Algebraic K–theory over the infinite dihedral group: an algebraic approach

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Two types of Nil-groups arise in the codimension 1 splitting obstruction theory for homotopy equivalences of finite CW–complexes: the Farrell–Bass Nil-groups in the nonseparating case when the fundamental group is an HNN extension and the Waldhausen Nil-groups in the separating case when the fundamental group is an amalgamated free product. We obtain a general Nil-Nil theorem in algebraic K–theory relating the two types of Nil-groups.

The infinite dihedral group is a free product and has an index 2 subgroup which is an HNN extension, so both cases arise if the fundamental group surjects onto the infinite dihedral group. The Nil-Nil theorem implies that the two types of the reduced $\widetilde{\mathrm{Nil}}$–groups arising from such a fundamental group are isomorphic. There is also a topological application: in the finite-index case of an amalgamated free product, a homotopy equivalence of finite CW–complexes is semisplit along a separating subcomplex.

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Introduction

The infinite dihedral group is both a free product and an extension of the infinite cyclic group $\mathbb{Z}$ by the cyclic group $\mathbb{Z}_2$ of order 2

$$D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2 = \mathbb{Z} \rtimes \mathbb{Z}_2$$

with $\mathbb{Z}_2$ acting on $\mathbb{Z}$ by -1. A group G is said to be *over* D_∞ if it is equipped with an epimorphism $p\colon G \to D_\infty$. We study the algebraic K–theory of $R[G]$, for any ring R and any group G over D_∞. Such a group G inherits from D_∞ an injective amalgamated free product structure $G = G_1 *_H G_2$ with H an index 2 subgroup of G_1 and G_2. Furthermore, there is a canonical index 2 subgroup $\overline{G} \subset G$ with an

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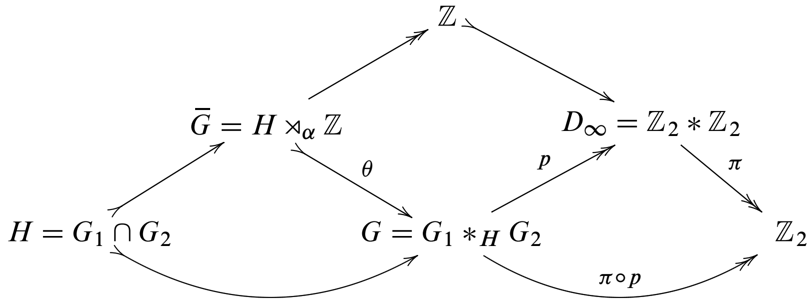
injective HNN structure $\overline{G} = H \rtimes_\alpha \mathbb{Z}$ for an automorphism $\alpha\colon H \to H$. The various groups fit into a commutative braid of short exact sequences:



The algebraic K–theory decomposition theorems of Waldhausen for injective amalgamated free products and HNN extensions give

(1) $K_*(R[G]) = K_*(R[H] \to R[G_1] \times R[G_2])$
$$\oplus\, \widetilde{\mathrm{Nil}}_{*-1}(R[H]; R[G_1 - H], R[G_2 - H])\ ,$$

(2) $K_*(R[\overline{G}]) = K_*(1-\alpha\colon R[H] \to R[H])$
$$\oplus\, \widetilde{\mathrm{Nil}}_{*-1}(R[H],\alpha) \oplus \widetilde{\mathrm{Nil}}_{*-1}(R[H],\alpha^{-1})\ .$$

We establish isomorphisms

$$\widetilde{\mathrm{Nil}}_*(R[H]; R[G_1 - H], R[G_2 - H]) \cong \widetilde{\mathrm{Nil}}_*(R[H],\alpha) \cong \widetilde{\mathrm{Nil}}_*(R[H],\alpha^{-1})\ .$$

A homotopy equivalence $f\colon M \to X$ of finite CW–complexes is *split along* a subcomplex $Y \subset X$ if it is a cellular map and the restriction $f|\colon N = f^{-1}(Y) \to Y$ is also a homotopy equivalence. The $\widetilde{\mathrm{Nil}}_*$–groups arise as the obstruction groups to splitting homotopy equivalences of finite CW–complexes for codimension 1 $Y \subset X$ with injective $\pi_1(Y) \to \pi_1(X)$, so that $\pi_1(X)$ is either an HNN extension or an amalgamated free product (Farrell–Hsiang, Waldhausen) – see Section 4 for a brief review of the codimension 1 splitting obstruction theory in the separating case of an amalgamated free product. In this paper we introduce the considerably weaker notion of a homotopy equivalence in the separating case being *semisplit* (Definition 4.4). By way of geometric application we prove in Theorem 4.5 that there is no obstruction to topological semisplitting in the finite-index case.

0.1 Algebraic semisplitting

The following is a special case of our main algebraic result (Theorems 1.1, 2.7) which shows that there is no obstruction to algebraic semisplitting.

Theorem 0.1 *Let G be a group over D_∞, with*

$$H = G_1 \cap G_2 \ \subset\ \bar{G} = H \rtimes_\alpha \mathbb{Z} \subset G = G_1 *_H G_2 \ .$$

(1) *For any ring R and $n \in \mathbb{Z}$ the corresponding reduced* Nil*–groups are isomorphic:*

$$\widetilde{\mathrm{Nil}}_n(R[H]; R[G_1 - F], R[G_2 - H]) \cong \widetilde{\mathrm{Nil}}_n(R[H], \alpha) \cong \widetilde{\mathrm{Nil}}_n(R[H], \alpha^{-1}) \ .$$

(2) *The inclusion $\theta\colon R[\bar{G}] \to R[G]$ determines induction and transfer maps*

$$\theta_!\colon K_n(R[\bar{G}]) \to K_n(R[G]) \ , \quad \theta^!\colon K_n(R[G]) \to K_n(R[\bar{G}]) \ .$$

For all integers $n \leqslant 1$, the $\widetilde{\mathrm{Nil}}_n(R[H], \alpha)$–$\widetilde{\mathrm{Nil}}_n(R[H]; R[G_1 - H], R[G_2 - H])$–components of the maps $\theta_!$ and $\theta^!$ in the decompositions (2) *and* (1) *are isomorphisms.*

Proof Part (i) is a special case of Theorem 0.4.

Part (ii) follows from Theorem 0.4, Lemma 3.20 and Proposition 3.23. □

The $n = 0$ case will be discussed in more detail in Sections 0.2 and 3.1.

Remark 0.2 We do not seriously doubt that a more assiduous application of higher K–theory would extend Theorem 0.1(2) to all $n \in \mathbb{Z}$ (see also Davis, Quinn and Reich [5]).

As an application of Theorem 0.1, we shall prove the following theorem. It shows that the Farrell–Jones Isomorphism Conjecture in algebraic K–theory can be reduced (up to dimension one) to the family of finite-by-cyclic groups, so that virtually cyclic groups of infinite dihedral type need not be considered.

Theorem 0.3 *Let Γ be any group, and let R be any ring. Then, for all integers $n < 1$, the following induced map of equivariant homology groups, with coefficients in the algebraic K–theory functor $\mathbf{K}_R$, is an isomorphism:*

$$H_n^\Gamma(E_{\mathsf{fbc}}\Gamma; \mathbf{K}_R) \longrightarrow H_n^\Gamma(E_{\mathsf{vc}}\Gamma; \mathbf{K}_R) \ .$$

Furthermore, this map is an epimorphism for $n = 1$.

This is a special case of a more general fibered version, Theorem 3.29. Theorem 0.3 has been proved for all degrees n by Davis, Quinn and Reich [5]; however our proof here uses only algebraic topology, avoiding the use of controlled topology.

The original reduced Nil–groups $\widetilde{\mathrm{Nil}}_*(R) = \widetilde{\mathrm{Nil}}_*(R, \mathrm{id})$ feature in the decompositions of Bass [2] and Quillen [9]:

$$\begin{aligned} K_*(R[t]) &= K_*(R) \oplus \widetilde{\mathrm{Nil}}_{*-1}(R) \\ K_*(R[\mathbb{Z}]) &= K_*(R) \oplus K_{*-1}(R) \oplus \widetilde{\mathrm{Nil}}_{*-1}(R) \oplus \widetilde{\mathrm{Nil}}_{*-1}(R) \ . \end{aligned}$$

In Section 3 we shall compute several examples which require Theorem 0.1:

$$\begin{aligned} K_*(R[\mathbb{Z}_2 * \mathbb{Z}_2]) &= \frac{K_*(R[\mathbb{Z}_2]) \oplus K_*(R[\mathbb{Z}_2])}{K_*(R)} \oplus \widetilde{\mathrm{Nil}}_{*-1}(R) \\ K_*(R[\mathbb{Z}_2 * \mathbb{Z}_3]) &= \frac{K_*(R[\mathbb{Z}_2]) \oplus K_*(R[\mathbb{Z}_3])}{K_*(R)} \oplus \widetilde{\mathrm{Nil}}_{*-1}(R)^{\infty} \\ \mathrm{Wh}(G_0 \times \mathbb{Z}_2 *_{G_0} G_0 \times \mathbb{Z}_2) &= \frac{\mathrm{Wh}(G_0 \times \mathbb{Z}_2) \oplus \mathrm{Wh}(G_0 \times \mathbb{Z}_2)}{\mathrm{Wh}(G_0)} \oplus \widetilde{\mathrm{Nil}}_0(\mathbb{Z}[G_0]) \end{aligned}$$

where $G_0 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}$. The point here is that $\widetilde{\mathrm{Nil}}_0(\mathbb{Z}[G_0])$ is an infinite torsion abelian group. This provides the first example 3.28 of a nonzero $\widetilde{\mathrm{Nil}}$–group in the amalgamated product case and hence the first example of a nonzero obstruction to splitting a homotopy equivalence in the separating case (A).

0.2 The Nil-Nil Theorem

We establish isomorphisms between two types of codimension 1 splitting obstruction nilpotent class groups, for any ring R. The first type, for separated splitting, arises in the decompositions of the algebraic K–theory of the R–coefficient group ring $R[G]$ of a group G over D_∞, with an epimorphism $p\colon G \to D_\infty$ onto the infinite dihedral group D_∞. The second type, for nonseparated splitting, arises from the α–twisted polynomial ring $R[H]_\alpha[t]$, with $H = \ker(p)$ and $\alpha\colon F \to F$ an automorphism such that

$$\bar{G} = \ker(\pi \circ p\colon G \to \mathbb{Z}_2) = H \rtimes_\alpha \mathbb{Z} \ ,$$

where $\pi\colon D_\infty \to \mathbb{Z}_2$ is the unique epimorphism with infinite cyclic kernel. Note:

(A) $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$ is the free product of two cyclic groups of order 2, whose generators will be denoted t_1, t_2.

(B) $D_\infty = \langle t_1, t_2 \mid t_1^2 = 1 = t_2^2 \rangle$ contains the infinite cyclic group $\mathbb{Z} = \langle t \rangle$ as a subgroup of index 2 with $t = t_1 t_2$. In fact there is a short exact sequence with a split epimorphism

$$\{1\} \longrightarrow \mathbb{Z} \longrightarrow D_\infty \xrightarrow{\ \pi\ } \mathbb{Z}_2 \longrightarrow \{1\} \ .$$

More generally, if G is a group over D_∞, with an epimorphism $p\colon G \to D_\infty$, then:

(A) $G = G_1 *_H G_2$ is a free product with amalgamation of two groups

$$G_1 = \ker(p_1\colon G \to \mathbb{Z}_2)\ , \quad G_2 = \ker(p_2\colon G \to \mathbb{Z}_2) \subset G$$

amalgamated over their common subgroup $H = \ker(p) = G_1 \cap G_2$ of index 2 in both G_1 and G_2.

(B) G has a subgroup $\overline{G} = \ker(\pi \circ p\colon G \to \mathbb{Z}_2)$ of index 2 which is an HNN extension $\overline{G} = H \rtimes_\alpha \mathbb{Z}$ where $\alpha\colon H \to H$ is conjugation by an element $t \in \overline{G}$ with $p(t) = t_1 t_2 \in D_\infty$.

The K–theory of type (A) For any ring S and S–bimodules $\mathcal{B}_1, \mathcal{B}_2$, we write the S–bimodule $\mathcal{B}_1 \otimes_S \mathcal{B}_2$ as $\mathcal{B}_1\mathcal{B}_2$, and we suppress left-tensor products of maps with the identities $\mathrm{id}_{\mathcal{B}_1}$ or $\mathrm{id}_{\mathcal{B}_2}$. The exact category $\mathrm{NIL}(S;\mathcal{B}_1,\mathcal{B}_2)$ has objects being quadruples $(P_1, P_2, \rho_1, \rho_2)$ consisting of finitely generated (=finitely generated) projective S–modules P_1, P_2 and S–module morphisms

$$\rho_1\colon P_1 \longrightarrow \mathcal{B}_1 P_2\ , \quad \rho_2\colon P_2 \longrightarrow \mathcal{B}_2 P_1$$

such that $\rho_2\rho_1\colon P_1 \to \mathcal{B}_1\mathcal{B}_2 P_1$ is *nilpotent* in the sense that

$$(\rho_2 \circ \rho_1)^k \ = 0 : P_1 \longrightarrow (\mathcal{B}_1\mathcal{B}_2)(\mathcal{B}_1\mathcal{B}_2)\cdots(\mathcal{B}_1\mathcal{B}_2)P_1$$

for some $k \geqslant 0$. The morphisms are pairs $(f_1\colon P_1 \to P_1', f_2\colon P_2 \to P_2')$ such that $f_2 \circ \rho_1 = \rho_1' \circ f_1$ and $f_1 \circ \rho_2 = \rho_2' \circ f_2$. Recall the Waldhausen Nil–groups $\mathrm{Nil}_*(S;\mathcal{B}_1,\mathcal{B}_2) := K_*(\mathrm{NIL}(S;\mathcal{B}_1,\mathcal{B}_2))$, and the reduced Nil–groups $\widetilde{\mathrm{Nil}}_*$ satisfy

$$\mathrm{Nil}_*(S;\mathcal{B}_1,\mathcal{B}_2) = K_*(S) \oplus K_*(S) \oplus \widetilde{\mathrm{Nil}}_*(S;\mathcal{B}_1,\mathcal{B}_2)\ .$$

An object $(P_1, P_2, \rho_1, \rho_2)$ in $\mathrm{NIL}(S;\mathcal{B}_1,\mathcal{B}_2)$ is *semisplit* if the S–module isomorphism $\rho_2\colon P_2 \to \mathcal{B}_2 P_1$ is an isomorphism.

Let R be a ring which is an amalgamated free product

$$R = R_1 *_S R_2$$

with $R_k = S \oplus \mathcal{B}_k$ for S–bimodules $\mathcal{B}_k$ which are free S–modules, $k = 1, 2$. The algebraic K–groups were shown by Waldhausen [24; 25; 26] to fit into a long exact sequence

$$\cdots \to K_n(S) \oplus \widetilde{\mathrm{Nil}}_n(S;\mathcal{B}_1,\mathcal{B}_2) \to K_n(R_1) \oplus K_n(R_2) \to K_n(R) \\ \to K_{n-1}(S) \oplus \widetilde{\mathrm{Nil}}_{n-1}(S;\mathcal{B}_1,\mathcal{B}_2) \to \cdots$$

with $K_n(R) \to \widetilde{\mathrm{Nil}}_{n-1}(S;\mathcal{B}_1,\mathcal{B}_2)$ a split surjection.

For any ring R a based finitely generated free R–module chain complex C has a torsion $\tau(C) \in K_1(R)$. The torsion of a chain equivalence $f\colon C \to D$ of based finitely generated free R–module chain complexes is the torsion of the algebraic mapping cone

$$\tau(f) = \tau(\mathscr{C}(f)) \in K_1(R) .$$

By definition, the chain equivalence is *simple* if $\tau(f) = 0 \in K_1(R)$.

For $R = R_1 *_S R_2$ the algebraic analogue of codimension 1 manifold transversality shows that every based finitely generated free R–module chain complex C admits a Mayer–Vietoris presentation

$$\mathscr{C}\colon 0 \to R \otimes_S D \to (R \otimes_{R_1} C_1) \oplus (R \otimes_{R_2} C_2) \to C \to 0$$

with C_k a based finitely generated free R_k–module chain complex, D a based finitely generated free S–module chain complex with R_k–module chain maps $R_k \otimes_S D \to C_k$, and $\tau(\mathscr{C}) = 0 \in K_1(R)$. This was first proved in [24; 25]; see also Ranicki [18, Remark 8.7; 19]. A contractible C *splits* if it is simple chain equivalent to a chain complex (also denoted by C) with a Mayer–Vietoris presentation $\mathscr{C}$ with D contractible, in which case C_1, C_2 are also contractible and the torsion $\tau(C) \in K_1(R)$ is such that

$$\tau(C) = \tau(R \otimes_{R_1} C_1) + \tau(R \otimes_{R_2} C_2) - \tau(R \otimes_S D) \in \operatorname{im}(K_1(R_1) \oplus K_1(R_2) \to K_1(R)) .$$

By the algebraic obstruction theory of [24] C splits if and only if

$$\tau(C) \in \operatorname{im}(K_1(R_1) \oplus K_1(R_2) \to K_1(R)) = \ker(K_1(R) \to K_0(S) \oplus \widetilde{\operatorname{Nil}}_0(S; \mathscr{B}_1, \mathscr{B}_2)) .$$

For any ring R the group ring $R[G]$ of an amalgamated free product of groups $G = G_1 *_H G_2$ is an amalgamated free product of rings

$$R[G] = R[G_1] *_{R[H]} R[G_2] .$$

If $H \to G_1$, $H \to G_2$ are injective then the $R[H]$–bimodules $R[G_1 - H]$, $R[G_2 - H]$ are free, and Waldhausen [26] decomposed the algebraic K–theory of $R[G]$ as

$$K_*(R[G]) = K_*(R[H] \to R[G_1] \times R[G_2]) \oplus \widetilde{\operatorname{Nil}}_{*-1}(R[F]; R[G_1 - H], R[G_2 - H]) .$$

In particular, there is defined a split monomorphism

$$\sigma_A\colon \widetilde{\operatorname{Nil}}_{*-1}(R[H]; R[G_1 - H], R[G_2 - H]) \longrightarrow K_*(R[G]) ,$$

which for $* = 1$ is given by

$$\begin{aligned} \sigma_A\colon \widetilde{\operatorname{Nil}}_0(R[H]; R[G_1 - H], R[G_2 - H]) &\longrightarrow K_1(R[G]) , \\ [P_1, P_2, \rho_1, \rho_2] &\longmapsto \left[R[G] \otimes_{R[H]} (P_1 \oplus P_2), \begin{pmatrix} 1 & \rho_2 \\ \rho_1 & 1 \end{pmatrix} \right] . \end{aligned}$$

The K–theory of type (B) Given a ring R and an R–bimodule $\mathcal{B}$, consider the tensor algebra $T_R(\mathcal{B})$ of $\mathcal{B}$ over R:

$$T_R(\mathcal{B}) := R \oplus \mathcal{B} \oplus \mathcal{B}\mathcal{B} \oplus \cdots .$$

The Nil–groups $\mathrm{Nil}_*(R;\mathcal{B})$ are defined to be the algebraic K–groups $K_*(\mathrm{NIL}(R;\mathcal{B}))$ of the exact category $\mathrm{NIL}(R;\mathcal{B})$ with objects pairs (P,ρ) with P a finitely generated projective R–module and $\rho\colon P \to \mathcal{B}P$ an R–module morphism, *nilpotent* in the sense that for some $k \geqslant 0$, we have

$$\rho^k = 0\colon P \longrightarrow \mathcal{B}P \longrightarrow \cdots \longrightarrow \mathcal{B}^k P .$$

The reduced Nil–groups $\widetilde{\mathrm{Nil}}_*$ are such that

$$\mathrm{Nil}_*(R;\mathcal{B}) = K_*(R) \oplus \widetilde{\mathrm{Nil}}_*(R;\mathcal{B}) .$$

Waldhausen [26] proved that if $\mathcal{B}$ is finitely generated projective as a left R–module and free as a right R–module, then

$$K_*(T_R(\mathcal{B})) = K_*(R) \oplus \widetilde{\mathrm{Nil}}_{*-1}(R;\mathcal{B}) .$$

There is a split monomorphism

$$\sigma_B\colon \widetilde{\mathrm{Nil}}_{*-1}(R;\mathcal{B}) \longrightarrow K_*(T_R(\mathcal{B}))$$

which for $* = 1$ is given by

$$\sigma_B\colon \widetilde{\mathrm{Nil}}_0(R;\mathcal{B}) \longrightarrow K_1(T_R(\mathcal{B})) , \quad [P,\rho] \longmapsto [T_R(\mathcal{B})P, 1-\rho] .$$

In particular, for $\mathcal{B} = R$, we have

$$\begin{aligned} \mathrm{Nil}_*(R;R) &= \mathrm{Nil}_*(R) , & \widetilde{\mathrm{Nil}}_*(R;R) &= \widetilde{\mathrm{Nil}}_*(R) , \\ T_R(\mathcal{B}) &= R[t] , & K_*(R[t]) &= K_*(R) \oplus \widetilde{\mathrm{Nil}}_{*-1}(R) . \end{aligned}$$

Relating the K–theory of types (A) and (B) Recall that a small category I is *filtered* if:

- For any pair of objects α, α' in I, there exist an object β and morphisms $\alpha \to \beta$ and $\alpha' \to \beta$ in I.
- For any pair of morphisms $u, v\colon \alpha \to \alpha'$ in I, there exists an object β and morphism $w\colon \alpha' \to \beta$ such that $w \circ u = w \circ v$.

