of $\mathbf{H}_*(L_f Y; \mathbb{Q})$ which is noncommutative.

Example 6.2 Let $\mathbb{C}P^n$ be the complex projective space and $i: \mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$ the inclusion for $n \geq 2$. Recall that the commutative differential graded algebra $M(\mathbb{C}P^n) := (\Lambda(x_2, x_{2n+1}), dx_{2n+1} = x_2^{n+1})$ is a minimal Sullivan model for $\mathbb{C}P^n$ and a map

$$\tilde{\tau}: M(\mathbb{C}P^n) \longrightarrow M(\mathbb{C}P^{n-1}), \quad \tilde{\tau}(x_2) = x_2, \quad \tilde{\tau}(x_{2n+1}) = x_2 x_{2n-1}$$

is a Sullivan model for i , where the degree of x_j is j . By [2, Theorem 2], a $\tilde{\tau}$ -derivation of degree -3 , $[\theta, \theta]$, defined by

$$[\theta, \theta]: M(\mathbb{C}P^n) \longrightarrow M(\mathbb{C}P^{n-1}), \quad [\theta, \theta](x_2) = 0, \quad [\theta, \theta](x_{2n+1}) = x_2^{n-1}$$

is a nontrivial Whitehead product of

$$H^{-3}(\text{Der}^*(M(\mathbb{C}P^n), M(\mathbb{C}P^{n-1}); \tilde{\tau})) \cong \pi_3(\text{map}(\mathbb{C}P^{n-1}, \mathbb{C}P^n; i)) \otimes \mathbb{Q},$$

where θ is a $\tilde{\tau}$ -derivation of degree -2 defined by $\theta(x_2) = 1$ and $\theta(x_{2n+1}) = 0$. The existence of a nonzero Whitehead product in $\pi_*(\text{map}(\mathbb{C}P^{n-1}, \mathbb{C}P^n; i)) \otimes \mathbb{Q}$ is also showed by the results of Møller and Raussen [17, Example 3.4]. They proved that $\text{map}(\mathbb{C}P^{n-1}, \mathbb{C}P^n; i)$ is of the rational homotopy type of $S^2 \times S^5 \times S^7 \times \dots \times S^{2n+1}$ and the nonzero Whitehead product comes from the S^2 factor. Therefore, by Proposition 6.1, $\mathbf{H}_*(L_i \mathbb{C}P^n; \mathbb{Q})$ is a noncommutative algebra.

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