Note that any directed poset I is a filtered category. A *filtered colimit* is a colimit over a filtered category.

Theorem 0.4 (The Nil-Nil Theorem) *Let R be a ring. Let $\mathcal{B}_1, \mathcal{B}_2$ be R–bimodules. Suppose that $\mathcal{B}_2 = \operatorname{colim}_{\alpha \in I} \mathcal{B}_2^\alpha$ is a filtered colimit limit of R–bimodules such that each $\mathcal{B}_2^\alpha$ is a finitely generated projective left R–module. Then, for all $n \in \mathbb{Z}$, the* Nil*–groups of the triple $(R; \mathcal{B}_1, \mathcal{B}_2)$ are related to the* Nil*–groups of the pair $(R; \mathcal{B}_1\mathcal{B}_2)$ by isomorphisms*

$$\begin{aligned} \mathrm{Nil}_n(R; \mathcal{B}_1, \mathcal{B}_2) &\cong \mathrm{Nil}_n(R; \mathcal{B}_1\mathcal{B}_2) \oplus K_n(R) \\ \widetilde{\mathrm{Nil}}_n(R; \mathcal{B}_1, \mathcal{B}_2) &\cong \widetilde{\mathrm{Nil}}_n(R; \mathcal{B}_1\mathcal{B}_2) \ . \end{aligned}$$

In particular, for $n = 0$ and $\mathcal{B}_2$ a finitely generated projective left R–module, there are defined inverse isomorphisms

$$\begin{aligned} i_*\colon \mathrm{Nil}_0(R; \mathcal{B}_1\mathcal{B}_2) \oplus K_0(R) &\xrightarrow{\cong} \mathrm{Nil}_0(R; \mathcal{B}_1, \mathcal{B}_2) \ , \\ ([P_1, \rho_{12}\colon P_1 \to \mathcal{B}_1\mathcal{B}_2 P_1], [P_2]) &\longmapsto \left[P_1, \mathcal{B}_2 P_1 \oplus P_2, \begin{pmatrix} \rho_{12} \\ 0 \end{pmatrix}, (1\ 0) \right] \\ j_*\colon \mathrm{Nil}_0(R; \mathcal{B}_1, \mathcal{B}_2) &\xrightarrow{\cong} \mathrm{Nil}_0(R; \mathcal{B}_1\mathcal{B}_2) \oplus K_0(R) \ , \\ [P_1, P_2, \rho_1\colon P_1 \to \mathcal{B}_1 P_2, \rho_2\colon P_2 \to \mathcal{B}_2 P_1] &\longmapsto ([P_1, \rho_2 \circ \rho_1], [P_2] - [\mathcal{B}_2 P_1]) \ . \end{aligned}$$

The reduced versions are the inverse isomorphisms

$$\begin{aligned} i_*\colon \widetilde{\mathrm{Nil}}_0(R; \mathcal{B}_1\mathcal{B}_2) &\xrightarrow{\cong} \widetilde{\mathrm{Nil}}_0(R; \mathcal{B}_1, \mathcal{B}_2) \ , & [P_1, \rho_{12}] &\longmapsto [P_1, \mathcal{B}_2 P_1, \rho_{12}, 1] \\ j_*\colon \widetilde{\mathrm{Nil}}_0(R; \mathcal{B}_1, \mathcal{B}_2) &\xrightarrow{\cong} \widetilde{\mathrm{Nil}}_0(R; \mathcal{B}_1\mathcal{B}_2) \ , & [P_1, P_2, \rho_1, \rho_2] &\longmapsto [P_1, \rho_2 \circ \rho_1] \end{aligned}$$

with $i_*(P_1, \rho_{12}) = (P_1, \mathcal{B}_2 P_1, \rho_{12}, 1)$ semisplit.

Proof This follows immediately from Theorem 1.1 and Theorem 2.7. □

Remark 0.5 Theorem 0.4 was already known to Pierre Vogel in 1990 [22].

1 Higher Nil–groups

In this section, we shall prove Theorem 0.4 for nonnegative degrees.

Quillen [17] defined the K–theory space $K\mathcal{E} := \Omega BQ(\mathcal{E})$ of an exact category $\mathcal{E}$. The space $BQ(\mathcal{E})$ is the geometric realization of the simplicial set $N_\bullet Q(\mathcal{E})$, which is the nerve of a certain category $Q(\mathcal{E})$ associated to $\mathcal{E}$. The algebraic K–groups of $\mathcal{E}$ are defined for $* \in \mathbb{Z}$ as

$$K_*(\mathcal{E}) := \pi_*(K\mathcal{E})$$

using a nonconnective delooping for $* \leqslant -1$. In particular, the algebraic K–groups of a ring R are the algebraic K–groups

$$K_*(R) := K_*(\mathrm{PROJ}(R))$$

of the exact category $\mathrm{PROJ}(R)$ of finitely generated projective R–modules. The NIL–categories defined in the Introduction all have the structure of exact categories.

Theorem 1.1 *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bimodules over a ring R. Let j be the exact functor*

$$j\colon \mathrm{NIL}(R;\mathcal{B}_1,\mathcal{B}_2) \longrightarrow \mathrm{NIL}(R;\mathcal{B}_1\mathcal{B}_2)\ , \quad (P_1,P_2,\rho_1,\rho_2) \longmapsto (P_1,\rho_2\circ\rho_1)\ .$$

(1) *If $\mathcal{B}_2$ is finitely generated projective as a left R–module, then there is an exact functor*

$$i\colon \mathrm{NIL}(R;\mathcal{B}_1\mathcal{B}_2) \longrightarrow \mathrm{NIL}(R;\mathcal{B}_1,\mathcal{B}_2)\ , \quad (P,\rho) \longmapsto (P,\mathcal{B}_2P,\rho,1)$$

such that $i(P,\rho) = (P,\mathcal{B}_2P,\rho,1)$ is semisplit, $j\circ i = 1$, and i_* and j_* induce inverse isomorphisms on the reduced Nil-groups

$$\widetilde{\mathrm{Nil}}_*(R;\mathcal{B}_1\mathcal{B}_2) \cong \widetilde{\mathrm{Nil}}_*(R;\mathcal{B}_1,\mathcal{B}_2)\ .$$

(2) *If $\mathcal{B}_2 = \mathrm{colim}_{\alpha\in I}\,\mathcal{B}_2^\alpha$ is a filtered colimit of bimodules each of which is finitely generated projective as a left R–module, then there is a unique exact functor i so that the following diagram commutes for all $\alpha \in I$:*

$$\begin{array}{ccc} \mathrm{NIL}(R;\mathcal{B}_1\mathcal{B}_2) & \xrightarrow{\ i\ } & \mathrm{NIL}(R;\mathcal{B}_1,\mathcal{B}_2) \\ \uparrow & & \uparrow \\ \mathrm{NIL}(R;\mathcal{B}_1\mathcal{B}_2^\alpha) & \xrightarrow{\ i^\alpha\ } & \mathrm{NIL}(R;\mathcal{B}_1,\mathcal{B}_2^\alpha)\ . \end{array}$$

Then $j\circ i = 1$ and i_* and j_* induce inverse isomorphisms on the reduced Nil-groups

$$\widetilde{\mathrm{Nil}}_*(R;\mathcal{B}_1\mathcal{B}_2) \cong \widetilde{\mathrm{Nil}}_*(R;\mathcal{B}_1,\mathcal{B}_2)\ .$$

Proof (1) Note that there are split injections of exact categories

$$\begin{aligned} &\mathrm{PROJ}(R)\times\mathrm{PROJ}(R) \to \mathrm{NIL}(R;\mathcal{B}_1,\mathcal{B}_2)\ , && (P_1,P_2) \longmapsto (P_1,P_2,0,0) \\ &\mathrm{PROJ}(R) \to \mathrm{NIL}(R;\mathcal{B}_1\mathcal{B}_2)\ , && (P) \longmapsto (P,0) \end{aligned}$$

which underlie the definition of the reduced Nil groups. Since both i and j take the image of the split injection to the image of the other split injection, they induce maps i_* and j_* on the reduced Nil groups. Since $j\circ i = 1$, it follows that $j_*\circ i_* = 1$.

In preparation for the proof that $i_* \circ j_* = 1$, consider the following objects of $\mathrm{NIL}(R;\mathcal{B}_1,\mathcal{B}_2)$:

$$\begin{aligned}
x &:= (P_1, P_2, \rho_1, \rho_2)\\
x' &:= \left(P_1, \mathcal{B}_2 P_1 \oplus P_2, \begin{pmatrix} 0 \\ \rho_1 \end{pmatrix}, \begin{pmatrix} 1 & \rho_2 \end{pmatrix}\right)\\
x'' &:= (P_1, \mathcal{B}_2 P_1, \rho_2 \circ \rho_1, 1)\\
a &:= (0, P_2, 0, 0)\\
a' &:= (0, \mathcal{B}_2 P_1, 0, 0)
\end{aligned}$$

with x'' semisplit. Note that $(i \circ j)(x) = x''$. Define morphisms

$$\begin{aligned}
f &:= \left(1, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \colon x \longrightarrow x'\\
f' &:= \left(1, \begin{pmatrix} 1 & \rho_2 \end{pmatrix}\right) \colon x' \longrightarrow x''\\
g &:= \left(0, \begin{pmatrix} -\rho_2 \\ 1 \end{pmatrix}\right) \colon a \longrightarrow x'\\
g' &:= \left(0, \begin{pmatrix} 1 & 0 \end{pmatrix}\right) \colon x' \longrightarrow a'\\
h &:= (0, \rho_2) \colon a \longrightarrow a' .
\end{aligned}$$

There are exact sequences

$$0 \longrightarrow x \oplus a \xrightarrow{\begin{pmatrix} f & g \\ 0 & 1 \end{pmatrix}} x' \oplus a \xrightarrow{\begin{pmatrix} g' & h \end{pmatrix}} a' \longrightarrow 0$$
$$0 \longrightarrow a \xrightarrow{g} x' \xrightarrow{f'} x'' \longrightarrow 0 .$$

Define exact functors $F', F'', G, G' \colon \mathrm{NIL}(R;\mathcal{B}_1,\mathcal{B}_2) \to \mathrm{NIL}(R;\mathcal{B}_1,\mathcal{B}_2)$ by

$$F'(x) = x' , \quad F''(x) = x'' , \quad G(x) = a , \quad G'(x) = a' .$$

Thus we have two exact sequences of exact functors

$$0 \longrightarrow 1 \oplus G \longrightarrow F' \oplus G \longrightarrow G' \longrightarrow 0$$
$$0 \longrightarrow G \longrightarrow F' \longrightarrow F'' \longrightarrow 0 .$$

Recall $j \circ i = 1$, and note $i \circ j = F''$. By Quillen's Additivity Theorem [17, page 98, Corollary 1], we obtain homotopies $KF' \simeq 1 + KG'$ and $KF' \simeq KG + KF''$. Then

$$Ki \circ Kj = KF'' \simeq 1 + (KG' - KG) ,$$

where the subtraction uses the loop space structure. Observe that both G and G' send $\mathrm{NIL}(R;\mathcal{B}_1,\mathcal{B}_2)$ to the image of $\mathrm{PROJ}(R)\times\mathrm{PROJ}(R)$. Thus $i_*\circ j_*=1$ as desired.

(2) It is straightforward to show that tensor product commutes with colimits over a category. Moreover, for any object $x=(P_1,P_2,\rho_1\colon P_1\to\mathcal{B}_1P_2,\rho_2\colon P_2\to\mathcal{B}_2P_1)$, since P_2 is finitely generated, there exists $\alpha\in I$ such that ρ_2 factors through a map $P_2\to\mathcal{B}_2^\alpha P_1$, and similarly for short exact sequences of nil-objects. We thus obtain induced isomorphisms of exact categories:

$$\operatorname*{colim}_{\alpha\in I}\mathrm{NIL}(R;\mathcal{B}_1\mathcal{B}_2^\alpha)\longrightarrow\mathrm{NIL}(R;\mathcal{B}_1\mathcal{B}_2)$$
$$\operatorname*{colim}_{\alpha\in I}\mathrm{NIL}(R;\mathcal{B}_1,\mathcal{B}_2^\alpha)\longrightarrow\mathrm{NIL}(R;\mathcal{B}_1,\mathcal{B}_2)\ .$$

This justifies the existence and uniqueness of the functor i.

By Quillen's colimit observation [17, Section 2, Equation (9), page 20], we obtain induced weak homotopy equivalences of K–theory spaces:

$$\operatorname*{colim}_{\alpha\in I}K\,\mathrm{NIL}(R;\mathcal{B}_1\mathcal{B}_2^\alpha)\longrightarrow K\,\mathrm{NIL}(R;\mathcal{B}_1\mathcal{B}_2)$$
$$\operatorname*{colim}_{\alpha\in I}K\,\mathrm{NIL}(R;\mathcal{B}_1,\mathcal{B}_2^\alpha)\longrightarrow K\,\mathrm{NIL}(R;\mathcal{B}_1,\mathcal{B}_2)\ .$$

The remaining assertions of part (2) then follow from part (1). □

Remark 1.2 The proof of Theorem 1.1 is best understood in terms of finite chain complexes $x=(P_1,P_2,\rho_1,\rho_2)$ in the category $\mathrm{NIL}(R;\mathcal{B}_1,\mathcal{B}_2)$, assuming that $\mathcal{B}_2$ is a finitely generated projective left R–module. Any such x represents a class

$$[x]=\sum_{r=0}^{\infty}(-1)^r[(P_1)_r,(P_2)_r,\rho_1,\rho_2]\in\mathrm{Nil}_0(R;\mathcal{B}_1,\mathcal{B}_2)\ .$$

The key observation is that x determines a finite chain complex $x'=(P_1',P_2',\rho_1',\rho_2')$ in $\mathrm{NIL}(R;\mathcal{B}_1,\mathcal{B}_2)$ which is semisplit in the sense that $\rho_2'\colon P_2'\to\mathcal{B}_2P_1'$ is a chain equivalence, and such that

$$[x]=[x']\in\widetilde{\mathrm{Nil}}_0(R;\mathcal{B}_1,\mathcal{B}_2)\ . \tag{3}$$

Specifically, let $P_1'=P_1$, $P_2'=\mathcal{M}(\rho_2)$, the algebraic mapping cylinder of the chain map $\rho_2\colon P_2\to\mathcal{B}_2P_1$, and let

$$\rho_1'=\begin{pmatrix}0\\0\\\rho_1\end{pmatrix}:\ P_1'=P_1\longrightarrow\mathcal{B}_1P_2'=\mathcal{M}(1_{\mathcal{B}_1}\otimes\rho_2)$$
$$\rho_2'=\begin{pmatrix}1&0&\rho_2\end{pmatrix}\colon\ P_2'=\mathcal{M}(\rho_2)\longrightarrow\mathcal{B}_2P_1\ ,$$

so that $P_2'/P_2 = \mathcal{C}(\rho_2)$ is the algebraic mapping cone of ρ_2. Moreover, the proof of (3) is sufficiently functorial to establish not only that the following maps of the reduced nilpotent class groups are inverse isomorphisms:

$$\begin{aligned} i\colon \widetilde{\mathrm{Nil}}_0(R;\mathcal{B}_1\mathcal{B}_2) &\longrightarrow \widetilde{\mathrm{Nil}}_0(R;\mathcal{B}_1,\mathcal{B}_2)\ , \quad (P,\rho)\longmapsto (P,\mathcal{B}_2P,\rho,1) \\ j\colon \widetilde{\mathrm{Nil}}_0(R;\mathcal{B}_1,\mathcal{B}_2) &\longrightarrow \widetilde{\mathrm{Nil}}_0(R;\mathcal{B}_1\mathcal{B}_2)\ , \qquad\quad [x]\longmapsto [x']\ , \end{aligned}$$

but also that there exist isomorphisms of $\widetilde{\mathrm{Nil}}_n$ for all higher dimensions $n>0$, as shown above. In order to prove Equation (3), note that x fits into the sequence

$$0 \longrightarrow x \xrightarrow{(1,u)} x' \xrightarrow{(0,v)} y \longrightarrow 0 \tag{4}$$

with
$$\begin{aligned} y &= (0,\mathcal{C}(\rho_2),0,0) \\ u &= \begin{pmatrix} 0\\0\\1\end{pmatrix}\colon P_2 \to P_2' = \mathcal{M}(\rho_2) \\ v &= \begin{pmatrix} 1&0&0\\0&1&0\end{pmatrix}\colon P_2' = \mathcal{M}(\rho_2)\to \mathcal{C}(\rho_2) \end{aligned}$$

and
$$[y] = \sum_{r=0}^{\infty}(-)^r[0,(\mathcal{B}_2P_1)_{r-1}\oplus(P_2)_r,0,0] = 0 \in \widetilde{\mathrm{Nil}}_0(R;\mathcal{B}_1,\mathcal{B}_2)\ .$$

The projection $\mathcal{M}(\rho_2)\to\mathcal{B}_2P_1$ defines a chain equivalence

$$x' \simeq (P_1,\mathcal{B}_2P_1,\rho_2\circ\rho_1,1) = ij(x)$$

so that $[x]=[x']-[y]=[P_1,\mathcal{B}_2P_1,\rho_2\circ\rho_1,1] = ij[x]\in\widetilde{\mathrm{Nil}}_0(R;\mathcal{B}_1,\mathcal{B}_2)$.

Now suppose that x is a 0–dimensional chain complex in $\mathrm{NIL}(R;\mathcal{B}_1,\mathcal{B}_2)$, that is, an object as in the proof of Theorem 1.1. Let x',x'',a,a',f,f',g,g',h be as defined there. The exact sequence of (4) can be written as the short exact sequence of chain complexes

$$\begin{array}{ccccccccc} & & & & a & = & a & & \\ & & & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle -h} & & \\ 0 & \longrightarrow & x & \xrightarrow{f} & x' & \xrightarrow{g'} & a' & \longrightarrow & 0\ . \end{array}$$

The first exact sequence of the proof of Theorem 1.1 is now immediate:

$$0 \longrightarrow x\oplus a \xrightarrow{\begin{pmatrix} f & g\\ 0 & 1\end{pmatrix}} x'\oplus a \xrightarrow{(g'\ h)} a' \longrightarrow 0\ .$$

The second exact sequence is self-evident:

$$0 \longrightarrow a \xrightarrow{g} x' \xrightarrow{f'} x'' \longrightarrow 0\ .$$

2 Lower Nil–groups

2.1 Cone and suspension rings

Let us recall some additional structures on the tensor product of modules.

Originating from ideas of Karoubi and Villamayor [11], the following concept was studied independently by S M Gersten [8] and J B Wagoner [23] in the construction of the nonconnective K–theory spectrum of a ring.

Definition 2.1 (Gersten, Wagoner) The *cone ring* $\Lambda\mathbb{Z}$ is the subring of $(\omega\times\omega)$–matrices over $\mathbb{Z}$ such that each row and column have only a finite number of nonzero entries. The *suspension ring* $\Sigma\mathbb{Z}$ is the quotient ring of $\Lambda\mathbb{Z}$ by the two-sided ideal of matrices with only a finite number of nonzero entries. For each $n \in \mathbb{N}$, define the rings

$$\Sigma^n\mathbb{Z} := \underbrace{\Sigma\mathbb{Z} \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \Sigma\mathbb{Z}}_{n \text{ copies}} \quad \text{with } \Sigma^0\mathbb{Z} = \mathbb{Z} .$$

For a ring R and for $n \in \mathbb{N}$, define the ring $\Sigma^n R := \Sigma^n\mathbb{Z} \otimes_{\mathbb{Z}} R$.

Roughly speaking, the suspension should be regarded as the ring of "bounded modulo compact operators." Gersten and Wagoner showed that $K_i(\Sigma^n R)$ is naturally isomorphic to $K_{i-n}(R)$ for all $i, n \in \mathbb{Z}$, in the sense of Quillen when the subscript is positive, in the sense of Grothendieck when the subscript is zero, and in the sense of Bass when the subscript is negative.

For an R–bimodule $\mathcal{B}$, define the $\Sigma^n R$–bimodule $\Sigma^n\mathcal{B} := \Sigma^n\mathbb{Z} \otimes_{\mathbb{Z}} \mathcal{B}$.

Lemma 2.2 *Let R be a ring. Let $\mathcal{B}_1, \mathcal{B}_2$ be R–bimodules. Then, for each $n \in \mathbb{N}$, there is a natural isomorphism of $\Sigma^n R$–bimodules:*

$$t_n\colon \Sigma^n(\mathcal{B}_1\mathcal{B}_2) \longrightarrow \Sigma^n\mathcal{B}_1 \otimes_{\Sigma^n R} \Sigma^n\mathcal{B}_2 , \quad s \otimes (b_1 \otimes b_2) \longmapsto (s \otimes b_1) \otimes (1_{\Sigma^n\mathbb{Z}} \otimes b_2) .$$

Proof By transposition of the middle two factors, note that

$$\Sigma^n\mathcal{B}_1 \otimes_{\Sigma^n R} \Sigma^n\mathcal{B}_2 = (\Sigma^n\mathbb{Z} \otimes_{\mathbb{Z}} \mathcal{B}_1) \otimes_{(\Sigma^n\mathbb{Z}\otimes_{\mathbb{Z}} R)} (\Sigma^n\mathbb{Z} \otimes_{\mathbb{Z}} \mathcal{B}_2)$$

is isomorphic to

$$(\Sigma^n\mathbb{Z} \otimes_{\Sigma^n\mathbb{Z}} \Sigma^n\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathcal{B}_1\mathcal{B}_2) = \Sigma^n\mathbb{Z} \otimes_{\mathbb{Z}} (\mathcal{B}_1\mathcal{B}_2) = \Sigma^n(\mathcal{B}_1\mathcal{B}_2) . \qquad \square$$

2.2 Definition of lower Nil–groups

Definition 2.3 Let R be a ring. Let $\mathcal{B}$ be an R–bimodule. For all $n \in \mathbb{N}$, define

$$\begin{aligned}\mathrm{Nil}_{-n}(R;\mathcal{B}) &:= \mathrm{Nil}_0(\Sigma^n R;\Sigma^n\mathcal{B})\\ \widetilde{\mathrm{Nil}}_{-n}(R;\mathcal{B}) &:= \widetilde{\mathrm{Nil}}_0(\Sigma^n R;\Sigma^n\mathcal{B})\ .\end{aligned}$$

Definition 2.4 Let R be a ring. Let $\mathcal{B}_1,\mathcal{B}_2$ be R–bimodules. For all $n \in \mathbb{N}$, define

$$\begin{aligned}\mathrm{Nil}_{-n}(R;\mathcal{B}_1,\mathcal{B}_2) &:= \mathrm{Nil}_0(\Sigma^n R;\Sigma^n\mathcal{B}_1,\Sigma^n\mathcal{B}_2)\\ \widetilde{\mathrm{Nil}}_{-n}(R;\mathcal{B}_1,\mathcal{B}_2) &:= \widetilde{\mathrm{Nil}}_0(\Sigma^n R;\Sigma^n\mathcal{B}_1,\Sigma^n\mathcal{B}_2)\ .\end{aligned}$$

The next two theorems follow from the definitions and [26, Theorems 1,3].

Theorem 2.5 (Waldhausen) *Let R be a ring and $\mathcal{B}$ be an R–bimodule. Consider the tensor ring*

$$T_R(\mathcal{B}) := R \oplus \mathcal{B} \oplus \mathcal{B}^2 \oplus \mathcal{B}^3 \oplus \cdots\ .$$

Suppose $\mathcal{B}$ is finitely generated projective as a left R–module and free as a right R–module. Then, for all $n \in \mathbb{N}$, there is a split monomorphism

$$\sigma_B\colon \widetilde{\mathrm{Nil}}_{-n}(R;\mathcal{B}) \longrightarrow K_{1-n}(T_R(\mathcal{B}))$$

given for $n = 0$ by the map

$$\sigma_B\colon \mathrm{Nil}_0(R;\mathcal{B}) \longrightarrow K_1(T_R(\mathcal{B}))\ ,\quad [P,\rho] \longmapsto [\, T_R(\mathcal{B})P, 1-\widehat{\rho}\,]\ ,$$

where $\widehat{\rho}$ is defined using ρ and multiplication in $T_R(\mathcal{B})$.

Furthermore, there is a natural decomposition

$$K_{1-n}(T_R(\mathcal{B})) = K_{1-n}(R) \oplus \widetilde{\mathrm{Nil}}_{-n}(R;\mathcal{B})\ .$$

For example, the last assertion of the theorem follows from the equations

$$\begin{aligned}K_{1-n}(T_R(\mathcal{B})) &= K_1(\Sigma^n T_R(\mathcal{B}))\\ &= K_1(T_{\Sigma^n R}(\Sigma^n\mathcal{B}))\\ &= K_1(\Sigma^n R) \oplus \widetilde{\mathrm{Nil}}_0(\Sigma^n R;\Sigma^n\mathcal{B})\\ &= K_{1-n}(R) \oplus \widetilde{\mathrm{Nil}}_{-n}(R;\mathcal{B})\ .\end{aligned}$$

Theorem 2.6 (Waldhausen) *Let R, A_1, A_2 be rings. Let $R \to A_i$ be ring monomorphisms such that $A_i = R \oplus \mathcal{B}_i$ for R–bimodules $\mathcal{B}_i$. Consider the pushout of rings*

$$\begin{aligned} A &= A_1 *_R A_2 \\ &= R \oplus (\mathcal{B}_1 \oplus \mathcal{B}_2) \oplus (\mathcal{B}_1\mathcal{B}_2 \oplus \mathcal{B}_2\mathcal{B}_1) \oplus (\mathcal{B}_1\mathcal{B}_2\mathcal{B}_1 \oplus \mathcal{B}_2\mathcal{B}_1\mathcal{B}_2) \oplus \cdots . \end{aligned}$$

Suppose each $\mathcal{B}_i$ is free as a right R–module. Then, for all $n \in \mathbb{N}$, there is a split monomorphism

$$\sigma_A \colon \widetilde{\mathrm{Nil}}_{-n}(R; \mathcal{B}_1, \mathcal{B}_2) \longrightarrow K_{1-n}(A) ,$$

given for $n = 0$ by the map

$$\begin{aligned} \mathrm{Nil}_0(R; \mathcal{B}_1, \mathcal{B}_2) &\longrightarrow K_1(A) , \\ [P_1, P_2, \rho_1, \rho_2] &\longmapsto \left[(AP_1) \oplus (AP_2), \begin{pmatrix} 1 & \widehat{\rho}_2 \\ \widehat{\rho}_1 & 1 \end{pmatrix} \right] , \end{aligned}$$

where $\widehat{\rho}_i$ is defined using ρ_i and multiplication in A_i for $i = 1, 2$.

Furthermore, there is a natural Mayer–Vietoris type exact sequence:

$$\begin{array}{ccccc} \cdots & \xrightarrow{\partial} & K_{1-n}(R) \longrightarrow K_{1-n}(A_1) \oplus K_{1-n}(A_2) \longrightarrow \\ \dfrac{K_{1-n}(A)}{\widetilde{\mathrm{Nil}}_{-n}(R; \mathcal{B}_1, \mathcal{B}_2)} & \xrightarrow{\partial} & K_{-n}(R) \longrightarrow & \cdots \end{array}$$

□

2.3 The isomorphism for lower Nil–groups

Theorem 2.7 *Let R be a ring. Let $\mathcal{B}_1, \mathcal{B}_2$ be R–bimodules. Suppose that $\mathcal{B}_2 = \mathrm{colim}_{\alpha \in I} \mathcal{B}_2^\alpha$ is a filtered colimit of R–bimodules $\mathcal{B}_2^\alpha$, each of which is a finitely generated projective left R–module. Then, for all $n \in \mathbb{N}$, there is an induced isomorphism*

$$\mathrm{Nil}_{-n}(R; \mathcal{B}_1\mathcal{B}_2) \oplus K_{-n}(R) \longrightarrow \mathrm{Nil}_{-n}(R; \mathcal{B}_1, \mathcal{B}_2) .$$

Proof Let $n \in \mathbb{N}$. By Lemma 2.2 and Theorem 1.1, there are induced isomorphisms

$$\begin{aligned} \mathrm{Nil}_{-n}(R; \mathcal{B}_1\mathcal{B}_2) \oplus K_{-n}(R) &= \mathrm{Nil}_0(\Sigma^n R; \Sigma^n(\mathcal{B}_1\mathcal{B}_2)) \oplus K_0\Sigma^n(R) \\ &\longrightarrow \mathrm{Nil}_0(\Sigma^n R; \Sigma^n\mathcal{B}_1 \otimes_{\Sigma^n R} \Sigma^n\mathcal{B}_2) \oplus K_0\Sigma^n(R) \\ &\longrightarrow \mathrm{Nil}_0(\Sigma^n R; \Sigma^n\mathcal{B}_1, \Sigma^n\mathcal{B}_2) = \mathrm{Nil}_{-n}(R; \mathcal{B}_1, \mathcal{B}_2) . \end{aligned}$$

□

3 Applications

We indicate some applications of our main theorem 0.4. In Section 3.1 we prove Theorem 0.1(ii), which describes the restrictions of the maps

$$\theta_!\colon K_*(R[\bar{G}]) \to K_*(R[G]) \ , \quad \theta^!\colon K_*(R[G]) \to K_*(R[\bar{G}])$$

to the $\widetilde{\mathrm{Nil}}$–terms, with $\theta\colon \bar{G} \to G$ the inclusion of the canonical index 2 subgroup $\bar{G}$ for any group G over D_∞. In Section 3.2 we give the first known example of a nonzero Nil–group occurring in the K–theory of an integral group ring of an amalgamated free product. In Section 3.3 we sharpen the Farrell–Jones Conjecture in K–theory, replacing the family of virtually cyclic groups by the smaller family of finite-by-cyclic groups. In Section 3.4 we compute the $K_*(R[\Gamma])$ for the modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$.

3.1 Algebraic K–theory over D_∞

The overall goal here is to show that the abstract isomorphisms i_* and j_* coincide with the restrictions of the induction and transfer maps $\theta_!$ and $\theta^!$ in the group ring setting.

3.1.1 Twisting We start by recalling the algebraic K–theory of twisted polynomial rings.

Statement 3.1 Consider any (unital, associative) ring R and any ring automorphism $\alpha\colon R \to R$. Let t be an indeterminate over R such that

$$rt = t\alpha(r) \quad (r \in R) \ .$$

For any R–module P, let $tP := \{tx \mid x \in P\}$ be the set with left R–module structure

$$tx + ty = t(x+y) \ , \quad r(tx) = t(\alpha(r)x) \in tP \ .$$

Further endow the left R–module tR with the R–bimodule structure

$$R \times tR \times R \longrightarrow tR \ , \quad (q, tr, s) \longmapsto t\alpha(q)rs \ .$$

The Nil–category of R with respect to α is the exact category defined by

$$\mathrm{NIL}(R,\alpha) := \mathrm{NIL}(R; tR) \ .$$

The objects (P,ρ) consist of any finitely generated projective R–module P and any nilpotent morphism $\rho\colon P \to tP = tRP$. The Nil–groups are written

$$\mathrm{Nil}_*(R,\alpha) := \mathrm{Nil}_*(R; tR) \ , \quad \widetilde{\mathrm{Nil}}_*(R,\alpha) := \widetilde{\mathrm{Nil}}_*(R; tR) \ ,$$

so that
$$\mathrm{Nil}_*(R,\alpha) = K_*(R) \oplus \widetilde{\mathrm{Nil}}_*(R,\alpha) \ .$$

Statement 3.2 The tensor algebra on tR is the α–twisted polynomial extension of R

$$T_R(tR) = R_\alpha[t] = \sum_{k=0}^{\infty} t^k R\ .$$

Given an R–module P there is induced an $R_\alpha[t]$–module

$$R_\alpha[t] \otimes_R P = P_\alpha[t]$$

whose elements are finite linear combinations $\sum_{j=0}^{\infty} t^j x_j$ $(x_j \in P)$. Given R–modules P, Q and an R–module morphism $\rho\colon P \to tQ$, define its extension as the $R_\alpha[t]$–module morphism

$$\widehat{\rho} = t\rho\colon P_\alpha[t] \longrightarrow Q_\alpha[t]\ ,\quad \sum_{j=0}^{\infty} t^j x_j \longmapsto \sum_{j=0}^{\infty} t^j \rho(x_j)\ .$$

Statement 3.3 Bass [2], Farrell and Hsiang [6] and Quillen [9] give decompositions

$$\begin{aligned} K_n(R_\alpha[t]) &= K_n(R) \oplus \widetilde{\mathrm{Nil}}_{n-1}(R,\alpha) \\ K_n(R_{\alpha^{-1}}[t^{-1}]) &= K_n(R) \oplus \widetilde{\mathrm{Nil}}_{n-1}(R,\alpha^{-1}) \\ K_n(R_\alpha[t,t^{-1}]) &= K_n(1-\alpha\colon R \to R) \oplus \widetilde{\mathrm{Nil}}_{n-1}(R,\alpha) \oplus \widetilde{\mathrm{Nil}}_{n-1}(R,\alpha^{-1})\ . \end{aligned}$$

In particular for $n = 1$, by Theorem 2.5, there are defined split monomorphisms

$$\begin{aligned} &\sigma_B^+\colon \widetilde{\mathrm{Nil}}_0(R,\alpha) \longrightarrow K_1(R_\alpha[t])\ , \qquad [P,\rho] \longmapsto [P_\alpha[t], 1-t\rho] \\ &\sigma_B^-\colon \widetilde{\mathrm{Nil}}_0(R,\alpha^{-1}) \longrightarrow K_1(R_{\alpha^{-1}}[t^{-1}])\ , \quad [P,\rho] \longmapsto \big[P_{\alpha^{-1}}[t^{-1}], 1-t^{-1}\rho\big] \\ &\sigma_B = \big(\psi^+\sigma_B^+ \ \ \psi^-\sigma_B^-\big)\colon \widetilde{\mathrm{Nil}}_0(R,\alpha) \oplus \widetilde{\mathrm{Nil}}_0(R,\alpha^{-1}) \longrightarrow K_1(R_\alpha[t,t^{-1}])\ , \\ &\qquad ([P_1,\rho_1],[P_2,\rho_2]) \longmapsto \left[(P_1 \oplus P_2)_\alpha[t,t^{-1}], \begin{pmatrix} 1-t\rho_1 & 0 \\ 0 & 1-t^{-1}\rho_2 \end{pmatrix}\right]. \end{aligned}$$

These extend to all integers $n \leqslant 1$ by the suspension isomorphisms of Section 2.

3.1.2 Scaling

Next, consider the effect an inner automorphism on α.

Statement 3.4 Suppose $\alpha, \alpha'\colon R \to R$ are automorphisms satisfying

$$\alpha'(r) = u\alpha(r)u^{-1} \in R \quad (r \in R)$$

for some unit $u \in R$, and that t' is an indeterminate over R satisfying

$$rt' = t'\alpha'(r) \quad (r \in R)\ .$$

Denote the canonical inclusions

$$\psi^+\colon R_\alpha[t] \longrightarrow R_\alpha[t,t^{-1}]\,, \qquad \psi^-\colon R_{\alpha^{-1}}[t^{-1}] \longrightarrow R_\alpha[t,t^{-1}]\,,$$
$$\psi'^+\colon R_{\alpha'}[t'] \longrightarrow R_{\alpha'}[t',t'^{-1}]\,, \quad \psi'^-\colon R_{\alpha'^{-1}}[t'^{-1}] \longrightarrow R_{\alpha'}[t',t'^{-1}]\,.$$

Statement 3.5 The various polynomial rings are related by *scaling isomorphisms*

$$\begin{aligned} \beta_u^+\colon R_\alpha[t] &\longrightarrow R_{\alpha'}[t']\,, & t &\longmapsto t'u \\ \beta_u^-\colon R_{\alpha^{-1}}[t^{-1}] &\longrightarrow R_{\alpha'^{-1}}[t'^{-1}]\,, & t^{-1} &\longmapsto u^{-1}t'^{-1} \\ \beta_u\colon R_\alpha[t,t^{-1}] &\longrightarrow R_{\alpha'}[t',t'^{-1}]\,, & t &\longmapsto t'u \end{aligned}$$

satisfying the equations

$$\beta_u \circ \psi^+ = \psi'^+ \circ \beta_u^+\colon R_\alpha[t] \longrightarrow R_{\alpha'}[t',t'^{-1}]$$
$$\beta_u \circ \psi^- = \psi'^- \circ \beta_u^-\colon R_{\alpha^{-1}}[t^{-1}] \longrightarrow R_{\alpha'}[t',t'^{-1}]\,.$$

Statement 3.6 There are corresponding scaling isomorphisms of exact categories

$$\beta_u^+\colon \mathrm{NIL}(R,\alpha) \longrightarrow \mathrm{NIL}(R,\alpha')\,, \qquad (P,\rho) \longmapsto (P, t'ut^{-1}\rho\colon P \to t'P)$$
$$\beta_u^-\colon \mathrm{NIL}(R,\alpha^{-1}) \longrightarrow \mathrm{NIL}(R,\alpha'^{-1})\,, \quad (P,\rho) \longmapsto (P, t'^{-1}ut\rho\colon P' \to t'^{-1}P')\,,$$

where we mean

$$(t'ut^{-1}\rho)(x) := t'(uy) \qquad \text{with } \rho(x) = ty$$
$$(t'^{-1}ut\rho)(x) := t'^{-1}(uy) \quad \text{with } \rho(x) = t^{-1}y\,.$$

Statement 3.7 For all $n \leqslant 1$, the various scaling isomorphisms are related by equations

$$(\beta_u^+)_* \circ \sigma_B^+ = \sigma_B'^+ \circ \beta_u^+\colon \widetilde{\mathrm{Nil}}_{n-1}(R,\alpha) \longrightarrow K_n(R_{\alpha'}[t'])$$
$$(\beta_u^-)_* \circ \sigma_B^- = \sigma_B'^- \circ \beta_u^-\colon \widetilde{\mathrm{Nil}}_{n-1}(R,\alpha^{-1}) \longrightarrow K_n(R_{\alpha'^{-1}}[t'^{-1}])$$
$$(\beta_u)_* \circ \sigma_B = \sigma'_B \circ \begin{pmatrix} \beta_u^+ & 0 \\ 0 & \beta_u^- \end{pmatrix}\colon \widetilde{\mathrm{Nil}}_{n-1}(R,\alpha) \oplus \widetilde{\mathrm{Nil}}_{n-1}(R,\alpha^{-1}) \longrightarrow K_n(R_{\alpha'}[t',t'^{-1}])\,.$$

3.1.3 Group rings We now adapt these isomorphisms to the case of group rings $R[G]$ of groups G over the infinite dihedral group D_∞. In order to prove Lemma 3.20 and Proposition 3.23, the overall idea is to transform information about the product t_2t_1 arising from the transposition $\mathcal{B}_2 \otimes \mathcal{B}_1$ into information about the product $t_2^{-1}t_1^{-1}$ arising in the second $\widetilde{\mathrm{Nil}}$–summand of the twisted Bass decomposition. We continue to discuss the ingredients in a sequence of statements.

Statement 3.8 Let F be a group, and let $\alpha\colon F \to F$ be an automorphism. Recall that the injective HNN extension $F \rtimes_\alpha \mathbb{Z}$ is the set $F \times \mathbb{Z}$ with group multiplication

$$(x,n)(y,m) := (\alpha^m(x)y, m+n) \in F \rtimes_\alpha \mathbb{Z} \ .$$

Then, for any ring R, writing $t = (1_F, 1)$ and $(x,n) = t^n x \in F \rtimes_\alpha \mathbb{Z}$, we have

$$R[F \rtimes_\alpha \mathbb{Z}] = R[F]_\alpha[t, t^{-1}] \ .$$

Statement 3.9 Consider any group $G = G_1 *_F G_2$ over D_∞, where

$$F = G_1 \cap G_2 \ \subset \ \overline{G} = F \rtimes_\alpha \mathbb{Z} = F \rtimes_{\alpha'} \mathbb{Z} \ \subset \ G = G_1 *_F G_2 \ .$$

Fix elements $t_1 \in G_1 - F$, $t_2 \in G_2 - F$, and define elements

$$t := t_1 t_2 \in \overline{G} \ , \quad t' := t_2 t_1 \in \overline{G} \ , \quad u := (t')^{-1} t^{-1} \in F \ .$$

Define the automorphisms

$$\begin{aligned} \alpha_1 &\colon F \longrightarrow F \ , \quad x \longmapsto (t_1)^{-1} x t_1 \\ \alpha_2 &\colon F \longrightarrow F \ , \quad x \longmapsto (t_2)^{-1} x t_2 \\ \alpha := \alpha_2 \circ \alpha_1 &\colon F \longrightarrow F \ , \quad x \longmapsto t^{-1} x t \\ \alpha' := \alpha_1 \circ \alpha_2 &\colon F \longrightarrow F \ , \quad x \longmapsto t'^{-1} x t' \end{aligned}$$

such that

$$xt = t\alpha(x) \ , \quad xt' = t'\alpha'(x) \ , \quad \alpha'(x) = u\alpha^{-1}(x)u^{-1} \quad (x \in F) \ .$$

In particular, note α' and α^{-1} (not α) are related by inner automorphism by u.

Statement 3.10 Denote the canonical inclusions

$$\begin{aligned} \psi^+ &\colon R_\alpha[t] \longrightarrow R_\alpha[t, t^{-1}] \ , & \psi^- &\colon R_{\alpha^{-1}}[t^{-1}] \longrightarrow R_\alpha[t, t^{-1}] \\ \psi'^+ &\colon R_{\alpha'}[t'] \longrightarrow R_{\alpha'}[t', t'^{-1}] \ , & \psi'^- &\colon R_{\alpha'^{-1}}[t'^{-1}] \longrightarrow R_{\alpha'}[t', t'^{-1}] \ . \end{aligned}$$

The inclusion $R[F] \to R[G]$ extends to ring monomorphisms

$$\theta\colon R[F]_\alpha[t, t^{-1}] \longrightarrow R[G] \ , \quad \theta'\colon R[F]_{\alpha'}[t', t'^{-1}] \longrightarrow R[G]$$

such that

$$\operatorname{im}(\theta) = \operatorname{im}(\theta') = R[\overline{G}] \subset R[G] = R[G_1] *_{R[F]} R[G_2] \ .$$

Furthermore, the inclusion $R[F] \to R[G]$ extends to ring monomorphisms

$$\phi = \theta \circ \psi^+\colon R[F]_\alpha[t] \longrightarrow R[G] \ , \quad \phi' = \theta' \circ \psi'^+\colon R[F]_{\alpha'}[t'] \longrightarrow R[G] \ .$$

Statement 3.11 By Statement 3.5, there are defined scaling isomorphisms of rings

$$\beta_u^+\colon R[F]_{\alpha^{-1}}[t^{-1}] \longrightarrow R[F]_{\alpha'}[t'] , \qquad t^{-1} \longmapsto t'u$$
$$\beta_u^-\colon R[F]_{\alpha}[t] \longrightarrow R[F]_{\alpha'^{-1}}[t'^{-1}] , \qquad t \longmapsto u^{-1}t'^{-1}$$
$$\beta_u\colon R[F]_{\alpha}[t,t^{-1}] \longrightarrow R[F]_{\alpha'}[t',t'^{-1}] , \qquad t \longmapsto u^{-1}t'^{-1}$$

which satisfy the equations

$$\beta_u \circ \psi^- = \psi'^+ \circ \beta_u^+\colon R[F]_{\alpha^{-1}}[t^{-1}] \longrightarrow R[F]_{\alpha'}[t',t'^{-1}]$$
$$\beta_u \circ \psi^+ = \psi'^- \circ \beta_u^-\colon R[F]_{\alpha}[t] \longrightarrow R[F]_{\alpha'}[t',t'^{-1}]$$
$$\theta = \theta' \circ \beta_u\colon R[F]_{\alpha}[t,t^{-1}] \longrightarrow R[G] .$$

Statement 3.12 By Statement 3.6, there are scaling isomorphisms of exact categories

$$\beta_u^+\colon \mathrm{NIL}(R[F],\alpha^{-1}) \longrightarrow \mathrm{NIL}(R[F],\alpha') \quad (P,\rho) \longmapsto (P, t'ut\rho) ,$$
$$\beta_u^-\colon \mathrm{NIL}(R[F],\alpha) \longrightarrow \mathrm{NIL}(R[F],\alpha'^{-1}) , \quad (P,\rho) \longmapsto (P, t'^{-1}ut^{-1}\rho) .$$

Statement 3.13 By Statement 3.7, for all $n \leqslant 1$, the various scaling isomorphisms are related by

$$(\beta_u^+)_* \circ \sigma_B^- = \sigma_B'^+ \circ \beta_u^+\colon \widetilde{\mathrm{Nil}}_{*-1}(R[F],\alpha^{-1}) \longrightarrow K_*(R[F]_{\alpha'}[t'])$$
$$(\beta_u^-)_* \circ \sigma_B^+ = \sigma_B'^- \circ \beta_u^-\colon \widetilde{\mathrm{Nil}}_{*-1}(R[F],\alpha) \longrightarrow K_*(R[F]_{\alpha'^{-1}}[t'^{-1}])$$
$$(\beta_u)_* \circ \sigma_B = \sigma_B' \circ \begin{pmatrix} 0 & \beta_u^+ \\ \beta_u^- & 0 \end{pmatrix}\colon \widetilde{\mathrm{Nil}}_{*-1}(R[F],\alpha) \oplus \widetilde{\mathrm{Nil}}_{*-1}(R[F],\alpha^{-1}) \longrightarrow K_*(R[F]_{\alpha'}[t',t'^{-1}]) .$$

3.1.4 Transposition

Next, we study the effect of transposition of the bimodules $\mathcal{B}_1$ and $\mathcal{B}_2$ in order to relate α and α'. In particular, there is no mention of α^{-1} in this section.

Statement 3.14 The $R[F]$–bimodules

$$\mathcal{B}_1 = R[G_1 - F] = t_1 R[F] , \quad \mathcal{B}_2 = R[G_2 - F] = t_2 R[F]$$

are free left and right $R[F]$–modules of rank one. The $R[F]$–bimodule isomorphisms

$$\mathcal{B}_1 \otimes_{R[F]} \mathcal{B}_2 \longrightarrow tR[F] , \quad t_1x_1 \otimes t_2x_2 \longmapsto t\alpha_2(x_1)x_2$$
$$\mathcal{B}_2 \otimes_{R[F]} \mathcal{B}_1 \longrightarrow t'R[F] , \quad t_2x_2 \otimes t_1x_1 \longmapsto t'\alpha_1(x_2)x_1$$

shall be used to make the identifications

$$\mathcal{B}_1 \otimes_{R[F]} \mathcal{B}_2 = tR[F] , \quad \mathrm{NIL}(R[F];\mathcal{B}_1 \otimes_{R[F]} \mathcal{B}_2) = \mathrm{NIL}(R[F],\alpha) ,$$
$$\mathcal{B}_2 \otimes_{R[F]} \mathcal{B}_1 = t'R[F] , \quad \mathrm{NIL}(R[F];\mathcal{B}_2 \otimes_{R[F]} \mathcal{B}_1) = \mathrm{NIL}(R[F],\alpha') .$$

Statement 3.15 Theorem 0.4 gives inverse isomorphisms

$$\begin{aligned} i_*\colon \widetilde{\mathrm{Nil}}_*(R[F],\alpha) &\longrightarrow \widetilde{\mathrm{Nil}}_*(R[F];\mathcal{B}_1,\mathcal{B}_2) \\ j_*\colon \widetilde{\mathrm{Nil}}_*(R[F];\mathcal{B}_1,\mathcal{B}_2) &\longrightarrow \widetilde{\mathrm{Nil}}_*(R[F],\alpha) \end{aligned}$$

which for $*=0$ are given by

$$\begin{aligned} i_*\colon \widetilde{\mathrm{Nil}}_0(R[F],\alpha) \longrightarrow \widetilde{\mathrm{Nil}}_0(R[F];\mathcal{B}_1,\mathcal{B}_2)\ , &\quad [P,\rho] \longmapsto [P,t_2P,\rho,1] \\ j_*\colon \widetilde{\mathrm{Nil}}_0(R[F];\mathcal{B}_1,\mathcal{B}_2) \longrightarrow \widetilde{\mathrm{Nil}}_0(R[F],\alpha)\ , &\quad [P_1,P_2,\rho_1,\rho_2] \longmapsto [P_1,\rho_2\circ\rho_1]\ . \end{aligned}$$

Statement 3.16 Similarly, there are defined inverse isomorphisms

$$\begin{aligned} i'_*\colon \widetilde{\mathrm{Nil}}_*(R[F],\alpha') &\longrightarrow \widetilde{\mathrm{Nil}}_*(R[F];\mathcal{B}_2,\mathcal{B}_1) \\ j'_*\colon \widetilde{\mathrm{Nil}}_*(R[F];\mathcal{B}_2,\mathcal{B}_1) &\longrightarrow \widetilde{\mathrm{Nil}}_*(R[F],\alpha') \end{aligned}$$

which for $*=0$ are given by

$$\begin{aligned} i'_*\colon \widetilde{\mathrm{Nil}}_0(R[F],\alpha') \longrightarrow \widetilde{\mathrm{Nil}}_0(R[F];\mathcal{B}_2,\mathcal{B}_1)\ , &\quad [P',\rho'] \longmapsto [P',t_1P',\rho',1] \\ j'_*\colon \widetilde{\mathrm{Nil}}_0(R[F];\mathcal{B}_2,\mathcal{B}_1) \longrightarrow \widetilde{\mathrm{Nil}}_0(R[F],\alpha')\ , &\quad [P_2,P_1,\rho_2,\rho_1] \longmapsto [P_2,\rho_1\circ\rho_2]\ . \end{aligned}$$

Statement 3.17 The transposition isomorphism of exact categories

$$\begin{aligned} \tau_A\colon \mathrm{NIL}(R[F];\mathcal{B}_1,\mathcal{B}_2) &\longrightarrow \mathrm{NIL}(R[F];\mathcal{B}_2,\mathcal{B}_1)\ , \\ (P_1,P_2,\rho_1,\rho_2) &\longmapsto (P_2,P_1,\rho_2,\rho_1) \end{aligned}$$

induces isomorphisms

$$\begin{aligned} \tau_A\colon \mathrm{Nil}_*(R[F];\mathcal{B}_1,\mathcal{B}_2) &\cong \mathrm{Nil}_*(R[F];\mathcal{B}_2,\mathcal{B}_1) \\ \tau_A\colon \widetilde{\mathrm{Nil}}_*(R[F];\mathcal{B}_1,\mathcal{B}_2) &\cong \widetilde{\mathrm{Nil}}_*(R[F];\mathcal{B}_2,\mathcal{B}_1)\ . \end{aligned}$$

Note, by Theorem 0.4, the composites

$$\begin{aligned} \tau_B &:= j'_*\circ\tau_A\circ i_*\colon \widetilde{\mathrm{Nil}}_*(R[F],\alpha) \longrightarrow \widetilde{\mathrm{Nil}}_*(R[F],\alpha') \\ \tau'_B &:= j_*\circ\tau_A^{-1}\circ i'_*\colon \widetilde{\mathrm{Nil}}_*(R[F],\alpha') \longrightarrow \widetilde{\mathrm{Nil}}_*(R[F],\alpha) \end{aligned}$$

are inverse isomorphisms, which for $*=0$ are given by

$$\begin{aligned} \tau_B\colon \widetilde{\mathrm{Nil}}_0(R[F],\alpha) \longrightarrow \widetilde{\mathrm{Nil}}_0(R[F],\alpha')\ , &\quad [P,\rho] \longmapsto [t_2P,t_2\rho] \\ \tau'_B\colon \widetilde{\mathrm{Nil}}_0(R[F],\alpha') \longrightarrow \widetilde{\mathrm{Nil}}_0(R[F],\alpha)\ , &\quad [P',\rho'] \longmapsto [t_1P',t_1\rho']\ . \end{aligned}$$

Furthermore, note that the various transpositions are related by the equation

$$\tau_A\circ i_* = i'_*\circ\tau_B\colon \widetilde{\mathrm{Nil}}_*(R[F],\alpha) \longrightarrow \widetilde{\mathrm{Nil}}_*(R[F];\mathcal{B}_2,\mathcal{B}_1)\ .$$

Statement 3.18 Recall from Theorem 2.6 that there is a split monomorphism

$$\sigma_A\colon \widetilde{\mathrm{Nil}}_{n-1}(R[F];\mathcal{B}_1,\mathcal{B}_2) \longrightarrow K_n(R[G])$$

such that the $n=1$ case is given by

$$\begin{aligned}\sigma_A\colon \widetilde{\mathrm{Nil}}_0(R[F];\mathcal{B}_1,\mathcal{B}_2) &\longrightarrow K_1(R[G])\ ,\\ [P_1,P_2,\rho_1,\rho_2] &\longmapsto \left[P_1[G]\oplus P_2[G], \begin{pmatrix}1 & t_2\rho_2\\ t_1\rho_1 & 1\end{pmatrix}\right].\end{aligned}$$

Elementary row and column operations produce an equivalent representative:

$$\begin{pmatrix}1 & -t_2\rho_2\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & t_2\rho_2\\ t_1\rho_1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -\rho_1 & 1\end{pmatrix} = \begin{pmatrix}1-t\rho_2\rho_1 & 0\\ 0 & 1\end{pmatrix}.$$

Thus the $n=1$ case satisfies the equations (similarly for the second equality)

$$\sigma_A[P_1,P_2,\rho_1,\rho_2] = [P_1[G],1-t\rho_2\rho_1] = \left[P_2[G],1-t'\rho_1\rho_2\right].$$

Therefore for all $n\leqslant 1$, the split monomorphism σ_A', associated to the amalgamated free product $G=G_2 *_F G_1$, satisfies the equation

$$\sigma_A = \sigma_A'\circ\tau_A\colon \widetilde{\mathrm{Nil}}_{n-1}(R[F];\mathcal{B}_1,\mathcal{B}_2) \longrightarrow K_n(R[G]).$$

3.1.5 Induction

We analyze the effect of induction maps on $\widetilde{\mathrm{Nil}}$–summands.

Statement 3.19 Recall from Theorem 0.4 the isomorphism

$$\begin{aligned}i_*\colon \widetilde{\mathrm{Nil}}_{*-1}(R[F],\alpha) = \widetilde{\mathrm{Nil}}_{*-1}(R[F];\mathcal{B}_1\otimes_{R[F]}\mathcal{B}_2) &\longrightarrow \widetilde{\mathrm{Nil}}_{*-1}(R[F];\mathcal{B}_1,\mathcal{B}_2)\ ,\\ [P,\rho] &\longmapsto [P,t_2P,\rho,1].\end{aligned}$$

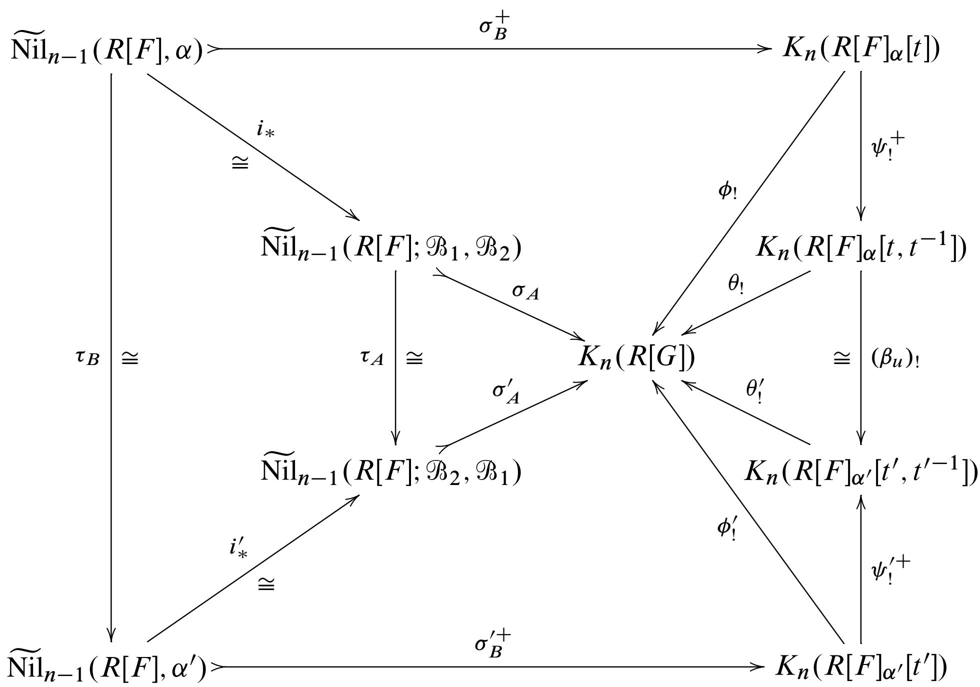
Let (P,ρ) be an object in the exact category $\mathrm{NIL}(R[F],\alpha)$. By Statement 3.18, note

$$\sigma_A i_*[P,\rho] = \sigma_A[P,t_2P,\rho,1] = [P[G],1-t\rho] = \phi_!\sigma_B^+[P,\rho].$$

Thus, for all $n\leqslant 1$, we obtain the key equality

$$\sigma_A\circ i_* = \phi_!\circ\sigma_B^+\colon \widetilde{\mathrm{Nil}}_{n-1}(R[F],\alpha) \longrightarrow K_n(R[G]).$$

Lemma 3.20 *Let $n \leqslant 1$ be an integer. The split monomorphisms $\sigma_A, \sigma'_A, \sigma^+_B, \sigma'^+_B$ are related by a commutative diagram*



Proof Commutativity of the various parts follow from the following implications:

- Statement 3.10 gives $\phi_! = \theta_! \circ \psi^+_!$ and $\phi'_! = \theta'_! \circ \psi'^+_!$.
- Statement 3.11 gives $\theta_! = \theta'_! \circ (\beta_u)_!$.
- Statement 3.17 gives $\tau_A \circ i_* = i'_* \circ \tau_B$.
- Statement 3.18 gives $\sigma_A = \sigma'_A \circ \tau_A$.
- Statement 3.19 gives $\sigma_A \circ i_* = \phi_! \circ \sigma^+_B$ and $\sigma'_A \circ i'_* = \phi'_! \circ \sigma'^+_B$. □

Observe the action of $G/\overline{G}$ on $K_n(R[G])$ is inner, hence is trivial. However, the action of $C_2 = G/\overline{G}$ on $K_n(R[\overline{G}])$ is outer, induced by, say $c_1\colon \overline{G} \to \overline{G}$, $y \mapsto t_1 y (t_1)^{-1}$. (Note that c_1 may not have order two.) This C_2–action on $K_n(R[\overline{G}])$ is nontrivial, as follows.

Proposition 3.21 *Let $n \leq 1$ be an integer. The induced map $\theta_!$ is such that there is a commutative diagram*

$$\begin{array}{ccc} \widetilde{\mathrm{Nil}}_{n-1}(R[F],\alpha) \oplus \widetilde{\mathrm{Nil}}_{n-1}(R[F],\alpha^{-1}) & \xrightarrow{\sigma_B} & K_n(R[\overline{G}]) \\ \Big\downarrow {\scriptstyle (i_* \ \tau_A^{-1} i'_* \beta_u^+)} & & \Big\downarrow {\scriptstyle \theta_!} \\ \widetilde{\mathrm{Nil}}_{n-1}(R[F];\mathcal{B}_1,\mathcal{B}_2) & \xrightarrow{\sigma_A} & K_n(R[G]) \ . \end{array}$$

Furthermore, there is a C_2–action on the upper left hand corner which interchanges the two Nil*–summands, and all maps are C_2–equivariant. Here, the action of $C_2 = G/\overline{G}$ on the upper right is given by $(c_1)_!$, and the C_2–action on each lower corner is trivial.*

Proof First, we check commutativity of the square on each Nil–summand:

- Lemma 3.20 gives $\sigma_A \circ i_* = \phi_! \circ \sigma_B^+ = \theta_! \circ \psi_!^+ \circ \sigma_B^+ = \theta_! \circ \sigma_B | \widetilde{\mathrm{Nil}}_{n-1}(R[F],\alpha)$.
- Statements 3.13 and 3.11 give $\sigma_A \circ \tau_A^{-1} \circ i'_* \circ \beta_u^+ = \sigma'_A \circ i'_* \circ \underline{\beta_u^+} = \phi'_! \circ \sigma_B'^+ \circ \beta_u^+ = \theta_! \circ (\beta_u)_!^{-1} \circ \psi_!'^+ \circ (\beta_u^+)_! \circ \sigma_B^- = \theta_! \circ \psi_!^- \circ \sigma_B^- = \theta_! \circ \sigma_B | \widetilde{\mathrm{Nil}}_{n-1}(R[F],\alpha^{-1})$.

Next, define the involution

$$\begin{pmatrix} 0 & \varepsilon_* \\ \varepsilon_*^{-1} & 0 \end{pmatrix}$$

on $\widetilde{\mathrm{Nil}}_{n-1}(R[F],\alpha) \oplus \widetilde{\mathrm{Nil}}_{n-1}(R[F],\alpha^{-1})$ by

$$\varepsilon := (\alpha_1^{-1})_! \circ \beta_u^+ \colon \mathrm{NIL}(R[F],\alpha^{-1}) \longrightarrow \mathrm{NIL}(R[F],\alpha) \ .$$

Here, the automorphism $\alpha_1^{-1} \colon F \to F$ was defined in Statement 3.9 by $x \mapsto t_1 x (t_1)^{-1}$ and is the restriction of c_1. It remains to show σ_B and $(i_* \ \tau_{A*}^{-1} i'_* \beta_u^+)$ are C_2–equivariant, that is,

$$(c_1)_! \circ \psi^- \sigma_B^- = \psi^+ \sigma_B^+ \circ \varepsilon \tag{5}$$

$$\tau_{A*}^{-1} i'_* \beta_u^+ = i_* \circ \varepsilon \ . \tag{6}$$

Observe that the induced ring automorphism $(c_1)_! \colon R[F]_\alpha[t,t^{-1}] \to R[F]_\alpha[t,t^{-1}]$ restricts to a ring isomorphism

$$(c_1)_!^+ \colon R[F]_{\alpha^{-1}}[t^{-1}] \longrightarrow R[F]_\alpha[t] \ , \quad x \longmapsto \alpha_1^{-1}(x) \ , \quad t^{-1} \longmapsto t\, \alpha_1^{-1}(u) \ .$$

Then $(c_1)_! \circ \psi^- = \psi^+ \circ (c_1)_!^+$. So (5) follows from the commutative square

$$(c_1)_!^+ \circ \sigma_B^- = \sigma_B^+ \circ (\alpha_1^{-1})_! \beta_u^+ \colon \widetilde{\mathrm{Nil}}_{n-1}(R[F],\alpha^{-1}) \longrightarrow K_n(R[F]_\alpha[t]) \ ,$$

which can be verified by formulas for $n = 1$ and extends to $n < 1$ by variation of R.

Observe that (6) follows from the existence of an exact natural transformation

$$T\colon \tau_A^{-1}\circ i' \to i\circ(\alpha_1^{-1})_!\colon \mathrm{NIL}(R[F],\alpha') \longrightarrow \mathrm{NIL}(R[F];t_1R[F],t_2R[F])$$

defined on objects $(P,\rho\colon P\to t'P = t_2t_1P)$ by the rule

$$T_{(P,\rho)} := (1,\rho)\colon (t_1P,P,1,\rho) \longrightarrow (t_1P,t'P,t_1\rho,1)\ .$$

A key observation from Statement 3.9 is the isomorphism $R[F]\otimes_{c_1} P \to t_1P$ sending $x\otimes p \mapsto \alpha_1(x)p$. □

3.1.6 Transfer We analyze the effect of transfer maps on $\widetilde{\mathrm{Nil}}$–summands.

Statement 3.22 Given an $R[G]$–module M, let $M^!$ be the abelian group M with $R[\overline{G}]$–action the restriction of the $R[G]$–action. The transfer functor of exact categories

$$\theta^!\colon \mathrm{PROJ}(R[G]) \longrightarrow \mathrm{PROJ}(R[\overline{G}])\ ,\quad M \longmapsto M^!$$

induces the transfer maps in algebraic K–theory

$$\theta^!\colon K_*(R[G]) \longrightarrow K_*(R[\overline{G}])\ .$$

The exact functors of Theorem 0.4 combine to give an exact functor

$$\begin{pmatrix} j \\ j' \end{pmatrix}\colon \mathrm{NIL}(R[F];\mathcal{B}_1,\mathcal{B}_2) \longrightarrow \mathrm{NIL}(R[F],\alpha)\times\mathrm{NIL}(R[F],\alpha'),$$
$$[P_1,P_2,\rho_1,\rho_2] \longmapsto \big([P_1,\rho_2\circ\rho_1],[P_2,\rho_1\circ\rho_2]\big)$$

inducing a map between reduced Nil–groups

$$\begin{pmatrix} j_* \\ j'_* \end{pmatrix}\colon \widetilde{\mathrm{Nil}}_*(R[F];\mathcal{B}_1,\mathcal{B}_2) \longrightarrow \widetilde{\mathrm{Nil}}_*(R[F],\alpha)\oplus\widetilde{\mathrm{Nil}}_*(R[F],\alpha')\ .$$

Proposition 3.23 *Let $n\leqslant 1$ be an integer. The transfer map $\theta^!$ restricts to the isomorphism j_* in a commutative diagram*

$$\begin{array}{ccc} \widetilde{\mathrm{Nil}}_{n-1}(R[F];\mathcal{B}_1,\mathcal{B}_2) & \xrightarrow{\ \sigma_A\ } & K_n(R[G]) \\ \Big\downarrow {\scriptstyle \begin{pmatrix} j_* \\ (\beta_u^+)^{-1}j'_* \end{pmatrix}} & & \Big\downarrow {\scriptstyle \theta^!} \\ \widetilde{\mathrm{Nil}}_{n-1}(R[F],\alpha)\oplus\widetilde{\mathrm{Nil}}_{n-1}(R[F],\alpha^{-1}) & \xrightarrow{\ (\psi^+\sigma_B^+\ \ \beta_u\psi^-\sigma_B^-)\ } & K_n(R[\overline{G}])\ . \end{array}$$

Proof Using the suspension isomorphisms of Section 2, we may assume $n = 1$. Let $(P_1, P_2, \rho_1, \rho_2)$ be an object in $\mathrm{NIL}(R[F]; \mathcal{B}_1, \mathcal{B}_2)$. Define an $R[G]$–module automorphism

$$f := \begin{pmatrix} 1 & t_2\rho_2 \\ t_1\rho_1 & 1 \end{pmatrix} : P_1[G] \oplus P_2[G] \longrightarrow P_1[G] \oplus P_2[G] .$$

By Theorem 2.6, we have $[f] = \sigma_A[P_1, P_2, \rho_1, \rho_2] \in K_1(R[G])$. Note the transfer is

$$\theta^!(f) = \begin{pmatrix} 1 & t_2\rho_2 & 0 & 0 \\ t_1\rho_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & t_1\rho_1 \\ 0 & 0 & t_2\rho_2 & 1 \end{pmatrix}$$

as an $R[\bar{G}]$–module automorphism of $P_1[\bar{G}] \oplus t_1 P_2[\bar{G}] \oplus P_2[\bar{G}] \oplus t_1 P_1[\bar{G}]$. Furthermore, elementary row and column operations produce a diagonal representation:

$$\begin{pmatrix} 1 & -t_2\rho_2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -t_1\rho_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \theta^!(f) \begin{pmatrix} 1 & 0 & 0 & 0 \\ -t_1\rho_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -t_2\rho_2 & 1 \end{pmatrix} = \begin{pmatrix} 1-t'\rho_2\rho_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1-t\rho_1\rho_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

So $\theta^![f] = [1 - t'\rho_2\rho_1] + [1 - t\rho_1\rho_2]$. Thus we obtain a commutative diagram

$$\begin{array}{ccc} \widetilde{\mathrm{Nil}}_0(R[F]; \mathcal{B}_1, \mathcal{B}_2) & \xrightarrow{\quad \sigma_A \quad} & K_1(R[G]) \\ \Big\downarrow {\scriptstyle \begin{pmatrix} j_* \\ j'_* \end{pmatrix}} & & \Big\downarrow {\scriptstyle \theta^!} \\ \widetilde{\mathrm{Nil}}_0(R[F], \alpha) \oplus \widetilde{\mathrm{Nil}}_0(R[F], \alpha') & \xrightarrow{(\psi^+\sigma_B^+ \ \ \psi'^+\sigma_B'^+)} & K_1(R[\bar{G}]) . \end{array}$$

Finally, by Statement 3.13 and Statement 3.11, note

$$\psi'^+ \circ \sigma_B'^+ \circ \beta_u^+ = \psi'^+ \circ \beta_u^+ \circ \sigma_B^- = \beta_u \circ \psi^- \circ \sigma_B^- . \qquad \square$$

3.2 Waldhausen Nil

Examples of bimodules originate from group rings of amalgamated product of groups.

Definition 3.24 A subgroup H of a group G is *almost-normal* if $|H : H \cap xHx^{-1}| < \infty$ for every $x \in G$. In other words, H is commensurate with all its conjugates. Equivalently, H is an almost-normal subgroup of G if every (H, H)–double coset HxH is both a union of finitely many left cosets gH and a union of finitely many right cosets Hg.

Remark 3.25 Almost-normal subgroups arise in the Shimura theory of automorphic functions, with (G, H) called a Hecke pair. Here are two sufficient conditions for a subgroup $H \subset G$ to be almost-normal: if H is a finite-index subgroup of G, or if H is a normal subgroup of G. Examples of almost-normal subgroups are given by Kreig [12, page 9].

Here is our reduction for a certain class of group rings, specializing the General Algebraic Semi-splitting of Theorem 0.4.

Corollary 3.26 *Let R be a ring. Let $G = G_1 *_F G_2$ be an injective amalgamated product of groups over a subgroup F of G_1 and G_2. Suppose F is an almost-normal subgroup of G_2. Then, for all $n \in \mathbb{Z}$, there is an isomorphism of abelian groups*

$$j_*\colon \widetilde{\mathrm{Nil}}_n(R[F]; R[G_1-F], R[G_2-F]) \longrightarrow \widetilde{\mathrm{Nil}}_n(R[F]; R[G_1-F] \otimes_{R[F]} R[G_2-F]) .$$

Proof Consider the set $J := (F \backslash G_2/F) - F$ of nontrivial double cosets. Let $\mathcal{I}$ be the poset of all finite subsets of J, partially ordered by inclusion. Note, as $R[F]$–bimodules,

$$R[G_2 - F] = \operatorname*{colim}_{I \in \mathcal{I}} R[I] \quad \text{where } R[I] := \bigoplus_{FgF \in I} R[FgF] .$$

Since F is an almost-normal subgroup of G_2, each $R[F]$–bimodule $R[I]$ is a finitely generated free (hence projective) left $R[F]$–module. Observe that $\mathcal{I}$ is a filtered poset: if $I, I' \in \mathcal{I}$ then $I \cup I' \in \mathcal{I}$. Therefore we are done by Theorem 0.4. □

The case of $G = D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$ has a particularly simple form.

Corollary 3.27 *Let R be a ring and $n \in \mathbb{Z}$. There are natural isomorphisms:*

(1) $\widetilde{\mathrm{Nil}}_n(R; R, R) \cong \widetilde{\mathrm{Nil}}_n(R)$.

(2) $K_n(R[D_\infty]) \cong (K_n(R[\mathbb{Z}_2]) \oplus K_n(R[\mathbb{Z}_2]))/K_n(R) \oplus \widetilde{\mathrm{Nil}}_{n-1}(R)$.

Proof Part (i) follows from Corollary 3.26 with $F = 1$ and $G_i = \mathbb{Z}_2$. Then Part (ii) follows from Waldhausen's exact sequence 2.6, where the group retraction $\mathbb{Z}_2 \to 1$ induces a splitting of the map $K_n(R) \to K_n(R[\mathbb{Z}_2]) \times K_n(R[\mathbb{Z}_2])$. □

Example 3.28 Consider the group $G = G_0 \times D_\infty$ where $G_0 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}$. Since G surjects onto the infinite dihedral group, there is an amalgamated product decomposition

$$G = (G_0 \times \mathbb{Z}_2) *_{G_0} (G_0 \times \mathbb{Z}_2)$$

with the corresponding index 2 subgroup

$$\bar{G} = G_0 \times \mathbb{Z} .$$

Corollary 3.27(1) gives an isomorphism

$$\widetilde{\mathrm{Nil}}_{-1}(\mathbb{Z}[G_0]; \mathbb{Z}[G_0], \mathbb{Z}[G_0]) \cong \widetilde{\mathrm{Nil}}_{-1}(\mathbb{Z}[G_0]) .$$

On the other hand, Bass showed that the latter group is an infinitely generated abelian group of exponent a power of two [2, XII, 10.6]. Hence, by Waldhausen's algebraic K–theory decomposition result, $\mathrm{Wh}(G)$ is infinitely generated due to Nil elements. Now construct a codimension 1, finite CW–pair (X, Y) with $\pi_1 X = G$ realizing the above amalgamated product decomposition – for example, let $Y \to Z$ be a map of connected CW–complexes inducing the first factor inclusion $G_0 \to G_0 \times \mathbb{Z}_2$ on the fundamental group and let X be the double mapping cylinder of $Z \leftarrow Y \to Z$. Next construct a homotopy equivalence $f\colon M \to X$ of finite CW–complexes whose torsion $\tau(f) \in \mathrm{Wh}(G)$ is a nonzero Nil element. Then f is nonsplittable along Y by Waldhausen [24] (see Theorem 4.3). *This is the first explicit example of a nonzero Waldhausen* $\widetilde{\mathrm{Nil}}$ *group and a nonsplittable homotopy equivalence in the two-sided case.*

3.3 Farrell–Jones Conjecture

The Farrell–Jones Conjecture asserts the family of virtually cyclic subgroups is a "generating" family for $K_n(R[G])$. In this section we apply our main theorem to show the Farrell–Jones Conjecture holds up to dimension one if and only if the smaller family of finite-by-cyclic subgroups is a generating family for $K_n(R[G])$ up to dimension one.

Let $\mathrm{Or}\, G$ be the orbit category of a group G; objects are G–sets G/H where H is a subgroup of G and morphisms are G–maps. Davis and Lück [4] defined a functor $\mathbf{K}_R\colon \mathrm{Or}\, G \to \mathsf{Spectra}$ with the key property $\pi_n \mathbf{K}_R(G/H) = K_n(R[H])$. The utility of such a functor is that it allows the definition of an equivariant homology theory, indeed for a G–CW–complex X, one defines

$$H_n^G(X; \mathbf{K}_R) := \pi_n(\mathrm{map}_G(-, X)_+ \wedge_{\mathrm{Or}\, G} \mathbf{K}_R(-))$$

(see [4, Sections 4, 7] for basic properties). Note that the "coefficients" of the homology theory are given by $H_n^G(G/H; \mathbf{K}_R) = K_n(R[H])$.

A *family* $\mathcal{F}$ of subgroups of G is a nonempty set of subgroups closed under conjugation and taking subgroups. For such a family, $E_{\mathcal{F}}G$ is the classifying space for G–actions with isotropy in $\mathcal{F}$. It is characterized up to G–homotopy type as a G–CW–complex so that $(E_{\mathcal{F}}G)^H$ is contractible for subgroups $H \in \mathcal{F}$ and is empty for subgroups $H \notin \mathcal{F}$.

Four relevant families are $\mathtt{fin} \subset \mathtt{fbc} \subset \mathtt{vc} \subset \mathtt{all}$, the families of finite subgroups, finite-by-cyclic, virtually cyclic subgroups and all subgroups respectively. Here

$$\mathtt{fbc} := \mathtt{fin} \cup \{H < G \mid H \cong F \rtimes \mathbb{Z} \text{ with } F \text{ finite}\}$$
$$\mathtt{vc} := \{H < G \mid \exists \text{ cyclic } C < H \text{ with finite index}\} .$$

The Farrell–Jones Conjecture in K–theory for the group G [7; 4] asserts an isomorphism

$$H_n^G(E_{\mathtt{vc}}G; \mathbf{K}_R) \longrightarrow H_n^G(E_{\mathtt{all}}G; \mathbf{K}_R) = K_n(R[G]) .$$

We now state a more general version, the Fibered Farrell–Jones Conjecture. Let $\varphi\colon \Gamma \to G$ be a group homomorphism. If $\mathcal{F}$ is a family of subgroups of G, define the family of subgroups

$$\varphi^*\mathcal{F} := \{H < \Gamma \mid \varphi(H) \in \mathcal{F}\} .$$

The Fibered Farrell–Jones Conjecture in K–theory for the group G asserts, for every ring R and homomorphism $\varphi\colon \Gamma \to G$, the following induced map is an isomorphism:

$$H_n^\Gamma(E_{\varphi^*\mathtt{vc}(G)}\Gamma; \mathbf{K}_R) \longrightarrow H_n^\Gamma(E_{\varphi^*\mathtt{all}(G)}\Gamma; \mathbf{K}_R) = K_n(R[\Gamma]) .$$

The following theorem was proved for all n in [5] using controlled topology. We give a proof below up to dimension one using only algebraic topology.

Theorem 3.29 *Let $\varphi\colon \Gamma \to G$ be an homomorphism of groups. Let R be any ring. The inclusion-induced map*

$$H_n^\Gamma(E_{\varphi^*\mathtt{fbc}(G)}\Gamma; \mathbf{K}_R) \longrightarrow H_n^\Gamma(E_{\varphi^*\mathtt{vc}(G)}\Gamma; \mathbf{K}_R)$$

is an isomorphism for all integers $n < 1$ and an epimorphism for $n = 1$.

Hence we propose a sharpening of the Farrell–Jones Conjecture in algebraic K–theory.

Conjecture 3.30 Let G be a discrete group, and let R be a ring. Let n be an integer.

(1) There is an isomorphism

$$H_n^G(E_{\mathtt{fbc}}G; \mathbf{K}_R) \longrightarrow H_n^G(E_{\mathtt{all}}G; \mathbf{K}_R) = K_n(R[G]) .$$

(2) For any homomorphism $\varphi\colon \Gamma \to G$ of groups, there is an isomorphism

$$H_n^\Gamma(E_{\varphi^*\mathtt{fbc}(G)}\Gamma; \mathbf{K}_R) \longrightarrow H_n^\Gamma(E_{\mathtt{all}}\Gamma; \mathbf{K}_R) = K_n(R[\Gamma]) .$$

The proof of Theorem 3.29 will require three auxiliary results, some of which we quote from other sources. The first is a variant of Theorem A.10 of Farrell–Jones [7], whose proof is identical to the proof of Theorem A.10.

Transitivity Principle Let $\mathcal{F} \subset \mathcal{G}$ be families of subgroups of a group Γ. Let $\mathbf{E}$: $\mathrm{Or}\,\Gamma \to \mathtt{Spectra}$ be a functor. Let $N \in \mathbb{Z} \cup \{\infty\}$. If for all $H \in \mathcal{G} - \mathcal{F}$, the assembly map

$$H_n^H(E_{\mathcal{F}|H} H; \mathbf{E}) \longrightarrow H_n^H(E_{\mathtt{all}} H; \mathbf{E})$$

is an isomorphism for $n < N$ and an epimorphism if $n = N$, then the map

$$H_n^\Gamma(E_{\mathcal{F}} \Gamma; \mathbf{E}) \longrightarrow H_n^\Gamma(E_{\mathtt{all}} H; \mathbf{E})$$

is an isomorphism for $n < N$ and an epimorphism if $n = N$.

Of course, we apply this principle to the families $\mathtt{fbc} \subset \mathtt{vc}$. The second auxiliary result is a well-known lemma (see Scott and Wall [21, Theorem 5.12]), but we offer an alternative proof.

Lemma 3.31 *Let G be a virtually cyclic group. Then either*

(1) *G is finite.*

(2) *G maps onto $\mathbb{Z}$; hence $G = F \rtimes_\alpha \mathbb{Z}$ with F finite.*

(3) *G maps onto D_∞; hence $G = G_1 *_F G_2$ with $|G_i : F| = 2$ and F finite.*

Proof Assume G is an infinite virtually cyclic group. The intersection of the conjugates of a finite index, infinite cyclic subgroup is a normal, finite index, infinite cyclic subgroup C. Let Q be the finite quotient group. Embed C as a subgroup of index $|Q|$ in an infinite cyclic group C'. There exists a unique $\mathbb{Z}[Q]$–module structure on C' such that C is a $\mathbb{Z}[Q]$–submodule. Observe that the image of the obstruction cocycle under the map $H^2(Q; C) \to H^2(Q; C')$ is trivial. Hence G embeds as a finite index subgroup of a semidirect product $G' = C' \rtimes Q$. Note G' maps epimorphically to $\mathbb{Z}$ (if Q acts trivially) or to D_∞ (if Q acts nontrivially). In either case, G maps epimorphically to a subgroup of finite index in D_∞, which must be either infinite cyclic or infinite dihedral. □

In order to see how the reduced Nil–groups relate to equivariant homology (and hence to the Farrell–Jones Conjecture), we need [5, Lemma 3.1], the third auxiliary result.

Lemma 3.32 (Davis–Quinn–Reich) *Let G be a group of the form $G_1 *_F G_2$ with $|G_i : F| = 2$, and let $\mathtt{fac}$ be the smallest family of subgroups of G containing G_1 and G_2. Let $\overline{G}$ be a group of the form $F \rtimes_\alpha \mathbb{Z}$, and let $\overline{\mathtt{fac}}$ be the smallest family of subgroups of $\overline{G}$ containing F. (Note that F need not be finite.)*

(1) *The following exact sequences are split, and hence short exact:*

$$H_n^G(E_{\mathfrak{fac}}G;\mathbf{K}_R) \xrightarrow{f_A} H_n^G(E_{\mathfrak{all}}G;\mathbf{K}_R) \xrightarrow{\eta_A} H_n^G(E_{\mathfrak{all}}G, E_{\mathfrak{fac}}G;\mathbf{K}_R)$$

$$H_n^{\overline{G}}(E_{\overline{\mathfrak{fac}}}\overline{G};\mathbf{K}_R) \xrightarrow{f_B} H_n^{\overline{G}}(E_{\mathfrak{all}}\overline{G};\mathbf{K}_R) \xrightarrow{\eta_B} H_n^{\overline{G}}(E_{\mathfrak{all}}\overline{G}, E_{\overline{\mathfrak{fac}}}\overline{G};\mathbf{K}_R)\,.$$

Here f_A, η_A *and* f_B, η_B *are inclusion-induced maps.*

(2) *The maps*

$$\eta_A\circ\sigma_A\colon \widetilde{\mathrm{Nil}}_{n-1}(R[F]; R[G_1-F], R[G_2-F]) \xrightarrow{\cong} H_n^G(E_{\mathfrak{all}}G, E_{\mathfrak{fac}}G;\mathbf{K}_R)$$

$$\eta_B\circ\sigma_B\colon \widetilde{\mathrm{Nil}}_{n-1}(R[F],\alpha)\oplus\widetilde{\mathrm{Nil}}_{n-1}(R[F],\alpha^{-1}) \xrightarrow{\cong} H_n^{\overline{G}}(E_{\mathfrak{all}}\overline{G}, E_{\overline{\mathfrak{fac}}}\overline{G};\mathbf{K}_R)$$

are isomorphisms where σ_A *and* σ_B *are Waldhausen's split injections.*

The statement of Lemma 3.1 of [5] does not explicitly identify the isomorphisms in Part (2) above, but the identification follows from the last paragraph of the proof.

It is not difficult to compute $H_n^G(E_{\mathfrak{fac}}G;\mathbf{K}_R)$ and $H_n^{\overline{G}}(E_{\overline{\mathfrak{fac}}}\overline{G};\mathbf{K}_R)$ in terms of a Wang sequence and a Mayer–Vietoris sequence respectively. An example is in Section 3.4.

Next, we further assume $\overline{G}\subset G$ with $|G:\overline{G}|=2$. In this case, $C_2=G/\overline{G}$ acts on $K_n(R\overline{G}) = H_n^{\overline{G}}(E_{\mathfrak{all}}\overline{G};\mathbf{K}_R)$ by conjugation. By [5, Remark 3.21], there is a C_2–action on $H_n^{\overline{G}}(E_{\mathfrak{all}}\overline{G}, E_{\overline{\mathfrak{fac}}}\overline{G};\mathbf{K}_R)$ so that η_B and $\theta_{!!}$ below are C_2–equivariant.

Lemma 3.33 *Let* $n\le 1$ *be an integer. There is a commutative diagram of* C_2*–equivariant homomorphisms*

$$\begin{array}{ccc}
\widetilde{\mathrm{Nil}}_{n-1}(R[F],\alpha)\oplus\widetilde{\mathrm{Nil}}_{n-1}(R[F],\alpha^{-1}) & \xrightarrow{\eta_B\circ\sigma_B} & H_n^{\overline{G}}(E_{\mathfrak{all}}\overline{G}, E_{\overline{\mathfrak{fac}}}\overline{G};\mathbf{K}_R) \\
\Big\downarrow{\scriptstyle (i_*\ \tau_A^{-1}i'_*\beta_u^+)} & & \Big\downarrow{\scriptstyle \theta_{!!}} \\
\widetilde{\mathrm{Nil}}_{n-1}(R[F];\mathcal{B}_1,\mathcal{B}_2) & \xrightarrow{\eta_A\circ\sigma_A} & H_n^G(E_{\mathfrak{all}}G, E_{\mathfrak{fac}}G;\mathbf{K}_R)\,.
\end{array}$$

Here, the $C_2=G/\overline{G}$*–action on the upper left-hand corner is given in Proposition 3.21, on the upper right it is given by* [5, Remark 3.21], *and on each lower corner it is trivial.*

Proof This follows from Proposition 3.21 and the C_2–equivariance of η_B. □

Recall that if $C_2 = \{1, T\}$ and if M is a $\mathbb{Z}[C_2]$–module then the *coinvariant group* $M_{C_2} = H_0(C_2; M)$ is the quotient group of M modulo the subgroup $\{m - Tm \mid m \in M\}$.

Lemma 3.34 *Let $n \leq 1$ be an integer. There is an induction-induced isomorphism*

$$\big(H_n^{\bar{G}}(E_{\mathfrak{all}}\bar{G}, E_{\overline{\mathfrak{fac}}}\bar{G}; \mathbf{K}_R)\big)_{C_2} \longrightarrow H_n^G(E_{\mathfrak{fac}\, G \cup \mathfrak{sub}\, \bar{G}}G, E_{\mathfrak{fac}}G; \mathbf{K}_R) .$$

Proof Recall $G/\bar{G} = C_2$. Since $\overline{\mathfrak{fac}} = \mathfrak{fac} \cap \bar{G}$, by [5, Lemma 4.1(i)] there is a identification of $\mathbb{Z}[C_2]$–modules

$$H_n^{\bar{G}}(E_{\mathfrak{all}}\bar{G}, E_{\overline{\mathfrak{fac}}}\bar{G}; \mathbf{K}) = \pi_n(\mathbf{K}/\mathbf{K}_{\mathfrak{fac}})(G/\bar{G}) .$$

The C_2–coinvariants can be interpreted as a C_2–homology group:

$$(\pi_n(\mathbf{K}/\mathbf{K}_{\mathfrak{fac}})(C_2))_{C_2} = H_0^{C_2}(EC_2; \pi_n(\mathbf{K}/\mathbf{K}_{\mathfrak{fac}})(C_2)) .$$

By Lemma 3.32(2) and Lemma 3.33, the coefficient $\mathbb{Z}[C_2]$–module is induced from a $\mathbb{Z}$–module. By the Atiyah–Hirzebruch spectral sequence (which collapses at E^2), note

$$H_0^{C_2}(EC_2; \pi_n(\mathbf{K}/\mathbf{K}_{\mathfrak{fac}})(C_2)) = H_n^{C_2}(EC_2; (\mathbf{K}/\mathbf{K}_{\mathfrak{fac}})(C_2)) .$$

Therefore, by [5, Lemma 4.6, Lemma 4.4, Lemma 4.1], we conclude

$$\begin{aligned} H_n^{C_2}(EC_2; (\mathbf{K}/\mathbf{K}_{\mathfrak{fac}})(C_2)) &= H_n^G(E_{\mathfrak{sub}\bar{G}}G; \mathbf{K}/\mathbf{K}_{\mathfrak{fac}}) \\ &= H_n^G(E_{\mathfrak{fac}\, G \cup \mathfrak{sub}\, \bar{G}}G; \mathbf{K}/\mathbf{K}_{\mathfrak{fac}}) \\ &= H_n^G(E_{\mathfrak{fac}\, G \cup \mathfrak{sub}\, \bar{G}}G, E_{\mathfrak{fac}}G; \mathbf{K}) . \end{aligned}$$ □

The identifications in the above proof are extracted from the proof of [5, Theorem 1.5].

Proof of Theorem 3.29 Let $\varphi\colon \Gamma \to G$ be a homomorphism of groups. Using the Transitivity Principle applied to the families $\varphi^*\mathfrak{fbc} \subset \varphi^*\mathfrak{vc}$, it suffices to show that

$$H_n^H(E_{\mathfrak{all}}H, E_{\varphi^*\mathfrak{fbc}|H}H; \mathbf{K}_R) = 0$$

for all $n \leq 1$ and for all $H \in \varphi^*\mathfrak{vc} - \varphi^*\mathfrak{fbc}$. To identify the family $\varphi^*\mathfrak{fbc}|H$ we will use two facts, the proofs of which are left to the reader.

- If $q\colon A \to B$ is a group epimorphism with finite kernel, then $\mathfrak{fbc}\, A = q^*\mathfrak{fbc}\, B$. (The key step is to show that an epimorphic image of a finite-by-cyclic group is finite-by-cyclic.)
- If $q\colon A \to B = G_1 *_F G_2$ is a group epimorphism, then $A = q^{-1}G_1 *_{q^{-1}F} q^{-1}G_2$ and $\mathfrak{fac}\, A = q^*\mathfrak{fac}\, B$.

Let $\varphi|\colon H \to \varphi(H)$ denote the restriction of φ to H. By the definition of both sides,

$$\varphi^* \mathtt{fbc}|H = \varphi|^*(\mathtt{fbc}\,\varphi(H)) .$$

Since $H \in \varphi^* \mathtt{vc} - \varphi^* \mathtt{fbc}$, we have $\varphi(H) \in \mathtt{vc} - \mathtt{fbc}$. So, by Lemma 3.31, there is an epimorphism $p\colon \varphi(H) \to D_\infty = \mathbb{Z}_2 *_1 \mathbb{Z}_2$ with finite kernel. By the first fact above

$$\varphi|^*(\mathtt{fbc}\,\varphi(H)) = \varphi|^*(p^*(\mathtt{fbc}\,D_\infty)) .$$

Next, write $\bar{H}q := (p \circ \varphi|)^{-1}(\mathbb{Z})$. Note

$$\begin{aligned}\varphi|^*(p^*(\mathtt{fbc}\,D_\infty)) &= (p \circ \varphi|)^*(\mathtt{fbc}\,D_\infty)\\ &= (p \circ \varphi|)^*(\mathtt{fac}\,D_\infty \cup \mathtt{sub}\,\mathbb{Z})\\ &= (p \circ \varphi|)^*(\mathtt{fac}\,D_\infty) \cup (p \circ \varphi|)^*(\mathtt{sub}\,\mathbb{Z})\\ &= \mathtt{fac}\,H \cup \mathtt{sub}\,\bar{H} ,\end{aligned}$$

where the last equality uses the second fact above. Thus it suffices to prove, for any group H mapping epimorphically to D_∞ and for all $n \le 1$, that

$$H_n^H(E_{\mathtt{all}}H, E_{\mathtt{fac}\,H \cup \mathtt{sub}\,\bar{H}}; \mathbf{K}_R) = 0$$

where $\bar{H}$ is the inverse image of the maximal infinite cyclic subgroup of D_∞.

Consider the composite

$$\begin{aligned}\big(H_n^{\bar{H}}(E_{\mathtt{all}}\bar{H}, E_{\overline{\mathtt{fac}}}\bar{H}; \mathbf{K}_R)\big)_{H/\bar{H}} &\xrightarrow{\alpha} H_n^H(E_{\mathtt{fac}\,H \cup \mathtt{sub}\,\bar{H}}H, E_{\mathtt{fac}}H; \mathbf{K}_R)\\ &\xrightarrow{\beta} H_n^H(E_{\mathtt{all}}H, E_{\mathtt{fac}}H; \mathbf{K}_R) .\end{aligned}$$

The map α exists and is an isomorphism by Lemma 3.34. Apply C_2–covariants to the commutative diagram in the statement of Lemma 3.33. In this diagram of C_2–coinvariants, the top and bottom are isomorphisms by Lemma 3.32(2) and the left map is an isomorphism by Proposition 3.21 and Theorem 0.4. Hence the right-hand map, which is $\beta \circ \alpha$, is an isomorphism for all $n \le 1$. It follows that β is an isomorphism for all $n \le 1$. So, by the exact sequence of a triple, we obtain

$$H_n^H(E_{\mathtt{all}}H, E_{\mathtt{fac}\,H \cup \mathtt{sub}\,\bar{H}}H; \mathbf{K}_R) = 0$$

for all $n \le 1$ as desired. □

3.4 K–theory of the modular group

Let $\Gamma = \mathbb{Z}_2 * \mathbb{Z}_3 = \mathrm{PSL}_2(\mathbb{Z})$. The following theorem follows from applying our main theorem and the recent proof by Bartels, Lück and Reich [1] of the Farrell–Jones conjecture in K–theory for word hyperbolic groups.

The Cayley graph for $\mathbb{Z}_2 * \mathbb{Z}_3$ with respect to the generating set given by the nonzero elements of $\mathbb{Z}_2$ and $\mathbb{Z}_3$ has the quasi-isometry type of the usual Bass–Serre tree for the amalgamated product (Figure 1). This is an infinite tree with alternating vertices

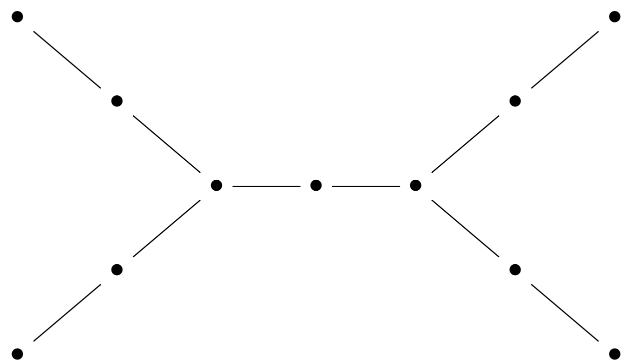


Figure 1: Bass–Serre tree for $\mathrm{PSL}_2(\mathbb{Z})$

of valence two and three. The group Γ acts on the tree, with the generator of order two acting by reflection through an valence two vertex and the generator of order three acting by rotation through an adjoining vertex of valence three.

Any geodesic triangle in the Bass–Serre tree has the property that the union of two sides is the union of all three sides. It follows that the Bass–Serre graph is δ–hyperbolic for any $\delta > 0$, the Cayley graph is δ–hyperbolic for some $\delta > 0$, and hence Γ is a hyperbolic group.

Theorem 3.35 *For any ring R and integer n,*

$$K_n(R[\Gamma]) = (K_n(R[\mathbb{Z}_2]) \oplus K_n(R[\mathbb{Z}_3]))/K_n(R) \oplus \bigoplus_{\mathcal{M}_C} \widetilde{\mathrm{Nil}}_{n-1}(R) \oplus \widetilde{\mathrm{Nil}}_{n-1}(R) \oplus \bigoplus_{\mathcal{M}_D} \widetilde{\mathrm{Nil}}_{n-1}(R) ,$$

where $\mathcal{M}_C$ and $\mathcal{M}_D$ are the set of conjugacy classes of maximal infinite cyclic subgroups and maximal infinite dihedral subgroups, respectively. Moreover, all virtually cyclic subgroups of Γ are cyclic or infinite dihedral.

Proof By Lemma 3.32, the exact sequence of $(E_{\mathrm{all}}\Gamma, E_{\mathrm{fin}}\Gamma)$ is short exact and split:

$$H_n^\Gamma(E_{\mathrm{fin}}\Gamma; \mathbf{K}_R) \to H_n^\Gamma(E_{\mathrm{all}}\Gamma; \mathbf{K}_R) \to H_n^\Gamma(E_{\mathrm{all}}\Gamma, E_{\mathrm{fin}}\Gamma; \mathbf{K}_R) .$$

Then, by the Farrell–Jones Conjecture [1] for word hyperbolic groups, we obtain

$$\begin{aligned} K_n(R[\Gamma]) &= H_n^\Gamma(E_{\mathrm{fin}}\Gamma; \mathbf{K}_R) \oplus H_n^\Gamma(E_{\mathrm{all}}\Gamma, E_{\mathrm{fin}}\Gamma; \mathbf{K}_R) \\ &= H_n^\Gamma(E_{\mathrm{fin}}\Gamma; \mathbf{K}_R) \oplus H_n^\Gamma(E_{\mathrm{vc}}\Gamma, E_{\mathrm{fin}}\Gamma; \mathbf{K}_R) . \end{aligned}$$

Observe $E_{\mathrm{fin}}\Gamma$ is constructed as a pushout of Γ–spaces

$$\begin{array}{ccc} \Gamma \sqcup \Gamma & \longrightarrow & \Gamma/\mathbb{Z}_2 \sqcup \Gamma/\mathbb{Z}_3 \\ \downarrow & & \downarrow \\ \Gamma \times D^1 & \longrightarrow & E_{\mathrm{fin}}\Gamma \ . \end{array}$$

Then $E_{\mathrm{fin}}\Gamma$ is the Bass–Serre tree for $\Gamma = \mathbb{Z}_2 * \mathbb{Z}_3$. Note that $H_*^{\Gamma}(\Gamma/H; \mathbf{K}_R) = K_*(R[H])$. The pushout gives, after canceling a $K_n(R)$ term, a split long exact sequence

$$\cdots \to K_n(R) \to K_n(R[\mathbb{Z}_2]) \oplus K_n(R[\mathbb{Z}_3]) \to H_n^{\Gamma}(E_{\mathrm{fin}}\Gamma; \mathbf{K}_R) \to K_{n-1}(R) \to \cdots .$$

Hence $$H_n^{\Gamma}(E_{\mathrm{fin}}\Gamma; \mathbf{K}_R) = (K_n(R[\mathbb{Z}_2]) \oplus K_n(R[\mathbb{Z}_3]))/K_n(R) \ .$$

Next, for a word hyperbolic group G,

$$H_n^G(E_{\mathrm{vc}}G, E_{\mathrm{fin}}G; \mathbf{K}) \cong \bigoplus_{[V]\in\mathcal{M}(G)} H_n^V(E_{\mathrm{vc}}V, E_{\mathrm{fin}}V; \mathbf{K}) \ ,$$

where $\mathcal{M}(G)$ is the set of conjugacy classes of maximal virtually cyclic subgroups of G (see Lück 16, Theorem 8.11 and Juan-Pineda and Leary [10]). The geometric interpretation of this result is that $E_{\mathrm{vc}}G$ is obtained by coning off each geodesic in the tree $E_{\mathrm{fin}}G$; then apply excision.

The Kurosh subgroup theorem implies that a subgroup of $\mathbb{Z}_2 * \mathbb{Z}_3$ is a free product of $\mathbb{Z}_2$'s, $\mathbb{Z}_3$'s, and $\mathbb{Z}$'s. Note that $\mathbb{Z}_2 * \mathbb{Z}_3 = \langle a, b \mid a^2 = 1 = b^3 \rangle$, $\mathbb{Z}_3 * \mathbb{Z}_3 = \langle c, d \mid c^3 = 1 = d^3 \rangle$, and $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 = \langle e, f, g \mid e^2 = f^2 = g^2 = 1 \rangle$ have free subgroups of rank 2, for example $\langle ab, ab^2 \rangle$, $\langle cd, cd^2 \rangle$, and $\langle ef, fg \rangle$. On the other hand, the free group F_2 rank 2 is not a virtually cyclic group since its first Betti number $\beta_1(F_2) = \operatorname{rank} H_1(F_2) = 2$, while for a virtually cyclic group V, transferring to the cyclic subgroup $C \subset V$ of finite index shows that $\beta_1(V)$ is 0 or 1. Subgroups of virtually cyclic groups are also virtually cyclic. Therefore all virtually cyclic subgroups of Γ are cyclic or infinite dihedral.

By the fundamental theorem of K–theory and Waldhausen's Theorem 3.32,

$$\begin{aligned} H_n^{\mathbb{Z}}(E_{\mathrm{vc}}\mathbb{Z}, E_{\mathrm{fin}}\mathbb{Z}; \mathbf{K}_R) &= \widetilde{\mathrm{Nil}}_{n-1}(R) \oplus \widetilde{\mathrm{Nil}}_{n-1}(R) \\ H_n^{D_\infty}(E_{\mathrm{vc}}D_\infty, E_{\mathrm{fin}}D_\infty; \mathbf{K}_R) &= \widetilde{\mathrm{Nil}}_{n-1}(R; R, R) \ . \end{aligned}$$

Finally, by Corollary 3.27(1), we obtain exactly one type of Nil-group:

$$\widetilde{\mathrm{Nil}}_{n-1}(R; R, R) \cong \widetilde{\mathrm{Nil}}_{n-1}(R) \ . \qquad \square$$

Remark 3.36 The sets $\mathcal{M}_C$ and $\mathcal{M}_D$ are countably infinite. This can be shown by parameterizing these subsets either: combinatorially (using that elements in Γ are words in a, b, b^2), geometrically (maximal virtually cyclic subgroups correspond to stabilizers of geodesics in the Bass–Serre tree $E_{\mathrm{fin}}\Gamma$, where the geodesic may or may not be invariant under an element of order 2), or number theoretically (using solutions to Pell's equation and Gauss' theory of binary quadratic forms [20]).

Let us give an overview and history of some related work. The Farrell–Jones Conjecture and the classification of virtually cyclic groups 3.31 focused attention on the algebraic K–theory of groups mapping to the infinite dihedral group. Several years ago James Davis and Bogdan Vajiac outlined a unpublished proof of Theorem 0.1 when $n \leqslant 0$ using controlled topology and hyperbolic geometry. Lafont and Ortiz [13] proved that $\widetilde{\mathrm{Nil}}_n(\mathbb{Z}[F]; \mathbb{Z}[V_1 - F], \mathbb{Z}[V_2 - F]) = 0$ if and only if $\widetilde{\mathrm{Nil}}_n(\mathbb{Z}[F], \alpha) = 0$ for any virtually cyclic group V with an epimorphism $V \to D_\infty$ and $n = 0, 1$. More recently, Lafont and Ortiz [15] have studied the more general case of the K–theory $K_n(R[G_1 *_F G_2])$ of an injective amalgam, where F, G_1, G_2 are finite groups. Finally, we mentioned the paper [5], which was written in parallel with this one; it an alternate proof of Theorem 0.1. Also, [5] provides several auxiliary results used in Section 3.3 of this paper. The Nil-Nil isomorphism of Theorem 0.1 has been used in a geometrically motivated computation of Lafont and Ortiz [14, Section 6.4].

4 Codimension 1 splitting and semisplitting

We shall now give a topological interpretation of the Nil-Nil Theorem 1.1, proving in Theorem 4.5 that every homotopy equivalence of finite CW–complexes $f\colon M \to X = X_1 \cup_Y X_2$ with X_1, X_2, Y connected and $\pi_1(Y) \to \pi_1(X)$ injective is "semisplit" along $Y \subset X$, assuming that $\pi_1(Y)$ is of finite index in $\pi_1(X_2)$. Indeed, the proof of Theorem 1.1 is motivated by the codimension 1 splitting obstruction theory of Waldhausen [24], and the subsequent algebraic K–theory decomposition theorems of Waldhausen [25; 26]. The papers [24; 25] developed both an algebraic splitting obstruction theory for chain complexes over injective generalized free products, and a geometric codimension 1 splitting obstruction theory; the geometric splitting obstruction is the algebraic splitting obstruction of the cellular chain complex. There are parallel theories for the separating type (A) (amalgamated free product) and the nonseparating type (B) (HNN extension). We first briefly outline the theory, mainly for type (A).

The cellular chain complex of the universal cover $\widetilde{X}$ of a connected CW–complex X is a based free $\mathbb{Z}[\pi_1(X)]$–module chain complex $C(\widetilde{X})$ such that $H_*(\widetilde{X}) = H_*(C(\widetilde{X}))$.

The *kernel* $\mathbb{Z}[\pi_1(X)]$*–modules* of a map $f\colon M \to X$ are defined by

$$K_*(M) := H_{*+1}(\widetilde{f}\colon \widetilde{M} \to \widetilde{X})$$

with $\widetilde{M} := f^*\widetilde{X}$ the pullback cover of M and $\widetilde{f}\colon \widetilde{M} \to \widetilde{X}$ a $\pi_1(X)$–equivariant lift of f. For a cellular map f of CW–complexes let

$$\mathcal{K}(M) := \mathcal{C}(\widetilde{f}\colon C(\widetilde{M}) \to C(\widetilde{X}))_{*+1}$$

be the algebraic mapping cone of the induced $\mathbb{Z}[\pi_1(X)]$–module chain map $\widetilde{f}$, with homology $\mathbb{Z}[\pi_1(X)]$–modules

$$H_*(\mathcal{K}(M)) = K_*(M) = H_{*+1}(\widetilde{f}\colon \widetilde{M} \to \widetilde{X}) .$$

For $n \geqslant 1$ the map $f\colon M \to X$ is n–connected if and only if $f_*\colon \pi_1(M) \cong \pi_1(X)$ and $K_r(M) = 0$ for $r < n$, in which case the Hurewicz map is an isomorphism:

$$\pi_{n+1}(f) = \pi_{n+1}(\widetilde{f}) \to K_n(M) = H_{n+1}(\widetilde{f}) .$$

By the theorem of J H C Whitehead, $f\colon M \to X$ is a homotopy equivalence if and only if $f_*\colon \pi_1(M) \cong \pi_1(X)$ and $K_*(M) = 0$ (if and only if $\mathcal{K}(M)$ is chain contractible).

Decompose the boundary of the $(n+1)$–disk as a union of upper and lower n–disks:

$$\partial D^{n+1} = S^n = D^n_+ \cup_{S^{n-1}} D^n_- .$$

Given a CW–complex M and a cellular map $\phi\colon D^n_+ \to M$ define a new CW–complex

$$M' = (M \cup_{\partial\phi} D^n_-) \cup_{\phi\cup 1} D^{n+1}$$

by attaching an n–cell and an $(n+1)$–cell, with

$$\partial\phi = \phi|\colon S^{n-1} \to M \ , \quad \phi \cup 1\colon S^n = D^n_+ \cup_{S^{n-1}} D^n_- \to M \cup_{\partial\phi} D^n_- .$$

The inclusion $M \subset M'$ is a homotopy equivalence called an *elementary expansion*. The cellular based free $\mathbb{Z}[\pi_1(M)]$–module chain complexes fit into a short exact sequence

$$0 \to C(\widetilde{M}) \to C(\widetilde{M}') \to C(\widetilde{M}', \widetilde{M}) \to 0$$

with $C(\widetilde{M}', \widetilde{M})\colon \cdots \longrightarrow 0 \longrightarrow \mathbb{Z}[\pi_1(M)] \xrightarrow{\;1\;} \mathbb{Z}[\pi_1(M)] \longrightarrow 0 \longrightarrow \cdots$

concentrated in dimensions $n, n+1$. For a commutative diagram of cellular maps

$$\begin{array}{ccc} D^n_+ & \xrightarrow{\;\phi\;} & M \\ \big\downarrow & & \big\downarrow{\scriptstyle f} \\ D^{n+1} & \xrightarrow{\;\delta\phi\;} & X \ , \end{array}$$

the map f extends to a cellular map

$$f' = (f \cup \delta\phi|_{D^n_-}) \cup \delta\phi\colon M' = (M \cup_{\partial\phi} D^n_-) \cup_{\phi\cup 1} D^{n+1} \to X$$

which is also called an *elementary expansion*, and there is defined a short exact sequence of based free $\mathbb{Z}[\pi_1(X)]$–module chain complexes

$$0 \to \mathcal{K}(M) \to \mathcal{K}(M') \to C(\widetilde{M}', \widetilde{M}) \to 0 .$$

Recall the *Whitehead group* of a group G is defined by

$$\mathrm{Wh}(G) := K_1(\mathbb{Z}[G])/\{\pm g \mid g \in G\} .$$

Suppose the CW–complexes M, M', X are finite. The *Whitehead torsion* of a homotopy equivalence $f\colon M \to X$ is

$$\tau(f) = \tau(\mathcal{K}(M)) \in \mathrm{Wh}(\pi_1(X)) .$$

Homotopy equivalences $f\colon M \to X$ and $f'\colon M' \to X$ are *simple-homotopic* if

$$\tau(f) = \tau(f') \in \mathrm{Wh}(\pi_1(X)) .$$

This is equivalent to being able to obtain f' from f by a finite sequence of elementary expansions and subdivisions and their formal inverses. For details, see Cohen's book [3].

A 2*–sided codimension* 1 *pair* $(X, Y \subset X)$ is a pair of spaces such that the inclusion $Y = Y \times \{0\} \subset X$ extends to an open embedding $Y \times \mathbb{R} \subset X$. We say that a homotopy equivalence $f\colon M \to X$ *splits along* $Y \subset X$ if the restrictions $f|\colon N = f^{-1}(Y) \to Y$, $f|\colon M - N \to X - Y$ are also homotopy equivalences.

In dealing with maps $f\colon M \to X$ and 2–sided codimension 1 pairs (X, Y) we shall assume that f is cellular and that both (X, Y) and $(M, N = f^{-1}(Y))$ are a 2–sided codimension 1 CW–pair.

A 2–sided codimension 1 CW–pair (X, Y) is π_1*–injective* if X, Y are connected and $\pi_1(Y) \to \pi_1(X)$ is injective. As usual, there are two cases, according as to whether Y separates X or not:

(A) The *separating* case: $X - Y$ is disconnected, so

$$X = X_1 \cup_Y X_2$$

with X_1, X_2 connected. By the Seifert–van Kampen theorem

$$\pi_1(X) = \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2)$$

is the amalgamated free product determined by the injections $i_k\colon \pi_1(Y) \to \pi_1(X_k)$ $(k = 1, 2)$.

(B) The *nonseparating* case: $X - Y$ is connected, so

$$X = X_1/\{y \sim ty \mid y \in Y\}$$

for a connected space X_1 (a deformation retract of $X - Y$) which contains two disjoint copies $Y \sqcup tY \subset X_1$ of Y. By the Seifert–van Kampen theorem,

$$\pi_1(X) = \pi_1(X_1) *_{i_1,i_2} \{t\}$$

is the HNN extension determined by the injections $i_1, i_2\colon \pi_1(Y) \to \pi_1(X_1)$, with $i_1(y)t = ti_2(y)$ $(y \in \pi_1(Y))$.

Remark 4.1 Let $\widetilde{X}$ be the universal cover of X, and let $\overline{X} := \widetilde{X}/\pi_1(Y)$, so that for both types (A) and (B), $(\overline{X}, Y)$ is a π_1–injective 2–sided codimension 1 pair of the separating type (A), with $\overline{X} = \overline{X}^- \cup_Y \overline{X}^+$ for connected subspaces $\overline{X}^-, \overline{X}^+ \subset \overline{X}$ such that

$$\pi_1(\overline{X}) = \pi_1(\overline{X}^-) = \pi_1(\overline{X}^+) = \pi_1(Y) .$$

Moreover for type (B), when i_1, i_2 are isomorphisms, the HNN extension simplifies to

$$1 \longrightarrow \pi_1(Y) \longrightarrow \pi_1(X) = \pi_1(Y) \rtimes_\alpha \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 1$$

with automorphism $\alpha = (i_1)^{-1} i_2$ of $\pi_1(Y)$, studied originally by Farrell and Hsiang [6].

From now on, we shall only consider the separating case (A) of $X = X_1 \cup_Y X_2$. Write

$$\pi_1(X) = G\ , \quad \pi_1(X_1) = G_1\ , \quad \pi_1(X_2) = G_2\ , \quad \pi_1(Y) = H\ ,$$
$$i_k\colon \mathbb{Z}[H] \to \mathbb{Z}[G_k] = \mathbb{Z}[H] \oplus \mathcal{B}_k\ , \quad \mathcal{B}_k = \mathbb{Z}[G_k - H]\ ,$$

with $\mathcal{B}_k$ free as both a right and a left $\mathbb{Z}[H]$–module, and

$$\mathbb{Z}[G] = \mathbb{Z}[G_1] *_{\mathbb{Z}[H]} \mathbb{Z}[G_2] = \mathbb{Z}[H] \oplus \mathcal{B}_1 \oplus \mathcal{B}_2 \oplus \mathcal{B}_1\mathcal{B}_2 \oplus \mathcal{B}_2\mathcal{B}_1 \oplus \cdots\ .$$

Use the injections $i_k\colon H \to G_k$ to define covers

$$\overline{X}_1 = \widetilde{X}_1/H \subset \overline{X}^-\ , \quad \overline{X}_2 = \widetilde{X}_2/H \subset \overline{X}^+$$

such that $\overline{X}_1 \cap \overline{X}_2 = Y$ and

$$\widetilde{X} = \Big(\bigcup_{g_1G_1 \in G/G_1} g_1\widetilde{X}_1\Big) \cup_{\left(\bigcup_{hH \in G/H} h\widetilde{Y}\right)} \Big(\bigcup_{g_2G_2 \in G/G_2} g_2\widetilde{X}_2\Big)$$
$$\overline{X} = \Big(\bigcup_{g_1G_1 \in G/G_1} g_1\overline{X}_1\Big) \cup_{\left(\bigcup_{hH \in G/H} hY\right)} \Big(\bigcup_{g_2G_2 \in G/G_2} g_2\overline{X}_2\Big)$$

with $\widetilde{X}_k$ the universal cover of X_k, and $\widetilde{Y}$ the universal cover of Y.

Let $(f,g)\colon (M,N)\to(X,Y)$ be a map of separating π_1–injective codimension 1 finite CW–pairs. This gives an exact sequence of based free $\mathbb{Z}[H]$–module chain complexes

$$(7)\quad 0 \longrightarrow \mathcal{K}(N) \longrightarrow \mathcal{K}(\overline{M}) \longrightarrow \mathcal{K}(\overline{M}^-,N)\oplus\mathcal{K}(\overline{M}^+,N) \longrightarrow 0$$

inducing a long exact sequence of homology modules

$$\cdots \longrightarrow K_r(N) \longrightarrow K_r(\overline{M}) \longrightarrow K_r(\overline{M}^+,N)\oplus K_r(\overline{M}^-,N) \longrightarrow K_{r-1}(N) \longrightarrow \cdots .$$

Note that $f\colon M\to X$ is a homotopy equivalence if and only if $f_*\colon \pi_1(M)\to\pi_1(X)$ is an isomorphism and $\mathcal{K}(\overline{M})$ is contractible. The map of pairs $(f,g)\colon (M,N)\to(X,Y)$ is a split homotopy equivalence if and only if any two of the chain complexes in (7) are contractible, in which case the third chain complex is also contractible.

Suppose $f\colon M\to X$ is a homotopy equivalence. Then

$$K_*(\overline{M}) = K_*(M) = 0\,,\quad K_*(N) = K_{*+1}(\overline{M}^-,N)\oplus K_{*+1}(\overline{M}^+,N)\,.$$

We obtain an exact sequence of $\mathbb{Z}[H]$–module chain complexes

$$0\to\mathcal{K}(\overline{M}_1,N)\to\mathcal{K}(\overline{M}^-,N)\xrightarrow{\rho_1}\mathcal{K}(\overline{M}^-,\overline{M}_1)=\mathcal{B}_1\otimes_{\mathbb{Z}[H]}\mathcal{K}(\overline{M}^+,N)\to 0$$
$$0\to\mathcal{K}(\overline{M}_2,N)\to\mathcal{K}(\overline{M}^+,N)\xrightarrow{\rho_2}\mathcal{K}(\overline{M}^+,\overline{M}_2)=\mathcal{B}_2\otimes_{\mathbb{Z}[H]}\mathcal{K}(\overline{M}^-,N)\to 0\,.$$

The pair (ρ_1,ρ_2) of intertwined chain maps is *chain homotopy nilpotent*, in the sense that the following chain map is a $\mathbb{Z}[G]$–module chain equivalence:

$$\begin{pmatrix}1&\rho_2\\ \rho_1&1\end{pmatrix}\colon \mathbb{Z}[G]\otimes_{\mathbb{Z}[H]}(\mathcal{K}(\overline{M}^-,N)\oplus\mathcal{K}(\overline{M}^+,N)) \longrightarrow \mathbb{Z}[G]\otimes_{\mathbb{Z}[H]}(\mathcal{K}(\overline{M}^-,N)\oplus\mathcal{K}(\overline{M}^+,N))\,.$$

Definition 4.2 Let $x=(P_1,P_2,\rho_1,\rho_2)$ be an object of $\mathrm{NIL}(\mathbb{Z}[H];\mathcal{B}_1,\mathcal{B}_2)$.

(1) Let $x'=(P_1',P_2',\rho_1',\rho_2')$ be another object. We say x and x' are *equivalent* if

$$[P_1]=[P_1']\,,\quad [P_2]=[P_2']\in\widetilde{K}_0(\mathbb{Z}[H])\,,\quad [x]=[x']\in\widetilde{\mathrm{Nil}}_0(\mathbb{Z}[H];\mathcal{B}_1,\mathcal{B}_2)\,,$$

or equivalently,

$$[x']-[x]\in K_0(\mathbb{Z})\oplus K_0(\mathbb{Z})\subseteq \mathrm{Nil}_0(\mathbb{Z}[H];\mathcal{B}_1,\mathcal{B}_2) = K_0(\mathbb{Z}[H])\oplus K_0(\mathbb{Z}[H])\oplus\widetilde{\mathrm{Nil}}_0(\mathbb{Z}[H];\mathcal{B}_1,\mathcal{B}_2)$$

with $K_0(\mathbb{Z})\oplus K_0(\mathbb{Z})$ the subgroup generated by $(\mathbb{Z}[H],0,0)$ and $(0,\mathbb{Z}[H],0,0)$.

(2) Let $k = 1$ or 2. Let $y_k \in \ker(\rho_k)$ generate a direct summand $\langle y_k \rangle \subseteq P_k$. Define an object x' in $\mathrm{NIL}(\mathbb{Z}[H]; \mathcal{B}_1, \mathcal{B}_2)$ by

$$x' = (P_1', P_2', \rho_1', \rho_2') = \begin{cases} (P_1/\langle y_1 \rangle, P_2, [\rho_1], [\rho_2]) & \text{if } k = 1, \\ (P_1, P_2/\langle y_2 \rangle, [\rho_1], [\rho_2]) & \text{if } k = 2, \end{cases}$$

with an exact sequence in $\mathrm{NIL}(\mathbb{Z}[H]; \mathcal{B}_1, \mathcal{B}_2)$

$$\begin{cases} 0 \longrightarrow (\mathbb{Z}[H], 0, 0, 0) \xrightarrow{(y_1, 0)} x \longrightarrow x' \longrightarrow 0 \\ 0 \longrightarrow (0, \mathbb{Z}[H], 0, 0) \xrightarrow{(0, y_2)} x \longrightarrow x' \longrightarrow 0 \ . \end{cases}$$

Thus x' is equivalent to x, obtained by the *algebraic cell-exchange* which *kills* $y_k \in P_1 \oplus P_2$.

It can be shown that two objects x and x' in $\mathrm{NIL}(\mathbb{Z}[H]; \mathcal{B}_1, \mathcal{B}_2)$ are equivalent if and only if x' can be obtained from x by a finite sequence of isomorphisms, algebraic cell-exchanges, and their formal inverses.

Geometric cell-exchanges (called *surgeries* in [24]) determine algebraic cell-exchanges. In the highly connected case, algebraic and geometric cell-exchanges occur in tandem:

Theorem 4.3 [24] *Let $(f, g)\colon (M, N) \to (X, Y)$ be a map of separating π_1–injective codimension 1 finite CW–pairs, with $f\colon M \to X$ a homotopy equivalence. Write $X = X_1 \cup_Y X_2$ with induced amalgam $\pi_1(X) = G = G_1 *_H G_2$ of fundamental groups.*

(i) *Let $k = 1, 2$. Suppose for some $n \geqslant 0$ that we are given a map*

$$(\phi, \partial\phi)\colon (D^{n+1}, S^n) \longrightarrow (M_k, N)$$

and a null-homotopy of pairs

$$(\theta, \partial\theta)\colon (f|_{M_k} \circ \phi, g \circ \partial\phi) \simeq (*, *)\colon (D^{n+1}, S^n) \longrightarrow (X_k, Y)\ .$$

Assume they represent an element in $\ker(\rho_k)$ (with $\epsilon = -$ if $k = 1$; $\epsilon = +$ if $k = 2$):

$$\begin{aligned} y_k = [\phi, \theta] &\in \mathrm{im}(K_{n+1}(\overline{M}_k, N) \to K_{n+1}(\overline{M}^{\epsilon}, N)) \\ &= \ker(\rho_k\colon K_{n+1}(\overline{M}^{\epsilon}, N) \to \mathcal{B}_k \otimes_{\mathbb{Z}[H]} K_{n+1}(\overline{M}^{-\epsilon}, N)) \subseteq K_n(N)\ . \end{aligned}$$

The map (f, g) extends to the map of codimension 1 pairs

$$\begin{aligned} (f', g') &:= ((f \cup f|_{M_k} \circ \phi) \cup \theta, g \cup \partial\theta)\colon \\ &(M', N') := ((M \cup_{\partial\phi} D^{n+1}) \cup_{\phi \cup 1} D^{n+2}, N \cup_{\partial\phi} D^{n+1}) \longrightarrow (X, Y) \end{aligned}$$

where the new $(n+2)$*–cell has attaching map*

$$\phi \cup 1\colon \partial D^{n+2} = D^{n+1} \cup_{S^n} D^{n+1} \longrightarrow M \cup_{\partial\phi} D^{n+1} .$$

The homological effect on (f,g) *of this* geometric cell-exchange *is no change in*

$$\begin{cases} K_r(\overline{M}'^{\epsilon}, N') = K_r(M^{\epsilon}, N) & \text{for } r \neq n+1, n+2 , \\ K_r(\overline{M}'^{-\epsilon}, N') = K_r(M^{-\epsilon}, N) & \text{for all } r \in \mathbb{Z}, \end{cases}$$

except there is a five-term exact sequence

$$\begin{aligned} 0 \longrightarrow\ & K_{n+2}(\overline{M}^{\epsilon}, N) \longrightarrow K_{n+2}(\overline{M}'^{\epsilon}, N') \longrightarrow \mathbb{Z}[H] \\ \xrightarrow{y_k}\ & K_{n+1}(\overline{M}^{\epsilon}, N) \longrightarrow K_{n+1}(\overline{M}'^{\epsilon}, N') \longrightarrow 0 . \end{aligned}$$

The inclusion $h\colon M \subset M'$ *is a simple homotopy equivalence with* $(f,g) \simeq (f'h, g'h|_N)$.

(ii) *Suppose for some* $n \geqslant 2$ *that* $K_r(N) = 0$ *for all* $r \neq n$. *Then*

$$K_r(\overline{M}^-, N) = 0 = K_r(\overline{M}^+, N) \quad \text{for all } r \neq n+1 ,$$

and $K_n(N)$ *is a stably finitely generated free* $\mathbb{Z}[H]$*–module. Moreover, we may define an object* x *in* $\mathrm{NIL}(\mathbb{Z}[H]; \mathcal{B}_1, \mathcal{B}_2)$ *by*

$$x := (K_{n+1}(\overline{M}^-, N), K_{n+1}(\overline{M}^+, N), \rho_1, \rho_2)$$

whose underlying modules satisfy

$$\begin{gathered} [K_{n+1}(\overline{M}^-, N)] + [K_{n+1}(\overline{M}^+, N)] = [K_n(N)] = 0 \in \widetilde{K}_0(\mathbb{Z}[H]) , \\ [\mathbb{Z}[G_k] \otimes_{\mathbb{Z}[H]} K_{n+1}(\overline{M}^{\epsilon}, N)] = 0 \in \widetilde{K}_0(\mathbb{Z}[G_k]) \quad (k = 1, 2) . \end{gathered}$$

If $(f', g')\colon (M', N') \to (X, Y)$ *is obtained from* (f,g) *by a geometric cell-exchange killing an element* $y_k \in K_{n+1}(\overline{M}^{\epsilon}, N)$ $((k,\epsilon) = (1,-)$ *or* $(2,+))$ *which generates a direct summand* $\langle y_k \rangle \subseteq K_{n+1}(\overline{M}^{\epsilon}, N)$, *then the corresponding object in* $\mathrm{NIL}(\mathbb{Z}[H]; \mathcal{B}_1, \mathcal{B}_2)$

$$x' := (K_{n+1}(\overline{M}'^-, N'), K_{n+1}(\overline{M}'^+, N'), \rho_1', \rho_2')$$

is obtained from x *by an algebraic cell-exchange. Since* $\pi_{n+1}(\overline{M}_k, N) = K_{n+1}(\overline{M}_k, N)$ *by the relative Hurewicz theorem, there is a one-one correspondence between algebraic and geometric cell-exchanges killing elements* y_k *generating direct summands* $\langle y_k \rangle$.

(iii) *For any* $n \geqslant 2$ *it is possible to modify the given* (f,g) *by a finite sequence of geometric cell-exchanges and their formal inverses to obtain a pair (also denoted by*

(f,g)) such that $K_r(N)=0$ for all $r\neq n$ as in (ii), *and hence a canonical equivalence class of nilpotent objects $x=(P_1,P_2,\rho_1,\rho_2)$ in $\mathrm{NIL}(\mathbb{Z}[H];\mathcal{B}_1,\mathcal{B}_2)$ such that*

$$[P_1]+[P_2]=0\in \widetilde{K}_0(\mathbb{Z}[H])\ ,\quad [\mathbb{Z}[G_k]\otimes_{\mathbb{Z}[H]}P_k]=0\in\widetilde{K}_0(\mathbb{Z}[G_k])$$

with $P_1:=K_{n+1}(\overline{M}^-,N)$, $P_2:=K_{n+1}(\overline{M}^+,N)$. Any x' in the equivalence class of x is realized by a map $(f',g')\colon (M',N')\to(X,Y)$ with f' simple-homotopic to f. The splitting obstruction of f is the image of the Whitehead torsion $\tau(f)\in\mathrm{Wh}(G)$, namely,

$$\partial(\tau(f))=([P_1],[x])=([P_1],[P_1,P_2,\rho_1,\rho_2])$$
$$\in\ker(\widetilde{K}_0(\mathbb{Z}[H])\to\widetilde{K}_0(\mathbb{Z}[G_1])\oplus\widetilde{K}_0(\mathbb{Z}[G_2]))\oplus\widetilde{\mathrm{Nil}}_0(\mathbb{Z}[H];\mathcal{B}_1,\mathcal{B}_2)\ .$$

Thus f is simply homotopic to a split homotopy equivalence if and only if $\partial(\tau(f))=0$, if and only if x is equivalent to 0.

(iv) *The Whitehead group of $G=G_1*_H G_2$ fits into an exact sequence*

$$\mathrm{Wh}(H)\longrightarrow \mathrm{Wh}(G_1)\oplus\mathrm{Wh}(G_2)\longrightarrow \mathrm{Wh}(G)$$
$$\xrightarrow{\ \partial\ }\widetilde{K}_0(\mathbb{Z}[H])\oplus\widetilde{\mathrm{Nil}}_0(\mathbb{Z}[H];\mathcal{B}_1,\mathcal{B}_2)\longrightarrow\widetilde{K}_0(\mathbb{Z}[G_1])\oplus\widetilde{K}_0(\mathbb{Z}[G_2])\ .$$

Furthermore, the homomorphism

$$\partial\colon \mathrm{Wh}(G)\longrightarrow\widetilde{K}_0(\mathbb{Z}[H])\oplus\widetilde{\mathrm{Nil}}_0(\mathbb{Z}[H];\mathcal{B}_1,\mathcal{B}_2)\ ,\quad \tau(f)\longmapsto([P_1],[P_1,P_2,\rho_1,\rho_2])$$

satisfies that $\mathrm{proj}_2\circ\partial\colon \mathrm{Wh}(G)\to\widetilde{\mathrm{Nil}}_0(\mathbb{Z}[H];\mathcal{B}_1,\mathcal{B}_2)$ *is an epimorphism split by*

$$\iota\colon \widetilde{\mathrm{Nil}}_0(\mathbb{Z}[H];\mathcal{B}_1,\mathcal{B}_2)\longrightarrow\mathrm{Wh}(G)\ ,\quad [P_1,P_2,\rho_1,\rho_2]\longmapsto\begin{bmatrix}1&\rho_2\\ \rho_1&1\end{bmatrix}\ .$$

Definition 4.4 Let (X,Y) be a separating π_1–injective codimension 1 finite CW–pair. A homotopy equivalence $f\colon M\to X$ from a finite CW–complex M is *semisplit along* $Y\subset X$ if f is simple homotopic to a map (also denoted by f) such that for the corresponding map of pairs $(f,g)\colon (M,N)\to(X,Y)$ the relative homology kernel $\mathbb{Z}[H]$–modules

$$K_*(\overline{M}_2,N)=H_{*+1}((\widetilde{M}_2,\widetilde{N})\to(\widetilde{X}_2,\widetilde{Y}))$$

vanish, which is equivalent to the induced $\mathbb{Z}[H]$–module morphisms

$$\rho_2\colon K_*(\overline{M}^+,N)\longrightarrow K_*(\overline{M}^+,\overline{M}_2)=\mathbb{Z}[G_2-H]\otimes_{\mathbb{Z}[H]}K_*(\overline{M}^-,N)\ ,$$

being isomorphisms. Equivalently, f is semisplit along Y if there is a semisplit object $x=(P_1,P_2,\rho_1,\rho_2)$ in the canonical equivalence class of Theorem 4.3, that is, with $\rho_2\colon P_2\to\mathcal{B}_1P_1$ a $\mathbb{Z}[H]$–module isomorphism.

In particular, a split homotopy equivalence f of separating pairs is semisplit.

Theorem 4.5 *Let (X, Y) be a separating π_1–injective codimension 1 finite CW–pair, with $X = X_1 \cup_Y X_2$. Suppose that $H = \pi_1(Y)$ is a finite-index subgroup of $G_2 = \pi_1(X_2)$. Every homotopy equivalence $f\colon M \to X$ with M a finite CW–complex is simple-homotopic to a homotopy equivalence which is semisplit along Y.*

Proof Let $x = (P_1, P_2, \rho_1, \rho_2)$ represent the canonical equivalence class of objects in $\mathrm{NIL}(\mathbb{Z}[H]; \mathcal{B}_1, \mathcal{B}_2)$ associated to f in Theorem 4.3(ii). Since H is of finite index in G_2, as in the proof of Theorem 1.1, we can define a semisplit object

$$x'' := (P_1, \mathcal{B}_2 P_1, \rho_2 \circ \rho_1, 1)$$

satisfying

$$[x''] - [x] = [0, \mathcal{B}_2 P_1, 0, 0] - [0, P_2, 0, 0] \in \mathrm{Nil}_0(\mathbb{Z}[H]; \mathcal{B}_1, \mathcal{B}_2)\ .$$

By Theorem 4.3(iii), the direct sum

$$\mathcal{B}_2 P_1 \oplus P_1 = (\mathbb{Z}[G_2 - H] \otimes_{\mathbb{Z}[H]} P_1) \oplus P_1 = \mathbb{Z}[G_2] \otimes_{\mathbb{Z}[H]} P_1$$

is a stably finitely generated free $\mathbb{Z}[G_2]$–module. Since $\mathbb{Z}[G_2]$ is a finitely generated free $\mathbb{Z}[H]$–module, $\mathcal{B}_2 P_1 \oplus P_1$ is a stably finitely generated free $\mathbb{Z}[H]$–module. So

$$[\mathcal{B}_2 P_1] - [P_2] = [\mathcal{B}_2 P_1] + [P_1] = [\mathbb{Z}[G_2] \otimes_{\mathbb{Z}[H]} P_1] = 0 \in \widetilde{K}_0(\mathbb{Z}[H])\ .$$

So x is equivalent to x''. Thus, by Theorem 4.3(iii), there is a homotopy equivalence $f''\colon M'' \to X$ simple-homotopic to f realizing x''; note it is semisplit. □

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